New cylindrical gravitational soliton waves and gravitational Faraday rotation

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In terms of gravitational solitons, we study the gravitational nonlinear effects of gravitational solitary waves such as Faraday rotation. Applying Pomeransky's procedure for the inverse scattering method, which has been recently used for constructing stationary black hole solutions in five dimensions, to a cylindrical spacetime in four dimensions, we construct a new cylindrically symmetric soliton solution. This is the first example to be applied to the cylindrically symmetric case. In particular, we clarify the difference from Tomimatsu's single soliton solution, which was constructed by Belinski-Zakharov's procedure.

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I. INTRODUCTION

Gravitational solitons in relativity are found in the framework of the inverse scattering method as exact solutions of the Einstein equations with two Killing vector fields [1]. These solutions describe self-reinforcing solitary waves (wave packets or pulses) that maintain their shape while propagating in various physically interesting background spacetimes. The gravitational soliton as a gravitational wave interacts with the background itself as its nonlinear effect of gravitation. On the other hand, in recent years, the gravitational solitons have been studied in the context of constructing black hole solutions in higher dimensions. In particular, Pomeransky [2] has improved the inverse scattering method which Belinski and Zakharov [1,3] established so that one can construct higher dimensional regular black hole solutions. Surprisingly, this improved method has succeeded in generating some physically interesting exact solutions which describe a certain kind of stationary and axisymmetric black hole [4,5]. The main aim of this paper is to construct a new cylindrically symmetric gravitational soliton which describes a gravitational wave by using Pomeransky's procedure rather than Belinski-Zakharov's original procedure [1].

Einstein and Rosen [6,7] provided the first cylindrical gravitational wave with a $+$ mode only, where the vacuum Einstein equation is reduced to a simple linear wave equation. Piran *et al*. [8] numerically studied time evolution and the nonlinear interaction of cylindrical gravitational waves of both polarization modes ($+$ and $\times$ modes) and showed that if an outgoing (ingoing) wave is linearly polarized when an ingoing (outgoing) $\times$-mode wave is present, the polarization for the outgoing (ingoing) wave rotates via the nonlinear interaction as it propagates. Today this effect is well known as the *gravitational Faraday effect*. Tomimatsu [9] studied the gravitational Faraday rotation for cylindrical gravitational solitons by using the inverse scattering technique. The outgoing $+$ waves emitted from an axis convert to the $\times$ mode completely in the immediate interaction region, and finally the outgoing waves contain both polarizations. For the single soliton solution, the soliton disturbance exists only in the interior region of a future light cone and a shock wave is present on the future light cone. Further, to avoid the shock wave–like structure on the light cone, he considered a two soliton solution with complex poles, which behaves like the single soliton field with a particular choice of the parameter. Moreover, the interaction of gravitational soliton waves with a cosmic string was also studied in [10–12].

In this paper, we study the nonlinear interaction of new cylindrically symmetric soliton solutions which can be generated by Pomeransky's procedure. As is well known, the most general metric for a cylindrically symmetric spacetime can be described by the Kompaneets-Jordan-Ehlers form [13]. Following Ref. [8], in terms of the metric form, we calculate the wave amplitudes for the ingoing (outgoing) gravitational waves with the $+$ and $\times$ modes. According to Refs. [8,9], for such a cylindrically symmetric spacetime, the polarization angles of gravitational waves can be defined by the ratio of $\times$ wave amplitudes to $+$ wave ones. Using the useful definition and some convenient formulas in Refs. [8,9], we will calculate the time development of the polarization angles. We will find that for the single soliton solution, the $\times$ mode which is initially dominant near the axis of symmetry decreases with time and at a certain time, fully converts to the $+$ mode. After

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that, in turns, it begins to increase, and finally again the $\times$ mode is completely dominant. Moreover, we will also show that the polarization vectors of two independent modes change as the waves propagate through a background spacetime along a null geodesic (gravitational Faraday rotation).

