PHYSICAL REVIEW D **90,** 044008 (2014)

One-dimensional action for simplicial gravity in three dimensions

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(Received 17 March 2014; published 5 August 2014)

We present a derivation of the Ponzano-Regge model from a one-dimensional spinor action. The construction starts from the first-order Palatini formalism in three dimensions. We introduce a simplicial decomposition of the three-dimensional manifold and study the discretized action in the spinorial representation of loop gravity. A one-dimensional refinement limit along the edges of the discretization brings us back to a continuum formulation. The three-dimensional action turns into a line integral over the one-skeleton of the simplicial manifold. All fields are continuous but have support only along the one-dimensional edges. We define the path integral and remove the redundant integrals over the local gauge orbits through the usual Faddeev-Popov procedure. The resulting state sum model reproduces the Ponzano-Regge amplitudes.

DOI: 10.1103/PhysRevD.90.044008 PACS numbers: 04.60.Pp, 04.60.Kz, 04.60.Nc

I. INTRODUCTION

Three-dimensional gravity is topological, there are no propagating degrees of freedom, and yet it is rich enough to make its quantization an intriguing problem [1–4]. Solving this problem is an important consistency check for any approach that aims at quantum gravity in the real world.

This article provides such a consistency check for the spinorial representation of loop gravity, recently developed by Freidel, Speziale, and collaborators [5–15]. The spinorial framework sits halfway in between the most familiar connection representation [16–19], and the dual Baratin-Oriti momentum representation [20,21]. The spinors simplify the kinematical structure of the theory. Can they also teach us something about the dynamics? Here, we study this question only for the case of Euclidean gravity in three dimensions, and find a neat derivation of the Ponzano-Regge model [3] from a one-dimensional spinor action.

The article develops two results. Section III gives the classical part. We discretize the first-order Palatini action S_M over a simplicial decomposition of the underlying manifold. A one-dimensional refinement limit brings us back to a continuum formulation. The resulting action is a line integral over the edges of the simplicial decomposition (Fig. 1). We can rearrange this action so as to get a sum over the elementary spin-foam faces f, each of which contributes as follows:

$$S_M = \sum_{f:\ \text{faces}} S_f, \quad \text{with} \quad S_f = -\mathrm{i}\hbar \int_{\partial f} (\langle z|D|z\rangle - \langle w|\mathrm{d}|w\rangle - \mathrm{i}\varphi \mathrm{d}t(\langle z|z\rangle - \langle w|w\rangle)).$$

The action S_M depends on two spinors for each face, but is also a functional of an $SU(2)$ connection A hiding in the covariant differential $D|z\rangle = \mathrm{d}|z\rangle + A|z\rangle$ of the spinor (and φ is a Lagrange multiplier, while t parametrizes the boundary of f). All fields are continuous, but are supported only along the one-dimensional edges of the discretization. Next, we study the local gauge symmetries of the theory and derive the equations of motion from the principle of least action. The resulting theory is a version of first order Regge calculus, with spinors as the fundamental configuration variables.

Section IV develops the second result and defines the transition amplitudes as a path integral over the spinorial variables,

$$Z_{\mathrm{PR}} = \int \mathcal{D}_{\mathrm{gf}}[z, w, \dots] \prod_{f:\ \text{faces}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_f}.$$

The integration measure includes a gauge fixing condition together with the corresponding Faddeev-Popov determinant. This removes the divergent integrals over the orbits of the local gauge symmetries. We evaluate the integral for a generic simplicial decomposition and establish the equivalence with the Ponzano-Regge spin-foam model. This is our final result. It proves that the Ponzano-Regge spin-foam model can be derived from a one-dimensional spinorial field theory over the[1] of the simplicial manifold.

II. EUCLIDEAN GRAVITY IN THREE DIMENSIONS

The entire section is a review, needed to make the article logically self-contained. Section II A introduces the most basic mathematical structures underlying three-dimensional

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[1] More precisely: The one-skeleton of the dual complex. This is the system of edges glued among the bounding vertices. See Fig. 1 for an illustration.

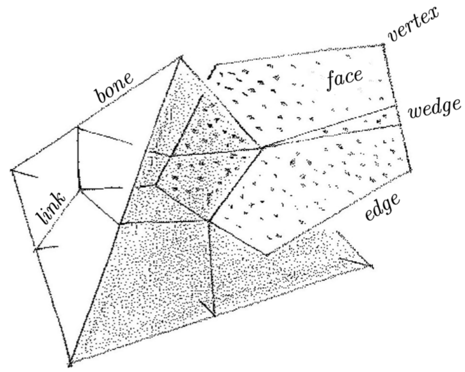

FIG. 1. A tetrahedron consists of four triangles glued together. Each tetrahedron contains its own dual, the *vertex*, a point inside. Three sides bound a triangle, and we call them the *bones* of the triangulation. Each bone belongs to many tetrahedra (vertices), but a triangle can be in only two of them. The surface dual to a bone is a *face*, and it touches all adjacent tetrahedra. An *edge*, the dual of a triangle, connects two vertices. A wedge is a "small" triangular part of a face: Two of its corners belong to an edge, and the third lies on the bone dual to the face. From the two-dimensional perspective of e.g. the boundary of a tetrahedron, a *link* is the dual of a bone. A wedge is thus bounded by two links and a short segment of an edge.

Euclidean gravity. Section II B gives the phase space for the discretized theory [22]. The concluding Sec. II C studies the spinorial representation of loop gravity as developed by Freidel, Speziale, and collaborators [5–14]. References [1,23–26] give further background material.

A. First-order action and symplectic structure

We are using first-order variables. The action for Euclidean gravity on a three-dimensional manifold M thus becomes

$$S_M[e,A] = \frac{\hbar}{2\ell_{\mathrm{P}}}\int_M \epsilon_{ijk} e^i \wedge F^{jk}, \tag{1}$$

where ℓ_{P} and $\hbar$ are the Planck length and Planck's constant, respectively, the flat Euclidean metric δ_{ij} moves all internal $\mathbb{R}^3$ indices $i, j, k, \ldots$, and ϵ_{ijk} is the Levi-Cività tensor in internal space. The action is a functional both of the $\mathfrak{so}(3)$ connection $A^i{}_{j\mu}$ and the cotriad $e^i{}_\mu$. The cotriad is an orthonormal frame, and it diagonalizes the Euclidean line element $g_{\mu\nu} = \delta_{ij} e^i{}_\mu e^j{}_\nu$. The $\mathfrak{so}(3)$ connection $A^i{}_j$ defines the curvature two-form $F^i{}_j = \mathrm{d}A^i{}_j + A^i{}_k \wedge A^k{}_j$. We can equally well work with an $\mathfrak{su}(2)$ connection instead. The isomorphism between $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ is given by $A^i{}_j = \epsilon^i{}_{kj} A^k \mapsto A = A^k \otimes \tau_k$, where τ_k is a basis in $\mathfrak{su}(2)$ such that $[\tau_i, \tau_j] = \epsilon_{ij}{}^k \tau_k$. If σ_i are the Pauli matrices, a possible choice is $\tau_i = (2\mathrm{i})^{-1}\sigma_i$. The action variation gives the equations of motion, namely,

