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Gribov ambiguity and degenerate systems

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The relation between Gribov ambiguity and degeneracies in the symplectic structure of physical systems is analyzed. It is shown that, in finite-dimensional systems, the presence of Gribov ambiguities in regular constrained systems (those where the constraints are functionally independent) always leads to a degenerate symplectic structure upon Dirac reduction. The implications for the Gribov-Zwanziger approach to QCD are discussed.

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I. INTRODUCTION

A. Gribov problem and the Zwanzinger restriction

In his seminal paper, Gribov showed that a standard gauge condition, such as the Coulomb or the Landau choices, fails to provide *proper gauge fixings*[1] in Yang-Mills theories [3]. This so-called Gribov problem, that affects non-Abelian gauge theories, means that a generic gauge fixing intersects the same gauge orbit more than once (Gribov copies) and may fail to intersect others. Algebraic gauge conditions free of Gribov ambiguities are possible, but those choices are affected by severe technical problems as, for instance, incompatibility with the boundary conditions that must be imposed on the gauge fields in order to properly define the configuration space for the theory [4]. Additionally, Singer [5] showed that Gribov ambiguities occur for all gauge-fixing conditions involving derivatives (see also Ref. [6]), and moreover, the presence of the Gribov problem breaks BRST symmetry at a nonperturbative level [7].[2]

The Gribov problem occurs because it is generically impossible to ensure positive definiteness of the Faddeev-Popov (FP) determinant everywhere in functional space. The configurations for which the FP operator develops a nontrivial zero mode are those where the gauge condition becomes "tangent" to the gauge orbits and it therefore fails to intersect them. The Gribov horizon (GH), where this happens, marks the boundary beyond which the gauge condition intersects the gauge orbits more than once (Gribov copies). The appearance of Gribov copies invalidates the usual approach to the path integral, and one way to avoid overcounting is to restrict the sum over field configurations to the so-called Gribov region around $A_\mu = 0$, where the FP operator is positive definite (see, in particular, Refs. [3,10–14]).

In the case of flat, topologically trivial space-time, the restriction to the Gribov region coincides with the usual perturbation theory around $A_\mu = 0$ (with respect to a suitable functional norm [14]). The restriction to the first Gribov region takes into account the infrared effects related to the *partial* elimination of the Gribov copies, in the sense that it only guarantees the exclusion of those copies obtained by ("small") gauge transformations perturbatively connected to $A_\mu = 0$ [11,15,16]. It has been shown that nonperturbatively accessible gauge copies may exist within the Gribov region as well if the space-time is not flat or is topologically nontrivial [17–19]. The complete elimination of the gauge copies if the space is not flat or its topology is nontrivial can be a very difficult issue, and here we do not consider those possibilities, restricting the path integral to the first Gribov region.

Remarkably enough, the partial elimination of Gribov copies in perturbation theory is related to the nonperturbative infrared physics [3]. The nonperturbative input in the modified path integral is the restriction to the Gribov region. When one takes into account the presence of suitable condensates [20–24], the agreement with lattice data is excellent [25,26]. Moreover, within this approach, it has been possible to solve an old issue on the Casimir energy and force for the Yang-Mills field in the MIT bag model [27].

A common criticism to the Gribov-Zwanziger approach that restricts the functional space to the Gribov region is that it goes against Feynman's postulate of summing over

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[1]A gauge fixing is called proper if it intersects all gauge orbits only once [1,2].
[2]See also Refs. [8,9].

all histories. There are various arguments that answer this criticism. First of all, the configurations outside the Gribov region are copies of some configuration inside the Gribov region [11,12]. Therefore, the Gribov restriction avoids overcounting, and no relevant physical configurations are lost. Second and most importantly, this framework considerably improves the analytic results of standard perturbation theory like the glueball spectrum, which closely reproduces the lattice results [24].

Hence, it is natural to look for examples in a context simpler than Yang-Mills theory, in which the issue of the Gribov restriction can be directly analyzed. Here we show in some toy models with a finite number of degrees of freedom that it may not be necessary to impose the Gribov restriction "by hand," but it arises naturally from the dynamics of the system.

B. Gribov problem and dynamic degeneracy

In Dirac's formalism for constrained systems [1,2], gauge-invariant mechanical systems are characterized by the presence of first-class constraints $\phi_i \approx 0$, $i = 1, \ldots, n$. Gauge fixing in those systems is achieved by the introduction of n gauge conditions $G_i \approx 0$, so that the $2n$ constraints $\{G_i, \phi_j\}$ become a second-class set. In this context, the Gribov problem is the statement that the second-class nature of this set cannot hold globally: the Dirac matrix defined by their Poisson brackets is not invertible everywhere in phase space; it is *degenerate*.

Degenerate Hamiltonian systems on the other hand, are those whose symplectic form is not invertible in a subset Σ of phase space Γ [28]. In classical degenerate systems, the evolution takes place over nonoverlapping causally disconnected subregions of the phase space separated by degenerate surfaces Σ. This means that if a system is prepared in one subregion, it never evolves to a state in a different subregion. This still holds in the quantum domain for some simple degenerate systems [29]. Degenerate systems are ubiquitous in many areas of physics, from fluid dynamics [30] to gravity theories in higher dimensions [31,32], in the strong electromagnetic fields of quasars [33], and in systems such as massive bigravity theory [34], which has been shown to possess degenerate sectors where the degrees of freedom change from one region of phase space to another [35].

Here we will show that Gribov ambiguity and the existence of degeneracies are related problems, and that the GH can be identified as a surface of degeneracy Σ. This means that the system would be naturally confined to a region surrounded by a horizon, exactly as proposed by Zwanziger [10]. This interpretation of the GH, as a surface of degeneracy that acts as a boundary beyond which the evolution cannot reach, makes the restriction in the sum over histories a natural prescription and not an *ad hoc* one.

II. DEGENERATE SYSTEMS

We now briefly review classical [28] and quantum [29] degenerate systems. In order to fix ideas, let us consider a system described by the first-order action:

$$I[u] = \int \mathrm{d}t(X_A(u)\dot{u}^A - H(u)), \quad \text{with} \quad A = 1, \ldots, N. \tag{1}$$

This action can be interpreted in two not exactly equivalent ways: A) The u^A's are N generalized coordinates, and $L(u, \dot{u}) = X(u)_A \dot{u}^A - H(u)$ is the Lagrangian. B) The u^A's are noncanonical coordinates in an N-dimensional phase space Γ, where N is necessarily even, and Eq. (1) gives the action in Hamiltonian form.

In the first approach, for each u there is a canonically conjugate momentum in the $2N$-dimensional canonical phase space $\tilde{\Gamma}$ given by

$$p_A = \frac{\partial L}{\partial \dot{u}^A}. \tag{2}$$

In this case, this definition gives a set of primary constraints,

$$\Phi_A = p_A - X_A(u) \approx 0, \tag{3}$$

whose (canonical) Poisson brackets define the antisymmetric matrix

$$[\Phi_A, \Phi_B] = \partial_A X_B - \partial_B X_A \equiv \Omega_{AB}(u). \tag{4}$$

If Ω_{AB} is invertible—which requires N to be even—the constraints $\Phi_A \approx 0$ are second class, and $\Omega_{AB}(u)$ gives the Dirac bracket necessary to eliminate them.[3] Elimination of these second-class constraints in the $2N$-dimensional canonical phase space $\tilde{\Gamma} = \{u^A, p_A\}$ corresponds to choosing half of the u's as coordinates and the rest as momenta, and $\Omega_{AB}(u)$ will be identified as the (not necessarily canonical) presymplectic form in the reduced N-dimensional phase space Γ. In fact, in the Hamiltonian approach, the presymplectic form can be read from the equations of motion for the action in Eq. (1),

$$\Omega_{AB}(u)\dot{u}^A + E_A(u) = 0, \tag{5}$$

where

$$\Omega_{AB} \equiv \partial_A X_B(u) - \partial_B X_A(u) \quad \text{and} \quad E_A \equiv \partial_A H(u). \tag{6}$$

This reasoning shows that in the open sets where Ω_{AB} is invertible, the Lagrangian and Hamiltonian versions of this

[3]Since Ω_{AB} is a curl, it satisfies the identity $\partial_A \Omega_{BC} + \partial_B \Omega_{CA} + \partial_C \Omega_{AB} = 0$ (or $\Omega = dX \Rightarrow d\Omega \equiv 0$).

