

**Snyder dynamics in a Schwarzschild spacetime**S. Mignemi<sup>1,2,\*</sup> and R. Štrajn<sup>1,†</sup><sup>1</sup>*Dipartimento di Matematica e Informatica, Università di Cagliari, viale Merello 92, 09123 Cagliari, Italy*<sup>2</sup>*INFN, Sezione di Cagliari, Cittadella Universitaria, 09042 Monserrato, Cagliari, Italy*

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We calculate the orbits of a particle in Schwarzschild spacetime, assuming that the dynamics is governed by a Snyder symplectic structure. With this assumption, the perihelion shift of the planets acquires an additional contribution with respect to the one predicted by general relativity. Moreover, the equivalence principle is violated. If one assumes that Snyder mechanics is valid also for macroscopic systems, these results impose strong constraints on the value of the coupling parameter of the Snyder model.

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**I. INTRODUCTION**

Noncommutative geometry is becoming a serious candidate to describe spacetime at Planck scales, where quantum gravity effects are sensible. In particular it accounts for the existence of a minimal measurable length, that seems to be a common outcome of different quantum gravity theories.

Among the many possible versions of noncommutative geometry, a special place is taken by its original formulation, proposed by Snyder [1], since, contrary to many of its rivals, this model preserves the Lorentz invariance, which is at the basis of the present understanding of physics.

Although the validity of noncommutative geometry is presumably limited to Planck-scale physics, it may be interesting to investigate if its effects can extend to macroscopic systems, where the classical limit holds, like for example the solar system. In our point of view, this is not plausible, since noncommutative geometry is supposed to hold only at scales where quantum gravity is effective, whereas extending its validity much beyond this realm one risks to be faced with problems analogous to the so-called soccer-ball problem of doubly special relativity [2], which shows that paradoxical effects arise if one tries to apply deformed momentum relations (analogous to those holding in Snyder mechanics) to macroscopic bodies.

This opinion is confirmed also by previous studies of planetary motion based on Snyder dynamics [3] that, when confronted with observations, predict for the coupling constant of the model a scale well below the Planck scale that would be expected on dimensional grounds.

These estimates have, however, been obtained from a Newtonian theory, while the effect of general relativity certainly cannot be neglected at these scales. For this reason in the present paper we repeat the calculation of Snyder planetary orbits in a relativistic setting. The results will

partially confirm those of previous works [3], since the corrections to relativistic dynamics due to Snyder mechanics will turn out to be of the same order of magnitude as the ones obtained in the Newtonian approximation, although numerically different.

We recall that the Snyder model, in its classical limit, is based on the noncanonical Poisson brackets [1]

$$\begin{aligned} \{x_\mu, p_\nu\} &= \eta_{\mu\nu} + \beta^2 p_\mu p_\nu, & \{x_\mu, x_\nu\} &= \beta^2 J_{\mu\nu}, \\ \{p_\mu, p_\nu\} &= 0, \end{aligned} \quad (1.1)$$

where  $J_{\mu\nu} = x_\nu p_\mu - x_\mu p_\nu$ ,  $\eta_{\mu\nu}$  is the flat metric with signature  $(-1, 1, 1, 1)$  and  $\beta$  a coupling constant that is assumed to be of order one in Planck units. In ordinary unities, this corresponds to  $\beta \sim \sqrt{\hbar}/cM_{\text{Pl}} \sim 10^{-17} (\text{s/kg})^{1/2}$ . The Poisson brackets (1.1) preserve the Lorentz invariance, but deform the action of translations on spacetime [4]. Moreover, spacetime coordinates satisfy nontrivial brackets, that are the classical mechanics counterpart of spacetime noncommutativity.

The implications of the Snyder model have been studied from several points of views, either in their classical or quantum aspects [5]. Also the generalization to spaces of constant curvature has been considered to some extent [6]. However, in most cases the investigations have been limited to the nonrelativistic version of the theory, essentially because the relativistic model poses several technical and conceptual problems. To our knowledge a concrete example of relativistic dynamics has only been considered in [7] in the case of the harmonic oscillator.