$$\text{the torsionless condition}: T^i{}_{\mu\nu} = 2D_{[\mu} e^i{}_{\nu]} = 0, \tag{2a}$$

$$\text{and the flatness constraint}: F^i{}_{\mu\nu} = 0, \tag{2b}$$

where $D = \mathrm{d} + [A, \cdot]$ is the exterior covariant derivative, and $[\mu \cdots]$ denotes antisymmetrization of all intermediate indices. The unique solution of the torsionless condition $De^i = 0$ determines the $SU(2)$ connection as a functional of the triad: The $SU(2)$ connection $A^i{}_\mu$ turns into the Levi-Cività spin connection $\Gamma^i{}_\mu[e]$. The equation $F^i{}_{\mu\nu} = 0$, on the other hand, tells us that the curvature of the connection vanishes, and hence the metric $g_{\mu\nu} = e_{i\mu} e^i{}_\nu$ is locally flat.

We want to eventually quantize the theory, so let us briefly recapitulate those aspects of its Hamiltonian formulation that will become important for us. We start with a $2+1$ split of the three-dimensional manifold M and foliate $M \simeq \Sigma \times \mathbb{R}$ into $t = \mathrm{const}$ equal "time" slices $\Sigma_t \simeq \Sigma \times \{t\}$. The $2+1$ decomposition requires a time-flow vector field[2] $t^\mu \in TM$, transversal to the $t = \mathrm{const}$ hypersurfaces: $t^\mu \partial_\mu t = 1$. Once we have chosen such a vector field, we can define the spatial and "temporal" components of the configuration variables,

$$N^i \coloneqq t^\mu e^i{}_\mu, \qquad \phi^i \coloneqq t^\mu A^i{}_\mu,$$
$$e^i{}_a \coloneqq [\mathrm{em}_t^* e^i]_a, \qquad A^i{}_a \coloneqq [\mathrm{em}_t^* A^i]_a. \tag{3}$$

We are working with Euclidean geometries; therefore this time function has no physical meaning whatsoever. Moreover, $\mathrm{em}_t\colon \Sigma \to \Sigma_t; x \mapsto (x,t) \in M$ is the canonical embedding of Σ into M, and em_t^* is the corresponding pullback: $e^i{}_a$ and $A^i{}_a$ are fields intrinsically defined on Σ, they are the pullback of the three-dimensional fields $e^i{}_\mu$ and $A^i{}_\mu$ to the $t = \mathrm{const}$ slice. We also define the velocity $\dot{A}^i{}_a = [\mathrm{em}_t^*(\mathcal{L}_t A^i)]_a$ of the connection as the pullback of the Lie derivative $\mathcal{L}_t A^i{}_\mu$. If we now also introduce the covariant derivative D_a with respect to $A^i{}_a$ on Σ, and call $F^i{}_{ab} = [\mathrm{em}_t^* F^i]_{ab} = [D_a, D_b]^i$ its curvature, then we can write down the action[3] in the following canonical form:

$$S[e,A] = \frac{\hbar}{\ell_{\mathrm{P}}}\int_0^1 \mathrm{d}t \int_\Sigma \tilde{\eta}^{ab}\left(e_{ia}(\dot{A}^i{}_b - D_b\phi^i) - \frac{1}{2} N_i F^i{}_{ab}\right). \tag{4}$$

Here $\tilde{\eta}^{ab}$ is the Levi-Cività density; its inverse (a density of weight minus one) is $\underset{\sim}{\eta}_{ab}$ (and $\underset{\sim}{\eta}_{ab}\tilde{\eta}^{bc} = \delta^b_a$). Looking at the first term in the action, we can identify the symplectic structure; the only nonvanishing Poisson brackets are

$$\{e^i{}_a(x), A^j{}_b(y)\} = \frac{\ell_{\mathrm{P}}}{\hbar}\delta^{ij}\underset{\sim}{\eta}_{ab}\tilde{\delta}_\Sigma(x,y), \tag{5}$$

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[2] $\mu, \nu, \rho, \ldots$ $(a, b, c, \ldots)$ are abstract indices in TM $(T\Sigma)$.

[3] To evaluate the integral we need to speak about orientation. Assume that M is orientable, and so is Σ_t: If (t, X, Y) are positively oriented vector fields in M, we choose the orientation in Σ_t such that the duple (X, Y) has positive orientation.

where $\tilde{\delta}_\Sigma(x,y)$ is the Dirac distribution on the two-dimensional $t = \text{const}$ slice Σ, a scalar density of weight one.

The canonical coordinates in the phase space of the theory are thus an $\mathfrak{su}(2)$-valued one-form $e^i{}_a$ and an $SU(2)$ connection $A^i{}_a$. These are the spatial projections of the configuration variables e^i and A^i in the action (1). The temporal components N^i and ϕ^i play a different role; they appear as Lagrange multipliers and impose the constraints of the theory, which are nothing but the equations of motion (2) pulled back to the spatial slice.

B. Holonomy-flux variables

In loop gravity [18,19,25,26] we work on a truncated phase space of smeared variables. We can think of this truncation as the result of a discretization: All fields are discretized over the elementary building blocks of a triangulation[4] of the three-dimensional manifold [22,28].

We thus introduce a simplicial decomposition of M, which consists of *tetrahedra* T glued along their bounding *triangles* $\tau \subset \partial T$ (see Fig. 1 for an illustration). Each triangle bounds two tetrahedra and is itself bounded by three sides, which we call the *bones* $b \subset \partial\tau$ of the triangulation. It is also important to know about the dual picture: Each tetrahedron is dual to a *vertex* (a point $v \in M$), while each triangle is dual to an *edge* e (a one-dimensional line). Edges close to form two-dimensional surfaces. These are the *faces* f, each of which is dual to a bone. We can then use the three-dimensional discretization of M to triangulate a two-dimensional hypersurface $\Sigma \subset M$. This time, the elementary building blocks are just triangles glued along their bounding sides. From this two-dimensional perspective, every triangle is dual to a *node* (a point in Σ), and every bone $b, b', \ldots$ is dual to a *link* $\gamma, \gamma', \ldots$ (a path in Σ). At each node three links meet, and every link connects two adjacent nodes.