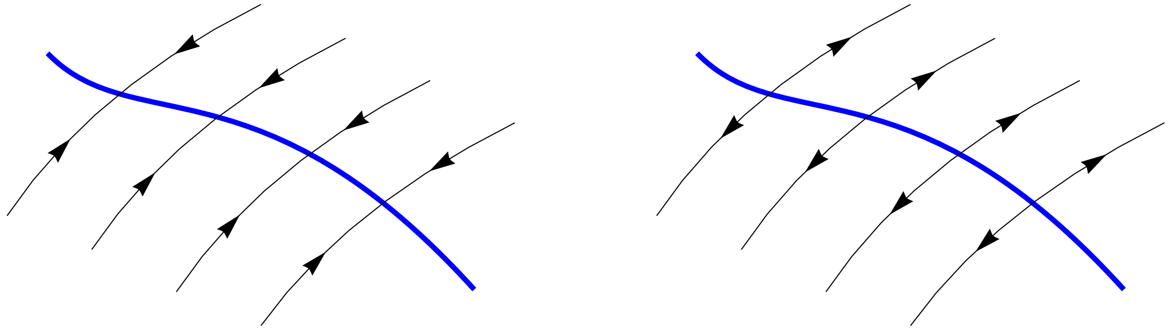

FIG. 1 (color online). Qualitative flow of the orbits near the degeneracy surface (thick line), which can act as a sink or as a source.

system are equivalent. In this case, the inverse symplectic form, Ω^{AB}, defines the Poisson bracket for the theory in (not necessarily canonical) coordinates:

$$\Omega^{AB} = [u^A, u^B]. \tag{7}$$

In what follows, we will refer to Γ as the phase space where u are the coordinates. The presymplectic form $\Omega_{AB}(u)$ is a function of the phase space coordinates u^A, and its determinant can vanish on some subset $\Sigma \subset \Gamma$ of measure 0. Degenerate systems are characterized by having a presymplectic form whose rank is not constant throughout phase space. Moreover, in its evolution a degenerate system can reach a degenerate surface Σ where $\det[\Omega_{AB}] = 0$ in a finite time,

$$\Sigma = \{u \in \Gamma | \Upsilon(u) = 0\}, \tag{8}$$

where $\Upsilon(u) = \epsilon^{A_1 A_2 \ldots A_N} \Omega_{A_1 A_2} \ldots \Omega_{A_{N-1} A_N}$ is the Pfaffian of Ω_{AB}, and $\det[\Omega_{AB}] = (\Upsilon)^2$.

Generically, a degenerate surface represents a codimension-1 submanifold in phase space and, as shown in Ref. [28], the classical evolution cannot take the system across Σ. The equations of motion [Eq. (5)] can be solved for $\dot{u}^A$ provided Ω_{AB} can be inverted. Moreover, the velocity diverges in the vicinity of Σ, and if $\Delta(u)|_\Sigma = 0$ is a simple zero, the velocity changes sign across Σ. Therefore, an initial state on one side of Σ could never reach the other: there is no causal connection between configurations on opposite sides of Σ. This degeneracy surface Σ acts as a source or sink of orbits, splitting the phase space into causally disconnected, nonoverlapping regions (see Fig. 1).

In the quantum case, the degeneracy of the symplectic form becomes the singular set of the Hamiltonian, and the corresponding Hilbert space $\mathcal{H}$ is endowed with a weighted scalar product,

$$\langle \psi_1, \psi_2 \rangle = \int \mathrm{d}V \psi_1^* w \psi_2, \tag{9}$$

where $\mathrm{d}V = \sqrt{g}\mathrm{d}^n u$ is the volume form, and the weight $w(u)$ is the Pfaffian $\Upsilon(u)$ of the symplectic form Ω_{AB}:

$$w(u) \coloneqq \sqrt{\det[\Omega_{AB}(u)]} = \Upsilon(u), \tag{10}$$

defined in order for the Hamiltonian to be symmetric and for the norm in $\mathcal{H}$ to be positive definite.

Since singular points must be excluded from the domain of the Hamiltonian operator, for consistency they should also be excluded from the domain of the wave functions. This means that the Hilbert space includes wave functions that can be discontinuous at the degenerate surfaces. Allowing discontinuous wave functions implies that the solutions can have support restricted to a single region bounded by Σ. Therefore, the Hilbert space is a direct sum of orthogonal subspaces of functions defined on each side of the degenerate surface and, in complete analogy with the classical picture, there is no quantum tunneling across Σ.

III. GAUGE FIXING AND GRIBOV AMBIGUITY

The quantum description of a gauge-invariant system can be achieved by first fixing the gauge and then applying the quantization prescription to the remaining classical degrees of freedom. Let Γ be a phase space described by generalized coordinates u^A $(A = 1, 2, \ldots, N)$, endowed with a symplectic form $\Omega_{AB}(u)$ everywhere invertible. Consider now an open patch of the phase space Γ where a system has local symmetries generated by a set of first-class constraints $\phi_i(u) \approx 0$, $(i = 1, \ldots, n < N/2)$. Following Dirac's procedure, for a system with n first-class constraints, an equal number of gauge-fixing conditions,

$$G_i(u) \approx 0, \qquad i = 1, \ldots, n, \tag{11}$$

must be included so that the whole set of constraints,

$$\{\gamma_I\} = \{G_i, \phi_j\}, \qquad I = 1, \ldots, 2n < N, \tag{12}$$

is second class (see Ref. [1]). In order to define a proper gauge fixing, two conditions must be fulfilled: every orbit must intersect the surface defined by the set $\{G_i\}$ in Γ (*accessibility*), and orbits cannot intersect the surface defined by $\{G_i\}$ more than once (*complete gauge fixation*) [2]. In other words, the surface in phase space defined by the gauge conditions in Eq. (11) must intersect every orbit once and only once.

The submanifold defined by setting the constraints $\{\gamma_I\}$ strongly equal to zero corresponds to the reduced gauge-fixed phase space of the system, which will be denoted by Γ_0:

$$\Gamma_0 \coloneqq \{u^A \in \Gamma | \gamma_I(u) = 0, I = 1, \ldots, 2n\}. \tag{13}$$

In Γ_0, a new Poisson structure is introduced by the Dirac bracket $[,]^*$:

$$[M,N]^* = [M,N] - [M,\gamma_I]C^{IJ}[\gamma_J,N], \tag{14}$$

where C^{IJ} is the inverse of the Dirac matrix constructed from the second-class constraints $\{\gamma_I\}$,

$$C_{IJ} = [\gamma_I,\gamma_J] = \Omega^{AB}\partial_A\gamma_I\partial_B\gamma_J. \tag{15}$$

