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Positive energy in quantum gravity

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This paper addresses the question of whether Witten's proof of positive Arnowitt-Deser-Misner (ADM) energy for classical general relativity [E. Witten, Commun. Math. Phys. 80, 381 (1981)] can be extended to give a proof of positive energy for a nonperturbative quantization of general relativity. To address this question, a set of conditions is shown to be sufficient for showing the positivity of a Hamiltonian operator corresponding to the ADM energy. One of these conditions is a particular factor ordering for the constraints of general relativity, in a representation where the states are functionals of the Ashtekar connection, and the auxiliary, Witten spinor.

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I. INTRODUCTION

One of the most evident facts about the world is the stability of empty space-time. In classical general relativity we can explain this as a consequence of the positive energy theorem, which establishes, in the asymptotically flat context, that, when the constraints of the theory are satisfied, and matter satisfies the positive energy condition, the Arnowitt-Deser-Misner (ADM) mass is positive definite. Further, the ADM mass only vanishes when the space-time is Minkowski space-time. This theorem was proved first by Schoen and Yau [1], although here we will be interested in a slightly later proof of Witten [2].

In this paper we discuss a corresponding result for the quantum theory of gravity. Certainly the positive energy theorem must extend in some suitable form to any viable quantum theory of gravity. This is highly nontrivial in a background-independent approach because, as a consequence of the equivalence principle, the ADM Hamiltonian comprises a bulk term, which is proportional to constraints, and a boundary term, which is not positive definite off the constraint surface.

To make progress towards such a quantum positive energy theorem, we study a particular class of theories, where the quantum state is a functional of the Ashtekar connection [which is the chiral $SU(2)_{\text{Left}}$ part of the space-time connection] and an auxiliary spinor variable, the Witten spinor. Working within this class of representations, we establish a set of sufficient conditions for a quantization of general relativity to have such a theorem. To do this, we work at a formal level in which we pay attention to operator ordering but not the details of a regularization scheme for operator products.

One crucial issue that emerges is the requirement that the spatial metric and frame fields be nondegenerate. This is a necessary condition of the classical proof [3], and the quantum proof requires correspondingly that $\frac{\hat{1}}{\sqrt{\det(q)}}$ be well defined as a quantum operator. This is a challenge for the standard Ashtekar-Lewandowski representation of loop quantum gravity, which allows for states where the metric operators are degenerate. This is indeed a crucial issue because the fact that the configuration space of the theory extends to degenerate three metrics is a consequence of the fact that the action, equations of motion and constraints of the connection-based form of general relativity are all low order polynomials, the same circumstance which makes possible exact and nontrivial results in the quantum theory.

As a by-product of our work, we show that some known classical results have particularly simple derivations within the Ashtekar formalism. These include Witten's positive energy theorem itself and the demonstration that there exists a positive definite bulk Hamiltonian which is, however, only equal to the ADM Hamiltonian on the constraint surface.

A. Heuristic motivation

The positive energy theorem was for half a century or more an open challenge to relativists. Many attempts were made to prove flat space-time was stable, but none completely succeeded completely until a majestic tour de force of geometric reasoning of Schoen and Yau [1]. This was followed two years later by a proof of Witten [2], which was as elegant as it was short. It is this proof of Witten's that we take as a template here for the quantum theory.

Witten's proof was inspired by an observation about supergravity made by Grisaru [4] and Deser and Teitelboim [5]. This was that the Hamiltonian of supergravity is positive definite because the ADM Hamiltonian is the square of the supersymmetry charge. In informal notation,[1]

[1]In this paper indices $A, B, C, \ldots = 0, 1$ are left-handed Weyl spinor indices, while primed indices $A', B', C', \ldots = 0', 1'$ signify the complex conjugate representation spanned by right-handed Weyl spinors. $a, b, c = 1, 2, 3$ are three-dimensional space-time indices.

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$$H_{\mathrm{ADM}} = \mathcal{Q}_A^\dagger \mathcal{Q}^A \geq 0. \tag{1}$$

The suggestion was that a positive energy proof for general relativity could be gotten by restricting supergravity to its bosonic sector, which is general relativity. Witten realized this suggestion in a very clever way which can be explained as follows.