The elementary building blocks of the triangulation are oriented: Each face f carries an orientation, but the orientation of its bounding edges e is independent and may not match the induced orientation of ∂f. Furthermore, every bone has an orientation such that it is positively oriented relative[5] to its dual face f. There are also the oppositely oriented elements; we denote them f^{-1}, b^{-1} and so on. Consider now a two-dimensional oriented hypersurface $\Sigma \subset M$ formed by glueing together adjacent triangles. We have already introduced the links $\gamma, \gamma', \ldots \subset \Sigma$, each of which is dual (from the two-dimensional perspective of Σ) to a bone $b, b', \ldots \subset \Sigma$, and we now give them an orientation and demand that the duple $(\dot{\gamma}, \dot{b})$ of corresponding tangent vectors be positively oriented in Σ.

[4]In principle we do not have to stick to triangulations; the kinematics of loop gravity allows arbitrary polytopes [27].

[5]If Z is a tangent vector in b, and $(X,Y) \in TM \times TM$ is a positively oriented duple in f, then the triple (Z,X,Y) shall be positively oriented in M.

Next, we introduce the smearing. We take the oriented bones $b, b', \ldots$ and their dual in Σ (the links $\gamma, \gamma', \ldots$), and smear the elementary phase space variables $e^i{}_a$ and $A^i{}_a$ over these lower dimensional structures. The connection defines the parallel propagator between any two nodes as the path-ordered exponential,

$$\text{holonomy}: h[b] = \text{Pexp}\left(-\int_\gamma A\right) \in SU(2), \qquad (6)$$

where b is the bone dual to the link γ. This gives the smearing of the connection. For the triad the situation is a little more complicated. The triad is a one-form, and we can smear it over the bones of the triangulation. The naive definition $\ell^i[b] = \int_b e^i$ breaks, however, $SU(2)$ gauge invariance, because it does not make sense to add internal vectors that belong to different points in $b \subset \Sigma$. The solution is to introduce additional holonomies $h_{\delta(x\to\gamma(0))} \in SU(2)$ that transport any internal vector in $x \in b$ into the frame at the initial point of γ,

$$\text{flux}: \ell[b] = \int_b e^i(x) h_{\delta(x\to\gamma(0))} \tau_i h^{-1}_{\delta(x\to\gamma(0))} \in \mathfrak{su}(2). \qquad (7)$$

The underlying path $\delta(x \to \gamma(0))$ starts at $x \in b$, follows the bone to the intersection point $b \cap \gamma$, where it then leaves b, and goes along γ^{-1} until it reaches the source $\gamma(0)$.

Let us also mention the oppositely oriented elements. Changing the orientation amounts to replacing the loop variables according to the following scheme:

$$h[b^{-1}] = h[b]^{-1},$$
$$\underset{\sim}{\ell}[b] := \ell[b^{-1}] \equiv \ell^i[b^{-1}]\tau_i = -h[b]\ell[b]h[b]^{-1}. \qquad (8)$$

The commutation relations of the continuum theory (5) induce commutation relations for holonomies and fluxes,

$$\{h^A{}_B[b], h^C{}_D[b']\} = 0, \qquad (9a)$$

$$\{\ell_i[b], h^A{}_B[b']\} = +\frac{\ell_\text{P}}{\hbar}\delta_{bb'} h^A{}_C[b]\tau^C{}_{Bi}, \qquad (9b)$$

$$\{\ell_i[b], \ell_j[b']\} = +\frac{\ell_\text{P}}{\hbar}\delta_{bb'}\epsilon_{ij}{}^m \ell_m[b]. \qquad (9c)$$

Variables belonging to different links commute, and the algebra closes. The resulting phase space is nothing but the cotangent bundle $T^*SU(2)^L$ equipped with its natural symplectic structure (L counts the number of links in the triangulation).

C. Spinors for loop gravity

Before we continue our review and speak about loop gravity in the spinorial representation [5,6], let us first fix some conventions. We will mostly use an index notation and denote the spinors as elements $z^A, w^A, \ldots$, of $\mathbb{C}^2$ with $A \in \{0,1\}$ labeling their "up" and "down" components. There is also the complex conjugate vector space $\bar{\mathbb{C}}^2$, and an overbar decorates the corresponding indices: $\bar{z}^{\bar{A}} \in \bar{\mathbb{C}}^2$. Spinors carry a natural action of $SU(2)$, and the group acts through its fundamental matrix representation: $SU(2) \ni U : z^A \mapsto (Uz)^A = U^A{}_B z^B$. Elements of $SU(2)$ are both unimodular and Hermitian, thus implying that both the antisymmetric ϵ tensor and the Hermitian metric $\delta_{A\bar{A}}$ commute with the group action. We can thus invariantly move the spinor indices according to the following scheme[6]:

$$[z| = z_A = \epsilon_{BA} z^B, \qquad |z\rangle = z^A = \epsilon^{AB} z_B,$$
$$|z] = z^A_\dagger = \delta^{A\bar{A}} \bar{z}_{\bar{A}}, \qquad \langle z| = z^\dagger_A = \delta_{A\bar{A}} \bar{z}^{\bar{A}}, \tag{10}$$

and $\langle z|z\rangle = [z|z] = \|z\|^2 = \delta_{A\bar{A}} z^A \bar{z}^{\bar{A}}$ denotes the corresponding $SU(2)$ norm. Notice also that the intertwining maps (10) generalize naturally to any higher rank spinor $T^{ABC\cdots}$.

We now use these $SU(2)$ spinors to parametrize both holonomy and flux. The flux $\ell[b]$ is an element of $\mathfrak{su}(2)$, it defines an anti-Hermitian 2×2 matrix $\ell[b] = \ell^A{}_B[b] = \ell^i[b]\tau^A{}_{Bi}$, and thus it as two orthogonal eigenspinors $|z\rangle = z^A$ and $|z] = z^A_\dagger$. Their normalization is free, and we can conveniently choose it to measure the metrical length of b in units of the Planck length ℓ_{P},

$$\ell[b] = +\frac{\ell_{\mathrm{P}}}{4\mathrm{i}}(|z\rangle\langle z| - |z][z|), \qquad \ell_{AB}[b] = +\frac{\ell_{\mathrm{P}}}{2\mathrm{i}} z_{(A} z^\dagger_{B)}, \tag{11}$$

where $(A\cdots)$ denotes symmetrization of all intermediate indices. Normalized like this, the spinors are unique up to an overall $U(1)$ transformation $z^A \mapsto \mathrm{e}^{\mathrm{i}\Omega} z^A$. They belong to the $SU(2)$ frame at the initial point. In the frame at the final point (8) we can find another pair of diagonalizing spinors,

$$\underset{\sim}{\ell}[b] = -\frac{\ell_{\mathrm{P}}}{4\mathrm{i}}(|\underset{\sim}{z}\rangle\langle \underset{\sim}{z}| - |\underset{\sim}{z}][\underset{\sim}{z}|), \qquad \underset{\sim}{\ell}_{AB}[b] = -\frac{\ell_{\mathrm{P}}}{2\mathrm{i}} \underset{\sim}{z}_{(A} \underset{\sim}{z}^\dagger_{B)}. \tag{12}$$