In the next section, we present the metric in a general cylindrical spacetime and reduce the Einstein equation with the symmetry to two-dimensional equations. In Sec. III, using the improved inverse scattering method of Pomeransky [2] rather than that of Belinski-Zakharov [1], we will derive a single soliton solution from a Minkowski metric. In particular, we compute the amplitudes and polarization angles for ingoing and outgoing waves and clarify the difference from the Tomimatsu solution [9] generated by the latter procedure. As discussed in Ref. [9], such a single soliton solution with real poles has a singular behavior on the light cone, which means that outgoing shock waves initially propagate at the speed of light. Section V is devoted to the summary of this paper and discussion of our results.

II. CYLINDRICALLY SYMMETRIC SPACETIMES

Throughout this paper, we assume that a four-dimensional spacetime admits two commuting Killing vector fields, an axisymmetric Killing vector $\partial/\partial\phi$ and a spatially translational Killing vector $\partial/\partial z$, where the polar angle coordinate ϕ and the coordinate z have the ranges $0 \leq \phi < 2\pi$ and $-\infty < z < \infty$, respectively. The metric with cylindrical symmetry $U(1) \times \boldsymbol{R}$ is generally written in the Kompaneets-Jordan-Ehlers form:

$$ds^2 = e^{2\psi}(dz + \omega d\phi)^2 + \rho^2 e^{-2\psi} d\phi^2 + e^{2(\gamma-\psi)}(d\rho^2 - dt^2), \tag{1}$$

where the functions ψ, ω, and γ depend on the time coordinate t and radial coordinate ρ only. Following Ref. [9], we find that the function γ is determined by

$$\gamma_{,t} = \frac{\rho}{8}(A_+^2 + B_+^2 + A_\times^2 + B_\times^2), \tag{2}$$

$$\gamma_{,\rho} = \frac{\rho}{8}(A_+^2 - B_+^2 + A_\times^2 - B_\times^2), \tag{3}$$

where

$$A_+ = 2\psi_{,v}, \tag{4}$$

$$B_+ = 2\psi_{,u}, \tag{5}$$

$$A_\times = \frac{e^{2\psi}\omega_{,v}}{\rho}, \tag{6}$$

$$B_\times = \frac{e^{2\psi}\omega_{,u}}{\rho}. \tag{7}$$

Here, the advanced ingoing and outgoing null coordinates u and v are defined by $u = (t-\rho)/2$ and $v = (t+\rho)/2$, respectively. The indices $+$ and $\times$ denote the respective polarizations. The ingoing and outgoing amplitudes are defined by, respectively,

$$A = \sqrt{A_+^2 + A_\times^2}, \qquad B = \sqrt{B_+^2 + B_\times^2}, \tag{8}$$

and the polarization angles θ_A and θ_B for the respective wave amplitudes are given by

$$\tan 2\theta_A = \frac{A_\times}{A_+}, \qquad \tan 2\theta_B = \frac{B_\times}{B_+}. \tag{9}$$

III. SINGLE SOLITON SOLUTIONS

Using Pomeransky's procedure [2] for the inverse scattering method, we construct a new single solitonic solution. Let us choose a Minkowski metric as a seed, whose 2×2 part is written as

$$g_0 = (1, \rho^2). \tag{10}$$

First, we remove a soliton with a vector $m^{(1)} = (1, 0)$ at $t = 0$ from this seed metric, and then we get the metric $g_0' = (\mu^2/\rho^2, \rho^2)$. Next, we add back a nontrivial soliton with $m^{(1)} = (1, a)$ at $t = 0$ to g_0', and then we can obtain the metric of a single soliton solution.