The symplectic form for the gauge-fixed system in the reduced phase space defines the Dirac bracket [Eq. (14)]. Suppose now that the set of gauge conditions $\{G_i\}$ fails to fix completely the gauge in a region of phase space, leading to a Gribov ambiguity (see Fig. 2). This means that if a configuration u^A satisfies the gauge conditions $G_i(u) \approx 0$, there exists a gauge-transformed configuration $u^A + \delta u^A$ that also satisfies it, namely

$$\delta G_i(q,p) \approx \partial_A G_i \delta u^A = 0. \tag{16}$$

Since gauge transformations are generated by first class constraints,

$$\delta u^A = \epsilon^j [u^A, \phi_j] = \epsilon^j \Omega^{AB}\partial_B\phi_j, \tag{17}$$

where ϵ^j are infinitesimal parameters, the condition for the existence of Gribov copies [Eq. (16)] takes the form

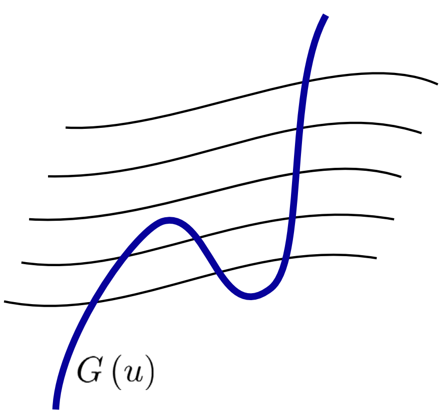

FIG. 2 (color online). The gauge condition $G_i(u) \approx 0$ (thick line) intersects the gauge orbits (thin lines) more than once, provided there exist points where the orbits run tangent to the gauge condition.

$$\epsilon^j\Omega^{AB}\partial_A G_i \partial_B\phi_j = \epsilon^j[G_i,\phi_j] = 0, \tag{18}$$

which has nontrivial solutions ($\epsilon^i \neq 0$) provided

$$\det[G_i,\phi_j] = 0.$$

The matrix $[G_i,\phi_j]$ corresponds to the FP operator in gauge field theory, whose definition is

$$\mathcal{M}_{ij} = [G_i,\phi_j] = \Omega^{AB}\partial_A G_i \partial_B \phi_j. \tag{19}$$

Gribov ambiguity occurs if the determinant of the FP operator $\mathcal{M}_{ij}$ vanishes. The Gribov copies continuously connected to a given configuration are related by the corresponding zero modes. The GH is defined to be the subset Ξ of phase space Γ where the FP determinant vanishes:

$$\Xi \coloneqq \{u^A \in \Gamma | \det[\mathcal{M}_{ij}] = 0\}. \tag{20}$$

Now, let us observe that the Dirac matrix [Eq. (15)] for the set of constraints $\{\gamma_I\}$ contains $\mathcal{M}_{ij}$ as a submatrix

$$C_{IJ} = [\gamma_I,\gamma_J] = \begin{pmatrix} \Omega^{AB}\partial_A G_i \partial_B G_j & \mathcal{M}_{ij} \\ -\mathcal{M}_{ij} & \Omega^{AB}\partial_A\phi_i\partial_B\phi_j \end{pmatrix}. \tag{21}$$

Hence, as the set $\{\phi_i\}$ is first class, the determinant of the Dirac matrix is given weakly by the square of the FP determinant,

$$\det[C_{IJ}] \approx (\det[\mathcal{M}_{ij}])^2. \tag{22}$$

In an open set where $\mathcal{M}_{ij}$ is invertible, the Dirac bracket in Eq. (14) can be safely defined. On the other hand, since at the GH $\det[\mathcal{M}_{ij}]$ vanishes, the determinant of the Dirac matrix vanishes as well, and the Dirac bracket becomes ill defined there. Moreover, in the next section, we will see that a Gribov ambiguity implies a degeneracy of the symplectic form for the gauge-fixed system at the GH.

IV. GRIBOV HORIZON AND DEGENERATE SURFACES

In general, the gauge generators $\phi_i \approx 0$, together with the gauge-fixing conditions $G_i \approx 0$ form a set of $2n$ second-class constraints. However, this is not globally true in the presence of a Gribov ambiguity, which can have nontrivial consequences in the symplectic structure of the reduced phase space. This can be seen when considering an open set where the Dirac matrix C_{IJ} is invertible, $C^{IJ}C_{JK} = \delta^I_K$. Setting the constraints strongly to zero defines the reduced gauge-fixed ("physical") phase space, which is generically a co-dimension-$2n$ surface Γ_0 embedded in phase space Γ.

Even though we started the analysis with a globally invertible symplectic form, implementing the gauge fixing changes the Poisson structure, and a new symplectic form for the reduced phase space must be found. In order to explicitly write the symplectic form in the reduced phase space, it is useful to take adapted coordinates $\{U^A\} = \{u^{*a}, v^I\}$, where $\{u^{*a}\}$ are "first-class" coordinates (in the sense that they have vanishing brackets with all second-class constraints; see Ref. [2])

$$u^{*a} = u^a - [u^a, \gamma_I]C^{IJ}\gamma_J \quad \text{with} \quad a = 1, \ldots, N-2n, \tag{23}$$

while $\{v^I\}$ is chosen as the set of second-class constraints [Eq. (12)]

$$v^I = \gamma_I \quad \text{with} \quad I = 1, \ldots, 2n. \tag{24}$$

Consequently, $\{u^{*a}\}$ and $\{v^I\}$ are canonically independent coordinates, i.e.,

$$[u^{*a}, v^I] = 0. \tag{25}$$

The conditions $v^I = 0$ define the reduced phase space, and the u^*'s fix the position within the reduced phase space Γ_0. The matrix of their Poisson brackets, given by

$$\hat{\Omega}^{AB} = [U^A, U^B] = \begin{pmatrix} \omega^{ab} & 0 \\ 0 & C_{IJ} \end{pmatrix}, \tag{26}$$

where

$$\omega^{ab} = [u^{*a}, u^{*b}] \approx [u^a, u^b]^*, \tag{27}$$

is the inverse of the symplectic form in the reduced phase space ω_{ab}.

The passage from the generic coordinates $\{u^A\}$ to the adapted ones, $\{U^A\} = \{u^{*a}, \gamma_I\}$, must be well defined. Then, the Jacobian for the transformation,

$$\mathcal{J}^A{}_B = \left(\frac{\partial U^A}{\partial u^B}\right) = \begin{pmatrix} \partial_B u^{*a} \\ \partial_B \gamma^I \end{pmatrix}, \tag{28}$$

is invertible. Assuming the original Poisson structure [Eq. (7)] to be well defined, i.e., $\det[\Omega^{AB}] = \Omega(u) \neq 0$, the new Poisson bracket in the adapted coordinates satisfies

$$\det[\hat{\Omega}^{AB}] = (\det[\mathcal{J}^A{}_B])^2\Omega. \tag{29}$$

Hence, we arrive at the following theorem.

Theorem: For a system with Gribov ambiguity, the symplectic form on the reduced phase space, ω_{ab}, necessarily degenerates at the Gribov horizon.

Proof: Since the coordinates U^A are globally well defined, the determinant of the Jacobian [Eq. (28)] is finite everywhere. In particular, it must approach a finite value $\mathcal{J}(\bar{u})$ on the GH,

$$\det[\mathcal{J}^A{}_B] \underset{u\to\bar{u}}{\longrightarrow} \mathcal{J}(\bar{u}) \neq 0, \tag{30}$$

where $\bar{u}$ stands for the values of the coordinates at the GH [Eq. (20)]. From Eq. (26), this means that

$$\det[\hat{\Omega}^{AB}] = \det[\omega^{ab}]\det[C_{IJ}] \underset{u\to\bar{u}}{\longrightarrow} \mathcal{J}(\bar{u})^2\Omega(\bar{u}). \tag{31}$$

On the other hand, from Eq. (22) we know that the determinant of the Dirac matrix vanishes at the GH, and therefore the determinant of the Poisson structure on the reduced phase space must be singular,

$$\det[\omega^{ab}] \underset{u\to\bar{u}}{\longrightarrow} \infty.$$

Consequently, the reduced phase space symplectic form necessarily degenerates at the GH,

$$\det[\omega_{ab}] \underset{u\to\bar{u}}{\longrightarrow} 0. \tag{32}$$

■

A well-defined Poisson structure ω^{ab} at the GH ($\det[\omega^{ab}(\bar{u})]$ finite) requires $\det[\hat{\Omega}^{AB}] \underset{u\to\bar{u}}{\longrightarrow} 0$, and consequently, the coordinates $\{U^A\}$ should be ill defined there. This might happen if the constraints [Eq. (12)] are not functionally independent at the GH—that is, if the constraints fail to be *regular*. If this problem is not produced by an erroneous choice of gauge fixing, it can only be due to an irregularity in the first-class constraints at the GH. Irregularity in dynamical systems is an independent problem from degeneracy and requires special handling to define the system in a consistent manner [36]. An example of a system with Gribov ambiguity where the reduced symplectic form is nondegenerate due to irregularities will be analyzed in Sec. VI.