Let us work in the chiral Hamiltonian formulation of $N = 1$ supergravity, as presented by Jacobson [6]. There, both the Hamiltonian and the supersymmetry charge are a sum of a bulk term proportional to constraints and a surface integral taken at the boundary at spatial infinity. We will not need the full formulation here, but to motivate Witten's proof we need to know two things about it. First, it extends the Ashtekar formulation of general relativity. Its canonical coordinates are the left-handed part of the gravitational connection, or Ashtekar connection, A_a^{AB} and the left-handed gravitino field ψ_a^A. Their conjugate momenta are the densitized frame field $\tilde{E}_{AB}^a$ and the gravitino momenta $\tilde{\pi}_A^a$. The nonvanishing Poisson brackets are

$$\{A_a^{AB}(x), \tilde{E}_{CD}^a(y)\} = \delta^3(x,y)\delta_a^b\delta_{CD}^{AB}, \tag{2}$$

$$\{\psi_a^A(x), \tilde{\pi}_C^a(y)\}_+ = \delta^3(x,y)\delta_a^b\delta_A^C. \tag{3}$$

Second, the constraint that generates left-handed supersymmetry transformations has the form

$$\mathcal{S}^A = \mathcal{D}_a \pi^{aA} = 0, \tag{4}$$

where $\mathcal{D}_a$ is the left-handed part of the gravitational connection, known as the Ashtekar connection.

The general relativity sector of supergravity can be taken to be the configurations in which the spinor field ψ_a^A and its conjugate momenta π_B^a vanish. But there is a larger sector of the phase space which is gauge equivalent to general relativity under local supersymmetry transformations. The left-handed part of this is

$$\psi_a^A \to \delta_\xi \psi_a^A = \{\psi_a^A, \mathcal{S}(\xi)\} = \mathcal{D}_a \xi^A. \tag{5}$$

To fully parametrize this sector of supergravity, which is gauge equivalent to general relativity, in a way that gets as close to preserving the Poisson brackets as possible, we may try to take

$$\tilde{\pi}_A^a \to \tilde{E}_{AB}^a \xi^B. \tag{6}$$

Then[2]

$$\{\psi_a^A(x), \tilde{\pi}_C^a(y)\}_+ \to \xi_E \xi^E \delta^3(x,y)\delta_a^b\delta_A^C. \tag{7}$$

Then the supersymmetry constraint $\mathcal{S}^E$ becomes an elliptic equation for ξ^E:

$$\mathcal{S}^E \to \mathcal{G}^{EF}\xi_F + \mathcal{W}(\xi)^E = 0 \tag{8}$$

where

$$\mathcal{W}_E = \tilde{E}_{EF}^a \mathcal{D}_a \xi^F = 0 \tag{9}$$

is known as the Witten equation, as it plays a key role in Witten's proof.

The other term in the equation is

$$\mathcal{G}_{AB}^{\mathrm{gr}} = \mathcal{D}_a \tilde{E}_{AB}^a = 0, \tag{10}$$

which is the Gauss law constraint that generates local chiral $SU(2)_L$ frame rotations.

To complete the description of this sector we may add a conjugate momenta π_E to the theory, satisfying

$$\{\xi)_A(x), \tilde{\pi}^C(y)\}_+ = \delta_A^C \delta^3(x,y). \tag{11}$$

This does not play much of a role, except in one place below.

Let us call this sector of supergravity the *bosonic sector of supergravity*. It is locally supergauge equivalent to general relativity, although it might have novel topological effects.

An appropriate restriction of the supercharge squared in (1) to this sector gauge equivalent to general relativity is then to square the Witten equation. This is the starting point of Witten's proof, which is reproduced in the next section.

If we seek to extend the positive energy proof of Witten to the quantum theory, the first question to be confronted is what is the appropriate way to represent the Witten spinor and its equation in the Hilbert space?

A first thought (which was investigated in [7]) is to take the spinor as an operator on the quantum gravity Hilbert space. This means to solve the Witten equation as a strong operator equation

$$\hat{\mathcal{W}}_E = \tilde{E}_{EF}^a \mathcal{D}_a \hat{\xi}^F = 0, \tag{12}$$

which when solved expresses $\hat{\xi}^F = \hat{\xi}^F(\hat{A}, \hat{E})$ as a (very) nonlinear and nonlocal functional of the gravitational operators $\hat{A}$ and $\hat{E}$. However, it turns out that because of operator ordering issues in the proof, the spinor operator $\hat{\xi}^F$ would have to commute with the operators that represent the Hamiltonian and diffeomorphism constraints and so be what is called a Dirac observable. Given that the Witten equation does not commute with those constraints this seems to be too much to ask.

[2] If we want to preserve the precise Poisson bracket we should take, instead of (6),

$$\tilde{\pi}_A^a \to \frac{1}{\xi_E \xi^E} \tilde{E}_{AB}^a \xi^B,$$

but this runs afoul of the fact that Grassmann numbers do not have inverses. To make sense of this we could try to extend the Grassmann algebra to a nonassociative algebra, but this is too much novelty for a peripheral point.