This is how we can obtain the metric coefficients for the single soliton solution, which are given by

$$e^{2\psi} = \frac{(w^2-1)^4\rho^2 + a^2 w^4}{(w^2-1)^4\rho^2 + a^2 w^2}, \tag{11}$$

$$\omega = -\frac{a(w^2-1)^3\rho^2}{(w^2-1)^4\rho^2 + a^2 w^4}, \tag{12}$$

$$e^{2\gamma} = \frac{a^2 w^4 + (-1+w^2)^4\rho^2}{(-1+w^2)^4\rho^2}, \tag{13}$$

where $\mu := \sqrt{t^2-\rho^2} - t$ and $w := -\mu/\rho$. From Eqs. (4)–(7), it is straightforward to compute the respective wave amplitudes A_+, B_+, $A_\times$, and $B_\times$:

$$A_+ = -\frac{2a^2(w-1)w^3(a^2w^3 - \rho^2(w-2)(w^2-1)^4)}{\rho(w+1)(a^2w^2+\rho^2(w^2-1)^4)(a^2w^4+\rho^2(w^2-1)^4)}, \tag{14}$$

$$B_+ = -\frac{2a^2w^3(w+1)(\rho^2(w+2)(w^2-1)^4 - a^2w^3)}{\rho(w-1)(a^2w^2+\rho^2(w^2-1)^4)(a^2w^4+\rho^2(w^2-1)^4)}, \tag{15}$$

$$A_\times = -\frac{2a(w-1)^3w^2(w+1)(a^2w^2(2w-1)+\rho^2(w^2-1)^4)}{(a^2w^2+\rho^2(w^2-1)^4)(a^2w^4+\rho^2(w^2-1)^4)}, \tag{16}$$

$$B_\times = -\frac{2a(w-1)w^2(w+1)^3(a^2w^2(2w+1)-\rho^2(w^2-1)^4)}{(a^2w^2+\rho^2(w^2-1)^4)(a^2w^4+\rho^2(w^2-1)^4)}. \tag{17}$$

Hence, from Eqs. (8), the respective wave amplitudes are written as

$$A = \frac{2|a||w-1||w|^2}{\rho|w+1|\sqrt{(a^2w^4+\rho^2(w^2-1)^4)}}, \tag{18}$$

$$B = \frac{2|a||w+1||w|^2}{\rho|w-1|\sqrt{(a^2w^4+\rho^2(w^2-1)^4)}}. \tag{19}$$

From Eqs. (9), the polarization θ_A and θ_B angles are determined by

$$\tan 2\theta_A = \frac{\rho(w^2-1)^2(a^2w^2(2w-1)+\rho^2(w^2-1)^4)}{aw(a^2w^3-\rho^2(w-2)(w^2-1)^4)}, \tag{20}$$

$$\tan 2\theta_B = \frac{\rho(w^2-1)^2(a^2w^2(2w+1)-\rho^2(w^2-1)^4)}{aw(\rho^2(w+2)(w^2-1)^4-a^2w^3)}. \tag{21}$$

IV. ANALYSIS OF THE SINGLE SOLITON SOLUTION

Now let us investigate the new single soliton solution in detail. It is useful to know how the restrictive wave components A_+, B_+, $A_\times$, and $B_\times$ and the polarization angles θ_A, θ_B behave near the boundaries of the spacetime, A. the axis of symmetry $\rho = 0$, B. the light cone $t = \rho$, C. the timelike infinity $t \to \infty$, and D. null infinity $v \to \infty$. Throughout this paper, we assume $t > 0$. As mentioned later, we cannot extend analytically the portion of a spacetime region $t > 0$ to the region $t < 0$ because of the existence of a singularity on the light cone $t = \rho$. For the region $t < 0$, the waves may have a different behavior from that for $t > 0$ because on the axis $\rho = 0$, the function μ behaves differently for $t > 0$ and $t < 0$. Therefore, studying the region $t < 0$ may also be interesting. However, since in this paper we have an interest in how shock wave pulses propagate throughout a spacetime as time passes, we do not discuss this region.

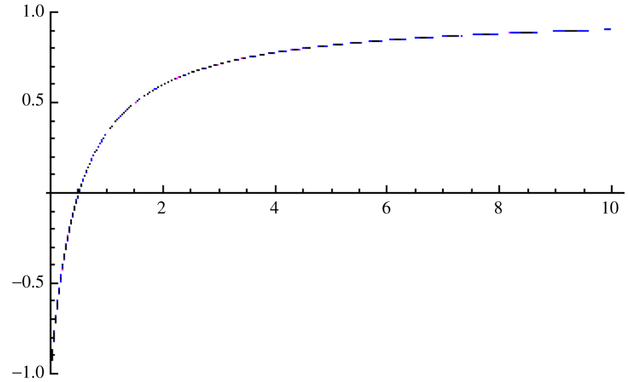

FIG. 1 (color online). The time dependence of $\tan\theta_A(=-\tan\theta_B)$ on the axis for ingoing (outgoing) waves, where we set $a = 1$.