The holonomy maps the flux $\ell[b]$ at the initial point into the flux $\underset{\sim}{\ell}[b]$ at the final point, Eq. (8) gives the precise relation. The spinors are unique up to an overall phase, and therefore Eq. (8) translates into the following condition:

$$\exists \Phi \in \mathbb{R} : \underset{\sim}{z}^A = \mathrm{e}^{\mathrm{i}\Phi} h^A{}_B[b] z^B. \tag{13}$$

[6]We give the only nonvanishing components of the invariant tensors: $\delta_{0\bar{0}} = 1 = \delta_{1\bar{1}}$ and $\epsilon_{01} = 1 = -\epsilon_{10}$; the inverse of the ϵ tensor is defined implicitly: $\epsilon^{BC}\epsilon_{AC} = \epsilon_A{}^B = \delta^B_A$.

There is thus an $SU(2)$ transformation that maps one spinor into the other, hence

$$C = \|\underset{\sim}{z}\|^2 - \|z\|^2 = 0. \tag{14}$$

This constraint imposes that the length of the bone is the same from whatever side we look at it, we call it the *length matching constraint*. We can now invert Eq. (13) thus providing a parametrization of the holonomy in terms of the spinors,

$$h = \frac{\mathrm{e}^{-\mathrm{i}\Phi}|\underset{\sim}{z}\rangle\langle z| + \mathrm{e}^{\mathrm{i}\Phi}|\underset{\sim}{z}][z|}{\|\underset{\sim}{z}\|\|z\|} \equiv h^A{}_B = \frac{\mathrm{e}^{-\mathrm{i}\Phi}\underset{\sim}{z}^A z^\dagger_B + \mathrm{e}^{-\mathrm{i}\Phi}\underset{\sim}{z}^A_\dagger z_B}{\|z\|\|\underset{\sim}{z}\|}. \tag{15}$$

So far we have just described a way to parametrize both holonomy and flux by a pair of spinors, but the spinorial formalism extends further. It can also capture the Poisson algebra of $T^*SU(2)$. The symplectic structure for two pairs of harmonic oscillators

$$\{z^\dagger_A, z^B\} = \frac{\mathrm{i}}{\hbar}\delta^B_A, \qquad \{\underset{\sim}{z}^\dagger_A, \underset{\sim}{z}^B\} = -\frac{\mathrm{i}}{\hbar}\delta^B_A, \tag{16}$$

induce commutation relations for holonomies (15) and fluxes (11),

$$\{h^A{}_B, h^C{}_D\} = +\frac{2}{\hbar\ell_{\mathrm{P}}}\|z\|^{-4}\|\underset{\sim}{z}\|^{-2} C\epsilon^{AC}\ell_{BD} - \frac{2}{\hbar\ell_{\mathrm{P}}}\|\underset{\sim}{z}\|^{-4}\|z\|^{-2} C\epsilon_{BD}\underset{\sim}{\ell}^{AC}, \tag{17a}$$

$$\{\ell_i, h^A{}_B\} = +\frac{\ell_{\mathrm{P}}}{\hbar} h^A{}_C \tau^C{}_{Bi}, \tag{17b}$$

$$\{\ell_i, \ell_j\} = +\frac{\ell_{\mathrm{P}}}{\hbar}\epsilon_{ij}{}^m \ell_m. \tag{17c}$$

On the constraint hypersurface $C = 0$ of the length matching constraint (14), these commutation relations reduce to the symplectic structure of $T^*SU(2)$, as given in (9). The Hamiltonian vector field $\mathfrak{X}_C = \{C, \cdot\}$ generates a flow inside the constraint hypersurface that leaves both holonomy (6) and flux (7) unchanged,

$$\exp(\varphi\hbar\mathfrak{X}_C) z^A = \mathrm{e}^{-\mathrm{i}\varphi} z^A, \qquad \exp(\varphi\hbar\mathfrak{X}_C)\underset{\sim}{z}^A = \mathrm{e}^{-\mathrm{i}\varphi}\underset{\sim}{z}^A. \tag{18}$$

Performing a symplectic quotient, thus projecting the orbits generated by C into a point, we arrive at almost all of the original phase space, exempt only of the submanifold $T_o := \{(h, X) \in SU(2) \times \mathfrak{su}(2) \simeq T^*SU(2) | X = 0\}$ of vanishing flux, where we reach a coordinate singularity. In other words $(\mathbb{C}^2 \times \mathbb{C}^2)//_C = T^*SU(2) - T_o$.

III. A SPINORIAL ACTION FOR DISCRETIZED GRAVITY

A. Discretization and partial continuum limit

The last section studied the kinematical structure of three-dimensional Euclidean gravity on a simplicial lattice. Now, we introduce the dynamics of the theory as derived from an action variation. This action is the key novelty of the paper and is based on what has been developed for the $(3+1)$-dimensional case in [13]. The derivation starts from a simplicial discretization of the action (1), but eventually yields again a continuum theory. This is possible through a partial continuum limit. The resulting action is a one-dimensional integral over the edges of the discretization. The three-dimensional action integral thus turns into a sum over one-dimensional line integrals.

We start with the discretization of the action (1) over the simplicial complex. This can be done with remarkable ease [18] and yields a sum over *wedges*,

$$S_M[e,A] = -\frac{\hbar}{\ell_{\mathrm{P}}}\int_M e_i \wedge F^i \approx -\frac{\hbar}{\ell_{\mathrm{P}}}\sum_{w:\ \mathrm{wedges}}\int_{b_w} e_i \int_w F^i. \tag{19}$$

Here, we have split every spin-foam face f into a sum over wedges, $f = \cup_{i=1}^N w_i$. Figure 1 gives an illustration of the geometry: A wedge w [29] is a triangular surface lying inside a spin-foam face f, two of its corners rest on an edge, and the third belongs to the bone b_w dual to w. Both b_w and w carry an orientation that agrees with the orientation of M: If the pair of tangent vectors (X,Y) is positively oriented in w, and Z is positively oriented in b_w, the triple (X,Y,Z) is positively oriented in M. The sum goes over only one of the two possible orientations of b_w.