So we try here something different, which is to put the Witten spinor into the wave functional, so that quantum states are functionals of A_a^{AB} and ξ^E:

$$\Phi = \Phi[A_a^{AB}, \xi^E]. \tag{13}$$

This does not change the number of degrees of freedom because the wave functionals are subject to an additional pair of constraints—the Witten constraint

$$\hat{\mathcal{W}}^E \Phi[A_a^{AB}, \xi^E] = 0. \tag{14}$$

This can be thought of two ways. First, we are used in theories with gauge invariance to writing quantum states on wave functionals on configuration spaces with auxiliary variables, which are then restricted to a dependence on the physical degrees of freedom by constraint equations. This is just one more instance of it.

We can also understand the quantum states of the form $\Phi[A_a^{AB}, \xi^E]$ as a restriction to the bosonic sector of quantum supergravity.

This however raises a difficult issue, which is that the first class nature of the constraint algebra is lost during the reduction from $\mathcal{S}^E$ to $\mathcal{W}^E$. As just mentioned, $\mathcal{W}^E$ fails to Poisson commute with the usual constraints of general relativity. This means that the others cannot be imposed as constraints on states as is usually done in loop quantum gravity. Instead, the positive energy proof demands a weaker condition which is that the constraints—when smeared with a particular lapse and shift constructed from ξ^E—have vanishing expectation value.

This brings us to the statement of the main result. After this in Sec. 2, I present Witten's classical proof of the positivity of the ADM energy, expressed in Ashtekar variables [7]. In Sec. 3, I present a sketch of a translation of the classical proof into the quantum context.

B. Statement of the main result

The main result of this paper is a set of sufficient conditions that a quantization of general relativity must satisfy to have an operator representing the ADM energy whose expectation values are positive.

Consider a representation of quantum general relativity whose states are functionals of the Ashtekar connection and the auxiliary spinor variables ξ^E:

$$\Phi = \Phi(A_a^{AB}, \xi^E), \tag{15}$$

defined by the usual Ashtekar relations

$$\hat{\tilde{E}}{}^a_{AB}\Phi[A,\xi] = -\hbar \frac{\delta}{\delta A_a^{AB}}\Phi[A,\xi],$$
$$\hat{A}^a_{AB}\Phi[A,\xi] = A_a^{AB}\Phi[A,\xi], \tag{16}$$

together with operators for the spinor ξ^E and its conjugate momenta $\tilde{\pi}_C$

$$\hat{\tilde{\pi}}_B \Phi[A,\xi] = -\hbar \frac{\delta}{\delta \xi^B}\Phi[A,\xi], \qquad \hat{\xi}^E\Phi[A,\xi] = \xi^E \Phi[A,\xi], \tag{17}$$

which satisfies the following conditions:

(1) The inner product is defined by

$$\langle \Phi(A,\xi)|\Psi(A,\xi)\rangle = \int dA d\bar{A} d\xi d\bar{\xi}\, \bar{\Phi}(\bar{A},\bar{\xi}) e^{I(A,\bar{A},\xi,\bar{\xi})}\Psi(A,\xi), \tag{18}$$

where $I(A,\bar{A},\xi,\bar{\xi})$ satisfies three conditions. The first two are reality conditions for the frame fields and their time derivatives, while the third is a positivity condition for a certain operator:

$$\frac{\delta e^{I(A,\bar{A},\xi,\bar{\xi})}}{\delta \bar{A}_a^{A'B'}(x)} n^{A'A} n^{B'B} - \frac{\delta e^{I(A,\bar{A},\xi,\bar{\xi})}}{\delta A_a^{AB}(x)} = 0, \tag{19}$$

$$n^{B'B}\nabla_a \left[\frac{\hat{1}}{e} \frac{\delta}{\delta \bar{A}_{[a}^{A'B'}(x)} \frac{\delta}{\delta A_{b]}^{AB}(x)} e^{I(A,\bar{A},\xi,\bar{\xi})}\right] = 0, \tag{20}$$

$$\mathcal{Q}^{ab}_{B'B} \equiv n^{A'A} \frac{\delta}{\delta \bar{A}_{(a}^{A'B'}(x)} \frac{\delta}{\delta A_{b)}^{AB}(x)} e^{I(A,\bar{A},\xi,\bar{\xi})} > 0. \tag{21}$$

Here $n^{AA'} = n^a \sigma_a^{AA'}$ is a timeline unit normal such that $n^a n_b = n^a n^b \eta_{ab} = -1$.

(2) The quantum Witten equation holds as a constraint on states

$$\hat{\mathcal{W}}^A \Phi[A,\xi] = \frac{\delta}{\delta A_a^{AB}} \mathcal{D}_a \xi^B \Phi[A,\xi] = 0. \tag{22}$$

We impose the boundary condition that as we go to infinity, the ξ^E approaches a constant spinor λ^E such that

$$\bar{\lambda}^{E'}\lambda^E = s^{E'E}, \tag{23}$$

where $s^{A'A} = s^a \sigma_a^{AA'}$ is a constant future-pointing null vector that is normalized to

$$s^a n_a = -1. \tag{24}$$

(3) The E^a_{AB} define an invertible metric, so that $\frac{1}{e}$ is a well-defined operator.