The importance of this result is that when the global coordinates are well defined, the induced symplectic form of the gauge-fixed theory degenerates at the GH. Consequently, as shown in Refs. [28,29], the dynamics is restricted to the regions of phase space bounded by the degeneracy surface. This argument puts the Gribov-Zwanziger restriction on a firm basis: the previous analysis (which strictly speaking only holds for finite-dimensional systems) suggests that the system cannot cross the GH (since it is a degenerate surface for the corresponding Hamiltonian system), and therefore, the Gribov-Zwanziger restriction would be naturally respected by the dynamics.

V. THE FLPR MODEL

In this section, we illustrate the previous discussion with a solvable model proposed by Friedberg, Lee, Pang, and Ren (FLPR), which presents a Gribov ambiguity for Coulomb-like gauge conditions [37]. This model has been extensively studied in attempts to understand how the

Gribov ambiguity could be circumvented [38–40]. We will show that, in this gauge, the symplectic form for the gauge-fixed system becomes degenerate at the GH. Closely related models, for which Dirac quantization is nontrivial, have been analyzed in Ref. [41].

The Lagrangian for the FLPR model is

$$L = \frac{1}{2}((\dot{x} + \alpha y q)^2 + (\dot{y} - \alpha x q)^2 + (\dot{z} - q)^2) - V(\rho), \tag{33}$$

where $\{x, y, z, q\}$ are Cartesian coordinates, $\rho = \sqrt{x^2 + y^2}$, and $\alpha > 0$ is a coupling constant. The velocity $\dot{q}$ is absent, and therefore the coordinate q plays the role of an auxiliary field or Lagrange multiplier. The associated canonical momenta are given by

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} + \alpha y q, \qquad p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} - \alpha x q,$$
$$p_z = \frac{\partial L}{\partial \dot{z}} = \dot{z} - q, \qquad p_q = \frac{\partial L}{\partial \dot{q}} = 0. \tag{34}$$

Following Dirac's procedure, we find one primary constraint:

$$\varphi = p_q \approx 0. \tag{35}$$

The total Hamiltonian is given by

$$H_T = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + [\alpha(x p_y - y p_x) + p_z]q + \xi\varphi + V(\rho), \tag{36}$$

where ξ is a Lagrange multiplier. Time preservation of the constraint φ leads to the secondary constraint

$$\phi = p_z + \alpha(x p_y - y p_x) \approx 0, \tag{37}$$

which leads to no new constraints for the system. Since φ and ϕ have vanishing Poisson bracket, they form a first-class set, reflecting the fact that they generate the local[4] gauge symmetries. The constraint φ generates arbitrary translations in q,

$$\delta_\varphi(x, y, z, q) = (0, 0, 0, \varepsilon(t)), \qquad \delta_\varphi(p_x, p_y, p_z, p_q) = 0, \tag{38}$$

while ϕ generates helicoidal motions,

$$\delta_\phi(x, y, z, q) = \epsilon(t)(-\alpha y, \alpha x, 1, 0),$$
$$\delta_\phi(p_x, p_y, p_z, p_q) = \alpha\epsilon(t)(-p_y, p_x, 0, 0), \tag{39}$$

[4]Locality here refers to time.

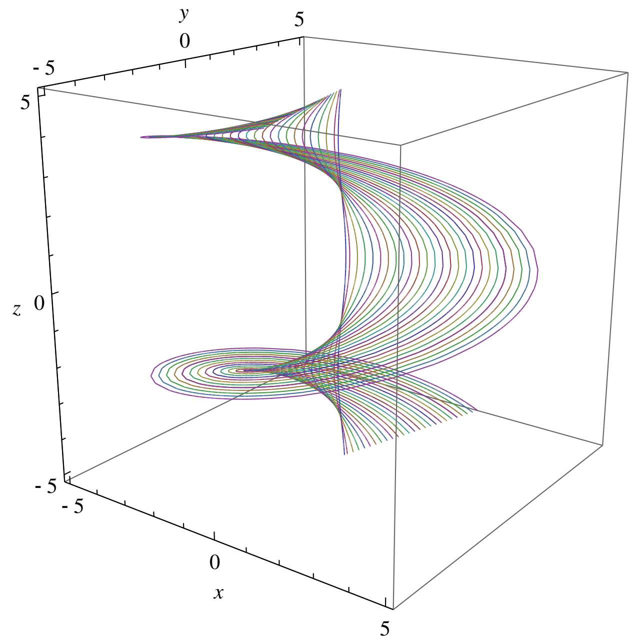

FIG. 3 (color online). The orbits generated by gauge transformations in the FLPR model are helicoids of the form $(x, y, z) = (\rho\cos[\alpha\epsilon(t)], \rho\sin[\alpha\epsilon(t)], \epsilon(t))$.

as shown in Fig. 3. Both transformations leave invariant the Hamiltonian [Eq. (36)] for arbitrary $\epsilon(t)$ and $\varepsilon(t)$. Note that the system is invariant under rotations in the x-y plane, translations in z, and time translations, but these are global symmetries that lead to conservation of the z components of the angular and the linear momenta, and the energy. Symmetries [Eqs. (38) and (39)], instead, are not rigid but local.

The gauge freedom generated by φ can be fixed by the gauge condition

$$\mathcal{G} = q \approx 0, \tag{40}$$

which is analogous to the temporal gauge $A_0 = 0$ in Maxwell theory. Thus, the coordinate q and its conjugate momentum p_q can be eliminated from phase space by an algebraic gauge choice, as happens with A_0 in electrodynamics, which also enters as a Lagrange multiplier. This partial gauge fixing eliminates the term $\xi\varphi$ from the Hamiltonian [Eq. (36)] and identifies q as a Lagrange multiplier. The result is a Hamiltonian system in the six-dimensional phase space Γ with coordinates $\{u^A\} = \{x, p_x, y, p_y, z, p_z\}$ and a single (necessarily first-class) constraint $\phi \approx 0$. The Poisson bracket in this phase space is given by

$$[M, N]_\Gamma = \Omega^{AB}\partial_A M \partial_A N, \tag{41}$$

where Ω^{AB} is the canonical Poisson bracket, and the canonical symplectic form is

$$\Omega_{AB} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{42}$$

Following Ref. [37], the gauge freedom generated by ϕ is to be eliminated by a gauge condition $G(x,y,z) \approx 0$, where G is a linear homogeneous function, which is in some sense analogous to the Coulomb gauge. Since the system is invariant under rotations in the x-y plane, we can choose the gauge condition to be independent of y. Hence, we take

$$G = z - \lambda x \approx 0, \tag{43}$$

which is called the "λ-gauge". As can be seen, for $\lambda \neq 0$ the condition in Eq. (43) does not fix the gauge globally (see Fig. 4). In the same way as the Coulomb gauge does in Yang-Mills theory, it has a Gribov ambiguity at $y = -(\alpha\lambda)^{-1}$. In fact, the nontrivial Poisson bracket,