The main idea of this section is to study a limiting process where the number of wedges goes to infinity. The result will turn the sum into an integral and give us a continuous action on each spin-foam face. For the moment let us study only one particular wedge w_o appearing in this sum. We then take the $SU(2)$ holonomy-flux variables and use them to parametrize the discretized action. For the flux the situation is immediate, and looking back[7] at (7) we trivially have

$$\int_{b_{w_o}} e = \ell[b_{w_o}]. \tag{20}$$

For the second piece, the curvature term $\int_{w_o} F$ in the action, we use the holonomy as an approximation. Consider first the differential of the holonomy under variations of the underlying path. Let $\gamma_\varepsilon\colon\ [0,1]\to M, s\mapsto\gamma_\varepsilon(s)$ be an ε-parameter family of paths. Taking derivatives with respect to s and ε we obtain the tangent vectors $\delta\gamma_\varepsilon(s) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\gamma_\varepsilon(s) \in T_{\gamma_\varepsilon(s)}M$ and $\gamma'_\varepsilon(s) = \frac{\mathrm{d}}{\mathrm{d}s}\gamma_\varepsilon(s)\in T_{\gamma_\varepsilon}(s)M$. Simplifying our notation we write $\delta\gamma \equiv \delta\gamma_{\varepsilon=0}$ and equally for all other quantities at $\varepsilon = 0$. We can then find the variation of the holonomy directly from its defining differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}s} h_{\gamma_\varepsilon(s)} = -A_{\gamma_\varepsilon(s)}(\gamma'_\varepsilon) h_{\gamma_\varepsilon(s)}. \tag{21}$$

This works as follows: We just take the differential of (21) with respect to ε, multiply everything by $h^{-1}_{\gamma_\varepsilon(s)}$, and integrate the resulting equation from $s=0$ to $s=1$. A partial integration eventually yields the desired variation of the holonomy,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} h_{\gamma_\varepsilon(1)} = -A_{\gamma(1)}(\delta\gamma)h_{\gamma(1)} + h_{\gamma(1)}A_{\gamma(0)}(\delta\gamma) + \int_0^1 \mathrm{d}s\, h_{\gamma(1)} h^{-1}_{\gamma(s)} F_{\gamma(s)}(\gamma',\delta\gamma) h_{\gamma(s)}. \tag{22}$$

Consider now the boundary ∂f of the underlying spin-foam face. This is a one-dimensional loop $\alpha\colon\ [0,1]\to M, t\mapsto\alpha(t)$, parametrized by some $t\in[0,1]$. The boundary of the wedge touches this loop in a small segment $\alpha(t_o, t_o+\Delta t)\subset\partial w_o$ corresponding to some interval $[t_o, t_o+\Delta t]$ in t. Two more sides bound the wedge w_o, these are the *half links* γ_{t_o} and $\gamma_{t_o+\Delta t}$: The path $\gamma_t\subset f$ connects the point $\alpha(t)$ on the boundary of f with the bone dual to the face: $\gamma_t(0) = \alpha(t)$ and $\gamma_t(1) = b_{w_o}\cap f$. Figures 1 and 2 should further clarify the situation.

Next, we take the holonomy $h_{\gamma_t} = \mathrm{Pexp}(-\int_{\gamma_t} A)$ along the connecting link and study its velocity as we move forward in t. This gives the infinitesimal change of h_{γ_t} under a variation $\gamma_t\to\gamma_t+\varepsilon\delta\gamma_t$ of the underlying path—a derivative just as in (22). The variation of the path vanishes at $t=1$, because all paths meet at the center of the spin-foam face: $\forall t,t'\in[0,1]\colon\ \gamma_t(1) = \gamma_{t'}(1)$; thus $\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t(1) = 0$, and hence

$$\frac{\mathrm{d}}{\mathrm{d}t} h_{\gamma_t(1)} - h_{\gamma_t(1)} A_{\gamma_t(0)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t\right) = \int_0^1 \mathrm{d}s\, h_{\gamma_t(1)} h^{-1}_{\gamma_t(s)} F_{\gamma_t(s)}\left(\frac{\mathrm{d}}{\mathrm{d}s}\gamma_t(s), \frac{\mathrm{d}}{\mathrm{d}t}\gamma_t(s)\right) h_{\gamma_t(s)}. \tag{23}$$

We can now use this equation to write the smeared curvature tensor as the covariant time derivative of the link holonomy,

$$h^{-1}_{\gamma_t(1)}\frac{D}{\mathrm{d}t} h_{\gamma_t(1)} \equiv h^{-1}_{\gamma_t(1)}\left(\frac{\mathrm{d}}{\mathrm{d}t} h_{\gamma_t(1)} - h_{\gamma_t(1)} A_{\alpha(t)}(\dot\alpha)\right) = \int_0^1 \mathrm{d}s\, h^{-1}_{\gamma_t(s)} F_{\gamma_t(s)}\left(\frac{\mathrm{d}}{\mathrm{d}s}\gamma_t(s), \frac{\mathrm{d}}{\mathrm{d}t}\gamma_t(s)\right) h_{\gamma_t(s)}. \tag{24}$$

Let us now isolate the contribution S_{w_o} to the discretized action (19) coming from the wedge w_o. Inserting our

[7] Equation (7) contains additional holonomies, and here we have dropped them to keep our formulas simple; adding them would not affect our final result.

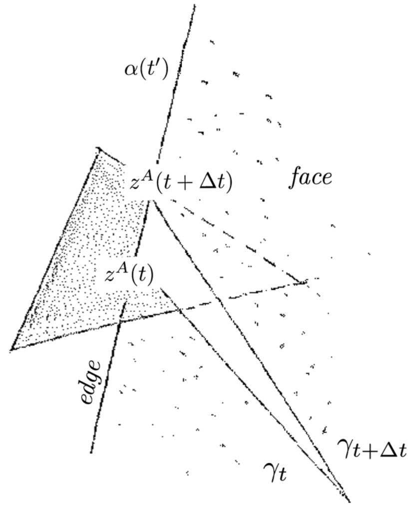

FIG. 2. Going from t to $t+\Delta t$ we can probe an infinitesimal wedge, the boundary of which has two parts. The first part belongs to the edge and has a tangent vector $\dot\alpha$. The second part (the triangular line in the picture) is a link inside the face and splits into two halves. Its "upper" part we call $\gamma_{t+\Delta t}$, while the lower half is γ_t, and putting them together determines $z^A(t+\Delta t)$: The spinor $z^A(t+\Delta t)$ is the parallel transport of $z^A(t)$ along the connecting link $\gamma^{-1}_{t+\Delta t}\circ\gamma_t$ modulo an overall phase $\Phi(t+\Delta t)$.