(4) The *expectation values of the scalar and vector quantum constraints hold*, when smeared against particular lapse and shift constructed as follows from the Witten spinor:

$$\langle \Phi| \int_\Sigma \bar{\xi}^{A'} n^A_{A'} \hat{\mathcal{C}}_{AB} \xi^B |\Phi\rangle = 0 \tag{25}$$

in a particular ordering

$$\hat{\mathcal{C}}^{AB} = \hat{\hat{E}}^{aA}_{C} \hat{\hat{E}}^{aC}_{D} F^{B}_{abD}. \tag{26}$$

The equivalence of these four constraints to the usual form of the Ashtekar constraints, for nondegenerate three geometries, was shown first by Jacobson in [6].

The main result is then that when these conditions are satisfied the expectation value of the ADM Hamiltonian for the null translation at infinity generated by $s^{A'A}$ is positive definite, where

$$\langle M_{\mathrm{ADM}}\rangle = -\int dA d\bar{A} d\xi d\bar{\xi} \int_{\partial\Sigma} d^2\sigma_a (n^D_{B'} e^{I(A,\bar{A},\xi,\bar{\xi})})(\bar{\xi}^{B'} \bar{\Phi}[\bar{A},\bar{\xi}]) \frac{1}{e}\left(\frac{\delta}{\delta A^D_{[aA}(x)}\frac{\delta}{\delta A^{AB}_{b]}(x)} \mathcal{D}_b \xi_B \Phi[A,\xi]\right) \geq 0. \tag{27}$$

II. CLASSICAL PROOF OF POSITIVE ENERGY

We first present Witten's proof of positive ADM energy, translated into chiral Ashtekar variables.[3]

We start by squaring the Witten equation

$$\begin{aligned} 0 = R &= \int_\Sigma \frac{n^{A'A}}{e} \bar{\mathcal{W}}_{A'} \mathcal{W}_A \\ &= \int_\Sigma \frac{n^{A'A}}{e} \bar{\tilde{E}}^a_{A'B'} \bar{\mathcal{D}}_a \bar{\xi}^{B'} \tilde{E}^b_{AB} \mathcal{D}_b \xi^B. \end{aligned} \tag{28}$$

Note that the $\frac{1}{e}$ is necessary because the Witten equation, (9), inherits a density weight of one from that of the $\tilde{E}^a_{AB}$.

In the presence of the Gauss's law constraint $\mathcal{G}^{AB}_{\mathrm{gr}}$ this is equivalent to squaring the supersymmetry generator

$$0 = R \approx \int_\Sigma \frac{n^{A'A}}{e} \bar{\mathcal{S}}_{A'} \mathcal{S}_A. \tag{29}$$

We can divide R into symmetric and antisymmetric parts:

$$R = R^{\mathrm{sym}} + R^{\mathrm{anti}} = 0, \tag{30}$$

where

$$\begin{aligned} R^{\mathrm{sym}} &= \int_\Sigma \frac{n^{A'A}}{e} \bar{\tilde{E}}^{(a}_{A'B'} \bar{\mathcal{D}}_a \bar{\xi}^{B'} \tilde{E}^{b)}_{AB} \mathcal{D}_b \xi^B \\ &= \int_\Sigma n_{B'B} e q^{ab} \bar{\mathcal{D}}_a \bar{\xi}^{B'} \mathcal{D}_b \xi^B \geq 0 \end{aligned} \tag{31}$$

is positive definite.

We then turn our attention to the antisymmetric part

$$R^{\mathrm{anti}} = \int_\Sigma \frac{n^{A'A}}{e} \bar{\tilde{E}}^{[a}_{A'B'} \bar{\mathcal{D}}_a \bar{\xi}^{B'} \tilde{E}^{b]}_{AB} \mathcal{D}_b \xi^B \leq 0. \tag{32}$$