Our approach to the problem of planetary orbits will be rather conservative: we write down the Hamilton equation of a free particle in a Schwarzschild background, and assume that the only changes in the dynamics are due to the Snyder noncanonical symplectic structure (1.1). In particular, we shall choose the same Hamiltonian as in general relativity, although the Snyder symmetries may allow for more general choices.

The Schwarzschild geodesics will be slightly deformed. In particular, a shift of the perihelion arises in addition to

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that predicted by general relativity, whose sign is however opposite to the one obtained from the calculation based on Newtonian gravity.

Another important outcome of our investigation is that the principle of equivalence is broken in Snyder mechanics, since the corrections to the equation of the geodesics depend on a parameter  $\beta^2 m^2$ , which is a function of the mass  $m$  of the particle. This effect is a consequence of the nontrivial dependence of the dynamics on the momenta of the particles, and it also puts strong limits on the value of the coupling constant  $\beta$  if the validity of Snyder mechanics at planetary scale is assumed.

Of course, the limitation of the validity of Snyder mechanics to microscopic physics should be justified. As mentioned before, this problem can be related to the soccer-ball problem of doubly special relativity: in fact, in Snyder spacetime the summation rules for the momenta must be nonlinear, since the translation invariance is deformed [4], and, following a reasoning analogous to that of Ref. [2], should be arranged in such a way that classical mechanics holds at macroscopic scales. A related argument, that has not been thoroughly investigated yet, is that passing from the quantum-gravity regime to its classical limit some kind of decoherence should occur and hence classical mechanics

is recovered, as in the classical limit of quantum mechanics. A discussion of this idea would however require a more definite theory of quantum gravity than available at present.

## II. PARTICLE MOTION IN FLAT SPACETIME

In order to set the formalism, we start by considering the free motion of a particle in three-dimensional flat Snyder spacetime. We parametrize the spatial sections with polar coordinates, defined in terms of cartesian coordinates as

$$\begin{aligned} t = x_0 = -x^0, \quad \rho &= \sqrt{(x^1)^2 + (x^2)^2}, \\ \theta &= \arctan \frac{x^2}{x^1}. \end{aligned} \quad (2.1)$$

The corresponding momentum components read

$$\begin{aligned} p_t = p_0, \quad p_\rho &= \frac{x^1 p_1 + x^2 p_2}{\sqrt{(x^1)^2 + (x^2)^2}}, \\ p_\theta &\equiv J_{12} = x_1 p_2 - x_2 p_1. \end{aligned} \quad (2.2)$$

With these definitions, the Poisson brackets for polar coordinates in Snyder space following from (1.1) are

$$\begin{aligned} \{t, p_t\} &= -1 + \beta^2 p_t^2, \quad \{\rho, p_\rho\} = 1 + \beta^2 \left( p_\rho^2 + \frac{p_\theta^2}{\rho^2} \right), \quad \{\theta, p_\theta\} = 1, \\ \{\rho, \theta\} &= \beta^2 \frac{p_\theta}{\rho}, \quad \{t, \rho\} = \beta^2 (t p_\rho - \rho p_t), \quad \{t, \theta\} = \beta^2 \frac{t p_\theta}{\rho^2}, \\ \{p_t, p_\rho\} &= -\beta^2 \frac{p_t p_\rho^2}{\rho^3}, \quad \{p_t, p_\theta\} = \{p_\rho, p_\theta\} = \{t, p_\theta\} = \{\rho, p_\theta\} = 0, \\ \{t, p_\rho\} &= \beta^2 \left( p_t p_\rho + \frac{t p_\theta^2}{\rho^3} \right), \quad \{\rho, p_t\} = \beta^2 p_t p_\rho, \quad \{\theta, p_t\} = \beta^2 \frac{p_t p_\theta}{\rho^2}, \quad \{\theta, p_\rho\} = \beta^2 \frac{p_\rho p_\theta}{\rho^2}. \end{aligned} \quad (2.3)$$

Note that, contrary to the canonical case, the choice of polar coordinates changes the symplectic structure.