curvature formula (24) into the discretized action (19) we find that each wedge adds the term

$$S_{w_o} \approx -\frac{2\hbar\Delta t}{\ell_{\mathrm{P}}}\ell_{AB}[b_{w_o}]\Big(h^{-1}_{\gamma_{t_o}(1)}\frac{D}{\mathrm{d}t}h_{\gamma_{t_o}(1)}\Big)^{AB} \tag{25}$$

to the total action (19). This approximation improves as the wedge shrinks to a line, where it becomes exact. For the flux $\ell_{AB}[b_{w_o}]$, we can now find a diagonalizing spinor z^A just as in Eq. (11) above. Since this spinor belongs to the frame at $t=t_o$, it is better to write $z^A=z^A(t_o)$, and we get

$$\ell_{AB}[b_{w_o}] = \frac{\ell_{\mathrm{P}}}{2\mathrm{i}}z_{(A}(t_o)z^\dagger_{B)}(t_o). \tag{26}$$

We can repeat this construction for all other values of t, thus obtaining a map $z^A\colon [0,1]\to\mathbb{C}^2, t\mapsto z^A(t)$. For each value of t, the spinor is unique up to an overall phase. We can always choose this phase such that the spinor $z^A(t)$ is continuous in t. It should also respect the periodicity of the underlying loop α, and hence $z^A(0)\overset{!}{=}z^A(1)$.

We now turn to the link holonomy h_{γ_t} connecting $\alpha(t)$ with the center of the spin-foam face f. We introduce an additional spinorial field $w^A\colon [0,1]\to\mathbb{C}^2, t\mapsto w^A(t)$ in the frame at the center $b_{w_o}\cap f$ of the face and use the pair (z^A,w^A) of spinors to parametrize the connecting holonomy. Going back to (15) we get the precise relation

$$[h_{\gamma_t(1)}]^A{}_B = \frac{w^A(t)z^\dagger_B(t)+w^A_\dagger(t)z_B(t)}{\|w(t)\|\|z(t)\|}. \tag{27}$$

Just as $z^A(t)$ also $w^A(t)$ shall be both continuous and periodic in t: $w^A(0)=w^A(1)$. Compared to (15) we have ignored the possibility of a relative phase Φ between z^A and w^A. Setting $\Phi=0$ does not affect our final result. The spinors z^A and w^A are not independent, and once again we must respect the length matching constraint (14),

$$C=\|w\|^2-\|z\|^2\overset{!}{=}0. \tag{28}$$

There is a subtlety with the covariant time derivative of these spinors: $z^A(t)$ belongs to the frame at $\alpha(t)$, while $w^A(t)$ is a spinor living at the center of the spin-foam face. The tangent vector $\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t(s)$ vanishes at $s=1$ because $\gamma_t(s=1)$ is the same for all values of t—this is just a point in the center of the spin-foam face. On the other hand, $\gamma_t(0)=\alpha(t)$, and hence

$$\frac{D}{\mathrm{d}t}z^A(t) = \dot z^A(t) + A^i_\mu(\alpha(t))\dot\alpha^\mu(t)\tau^A{}_{Bi}z^B(t),$$
$$\text{but } \frac{D}{\mathrm{d}t}w^A(t)=\dot w^A(t). \tag{29}$$

Once again $\alpha(t)$ denotes the loop bounding the spin-foam face f, $\dot\alpha^\mu(t)$ is its tangent vector, while $A^i{}_\mu$ are the $SU(2)$ connection components with respect to the canonical generators $\{\tau_i\}_{i=1,2,3}$ of $\mathfrak{su}(2)$ ($2\mathrm{i}\tau_i$ are the usual Pauli matrices).

Inserting the velocities (29) together with Eqs. (27) and (26) into our expression for the wedge action (25) we eventually get

$$S_{w_o} = \frac{\mathrm{i}\hbar\Delta t}{2}\Big(\frac{\|z\|^2}{\|w\|^2}w^\dagger_A\dot w^A - \frac{\|z\|^2}{\|w\|^2}w_A\dot w^A_\dagger + z_A\frac{D}{\mathrm{d}t}z^A_\dagger - z^\dagger_A\frac{D}{\mathrm{d}t}z^A\Big)\Big|_{t=t_o}. \tag{30}$$

Let us now repeat this construction for all wedges w_i appearing in the decomposition of the spin-foam face $f=\cup_{i=1}^N w_i$. The discretization should be uniform in t: We can always choose the t coordinate such that the ith wedge w_i intersects the boundary ∂f in the t interval $[\frac{i-1}{N},\frac{i}{N}]$. The difference Δt thus represents the fraction N^{-1}. Sending $N\to\infty$ leads us to an integral over the entire spin-foam face,

$$S_f = \frac{\mathrm{i}\hbar}{2}\int_0^1\mathrm{d}t\Big(\frac{\|z\|^2}{\|w\|^2}w^\dagger_A\dot w^A - \frac{\|z\|^2}{\|w\|^2}w_A\dot w^A_\dagger + z_A\frac{D}{\mathrm{d}t}z^A_\dagger - z^\dagger_A\frac{D}{\mathrm{d}t}z^A\Big). \tag{31}$$

Functional variations of the spinors must respect the length matching constraint (28). We can account for this $C=0$ constraint by introducing a Lagrange multiplier φ and adding the term φC to the action. Notice now that on the constraint hypersurface $C = 0$, the variation of the fraction $\|z\|^2/\|w\|^2$ turns into the variation of the constraint itself,

$$\delta\left(\frac{\|z\|^2}{\|w\|^2}\right)\Big|_{C=0} = -\frac{\delta C}{\|z\|^2}. \tag{32}$$

Therefore, variations of $\|z\|^2/\|w\|^2$ just shift the value of the Lagrange multiplier φ. In other words, we can ignore these two fractions and work with a simplified action instead,

$$S_f[z,w,\varphi,A] = \frac{\mathrm{i}\hbar}{2}\int_0^1 \mathrm{d}t\left(w^\dagger_A \dot{w}^A - w_A \dot{w}^A_\dagger + z_A \frac{D}{\mathrm{d}t} z^A_\dagger - z^\dagger_A \frac{D}{\mathrm{d}t} z^A + 2\mathrm{i}\varphi(\|z\|^2 - \|w\|^2)\right)$$
$$= -\mathrm{i}\hbar\int_0^1 \mathrm{d}t\left(z^\dagger_A \frac{D}{\mathrm{d}t} z^A - w^\dagger_A \dot{w}^A - \mathrm{i}\varphi(\|z\|^2 - \|w\|^2)\right). \tag{33}$$

The last step involved a partial integration, which, thanks to the periodicity of the spinors, does not yield any additional boundary terms.