We note the reality conditions

$$n^{A'}_A \bar{\tilde{E}}^a_{A'B'} n^{B'}_B = \tilde{E}^a_{AB} \tag{33}$$

and

$$n^B_{B'} \nabla_a [\tilde{E}^{[aB'}_{A'} \tilde{E}^{b]C}_{B}] = 0. \tag{34}$$

We make an integration by parts

$$R^{\mathrm{anti}} = \int_{\partial\Sigma} d^2\sigma_a \mu^a - \int_\Sigma \frac{n^{A'A}}{e} \bar{\xi}_{A'} \mathcal{C}^C_A \xi_C \leq 0, \tag{35}$$

where

$$\mu^a = \frac{n^{A'A}}{e} \bar{\xi}_{A'} [\tilde{E}^{[a} \tilde{E}^{b]}]_{AB} \mathcal{D}_b \xi^B \tag{36}$$

and

$$\mathcal{C}^C_A = [\tilde{E}^{[a} \tilde{E}^{b]}]_{AB} F^{BC}_{ab} = 0 \tag{37}$$

are four equations, equivalent to the four Ashtekar constraints. When they are satisfied we have

$$\begin{aligned} -\int_{\partial\Sigma} d^2\sigma_a \mu^a &= -\int_{\partial\Sigma} d^2\sigma_a \frac{n^{A'A}}{e} \bar{\xi}_{A'} [\tilde{E}^{[a} \tilde{E}^{b]}]_{AB} \mathcal{D}_b \xi^B \\ &\equiv M_{ADM} \geq 0. \end{aligned} \tag{38}$$

Also, in the presence of the constraints, we have a positive definite expression for the null ADM mass[4]:

$$M_{\mathrm{ADM}} = R^{\mathrm{symm}} = \int_\Sigma n_{B'B} e q^{ab} \bar{\mathcal{D}}_a \bar{\xi}^{B'} \mathcal{D}_b \xi^B \geq 0. \tag{39}$$

Three comments are in order.

(1) The argument must be completed by a proof that the Witten equation (9) has solutions asymptotic to any fixed null spinor at spatial infinity. This is supplied by Witten [2], to which I have nothing to add.

(2) To derive the positivity of the more usual timelike ADM energy we need two spinors ξ^A_I, where $I = 1, 2$, each a solution to the Witten equation,

[3]This was done first in [7].

[4]Jacobson has derived this expression directly [8].

chosen so that instead of (23), we require that at infinity ξ_I^A approach fixed spinors λ_I^E, such that

$$\sum_I \bar{\lambda}_I^{E'} \lambda_I^E = n^{E'E}. \tag{40}$$

(3) If we now impose the standard falloff conditions on $\tilde{E}^a_{AB}$ and A_a^{AB} then, as shown in [9], (38) is equal to the standard ADM mass. However, it is important and interesting to note that even when less stringent boundary conditions are imposed (38) still holds; only now what is proved to be positive is a highly nonlinear expression, which we may call the generalized ADM energy.

III. QUANTUM POSITIVE ENERGY

Our aim in the following is to *find conditions a representation of quantum gravity may satisfy which are sufficient to guarantee the positive definiteness of an operator for the ADM mass.*

We begin again by squaring the Witten equation, only now we use the quantum version:

$$\begin{aligned} 0 = \langle R \rangle &= \int_\Sigma d^3x n^{A'A} \langle \bar{\mathcal{W}}_{A'}(x) \bar{\Phi}(A,\xi) | \frac{1}{e} | \mathcal{W}_A(x) \Phi(A,\xi) \rangle \\ &= \int dA d\bar{A} d\xi d\bar{\xi} e^{I(A,\bar{A},\xi,\bar{\xi})} \int_\Sigma d^3x \frac{n^{A'A}}{e} \left(\frac{\delta}{\delta \bar{A}_a^{A'B'}(x)} \bar{\mathcal{D}}_a \bar{\xi}^{B'} \bar{\Phi}[\bar{A},\bar{\xi}] \right) \left(\frac{\delta}{\delta A_b^{AB}(x)} \mathcal{D}_b \xi^B \Phi[A,\xi] \right). \end{aligned} \tag{41}$$

Again we divide into symmetric and antisymmetric parts:

$$\langle R \rangle = \langle R^{\mathrm{sym}} \rangle + \langle R^{\mathrm{anti}} \rangle = 0. \tag{42}$$

We want to show that the symmetric part is again positive definite. To do this we integrate functionally by parts twice, and use (19), to find