A. On the axis $\rho = 0$

Near the axis of symmetry $\rho = 0$, the restrictive amplitudes of ingoing and outgoing waves become

$$A = B = \frac{|a|}{2t^2}, \tag{22}$$

and the restrictive polarization angles take the values

$$\tan\theta_A = -\tan\theta_B = \frac{a}{|a|}\frac{2t-|a|}{2t+|a|}. \tag{23}$$

Note that here $\tan\theta_A$ and $\tan\theta_B$ are used rather than $\tan 2\theta_A$ and $\tan 2\theta_B$. As shown in Fig. 1, the polarization angle $\theta_A(= -\theta_B)$ of ingoing (outgoing) waves propagating on the axis depends on time t. Initially, at $t = 0$, the $+$ mode is absent and the pure $\times$ mode only is present. After that, the $+$ mode waves come to exist. In particular, when $t = |a|/2$, the $\times$ mode completely vanishes and the $+$ mode only is present. Further, after that, the $+$ mode converts to the $\times$ mode and at $t \to \infty$, approaches the $\times$ mode completely. Note also that for the solution in [9], on the axis, the $\times$ mode is always absent.

We see that the C-energy density is proportional to

$$\gamma_{,\rho} = \frac{a^2w^4(1+6w^2+w^4)}{\rho(-1+w^2)^2(a^2w^4+(-1+w^2)^4\rho^2)} \tag{24}$$

$$\simeq \frac{a^2}{16t^4}\rho. \tag{25}$$

Therefore, in contrast to the single soliton in Ref. [9], the C energy does not diverge on the axis. It is regular on the axis (for the solution in [9], it diverges).

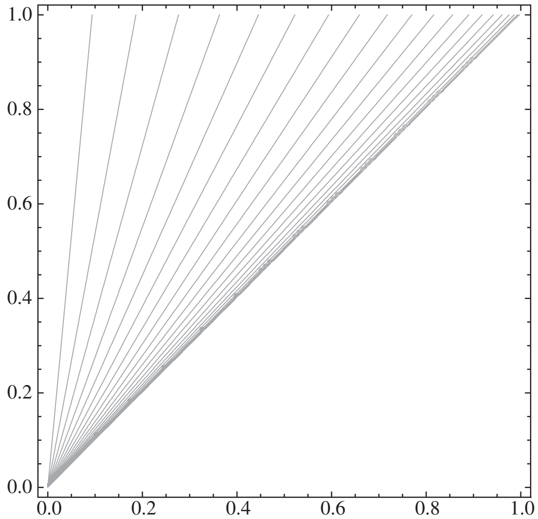

FIG. 2. $w =$ constant curves in the (t,ρ) plane: each $w =$ constant curve denotes a straight line. The axis $\rho = 0$ and the light cone $t = \rho (u = 0)$ correspond to the lines $w = 0$ and $w = 1$, respectively.

Near the axis, the metric behaves as

$$ds^2 \simeq \left(1+\frac{a^2}{4t^2}\right)^{-1}(dz+ad\phi)^2+\rho^2\left(1+\frac{a^2}{4t^2}\right)d\phi^2 + \left(1+\frac{a^2}{4t^2}\right)(-dt^2+d\rho^2). \tag{26}$$

Let us introduce a new coordinate $\tilde{z}{:=}z+a\phi$. Then the asymptotic behavior of the metric can be written as

$$ds^2 \simeq \left(1+\frac{a^2}{4t^2}\right)^{-1}d\tilde{z}^2+\rho^2\left(1+\frac{a^2}{4t^2}\right)d\phi^2 + \left(1+\frac{a^2}{4t^2}\right)(-dt^2+d\rho^2). \tag{27}$$

Therefore, under this choice of the periodicity $\Delta\phi = 2\pi$, there is no deficit angle for the solution.