$$\mathcal{M} = [G, \phi] = 1 + \alpha\lambda y, \tag{44}$$

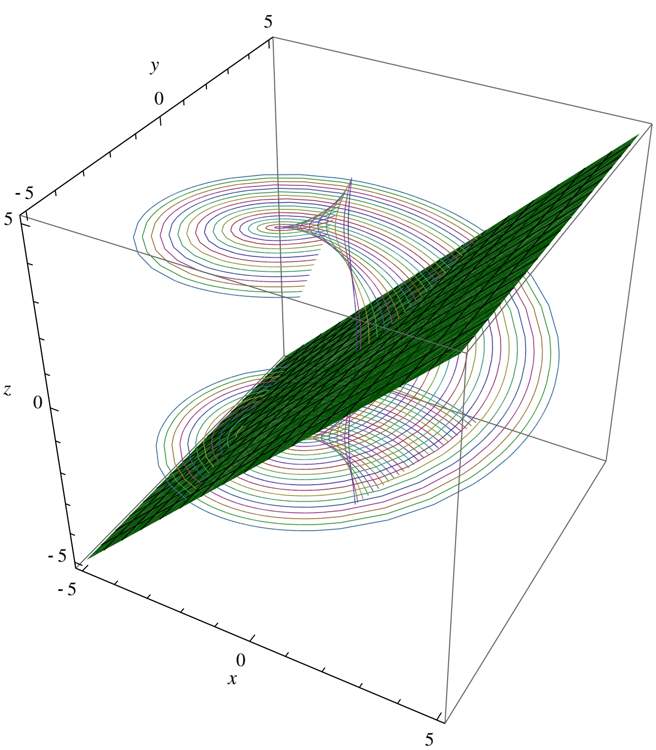

FIG. 4 (color online). The surface defined by the λ-gauge condition $G = z - \lambda x = 0$ is a plane (here plotted for $\lambda = 1$). The Gribov ambiguity in the FLPR model is reflected by the fact that this plane intersects some gauge orbits more than once.

which corresponds to the Faddeev-Popov determinant, indicates that these are second-class constraints everywhere in Γ_0, except at $y = -(\alpha\lambda)^{-1}$. Consequently, the determinant of the Dirac matrix (22) vanishes where the condition $G \approx 0$ fails to fix the gauge—that is, on the Gribov horizon:

$$\Xi = \{(x, p_x, y, p_y, z, p_z) \in \Gamma | \mathcal{M} = 0\}. \tag{45}$$

When the second-class constraints [Eqs. (37) and (43)] are set strongly equal to zero, z and p_z can be eliminated from the phase space. The four-dimensional reduced phase space Γ_0, parametrized with coordinates (x, p_x, y, p_y), acquires a noncanonical Poisson structure given by the Dirac bracket [Eq. (14)], where γ_I are the second-class constraints $\{G, \phi\}$,

$$\gamma_I\colon\ \gamma_1 = G = z - \lambda x, \qquad \gamma_2 = \phi = p_z + \alpha(xp_y - yp_x), \tag{46}$$

and C^{IJ} is the inverse of the Dirac matrix $C_{IJ} \equiv [\gamma_I, \gamma_J]$. In this case, the Dirac matrix is given by

$$C_{IJ} = [\gamma_I, \gamma_J]_\Gamma = \begin{pmatrix} 0 & \mathcal{M} \\ -\mathcal{M} & 0 \end{pmatrix}, \tag{47}$$

and the Dirac brackets are given by

$$[x, p_x]^* = \frac{1}{\mathcal{M}}, \qquad [x, y]^* = 0, \qquad [x, p_y]^* = 0,$$
$$[y, p_y]^* = 1, \qquad [y, p_x]^* = \frac{\alpha\lambda x}{\mathcal{M}}, \qquad [p_x, p_y]^* = -\frac{\alpha\lambda p_x}{\mathcal{M}}. \tag{48}$$

In the reduced phase space, the Poisson matrix [Eq. (27)] takes the form

$$\omega^{ab} = \begin{pmatrix} 0 & \frac{1}{\mathcal{M}} & 0 & 0 \\ -\frac{1}{\mathcal{M}} & 0 & -\frac{\alpha\lambda x}{\mathcal{M}} & -\frac{\alpha\lambda x}{\mathcal{M}} \\ 0 & \frac{\alpha\lambda x}{\mathcal{M}} & 0 & 1 \\ 0 & \frac{\alpha\lambda p_x}{\mathcal{M}} & -1 & 0 \end{pmatrix}, \tag{49}$$

and the corresponding symplectic form is

$$\omega_{ab} = \begin{pmatrix} 0 & -\mathcal{M} & -\alpha\lambda p_x & \alpha\lambda x \\ \mathcal{M} & 0 & 0 & 0 \\ \alpha\lambda p_x & 0 & 0 & -1 \\ -\alpha\lambda x & 0 & 1 & 0 \end{pmatrix}. \tag{50}$$

It can be checked that this symplectic form is closed, $\partial_a\omega_{bc} + \partial_b\omega_{ca} + \partial_c\omega_{ab} = 0$, and therefore in a local chart it can be expressed as the exterior derivative of a

1-form, $\omega_{ab} = \partial_a X_b - \partial_b X_a$ (or $\omega = dX$), which can be integrated as

$$X(x, p_x, y, p_y) = (p_x + \alpha\lambda[yp_x - xp_y])dx + p_y dy. \quad (51)$$

The determinant of the symplectic form in the reduced phase space can be read off from Eq. (50) and is given by

$$\det[\omega_{ab}] = \mathcal{M}^2. \quad (52)$$

Clearly, ω_{ab} degenerates precisely at the Gribov [Eq. (45)] restricted to the constraint surface, and the degeneracy surface [Eq. (8)] is given by

$$\Sigma = \{(x, p_x, y, p_y) \in \Gamma_0 | \Upsilon(u) \equiv \mathcal{M} = 0\}. \quad (53)$$

This corresponds to a particular realization of the behavior in Eq. (32). In fact, defining $\sigma^2 = 1 + \alpha^2\lambda^2\rho^2 > 0$, the eigenvalues of the reduced symplectic form are given by $\{\pm i\omega_+, \pm i\omega_-\}$, where

$$\omega_\pm = \frac{1}{\sqrt{2}}\left[\sigma^2 + \mathcal{M}^2 \pm \sqrt{(\sigma^2 + \mathcal{M}^2)^2 - 4\mathcal{M}^2}\right]^{1/2}. \quad (54)$$

Near the degeneracy, ω_+ and ω_- can be expanded in powers of $\mathcal{M}$, leading to

$$\omega_+ \approx \sigma, \qquad \omega_- \approx \frac{\mathcal{M}}{\sigma}. \quad (55)$$

Hence, as the system approaches the degeneracy, ω_+ goes linearly to zero, while ω_- never vanishes, which means that the symplectic form ω_{ab} has a simple zero in the degeneracy surface, and this system corresponds to the class of degenerate systems discussed in Refs. [28] and [29].