$$\begin{aligned} \langle R^{\mathrm{symm}} \rangle &= \int dA d\bar{A} d\xi d\bar{\xi} e^{I(A,\bar{A},\xi,\bar{\xi})} \int_\Sigma d^3x \frac{n^{A'A}}{e} \left(\frac{\delta}{\delta \bar{A}_{(a}^{A'B'}(x)} \bar{\mathcal{D}}_a \bar{\xi}^{B'} \bar{\Phi}[\bar{A},\bar{\xi}] \right) \left(\frac{\delta}{\delta A_{b)}^{AB}(x)} \mathcal{D}_b \xi^B \Phi[A,\xi] \right) \\ &= \int dA d\bar{A} d\xi d\bar{\xi} \int_\Sigma d^3x \frac{n^{A'A}}{e} \left(\frac{\delta}{\delta \bar{A}_{(a}^{A'B'}(x)} \frac{\delta}{\delta A_{b)}^{AB}(x)} e^{I(A,\bar{A},\xi,\bar{\xi})} \right) (\bar{\mathcal{D}}_a \xi^{B'} \bar{\Phi}[\bar{A},\bar{\xi}]) (\mathcal{D}_b \xi^B \Phi[A,\xi]) \\ &= \int dA d\bar{A} d\xi d\bar{\xi} \int_\Sigma d^3x \frac{1}{e} \mathcal{Q}^{ab}_{B'B} (\bar{\mathcal{D}}_a \xi^{B'} \bar{\Phi}[\bar{A},\bar{\xi}]) (\mathcal{D}_b \xi^B \Phi[A,\xi]) \geq 0, \end{aligned} \tag{43}$$

where

$$\mathcal{Q}^{ab}_{B'B} \equiv n^{A'A} \frac{\delta}{\delta \bar{A}_{(a}^{A'B'}(x)} \frac{\delta}{\delta A_{b)}^{AB}(x)} e^{I(A,\bar{A},\xi,\bar{\xi})}. \tag{44}$$

We now require that $\mathcal{Q}^{ab}_{B'B}$ be a positive Hermitian matrix, which is (21). This implies that (43) is positive definite.

We then study the antisymmetric part:

$$\langle R^{\mathrm{anti}} \rangle = \int dA d\bar{A} d\xi d\bar{\xi} \int_\Sigma d^3x \frac{n^{A'A}}{e} e^{I(A,\bar{A},\xi,\bar{\xi})} \left(\frac{\delta}{\delta \bar{A}_{[a}^{A'B'}(x)} \bar{\mathcal{D}}_a \bar{\xi}^{B'} \bar{\Phi}[\bar{A},\bar{\xi}] \right) \left(\frac{\delta}{\delta A_{b]}^{AB}(x)} \mathcal{D}_b \xi^B \Phi[A,\xi] \right) \leq 0. \tag{45}$$

We then functionally integrate by parts twice, but in a different way:

$$\begin{aligned} \langle R^{\mathrm{anti}} \rangle &= -\int dA d\bar{A} d\xi d\bar{\xi} \int_\Sigma d^3x \frac{n^{A'A}}{e} \left(\frac{\delta}{\delta \bar{A}_{[a}^{A'B'}(x)} e^{I(A,\bar{A},\xi,\bar{\xi})} \right) (\bar{\mathcal{D}}_a \bar{\xi}^{B'} \bar{\Phi}[\bar{A},\bar{\xi}]) \left(\frac{\delta}{\delta A_{b]}^{AB}(x)} \mathcal{D}_b \xi^B \Phi[A,\xi] \right) \\ &= -\int dA d\bar{A} d\xi d\bar{\xi} \int_\Sigma d^3x \frac{1}{e} \left(n^D_{B'} \frac{\delta}{\delta A_{[aA}^{D}(x)} e^{I(A,\bar{A},\xi,\bar{\xi})} \right) (\bar{\mathcal{D}}_a \bar{\xi}^{B'} \bar{\Phi}[\bar{A},\bar{\xi}]) \left(\frac{\delta}{\delta A_{b]}^{AB}(x)} \mathcal{D}_b \xi^B \Phi[A,\xi] \right) \\ &= \int dA d\bar{A} d\xi d\bar{\xi} \int_\Sigma d^3x \frac{1}{e} (n^D_{B'} e^{I(A,\bar{A},\xi,\bar{\xi})}) (\bar{\mathcal{D}}_a \bar{\xi}^{B'} \bar{\Phi}[\bar{A},\bar{\xi}]) \left(\frac{\delta}{\delta A_{[aA}^{D}(x)} \frac{\delta}{\delta A_{b]}^{AB}(x)} \mathcal{D}_b \xi^B \Phi[A,\xi] \right). \end{aligned} \tag{46}$$

We now integrate the $\bar{\mathcal{D}}_a$ by parts on Σ, which produces a boundary term

$$\langle R^{\rm anti}\rangle = \langle R^{\rm anti}\rangle^{\rm boundary} + \langle R^{\rm anti}\rangle^{\rm bulk}. \tag{47}$$