B. On light cone ($t = \rho$)

Since the amplitudes of the ingoing waves A_+ and $A_\times$ vanish on the light cone $w = 1(t = \rho)$ [see Fig. 2 and Eqs. (4), (6)], we see that no ingoing wave crosses it. On the other hand, for the outgoing waves, $B_\times$ vanishes on it but B_+ diverges [see Eqs. (5) and (7)]; therefore, the $+$ mode with infinite wave amplitude only exists there. Hence, from this fact and Eq. (25), we can read off that the C-energy density does also diverge on it. This singular behavior on the light cone is similar to that in [9], where the largest portion of the disturbance for the outgoing wave with both $+$ and $\times$ modes lies on the light cone. The largest portion of the outgoing gravitational radiation can be regarded as a gravitational wave pulse propagating at the speed of light from the axis $(t,\rho) = (0,0)$ to the null infinity. As will be seen later, it should be noted that the gravitational shock wave starting from the axis interacts with the background spacetime and generates the outgoing $\times$ mode wave and ingoing waves.

C. Timelike infinity $t \to \infty$

Let us consider the limit $t \to \infty$ while keeping the radial coordinate ρ constant. It is easy to confirm that the spacetime at the timelike infinity asymptotically behaves as Minkowski spacetime; actually, at $t \to \infty$ the metric behaves as

$$ds^2 \simeq \left(1-\frac{a^2}{4t^2}\right)\left\{dz+a\left(1+\frac{\rho^2}{4t^2}\right)d\phi\right\}^2 + \rho^2\left(1+\frac{a}{4t^2}\right)d\phi^2+\left(\rho^2\sin^2\theta+\frac{1}{t^2}\right)d\phi^2 + \left(1+\frac{a^2}{4t^2}\right)(-dt^2+d\rho^2). \tag{28}$$

Hence, this implies that both the amplitudes of the ingoing and outgoing gravitational waves gradually decay and finally vanish completely:

$$A \simeq \frac{a^2}{2t^2}+\mathcal{O}(t^{-3}), \tag{29}$$

$$B \simeq \frac{a^2}{2t^2}+\mathcal{O}(t^{-3}). \tag{30}$$

We also find that $\tan\theta_A$ and $\tan\theta_B$ for the ingoing and outgoing waves behave as, restrictively,

$$\tan\theta_A \simeq -\tan\theta_B \simeq 1, \tag{31}$$

which means that for the radial coordinate ρ fixed, the $\times$ mode for the restrictive waves becomes dominant as time passes.

D. Null infinity $v \to \infty$ with $u = u_0 > 0$

We can see that at the null infinity $v \to \infty$ with $u = u_0 > 0$, the spacetime approaches Minkowski spacetime. In this limit, the metric at $v \to \infty$ asymptotically behaves as

$$ds^2 \simeq \left(dz+\frac{a}{4u_0^{1/2}b^2}v^{1/2}d\phi\right)^2 +\rho^2 d\phi^2+b^2(-dt^2+d\rho^2), \tag{32}$$

where $b{:=}\sqrt{1+(a/16u_0)^2}$. This asymptotic metric form seems to be singular at the infinity because of the existence of the additional parameter a. To see that actually, the metric is asymptotically flat, let us introduce new coordinates $(\tilde{t}, x, y)$ defined by

$$\tilde{t} = bt, \tag{33}$$

$$x = b\rho \cos b\phi, \tag{34}$$

$$y = b\rho \sin b\phi, \tag{35}$$

and then we can see that at $v \to \infty$ the metric behaves as

$$ds^2 \simeq -d\tilde{t}^2 + dx^2 + dy^2 + dz^2 + \mathcal{O}(v^{-1/2}). \tag{36}$$

Therefore, at the null infinity, both the ingoing and outgoing waves asymptotically vanish.