The Hamiltonian is chosen as in special relativity,

$$H = \frac{\lambda}{2} \left( -p_t^2 + p_\rho^2 + \frac{p_\theta^2}{\rho^2} + m^2 \right) = 0, \quad (2.4)$$

with  $\lambda$  a Lagrange multiplier enforcing the mass shell constraint. The choice of the Hamiltonian is not unique, but (2.4) seems to be the most reasonable in this context.

The Hamilton equations derived from (2.3) and (2.4) are

$$\begin{aligned} \dot{t} &= \lambda \Delta p_t, \quad \dot{\rho} = \lambda \Delta p_\rho, \quad \dot{\theta} = \lambda \Delta \frac{p_\theta}{\rho^2}, \\ \dot{p}_t &= 0, \quad \dot{p}_\rho = \lambda \Delta \frac{p_\theta^2}{\rho^3}, \quad \dot{p}_\theta = 0, \end{aligned} \quad (2.5)$$

with  $\Delta = 1 - \beta^2 m^2$ . Hence, as in special relativity, the momenta  $p_\theta$  and  $p_t$  are constants of the motion, that according to the standard notations we denote  $ml$  and  $E$  respectively. They can be identified with the angular momentum and energy of the particle. As in 1+1 dimensions [7] all the equations are identical to those of classical relativity, except that they are multiplied by the common factor  $\Delta$ . Their solution can therefore be obtained as in special relativity, after a redefinition of the proper time.

In particular one should choose a gauge by fixing the time variable, in order to eliminate the Hamiltonian constraint (2.4) by means of the Dirac formalism. However, if one is only interested in the equation of the orbits, it is not necessary to fix the gauge since, like in special relativity,

$$\frac{d\rho}{d\theta} = \frac{\dot{\rho}}{\dot{\theta}} = \rho^2 \frac{p_\rho}{p_\theta}, \quad (2.6)$$

does not depend on  $\lambda$ . From the Hamiltonian constraint (2.4), it follows that

$$p_\rho = \sqrt{E^2 - m^2 \left(1 + \frac{l^2}{\rho^2}\right)}, \quad (2.7)$$

and hence

$$\rho' \equiv \frac{d\rho}{d\theta} = \frac{\rho}{l} \sqrt{\left(\frac{E^2}{m^2} - 1\right)\rho^2 - l^2}, \quad (2.8)$$

which is solved by

$$\rho = \frac{l}{\sqrt{E^2/m^2 - 1}} \frac{1}{\cos(\theta - \theta_0)}, \quad (2.9)$$

which describes a straight line in polar coordinates and coincides with the solution of classical special relativity.

### III. PARTICLE MOTION IN SCHWARZSCHILD SPACETIME

We now study the motion of a planet in the Schwarzschild spacetime with metric

$$ds^2 = -A(\rho)dt^2 + A^{-1}(\rho)d\rho^2 + \rho^2 d\Omega^2, \quad (3.1)$$

where

$$A(\rho) = 1 - \frac{2M}{\rho}, \quad (3.2)$$

and  $M$  is the mass of the sun. As in special relativity, due to the conservation of the angular momentum, the problem can be reduced to 2 + 1 dimensions.

The Hamiltonian is chosen as in standard relativity,

$$H = \frac{\lambda}{2} \left[ -\frac{p_t^2}{A} + A p_\rho^2 + \frac{p_\theta^2}{\rho^2} + m^2 \right] = 0, \quad (3.3)$$

where  $m$  is the mass of the planet.