We deal with the bulk first:

$$\begin{aligned}\langle R^{\rm anti}\rangle^{\rm bulk} &= -\int dA d\bar{A} d\xi d\bar{\xi} \int_\Sigma d^3x (n^D_{B'} e^{I(A,\bar{A},\xi,\bar{\xi})})(\bar{\xi}^{B'}\bar{\Phi}[\bar{A},\bar{\xi}]) \frac{1}{e}\left(\frac{\delta}{\delta A^D_{[aA}(x)}\frac{\delta}{\delta A^{AB}_{b]}(x)} F^{BE}_{ab}\xi_E \Phi[A,\xi]\right)\\ &= -\int dA d\bar{A} d\xi d\bar{\xi} \int_\Sigma d^3x (n^D_{B'} e^{I(A,\bar{A},\xi,\bar{\xi})})(\bar{\xi}^{B'}\bar{\Phi}[\bar{A},\bar{\xi}]) \frac{1}{e}\hat{\mathcal{C}}_{DB}\xi^B\Phi[A,\xi] \sim 0,\end{aligned} \tag{48}$$

where we use the second reality condition (20).

Equation (48) tells us that the quantum diffeomorphism and Hamiltonian constraints are imposed with specific lapse and shift given by the Witten spinor, and only in the expectation value sense:

$$\langle\Phi| \int_\Sigma \bar{\xi}^{A'} n^A_{A'} \hat{\mathcal{C}}_{AB}\xi^B|\Phi\rangle = 0. \tag{49}$$

In addition, note that we find the constraints in a particular ordering:

$$\hat{\mathcal{C}}^{\,E}_D = \frac{\delta}{\delta A^D_{[aA}(x)}\frac{\delta}{\delta A^{AB}_{b]}(x)} F^{BE}_{ab}. \tag{50}$$

Finally, we have

$$-\langle R^{\rm anti}\rangle^{\rm boundary} \equiv \langle M_{\rm ADM}\rangle \geq 0. \tag{51}$$

The operator is

$$\langle M_{\rm ADM}\rangle = -\int dA d\bar{A} d\xi d\bar{\xi} \int_{\partial\Sigma} d^2\sigma_a (n^D_{B'} e^{I(A,\bar{A},\xi,\bar{\xi})})(\bar{\xi}^{B'}\bar{\Phi}[\bar{A},\bar{\xi}]) \frac{1}{e}\left(\frac{\delta}{\delta A^D_{[aA}(x)}\frac{\delta}{\delta A^{AB}_{b]}(x)} \mathcal{D}_b\xi_B\Phi[A,\xi]\right) \geq 0, \tag{52}$$

where we use the boundary conditions (23) and (24). This establishes the main result outlined in the introduction.

IV. CONCLUSIONS

We conclude with some comments on future work.

(i) We so far have skirted the tricky issue of imposing asymptotically flat boundary conditions in the quantum theory. This is possible because even the classical theory the proof works for a more general class of boundary conditions, establishing the positivity of the generalized ADM energy (38).

(ii) The above calculation establishes that a quantum positive energy theorem may be possible using a representation based on the Ashtekar connection. Left open is a key question of whether this use of the Ashtekar connection is necessary or whether a quantum positive energy result can be achieved for representations based on other connections, i.e. for values of the Immirzi parameter besides $\gamma = \imath$. One possible obstacle is that the Lorentzian Hamiltonian constraint is not polynomial for other values of γ, making the operator ordering and regularization issues much more challenging.

(iii) Another important open question is whether there exist inner products which satisfy the reality conditions (19) and (20) and positivity condition (21).

(iv) The form of the constraints needed for the result (4) is very weak; it may be that a stronger condition can be imposed. However this cannot be that the $\mathcal{C}^{AB}$ annihilate the states, as those are not first class with the Witten equation (22). Whether there is a stronger condition consistent with (22) is unknown.

(v) The Gauss law constraint does not come into the proof, except that the constraints found here are only equivalent to the ADM Hamiltonian constraint and generators of spatial diffeomorphisms in the presence of the $SU(2)$ Gauss's law constraint, (10). Thus we have to decide how Gauss's law is to be imposed in the quantum theory. This is complicated by the fact that the Gauss law (10) does not commute under Poisson brackets with the Witten equation. Thus we have three choices. (i) We can gauge fix and reduce, in which case the present results will have to be reexamined. (ii) We can impose the expectation value of the Gauss law constraint, following (4):

$$\langle \widehat{\mathcal{D}_a\tilde{E}^a_{AB}}\rangle = 0. \tag{53}$$

Or (iii) we can extend the Gauss law to act on the spinor ξ^E, to make it first class with the Witten equation, $[\mathcal{G}^{\rm extend}_{AB}, \mathcal{W}^E] \approx 0$, where

$$\mathcal{G}^{\rm extend}_{AB} = \mathcal{D}_a \tilde{E}^a_{AB} + \xi_{(A}\pi_{B)}, \tag{54}$$

and then impose it on a constraint on states

$$\mathcal{G}^{AB}_{\rm extended}|\Phi >= 0. \tag{55}$$

In this case we get a stronger constraint at a cost of slightly weakening the equivalence of the Ashtekar constraints to the ADM constraints.