It is reassuring to confirm that the degeneracy is not an artifact introduced by the change of coordinates $\{U^A\} \to \{u^{*a}, v^I\}$ defined in Eq. (23), which in this case is given by

$$x^* = \frac{x + \alpha yz}{\mathcal{M}}, \qquad p_x^* = \frac{p_x + \alpha p_y z + \alpha\lambda p_z}{\mathcal{M}},$$
$$y^* = y - \frac{\alpha x(z - \lambda x)}{\mathcal{M}}, \qquad p_y^* = p_y - \frac{\alpha p_x(z - \lambda x)}{\mathcal{M}},$$
$$v^1 = \gamma_1 = z - \lambda x, \qquad v^2 = \gamma_2 p_z + \alpha(xp_y - yp_x). \quad (56)$$

In fact, the Jacobian [Eq. (28)] is given in this case by

$$\mathcal{J}^A{}_B = \begin{pmatrix} \frac{1}{\mathcal{M}} & 0 & 0 & 0 & \frac{\alpha y}{\mathcal{M}} & 0 \\ 0 & \frac{1}{\mathcal{M}} & -\frac{\alpha\lambda p_x}{\mathcal{M}} & \frac{\alpha\lambda x}{\mathcal{M}} & \frac{\alpha\lambda p_y}{\mathcal{M}} & \frac{\lambda}{\mathcal{M}} \\ \frac{\alpha\lambda x}{\mathcal{M}} & 0 & 1 & 0 & -\frac{\alpha x}{\mathcal{M}} & 0 \\ \frac{\alpha\lambda p_x}{\mathcal{M}} & 0 & 0 & 1 & -\frac{\alpha\lambda p_x}{\mathcal{M}} & 0 \\ -\lambda & 0 & 0 & 0 & 1 & 0 \\ \alpha p_y & -\alpha y & -\alpha p_x & \alpha x & 0 & 1 \end{pmatrix}, \quad (57)$$

which, in spite of the apparent singularities in its entries, has a unit determinant everywhere in phase space: $(\det\mathcal{J})|_\Gamma \equiv 1$.

A. Effective Lagrangian for the gauge-fixed system

The gauge-fixed system is a degenerate one described by a first-order Hamiltonian action, as presented in Eq. (1),

$$I_{gf}[u] = \int dt[\dot{u}^a X_a(u) - H_{gf}(u)], \quad (58)$$

where X_a is given by Eq. (51), H_{gf} is the gauge-fixed Hamiltonian,

$$H_{gf} = \frac{1}{2}(1 + \alpha^2 y^2)p_x^2 + \frac{1}{2}(1 + \alpha^2 x^2)p_y^2 - \alpha^2 xyp_xp_y + V(x^2 + y^2)$$
$$= \frac{1}{2}g^{ij}p_ip_j + V(x^2 + y^2). \quad (59)$$

Here, the matrix

$$g^{ij} := \begin{bmatrix} (1 + \alpha^2 y^2) & -\alpha^2 xy \\ -\alpha^2 xy & (1 + \alpha^2 x^2) \end{bmatrix} \quad (60)$$

is the inverse of the metric

$$g_{ij} := \frac{1}{1 + \alpha^2\rho^2}\begin{bmatrix} (1 + \alpha^2 x^2) & \alpha^2 xy \\ \alpha^2 xy & (1 + \alpha^2 y^2) \end{bmatrix}. \quad (61)$$

B. Gauge orbits and phase space

Gribov ambiguity results from the fact that the surface defined by a gauge condition does not intersect every gauge orbit once and only once. As was mentioned in Sec. III, this is a requirement to achieve a proper gauge fixing [2]. In the case of the FLPR model, this clearly happens because the plane defined by Eq. (43) intersects some gauge orbits many times for $\lambda > 0$, as can be seen in Fig. 4. The $G = 0$ plane intersects more than once any orbit such that $x^2 + y^2 > (\alpha\lambda)^{-2}$. The only way that this does not happen is if $\lambda = 0$.

Degenerate surfaces divide phase space into dynamically disconnected regions. In this case, the presence of the GH defines two regions in physical gauge-fixed space (see Fig. 5):

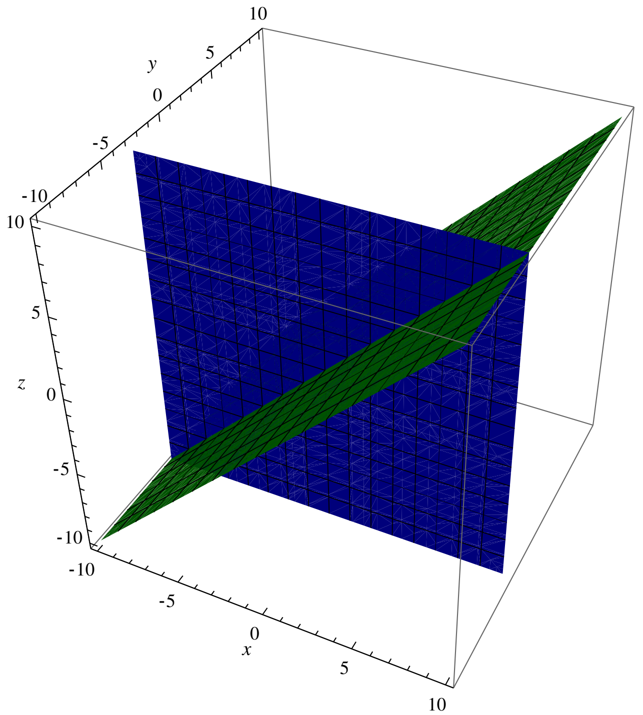

FIG. 5 (color online). In the case of the FLPR model, the Gribov horizon (vertical plane), $y = -(\alpha\lambda)^{-1}$, and the constraint surface (inclined plane), $G = 0$, are plotted for $\lambda = 1$ and $\alpha = 1/3$. The GH divides the constraint surface into two dynamically disconnected regions.

$$C_+ \coloneqq \{(x, y, z) | z - \lambda x = 0, 1 + \alpha\lambda y > 0\}, \quad (62)$$

$$C_- \coloneqq \{(x, y, z) | z - \lambda x = 0, 1 + \alpha\lambda y < 0\}. \quad (63)$$

These two regions are not equivalent, since only C_+ contains at least one representative of every gauge orbit, while not all gauge orbits pass through C_-. To restrict the analysis of the system to one region or the other is consistent in the sense that all states whose initial condition is in one region will remain there always (see Ref. [29]). In Yang-Mills theories, the Gribov region corresponds to the neighborhood of $A_\mu = 0$ in the functional space of connections where the FP operator is positive definite [4] and "small copies" (namely, points infinitesimally close which belong to the same gauge orbit) are absent. In the Yang-Mills case, all the gauge orbits cross the Gribov region at least once [12]. Similarly to what happens in the Yang-Mills case, within the region C_+ (which contains at least one representative of every gauge orbit) there are still large copies.[5]

[5]We remind the reader that large copies are points belonging to the same gauge orbit (and, of course, satisfying the same gauge condition) which are not infinitesimally close. This means that large copies (unlike the small ones) do not correspond to zero modes of the Faddeev-Popov operator. In Yang-Mills theory, the pattern of appearance of the large Gribov copies within the Gribov region is very complicated, and only a few examples are known [14].