Next, we see how the ingoing and outgoing waves behave along null rays $u = u_0 (> 0)$ by changing a value of u_0. Here, without loss of generality, we can assume $a > 0$ because under the transformations $a \to -a$ and $\phi \to -\phi$, the single soliton solution is invariant. Note that at the null infinity $v \to \infty$, the restrictive wave amplitudes behave as

$$A \simeq \frac{2|a|u_0}{(256u_0^2 + a^2)^{1/2} v^{3/2}}, \tag{37}$$

$$B \simeq \frac{2|a|}{\{(256u_0^2 + a^2)u_0\}^{3/2} v^{1/2}}, \tag{38}$$

and the restrictive polarization angles θ_A and θ_B behave as

$$\tan\theta_A \simeq \frac{-a + (256u_0^2 + a^2)^{1/2}}{16u_0}, \tag{39}$$

$$\tan\theta_B \simeq \frac{16u_0(256u_0^2 - 3a^2)}{a(a^2 - 768u_0^2) + \epsilon(256u_0^2 + a^2)^{3/2}}, \tag{40}$$

where ϵ takes 1 when $u_0 < |a|/(16\sqrt{3})$ and -1 when $u_0 > |a|/(16\sqrt{3})$.

E. Gravitational Faraday effect

It is physically interesting to see how the polarization angle θ_B of the outgoing wave propagating from the axis $\rho = 0$ changes along some null rays $u = u_0$ by changing a value of u_0. This effect is well known as gravitational Faraday rotation. Note that at $v \to \infty$, $\tan\theta_A$ is always positive and the signature of $\tan\theta_B$ depends on the value of u_0. Also note that on the axis $\rho = 0$, $\tan\theta_A = -\tan\theta_B$ is negative when $0 < u_0 < a/4$, positive when $u_0 > a/4$, and zero when $u_0 = a/4$.

(i) For $u_0 < |a|/16\sqrt{3}$ ($t < |a|/8\sqrt{3}$ on the axis $\rho = 0$), both $+$ and $\times$ modes are present on the axis. The ratio of the $\times$ mode wave to the $+$ mode wave is decreasing along a null ray $u = u_0$ and the outgoing wave converts to B_+ completely. After that, a part of the pure $+$ mode converts to the $\times$ mode and the ratio is increasing.

(ii) For $|a|/16\sqrt{3} < u_0 < \sqrt{3}|a|/16$ ($\sqrt{3}|a|/8 < t < \sqrt{3}|a|/8$ on the axis $\rho = 0$), both $+$ and $\times$ modes are present on the axis. The ratio of the $\times$ mode wave to the $+$ mode wave is decreasing along a null ray $u = u_0$ but does not vanish anywhere.

(iii) For $u_0 = \sqrt{3}|a|/16$ ($t = \sqrt{3}|a|/8$ on the axis $\rho = 0$), both $+$ and $\times$ modes are present on the axis. The ratio of the $\times$ mode wave to the $+$ mode wave is decreasing along a null ray $u = u_0$ and vanishes at infinity $v \to \infty$. Therefore, at null infinity, the $+$ mode wave only is present.

(iv) For $\sqrt{3}|a|/16 < u_0 < |a|/4$ ($\sqrt{3}|a|/8 < u_0 < |a|/2$ on the axis $\rho = 0$), the behavior of the outgoing wave is similar to the case of (i).

(v) For $u_0 = |a|/4$ ($t = |a|/2$ on the axis $\rho = 0$), the $\times$ mode wave vanishes on the axis. The pure $+$ mode wave emitted from the axis partially converts to the $\times$ mode and then the ratio is increasing along a null ray.

(vi) For $|a|/4 < u_0$ ($|a|/4 < t$ on the axis $\rho = 0$), both $+$ and $\times$ modes are present on the axis. The ratio of the $\times$ mode wave to the $+$ mode wave is increasing along a null ray $u = u_0$.

Next, see how the polarization angle θ_A of the ingoing wave from the axis $\rho = 0$ to the infinity $v \to \infty$ changes along some null rays $u = u_0$ by a value of u_0.