(vi) This sketch of a formal proof should be strengthened by fully regulating the operator products involved. This can be attempted, either within the context of the kind of point split, but $SU(2)$ gauge invariant, regularization originally used in loop quantum gravity, as described in [10], or the more rigorous approaches that have become standard since [11]. This will, however, require that one key issue can be addressed:

(vii) *The issue of* $\frac{1}{e}$.—Finally, we should comment on the problem of defining the inverse metric determinant operator $\frac{1}{e}$. This is a crucial issue for loop quantum gravity and related nonperturbative approaches whose naive ground state corresponds to $\langle \tilde{E}^a_{AB} \rangle \approx 0$. The problem is that, as shown by [3], there exist nonsingular but degenerate solutions to the classical constraints of the Ashtekar formulation which are asymptotically flat but have negative ADM energy.

We can note that in the classical proof, the antisymmetric part $\frac{1}{e}$ occurs in the combination

$$\frac{1}{e}\tilde{E}^{[aA}_D \tilde{E}^{b]}_{AB} = \epsilon^{abc} e^{DB}_c, \tag{56}$$

where e^{DB}_c is the one form frame field. In this case in loop quantum gravity we can use Thiemann's trick to write

$$e^{A\hat{B}}_c(x) = [\hat{A}^{AB}_c(x), \hat{V}], \tag{57}$$

where $\hat{V}$ is the volume operator and a regularization for the $\hat{A}^{AB}_c(x)$ operator can be constructed from a limit of short holonomies, as explained in [11,12].

Using this the ADM operator can be written in LQG as

$$\hat{M}_{\rm ADM} = \int_{\partial\Sigma} d^2\sigma_a \epsilon^{abc} [\hat{A}^{AB}_c(x), \hat{V}] A_{bBA}, \tag{58}$$

and the constraint operators, in the single densitized form, are

$$\frac{1}{e}\hat{\mathcal{C}}_A^{\ C} = \epsilon^{abc}[\hat{A}^B_{cA}(x), \hat{V}] F_{abB}{}^C. \tag{59}$$

To establish that this form of the constraints, (59), leads to positivity of the corresponding form of the ADM energy, (58), we must show that they are equivalent as operators to the forms that arise from squaring the Witten constraint. That is, one must show the operator identity

$$\frac{\hat{1}}{e}\left(\frac{\delta}{\delta A^D_{[aA}(x)}\frac{\delta}{\delta A^{AB}_{b]}(x)}\right) = \epsilon^{abc}[\hat{A}^{DB}_c(x), \hat{V}]. \tag{60}$$

This is challenging. Moreover, I am not aware of a similar identity which can be used to define $\frac{1}{e}$ by itself or in combination with the symmetric product of $\tilde{E}^a_{AB}$ which occur in the operator $\hat{R}^{\rm symm}$ in (43). This remains the chief open problem required to run the proof in the context of loop quantum gravity.

One promising approach is to modify loop quantum gravity to incorporate nondegenerate geometries along the lines of [13] or [14].

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[1] P. Schoen and S. T. Yau, Phys. Rev. Lett. **43**, 1457 (1979); R. Schoen and S.-T. Yau, Commun. Math. Phys. **79**, 231 (1981).
[2] E. Witten, Commun. Math. Phys. **80**, 381 (1981).
[3] M. Varadarajan, Classical Quantum Gravity **8**, L235 (1991).
[4] M. Grisaru, Phys. Lett. **73B**, 207 (1978).
[5] S. Deser and C. Teitelboim, Phys. Rev. Lett. **39**, 249 (1977).
[6] T. Jacobson, Classical Quantum Gravity **5**, 923 (1988).
[7] L. Smolin and A. Starodubtsev, Positive Energy Theorem and the Factor Ordering Problem in Quantum Gravity (unpublished).
[8] T. Jacobson (unpublished).
[9] A. Ashtekar, *Lectures on Non-perturbative Canonical Gravity (Notes Prepared in Collaboration with R.S. Tate)* (World Scientific, Singapore, 1991).
[10] L. Smolin, arXiv:hep-th/9202022.
[11] T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, Cambridge, England, 2007).
[12] T. Thiemann, Classical Quantum Gravity **15**, 839 (1998).
[13] T. Koslowski and H. Sahlmann, SIGMA **8**, 026 (2012); H. Sahlmann, Classical Quantum Gravity **27**, 225007 (2010); T. A. Koslowski, arXiv:0709.3465; M. Campiglia and M. Varadarajan, arXiv:1406.0579.
[14] B. Dittrich and M. Geiller, arXiv:1401.6441.