C. Quantization

In order to define the quantum theory, the Hilbert space for the system must be equipped with an inner product that provides a scalar product and a norm,

$$\|\psi(u)\| = \left(\int d^2u\sqrt{g}w(u)|\psi(u)|^2\right)^{1/2}. \quad (64)$$

In the FLPR model, $g = (1 + \alpha^2\rho^2)^{-1}$ is the determinant of the metric [Eq. (61)], and the weight $w(u)$ is such that the Hamiltonian is symmetric—that is,

$$\int d^2u\sqrt{g}w(u)\psi_1^*(u)(\hat{H}\psi_2(u)) = \int d^2u\sqrt{g}w(u)(\hat{H}\psi_1(u))^*\psi_2(u). \quad (65)$$

As discussed in Sec. II, the proper choice for the measure $w(u)$ corresponds to the Pfaffian of the symplectic form ω_{ab} [Eq. (50)], given in this case by Eq. (53),

$$w(u) = \Upsilon = \mathcal{M} = 1 + \alpha\lambda y, \quad (66)$$

whose zeros define the degeneracy surface [Eq. (8)]. In order to see this, let us define new variables $\{\pi_i\}$ canonically conjugate to the u's, so that

$$[u^i, \pi_j]^* = \delta^i_j, \qquad \{u^i\} = \{x, y\}. \quad (67)$$

A simple calculation using Eq. (48) leads to the following expression of the momenta in terms of the π's:

$$p_x = \frac{1}{1 + \alpha\lambda y}(\pi_x + \alpha\lambda x\pi_y), \qquad p_y = \pi_y. \quad (68)$$

The quantum operators are then obtained via the prescription

$$\begin{aligned} u^i &\longrightarrow \hat{u}^i = u^i, \\ \pi_i &\longrightarrow \hat{\pi}_i = -i\hbar\partial_i, \\ [,]^* &\longrightarrow \frac{1}{i\hbar}[,](\text{Commutator}). \end{aligned} \quad (69)$$

Using Eq. (68), the classical Hamiltonian [Eq. (59)] can be rewritten as

$$H = \frac{1}{2}h^{ij}\pi_i\pi_j + V, \quad (70)$$

where h^{ij} is the inverse of the metric

$$h_{ij} = \frac{1}{1+\alpha^2\rho^2} \times \begin{pmatrix} (1+\alpha\lambda y)^2 + \alpha^2(1+\lambda^2)x^2 & \alpha^2 xy - \alpha\lambda x \\ \alpha^2 xy - \alpha\lambda x & 1+\alpha^2 y^2 \end{pmatrix}. \tag{71}$$

At the quantum level, the correct ordering for the quantum operators in Eq. (69) that renders the Hamiltonian symmetric—and invariant under general coordinate transformations—is the one for which $\hat{H}$ is a Laplacian for the metric h_{ij} [42], i.e.,

$$\hat{H} = -\frac{\hbar^2}{2}\frac{1}{\sqrt{|h|}}\partial_i(\sqrt{|h|}h^{ij}\partial_j) + V(\rho), \tag{72}$$

where h is the determinant of Eq. (71) and where the integration measure in Eq. (64) is $\int d^2u\sqrt{h}$. A straightforward computation leads to

$$\sqrt{h} = \frac{(1+\alpha\lambda y)}{\sqrt{1+\alpha^2\rho^2}} = \sqrt{g}\Upsilon, \tag{73}$$

which confirms Eq. (66). Hence, the measure of the Hilbert space vanishes exactly where the symplectic form does. Then, according to the results in Ref. [29], this permits us to interpret the corresponding Hilbert space as a collection of causally disconnected subspaces: there is no tunneling from one side of the degenerate surface to the other. In turn, this confirms the dynamical correctness of imposing the restriction to the interior of the Gribov region, at least for the first quantization.

VI. IRREGULAR CASE

As mentioned in Sec. IV, there is an exceptional case in which the reduced symplectic form is nondegenerate at the GH. As will be shown in the following, this could happen if the constraints fail to be functionally independent—i.e., if they are irregular [2,36].

A set of constraints is regular if they are functionally independent on the constraints surface. For a set of constraints [Eq. (12)], this is ensured by demanding that the Jacobian

$$\mathcal{K}^I{}_B = \frac{\partial\gamma_I}{\partial u^B}\bigg|_{\Gamma_0} \tag{74}$$

have maximal rank. In particular, for a set of two constraints $\{G,\phi\}$, this means

$$dG\wedge d\phi|_{\Gamma_0} \neq 0 \Rightarrow \partial_{[A}G\partial_{B]}\phi|_{\Gamma_0} \neq 0, \tag{75}$$

while the Dirac matrix [Eq. (21)] takes the form

$$C_{IJ} = \begin{pmatrix} 0 & \mathcal{M} \\ -\mathcal{M} & 0 \end{pmatrix}, \tag{76}$$

where $\mathcal{M} = [G,\phi]$ is the FP determinant. Hence, using Eqs. (14) and (21), the reduced phase space symplectic form [Eq. (27)] can be expressed weakly as

$$\omega^{ab} \approx [u^a, u^b]^* = \Omega^{ab} + \mathcal{M}^{-1}\Omega^{aC}\Omega^{Db}\partial_{[C}G\partial_{D]}\phi. \tag{77}$$

This suggests that, if the constraints fail the regularity test [Eq. (75)] at the GH, the singularity in the inverse of the FP determinant $\mathcal{M}^{-1}$ can be canceled by the vanishing quantity $\partial_{[C}G\partial_{D]}\phi$, and no degeneracies would appear even in the presence of Gribov ambiguity.

Another way to see this picture for a general set of constraints [Eq. (12)], $\{\gamma_I\} = \{G_i,\phi_j\}$, is by noting that, as the original symplectic structure [Eq. (7)] is considered to be well defined ($\det[\Omega^{AB}] = \Omega$), the determinant of the Poisson bracket in the new coordinates $U^A = [u^{*a}, v^I]$, defined by Eqs. (23) and (24), is given by Eq. (29), which can be evaluated on the constraint surface Γ_0:

$$\det[\hat{\Omega}^{AB}]|_{\Gamma_0} = (\det[\mathcal{J}^A{}_B])^2\Omega|_{\Gamma_0}. \tag{78}$$

On the other hand, the Jacobian [Eq. (28)] evaluated on $\gamma_I = 0$ can be written in terms of Eq. (74) as

$$\mathcal{J}^A{}_B|_{\Gamma_0} = \begin{pmatrix} \partial_B u^{*a} \\ \mathcal{K}^I{}_B \end{pmatrix}. \tag{79}$$

Hence, if the constraints [Eq. (12)] are irregular at the GH, both $\mathcal{K}^I{}_B$ and $\mathcal{J}^A{}_B|_\Gamma$ have nonmaximal rank, implying that the determinant $\det[\mathcal{J}^A{}_B]$ vanishes at the intersection of the GH and Γ_0. Therefore,

$$\det[\hat{\Omega}^{AB}]|_{\Gamma_0} \underset{u\to\tilde{u}}{\longrightarrow} 0. \tag{80}$$

Then, looking again at Eq. (26), we see that in this case, the vanishing of $\det[C_{IJ}]$ at the GH does not imply that the reduced phase space Poisson structure should blow up and degeneracies in the symplectic structure of the gauge-fixed system can be overcome. However, this situation is even more pathological than the degenerate one, as the gauge-fixed system does not describe the dynamics of the original system. A detailed analysis of the consequences of irregularity in the abelianizability of a set of constraints and the gauge-fixing procedure can be found in Ref. [43]. In the following, an explicit example of this situation will be presented.

A. Example: Christ-Lee model

The Lagrangian for the Christ-Lee model [44] is given by

$$L = \frac{1}{2}(\dot{x} + \alpha y q)^2 + (\dot{y} - \alpha x q)^2 - V(x^2 + y^2),$$

where $\alpha > 0$ is a coupling constant. The canonical momenta of the system are given by

$$p_x = \dot{x} + \alpha y q, \qquad p_y = \dot{y} - \alpha x q, \qquad p_q = 0. \tag{81}$$

Dirac's method leads to the following first-class constraints:

$$\varphi = p_q \approx 0, \qquad \phi = x p_y - y p_x \approx 0, \tag{82}$$

which generate arbitrary translations in q and rotations in the x-y plane, respectively. The total Hamiltonian is given by

$$H_T = \frac{1}{2}(p_x^2 + p_y^2) + \alpha(x p_y - y p_x)q + \xi\varphi + V(\rho), \tag{83}$$

where ξ is a Lagrange multiplier. As before, the constraint φ can be trivially eliminated by the introduction of a gauge condition $\mathcal{G} = q \approx 0$. The Dirac bracket associated with this pair of constraints is just the Poisson bracket in the coordinates $\{x, p_x, y, p_y\}$, and using this we can set φ and $\mathcal{G}$ strongly to zero. Now we will focus on the constraint ϕ, whose action on the coordinates generates circular orbits in phase space.