(i) For $u_0 < |a|/4$ ($t < |a|/2$ on the axis $\rho = 0$), both $+$ and $\times$ modes are present on the axis. The ratio of the $\times$ mode wave to the $+$ mode wave is decreasing along a null ray $u = u_0$ and the outgoing wave converts to B_+ completely. Then, a part of the pure $+$ mode converts to $\times$ and the ratio is increasing.

(ii) For $u_0 = |a|/4$ ($t = |a|/2$ on the axis $\rho = 0$), the $\times$ mode wave vanishes on the axis. The pure $+$ mode wave emitted from the axis partially converts to the $\times$ mode and then the ratio is increasing along a null ray.

(iii) For $u_0 > |a|/4$ ($t > |a|/2$ on the axis $\rho = 0$), both $+$ and $\times$ modes are present on the axis. The ratio of the $\times$ mode wave to the $+$ mode wave is increasing along a null ray $u = u_0$ at large $v \gg |a|$ and does not vanish and does not reach 1 anywhere.

V. DISCUSSION

In this paper, applying Pomeransky's procedure for the inverse scattering method to a cylindrically symmetric spacetime, we have obtained the gravitational soliton as an exact solution to vacuum Einstein equations with cylindrical symmetry. We would like to emphasize that this work is the first example to be generated by Pomeransky's procedure in such a cylindrical context. In general, such a single soliton describes a gravitational wave pulse with time-depending polarization angles propagating through a cylindrically symmetric spacetime. In terms of the gravitational soliton, we have studied the effect of the gravitational Faraday rotation of gravitational waves. In particular, we have compared our single soliton solutions with Tomimatsu's single soliton [9] which was constructed by the use of the original Belinski-Zakharov procedure.

Here, we would like to point out that there are essential differences between the two solutions:

(i) For our solution, the C energy diverges on the light cone $u = 0$ as it does for Tomimatsu's solution. The outgoing wave with the largest portion of the disturbance initially propagates at light velocity and hence, we can physically interpret it as a gravitational wave pulse. It should be noted that for Tomimatsu's solution both the $+$ and $\times$ modes have infinite wave amplitude with the polarization fixed, while for our solution the outgoing wave with the $+$ mode only has infinite amplitude and the one with the $\times$ mode vanishes there.

(ii) For our solutions, the C energy does not diverge on an axis of symmetry, which is in contrast to the soliton solutions in Ref. [9]. This difference comes from the fact that for our solutions, the seed added back trivial soliton(s); i.e., Minkowski has no singular behavior on the axis. It is expected that Pomeransky's procedure does not change this structure by the solitonic transformations except at the point where the soliton is added. For Tomimatsu's solution, the singular source on the axis continues to absorb and emit gravitational waves constantly (therefore, amplitudes do not decay near the axis forever). On the other hand, for our solution, because of the absence of any sources on the axis, a shock wave is emitted initially from the origin of the spacetime and continues to scatter waves backward and make a tail, which is gradually damped and finally vanishes.

(iii) For our solutions, the polarization angles θ_A and θ_B of gravitational waves on the axis have time dependence. This behavior is in contrast to the soliton solutions in Ref. [9], where the $+$ mode amplitudes are dominant on the axis. At $t = 0$, the pure $\times$ mode amplitudes only are present on the axis. As time goes on, the $+$ mode gradually increases, and at some instance the $\times$ mode completely vanishes and only the $+$ mode is left. Then the $+$ mode is decreasing in turn and finally vanishes, while the $\times$ mode only remains. It should be noticed that the time dependence of the polarization of gravitational waves on the axis is determined by the behavior of the polarization of the ingoing waves near the axis, because the outgoing waves are derived just by reflection of the ingoing waves due to the regularity of the axis.

(iv) At $t \to \infty$ (with ρ constant), for our solutions, the spacetime asymptotically approaches Minkowski, and simultaneously both ingoing and outgoing gravitational waves fade into the background spacetime. We also find that the $\times$ mode for the ingoing and outgoing waves becomes dominant in the future. On the other hand, for Tomimatsu's solutions, the spacetime is not asymptotically Minkowski at timelike infinity, and the waves do not vanish anywhere due to the existence of the singular source on the axis as mentioned above in (ii). The polarizations approach 0 for both ingoing and outgoing waves, and therefore, the $+$ modes gradually become dominant in the future.