The field equations derived from (2.3) and (3.3) are

$$\begin{aligned} \dot{i} &= \lambda \left[ p_t \left( A^{-1} - \beta^2 m^2 - \beta^2 \frac{M}{\rho} \left( p_\rho^2 + \frac{p_t^2}{A^2} \right) \right) + \beta^2 \frac{M t p_\rho}{\rho} \left( p_\rho^2 + \frac{p_t^2}{A^2} - 2 \frac{p_\theta^2}{\rho^2} \right) \right], \\ \dot{\rho} &= \lambda \left[ A - \beta^2 m^2 - \frac{2\beta^2 M p_\theta^2}{\rho^3} \right] p_\rho, \quad \dot{\theta} = \lambda \frac{p_\theta}{\rho^2} \left[ 1 - \beta^2 m^2 - \frac{\beta^2 M}{\rho} \left( p_\rho^2 + \frac{p_t^2}{A^2} \right) \right], \\ \dot{p}_t &= -\lambda \left[ \frac{\beta^2 M p_t p_\rho}{\rho^2} \left( p_\rho^2 - \frac{2p_\theta^2}{\rho^2} + \frac{p_t^2}{A^2} \right) \right], \quad \dot{p}_\theta = 0, \\ \dot{p}_\rho &= \lambda \left[ (1 - \beta^2 m^2) \frac{p_\theta^2}{\rho^3} - \frac{M}{\rho^2} \left[ \left( p_\rho^2 + \frac{p_t^2}{A^2} \right) \left( 1 + \beta^2 \left( p_\rho^2 + \frac{p_\theta^2}{\rho^2} \right) \right) - 2\beta^2 \frac{p_\rho p_\theta^2}{\rho} \right] \right]. \end{aligned} \quad (3.4)$$

We are only interested in the equation of the orbits. To find it we proceed as in the previous section. While  $p_\theta$  is still a constant,  $p_t$  is no longer conserved. Instead, one can check that the quantity

$$E = \frac{p_t}{\sqrt{1 + \beta^2(-p_t^2 + p_\rho^2 + p_\theta^2/\rho^2)}} \quad (3.5)$$

is conserved and plays the role of the energy. It follows that

$$p_t^2 = \frac{E^2}{1 + \beta^2 E^2} [1 + \beta^2(p_\rho^2 + p_\theta^2/\rho^2)]. \quad (3.6)$$

Moreover, (3.3) and (3.6) imply that

$$p_\rho^2 = \frac{E^2(1 + \beta^2 m^2 l^2/\rho^2) - m^2(1 + \beta^2 E^2)(1 + l^2/\rho^2)A}{(1 + \beta^2 E^2)A^2 - \beta^2 E^2}, \quad (3.7)$$

where we have defined  $l = p_\theta/m$ .

The equation of the orbits is conveniently written in terms of the variable  $u = 1/\rho$  as

$$\frac{du}{d\theta} = -\frac{1}{\rho^2} \frac{\dot{\rho}}{\dot{\theta}} = -\frac{A - \beta^2 m^2(1 + 2Ml^2 u^3)}{1 - \beta^2 m^2 - \beta^2 M u(p_\rho^2 + p_t^2/A^2)} \frac{p_\rho}{ml}. \quad (3.8)$$

Substituting in (3.8) the values of  $p_\rho$  and  $p_t$  deduced from (3.6) and (3.7), one can write down a differential equation for the single variable  $u(\theta)$ .

The calculations are very involved, and the equation can only be solved perturbatively. One can first expand in the Snyder parameter  $\beta^2 m^2$  and then adopt the usual expansion used in standard textbooks on general relativity to solve for the Schwarzschild orbits. To this end, it is useful to define the dimensionless quantities  $v = \frac{l^2}{M} u$  and  $\epsilon = \frac{M^2}{l^2}$ . The parameter  $\epsilon$  is small for planetary orbits, and can be taken as an expansion parameter. We assume moreover that  $\beta^2 m^2 \ll \epsilon$  since the Snyder corrections are

expected to be small with respect to those of general relativity. Moreover, by the virial theorem, and the definition (3.5) of  $E$ ,  $E^2 - m^2 \sim m^2(\epsilon q + \beta^2 E^2)$ , with  $q$  a parameter of order unity.