As we are interested in Gribov ambiguity, we will pick the following gauge condition [39]:

$$G = y - \mu x \approx 0, \tag{84}$$

with μ a constant. The Dirac matrix for this set of constraints $\gamma_I = \{G, \phi\}$ with $I = 1, 2$ is given by Eq. (76), where

$$\mathcal{M} = [G, \phi] = x + \mu y$$

and there exists a GH [Eq. (20)] defined by

$$\Xi := \{(x, p_x, y, p_y) \in \Gamma | \mathcal{M} = 0\}. \tag{85}$$

The Poisson structure of the space is given via the Dirac bracket [Eq. (14)], where the γ_I's are the second-class constraints $\{G, \phi\}$. This leads to

$$[x, p_x]^* = \frac{x}{\mathcal{M}}, \qquad [x, y]^* = 0, \qquad [x, p_y]^* = \frac{y}{\mathcal{M}},$$
$$[y, p_y]^* = \frac{\mu y}{\mathcal{M}}, \qquad [y, p_x]^* = \frac{-\mu y}{\mathcal{M}},$$
$$[p_x, p_y]^* = \frac{\mu p_x p_y}{\mathcal{M}^2}. \tag{86}$$

Once the second-class constraints G and ϕ are strongly equal to zero,

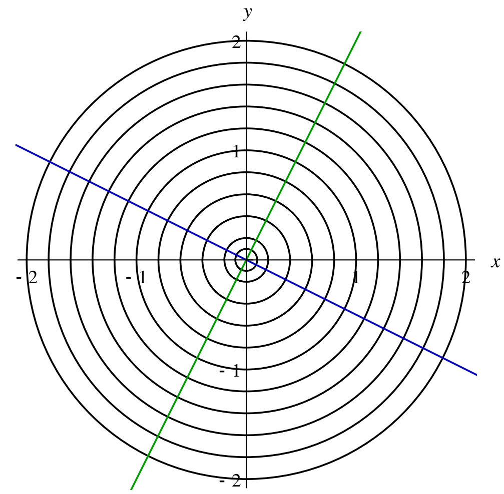

FIG. 6 (color online). Orbits for the Christ-Lee model are given by circles centered at the origin. The GH ($y = -x/\mu$) and the surface $G = 0$ ($y = \mu x$) are plotted for $\mu = 2$. The GH restricted to the constraint surface corresponds to the point $x = y = 0$.

$$y = \mu x, \qquad p_y = \mu p_x, \tag{87}$$

we are left with only one degree of freedom corresponding to the variable x. The Gribov horizon restricted to the constraint surface $G = 0$ is given by $x = 0$ (see Fig. 6). Then, the reduced phase space symplectic form [Eq. (27)] turns out to be nondegenerate:

$$\omega_{ab} = \begin{pmatrix} 0 & -(1+\mu^2) \\ 1+\mu^2 & 0 \end{pmatrix}, \qquad \det[\omega_{ab}] = (1+\mu^2)^2. \tag{88}$$

However, in this case, the constraints $\{G, \phi\}$ are not functionally independent at the GH. To see this, consider the sub-block Eq. (74) of Eq. (79), whose rank determines the functional independence of the constraints $\{G, \phi\}$,

$$\mathcal{K}^I{}_B = \frac{\partial \gamma_I}{\partial u^B}\bigg|_{\Gamma_0} = \begin{pmatrix} -\mu & 0 & 1 & 0 \\ \mu p_x & -\mu x & -p_x & x \end{pmatrix}. \tag{89}$$

This matrix has nonmaximal rank on the GH restricted to the constraint surface ($x = 0$); then the constraints are not regular there because their gradients are proportional.

The gauge-fixed Lagrangian now reads

$$L = \frac{1}{2}(1+\mu^2)\dot{x}^2 - V((1+\mu^2)x^2), \tag{90}$$

which seems to be free of degeneracy at the GH. However, this is an illusion because the absence of degeneracy results from the fact that the constraints are no longer functionally independent, so that the system, on the Gribov horizon, fails to be regular.

VII. CONCLUSIONS AND FURTHER COMMENTS

We have discussed the relation between Gribov ambiguity and degeneracy in Hamiltonian systems. In our analysis, the Gribov-Zwanziger restriction can be seen as a prescription consistent with the fact that it is respected by the dynamics, both classically and quantum mechanically, at least in finite-dimensional Hamiltonian systems.

In gauge systems with a finite number of degrees of freedom, the existence of Gribov ambiguity in the gauge-fixing conditions leads to a degenerate symplectic structure for the reduced system: the degenerate surface in the reduced phase space is the GH restricted to the constraint surface. It is important to observe that, although in the FLPR model the Gribov ambiguity can be circumvented by choosing $\lambda = 0$ (leading to the analog of the axial gauge in field theory), an analogous choice is not possible for Yang-Mills theories. In fact, as shown in Ref. [5], in order to include relevant nontrivial configurations—like instantons—in the function space of the theory, certain boundary conditions must be imposed on the fields, which rule out algebraic gauge conditions (see also Ref. [4]). In this sense, a consistent analog of the limit $\lambda \to 0$ for field theories does not exist, the Gribov ambiguity is unavoidable for gauge theories, and degeneracies should be expected in the gauge-fixed system. As we have shown, when the requirement of regularity is not imposed, a nondegenerate gauge-fixed system can be obtained. However, this is not a solution to the problem. Regularity is a key requirement for a set of constraints to be well defined, as irregularities lead to a Lagrangian that does not describe the real dynamics of the original system.

Even if the generalization of our results to field theories is conceptually straightforward, an interesting future direction for this work is to look for explicit degeneracies in the gauge-fixed symplectic form of Yang-Mills type theories. It is important to note that even though our analysis extends in a straightforward way to the cases with more than one Gribov horizon, which include the case of Yang-Mills theory in which there are infinite Gribov horizons [45] (for a detailed analysis, see also Ref. [46]), the problem involves additional important technical difficulties, such as, for instance, the definition of the reduced phase space when nonalgebraic gauge conditions are adopted. In particular, when set strongly to zero, this kind of gauge conditions do not allow us to express one field as local functions of the remaining ones, and a local action for the physical degrees of freedom with the reduced symplectic form is not available. These difficulties in the standard Hamiltonian formulation for Yang-Mills theories make the path integral formalism better suited. However, an interesting novel Hamiltonian approach to QCD, where Dirac reduction is considered, has been recently developed in Ref. [47], which could be worth studying within this context.

On the other hand, the question of whether the degeneracy surface can act as a sink or as a source in Yang-Mills theories is as interesting as it is extremely difficult and deserves further investigation. The difficulty stems from the fact that in Yang-Mills the Gribov horizon is an infinite-dimensional hypersurface with quite a complicated topology. Hence, in order to determine whether the degeneracy is a sink or a source, one should have a complete characterization of the geometry in the vicinity of the horizon, a task that seems out of reach for us for the time being. We hope to come back to this interesting issue in a future publication.

The fact that the GH is a degeneracy surface for the gauge-fixed system, which persists at the quantum level, strongly supports the consistency of the Gribov restriction for QCD, as the degeneracy divides phase space into causally disconnected regions. Even though the Gribov-Zwanziger idea is heuristic and supported by the fact that every orbit intersects the Gribov region [12] (which means that no physical information is lost if the restriction is applied), the results it yields have gained acceptance due to their match with the lattice data. Our results provide a novel point of view for the problem in support of the Gribov-Zwanziger proposal that makes it worth a deeper study within the Hamiltonian framework.

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