Among the properties mentioned above, one of the most peculiar phenomena may be the $\times$ mode dominance of the wave tail. As is well known, all of the Einstein-Rosen wave pulses, which bear the $+$ mode only, have the tails which approach the timelike infinity in the same way [14]. The asymptotic forms of the ingoing and outgoing $+$ mode amplitudes behave as $1/t^2$. On the other hand, for the wave pulse treated here, the $\times$ mode tails for the ingoing and outgoing waves have the same asymptotic form as the Einstein-Rosen wave pulses, but the $+$ mode tails decay more swiftly than the case of Einstein-Rosen wave pulses. It seems to be plausible that the $\times$ mode plays the role of the $+$ mode of the Einstein-Rosen wave pulses. Actually we may support this statement with the aid of the following basic equations of $+$ mode amplitudes (A_+, B_+) and $\times$ mode amplitudes $(A_\times, B_\times)$ given in [8]:

$$A_{+,u} = B_{+,v} = \frac{A_+ - B_+}{2\rho} + A_\times B_\times,$$

$$A_{\times,u} = \frac{A_\times + B_\times}{2\rho} - A_+ B_\times, B_{\times,v} = -\frac{A_\times + B_\times}{2\rho} - B_+ A_\times.$$

For the right-hand side of each equation above, the first term is the "linear" term which exists in the original equations of the Einstein-Rosen waves and the second one is the "nonlinear" term which gives new effects that are absent in Einstein-Rosen waves like the gravitational Faraday rotation. Substituting the exact form of the new solution into the right-hand side of the above equations, we can easily evaluate which term becomes dominant between the linear and nonlinear terms as time goes to infinity. From this simple evaluation, we know that for the $\times$ mode the first term dominates the second one so that the linear term controls the behavior of the $\times$ mode amplitudes, while for the $+$ mode the second term keeps the size comparable to the first term so that some nonlinear effect still has some influence on the behavior of the $+$ mode. As a result, we may suggest that the $\times$ mode behaves like the case of the Einstein-Rosen waves, but the $+$ mode rapidly decreases owing to the nonlinear effect originating from the second term of the right-hand side of the equations. To study this further, it would be interesting to clarify to what extent this phenomenon occurs generally. From this point, we must treat more general cases, including other soliton solutions and nonsoliton solutions.

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[1] V. A. Belinskii and E. Zakharov, Sov. Phys. JETP **48**, 985 (1979).
[2] A. A. Pomeransky, Phys. Rev. D **73**, 044004 (2006).
[3] V. A. Belinski and E. Verdaguer, *Gravitational Solitons* (Cambridge University Press, Cambridge, England, 2001).
[4] H. Iguchi, K. Izumi, and T. Mishima, Prog. Theor. Phys. Suppl. **189**, 93 (2011).
[5] R. Emparan and H. S. Reall, Living Rev. Relativity **11**, 6 (2008).
[6] A. Einstein and N. Rosen, J. Franklin Inst. **223**, 43 (1937).
[7] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, 2003), 2nd ed..
[8] T. Piran, P. N. Safier, and R. F. Stark, Phys. Rev. D **32**, 3101 (1985).
[9] A. Tomimatsu, Gen. Relativ. Gravit. **21**, 613 (1989).
[10] A. Economou and D. Tsoubelis, Phys. Rev. D **38**, 498 (1988).
[11] B. C. Xanthopoulos, Phys. Lett. B **178**, 163 (1986).
[12] B. C. Xanthopoulos, Phys. Rev. D **34**, 3608 (1986).
[13] P. Jordan, J. Ehlers, and W. Kundt, Abh. Akad. Wiss. Lit. Mainz, Abhandl. Math.-Nat. Kl. **2** (1960); A. S. Kompaneets, Zh. Eksp. Teor. Fiz. **34**, 953 (1958) [Sov. Phys. JETP **7**, 659 (1958)].
[14] N. Rosen, Bull. Res. Counc. Israel **3**, 328 (1954).