The first-order expansion in both  $\beta^2 m^2$  and  $\epsilon$  gives, after lengthy calculations,

$$v'' = q + 2v - v^2 + 2\epsilon v^3 + \beta^2 m^2 [2v + 4\epsilon(qv + v^2)]. \quad (3.9)$$

It is convenient to take the derivative of this expression. One has

$$v'' = 1 + \beta^2 m^2 - v + \epsilon[3v^2 + \beta^2 m^2(2q + 4v)]. \quad (3.10)$$

Expanding  $v = v_0 + \epsilon v_1 + \dots$ , at zeroth order one obtains a Newtonian approximation of the solution,

$$\begin{aligned} v_0 &= 1 + \beta^2 m^2 + e \cos \theta, \\ e &= 1 + \frac{q}{\epsilon} = 1 + \frac{l^2(E^2 - m^2)}{M^2 m^2}, \end{aligned} \quad (3.11)$$

while  $v_1$  satisfies

$$\begin{aligned} v_1'' + v_1 &= 3 + (10 + 2q)\beta^2 m^2 + 2(3 + 5\beta^2 m^2)e \cos \theta \\ &\quad + 3e^2 \cos^2 \theta, \end{aligned} \quad (3.12)$$

which is solved by

$$\begin{aligned} v_1 &= 3 \left( 1 + \frac{e^2}{2} \right) + 2\beta^2 m^2 (5 + q^2) + e(3 + 5\beta^2 m^2) \theta \sin \theta \\ &\quad - \frac{e^2}{2} \cos 2\theta. \end{aligned} \quad (3.13)$$

The solution at first order is therefore

$$\begin{aligned} v &\sim (1 + \beta^2 m^2) + \epsilon \left[ 3 \left( 1 + \frac{e^2}{2} \right) + 2\beta^2 m^2 (5 + q^2) \right] \\ &\quad + e \cos[(1 - \epsilon(3 + 5\beta^2 m^2))\theta] - \frac{\epsilon}{2} e^2 \cos 2\theta. \end{aligned}$$

From this expression one can easily obtain the perihelion shift as

$$\delta\theta = 2\pi\epsilon(3 + 5\beta^2 m^2) \sim \frac{6\pi M^2}{l^2} \left( 1 + \frac{5}{3}\beta^2 m^2 \right).$$

The first term is of course the one predicted by general relativity, while the second depends on the mass of the planet. This dependence is of course a consequence of the breaking of the equivalence principle in Snyder mechanics.

In a Newtonian setting, the shift due to Snyder mechanics is given by  $\delta\theta = -2\pi\beta^2 m^2 M^2 / l^2$  [3]. While the order of magnitude of the Snyder correction is the same as that obtained from the relativistic model, its sign is opposite. Therefore, calculations based on Newtonian mechanics are not much reliable in this context. In any case, it has been shown [3] that for these corrections to be compatible with the observed discrepancy of the perihelion shift of Mercury from the predictions of general relativity,  $\beta$  must be less than  $10^{-9}$  in Planck units. This estimate remains true in the relativistic case.

Another bound on the value of  $\beta$  can be obtained from the breaking of the equivalence principle caused by the presence of terms proportional to  $\beta^2 m^2$  in the corrections to the geodesics motion. Experimental data show that violation of the equivalence principle are less than one part in  $10^{12}$  [8]. It follows that  $\beta < 10^{-26}$  in Planck units for planetary masses of order  $10^{24} \text{ kg} = 10^{32} M_{\text{Pl}}$ . This bound is even stronger than the previous one.

These results seem to indicate that if one assumes that Snyder mechanics holds at scales compatible with the orbit of planets, the coupling constant  $\beta$  must be less than its natural value of order 1 in Planck units by many orders of magnitude.

As discussed in the Introduction, the most reasonable solution to this problem is that Snyder mechanics be valid only at Planck scales, while at larger scales the dynamics becomes classical, although the detailed mechanism of this transition has not been figured out yet.

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