Each spin-foam face contributes through Eq. (33) to the total action (19). Equation (33) is a functional that depends on three elements: the spinors z^A and w^A, the gauge connection A, and a $U(1)$ angle φ. Let us now repeat the construction for the entire simplicial decomposition. We thus have spinor fields $z^A_f\colon\ \partial f \to \mathbb{C}^2$, $w^A_f\colon\ \partial f \to \mathbb{C}^2$, and a $U(1)$ angle $\varphi_f\colon \partial f \to \mathbb{R}$ attached to each face f. The $SU(2)$ connection $A_e(t) = A_\mu(e(t))\dot{e}^\mu(t) \in \mathfrak{su}(2)$ belongs to the edges e of the discretization, where $\dot{e}(t) \in T_{e(t)}M$ denotes the corresponding tangent vector. The boundary conditions are such that all spinors are continuous once we go around the spin-foam face (the angle φ_f must only be periodic modulo 2π). We thus get the following action for the discretized manifold M,

$$\begin{aligned}
&S_M[z_{f_1}, z_{f_2},\ldots; w_{f_1}, w_{f_2},\ldots; \varphi_{f_1}, \varphi_{f_2},\ldots; A_{e_1}, A_{e_2},\ldots]\\
&\quad\equiv S_M[\underline{z},\underline{w},\underline{\varphi},\underline{A}]\\
&\quad= -\mathrm{i}\hbar\sum_f \oint_{\partial f}(z^\dagger_{fA} D z^A_f - w^\dagger_{fA}\mathrm{d}w^A_f\\
&\qquad - \mathrm{d}t\mathrm{i}\varphi_f(\|z_f\|^2 - \|w_f\|^2))\\
&\quad= -\mathrm{i}\hbar\sum_f \oint_{\partial f}(\langle z_f|D|z_f\rangle - \langle w_f|\mathrm{d}|w_f\rangle\\
&\qquad - \mathrm{d}t\mathrm{i}\varphi_f(\|z_f\|^2 - \|w_f\|^2)).
\end{aligned} \tag{34}$$

The action does not care about the value of the connection in the "bulk"; it only probes the connection along the edges through the covariant t derivative,

$$\frac{D}{\mathrm{d}t} z^A(t) = \dot{z}^A(t) + A^i_e(t)\tau^A{}_{Bi} z^B(t), \quad \text{with} \quad A^i_e(t) = A^i{}_\mu(e(t))\dot{e}^\mu(t), \tag{35}$$

where t parametrizes the edge e, and $\dot{e}^\mu(t)$ denotes its tangent vector.

B. Equations of motion and gauge symmetries

Now that we have a continuous action (34) for the discretized manifold we have to find its extremum, identify the equations of motion, and compare their solutions with those (2) of the continuum theory. All fields in the action (34)—the spinors, the gauge connection A^i, and the Lagrange multipliers φ—have support only on the one-dimensional edges of the discretization. The resulting equations of motion are therefore all local in t. This is a huge simplification compared to other discretization schemes, where one has to deal with difference equations instead (see for instance [30–32]). Here, all fields are continuous along the edges of the discretization.

Our analysis of the action variation splits into four steps: First of all, we give the evolution equations along the edges of the discretization, we then study the constraint equations, and eventually we speak about the canonical formalism and the gauge symmetries of the theory.

(i) Evolution equations. We start with the evolution equations for the spinors. The spinor fields z^A_f and w^A_f only appear in the corresponding face action $S_f[z_f, w_f, \varphi_f, A]$. Its action variation yields the evolution equations,

$$\frac{D}{\mathrm{d}t} z^A = \mathrm{i}\varphi z^A, \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} w^A = \mathrm{i}\varphi w^A, \tag{36}$$

where we have dropped the face label $z^A_f \equiv z^A$ for simplicity. We can immediately integrate these equations. The holonomy parallel transports the z spinors up to an overall phase, and this phase also turns the w spinors around,

$$z^A(t) = \mathrm{e}^{\mathrm{i}\Phi(t)} U^A{}_B(t) z^B(0), \quad w^A(t) = \mathrm{e}^{\mathrm{i}\Phi(t)} w^A(0). \tag{37}$$

We have introduced some new elements here: $U(t)$ is the $SU(2)$ holonomy around the boundary of the spin-foam face, from $t_0 = 0$ to $t_1 = t$, while the integral over the Lagrange multiplier φ gives the overall angle $\Phi(t)$,

$$U^A{}_B(t) = \mathrm{Pexp}\left(-\int_0^t \mathrm{d}s A_\mu(\alpha(s))\dot{\alpha}^\mu(s)\right)^A{}_B,$$
$$\text{and} \quad \Phi(t) = \int_0^t \mathrm{d}s\varphi(s). \tag{38}$$

Furthermore, $\alpha\colon [0,1] \to M$ bounds the spin-foam face f, and the orientation of α agrees with the induced orientation of ∂f.

Let us now see what happens once we go around the spin-foam face and close $\alpha(t)$, hence forming a loop. The w^A spinors are periodic in t, and looking back at (37) and (38), we see this immediately implies that

$$\forall f : \exists n_f \in \mathbb{Z}\colon \int_{\partial f} \mathrm{d}t \varphi_f = 2\pi n_f. \tag{39}$$

This, together with the periodic boundary conditions for the z^A spinors, gives us an eigenvalue equation for the $SU(2)$ holonomy around the bounding loop

$$z^A(0) = U^A{}_B(1) z^B(0). \tag{40}$$

Having one twice degenerate eigenvalue, this $SU(2)$ element can only be the identity: $U(1) = \mathbb{1}$. This must be true for all spin-foam faces appearing in the simplicial complex; hence

$$\forall f : h^A{}_B[\partial f] \coloneqq \mathrm{Pexp}\Big(-\oint_{\partial f} A\Big)^A{}_B = \delta^A_B. \tag{41}$$

This is the discrete analogue of the flatness condition $F^i{}_{\mu\nu} = 0$, i.e. Eq. (2b), because the holonomy well approximates the curvature in the spin-foam face f: $h^{(AB)}[\partial f] \approx \int_f F^{AB}$. We have thus recovered already one-half of the equations of motion (2) of the continuum theory. What about the other half, that is the vanishing of torsion $D_{[\mu} e^i{}_{\nu]} = 0$ as implied by Eq. (2a)? In the continuum, the vanishing of torsion follows from the connection variation. The same happens in the discrete: The variation of the spinor action (34) with respect to the $SU(2)$ connection $A_e(t)$ on the edges e will give us the discrete version of the torsionless condition. This is the Gauß law, which brings us to the next step of our analysis:

(ii) Constraint equations. We now study the constraint equations of the theory, and we start with the Gauß law. We obtain it from the variation of the action (34) with respect to the gauge potential $A^i_e(t)$ on the edges [as defined in (35) as the $SU(2)$ connection contracted with the tangent vector $\dot{e}^\mu$]. This gauge potential only appears in the covariant derivative D of the z spinors into the direction of the edge. Every edge bounds three faces f, each of which carries its own z_f spinor. There are thus three such differentials $D/\mathrm{d}t z^A_f$ for each value of t. Since we also have the w_f spinors, there are altogether six spinors per edge. To keep our notation simple, let us only study one edge e in the triangulation and call the corresponding spinors $(z^A_f, w^A_f)_{f=1,2,3}$, where $f = 1, 2, 3$ label the three adjacent faces. The edge e thus contributes to the full action (34) for the discretized manifold M through the expression

$$\begin{aligned} S_e[\underline{z}, \underline{w}, \underline{\varphi}, A_e] &= -\mathrm{i}\hbar \sum_{f=1}^{3} \int_{t_o}^{t_1} \mathrm{d}t \Big(z^\dagger_{fA} \frac{\mathrm{d}}{\mathrm{d}t} z^A_f + z^\dagger_{fA} \tau^A{}_{Bi} z^B_f A^i_e(t) - w^\dagger_{fA} \frac{\mathrm{d}}{\mathrm{d}t} w^A_f - \mathrm{i}\varphi_f (\|z_f\| - \|w_f\|^2) \Big) \\ &\equiv -\mathrm{i}\hbar \sum_{f=1}^{3} \int_{t_o}^{t_1} \mathrm{d}t \Big(\Big\langle z_f \Big| \frac{\mathrm{d}}{\mathrm{d}t} \Big| z_f \Big\rangle + \langle z_f | \tau_i | z_f \rangle A^i_e(t) - \Big\langle w_f \Big| \frac{\mathrm{d}}{\mathrm{d}t} \Big| w_f \Big\rangle - \mathrm{i}\varphi_f (\|z_f\| - \|w_f\|^2) \Big), \end{aligned} \tag{42}$$

which we shall call the *edge action*. Note that an implicit assumption is hiding here: There are three spin-foam faces meeting at the edge e (again we refer to Fig. 1 for an illustration), and each of them carries an orientation. These orientations may not match the orientation of the edge e, while in Eq. (42) we have implicitly assumed so. If the orientations did not match, relative sign factors would be necessary. We could then, however, always absorb those factors of ± 1 into a redefinition of the spinors: The replacement $z^A \to z^A_\dagger$ would bring us back to (42), modified only by a boundary term that is irrelevant for the following argument.

Variation of (42) with respect to the connection A^i_e gives us a constraint: The three internal vectors $\ell_i[b]$, defined as in (26), must close to form a triangle,

$$G_i \coloneqq \frac{\hbar}{\ell_{\mathrm{P}}} \sum_{f=1}^{3} \ell_i[b_f]_t = \mathrm{i}\hbar \sum_{f=1}^{3} \tau^{AB}{}_i z^\dagger_{fA}(t) z_{fB}(t) = 0. \tag{43}$$

The vanishing of G_i has a clean geometric interpretation. It gives us a discretization of the torsionless equation $D_{[\mu} e^i{}_{\nu]} = 0$, i.e. Eq. (2), smeared over the triangle dual to the edge. Indeed, there is the non-Abelian version of Stoke's theorem, and it tells us that for any triangle τ the fluxes through its bounding sides sum up to zero: $\int_\tau T^i = \int_\tau De^i = \int_{\partial\tau} e^i = \sum_{b\subset\partial T} \int_b e^i = \sum_{b\subset\partial T} \ell^i[b]$. This series of equations is true only in a small neighborhood, where we can map the internal $\mathfrak{su}(2)$ index i into a common frame, reached by a family of holonomies just as in Eq. (7). The geometric interpretation of (43) is immediate, for the vector $\ell^i[b]_t \in \mathbb{R}^3$ represents the bone b in the internal frame at the point $e(t)$ of the edge.

The three bounding sides close to form a triangle, but this is not any triangle: it is the triangle dual to the edge mapped into the local frame of reference. As we go along the edge and move forward in t, this triangle preserves its shape, and the evolution equations (42) for the spinors $z^A_f(t)$ just rigidly turn it around.

There is one more constraint to be studied. The variation of the Lagrange multiplier φ yields the length matching condition (28). Once again its geometrical meaning is immediate. The constraint $C = 0$ tells us that the w spinors and z spinors describe the very same geometrical object, the triad smeared over the bounding bones, evaluated just in two different frames, one at the center of the spinfoam face and the other attached to its boundary. There is a unique $SU(2)$ element (27) that maps one of these spinors into the other, and it gives us the parallel transport from the boundary of the spin-foam face toward its center.

(iii) Canonical formalism. Looking at the edge action, we can immediately read off the symplectic structure. The elementary Poisson brackets are

$$\{z^\dagger_{fA}, z^B_{f'}\} = +\frac{\mathrm{i}}{\hbar}\delta_{ff'}\delta^B_A, \qquad \{w^\dagger_{fA}, w^B_{f'}\} = -\frac{\mathrm{i}}{\hbar}\delta_{ff'}\delta^B_A, \tag{44}$$

while all mutual Poisson brackets between the w and z spinors vanish. Notice that this agrees with our conventions from our introductory section II C. Equation (16) introduced the spinorial Poisson brackets essentially by hand, and here they naturally fall out of the formalism.

The evolution equations (36) are generated by a Hamiltonian. This t-dependent Hamiltonian is a sum over both Gauß's law and the triple of length matching constraints,

$$H_t = -\mathrm{i}\hbar\sum_{f=1}^{3}[A_e(t)^i\tau^{AB}{}_i z^\dagger_{fA}z_{fB} + \mathrm{i}\varphi_f(\|z_f\|^2 - \|w_f\|^2)]. \tag{45}$$

That the Hamiltonian is a sum over constraints, and hence vanishes, should not surprise us. Indeed, the action is a prototypical example of timeless systems [19], invariant under reparametrizations in t. We are thus dealing with a general covariant system, systems for which the Hamiltonian always turns into a sum over constraints.

Although the Hamiltonian vanishes, this does not mean that the evolution equations are totally trivial. If $F\colon (\mathbb{C}^2\times\mathbb{C}^2)^3 \to \mathbb{C}, (z^A_f, w^A_f)_{f=1,2,3} \mapsto F[(z^A_f, w^A_f)_{f=1,2,3}]$ is a function on the phase space of an edge, the Hamilton equations imply, in fact,

$$\frac{\mathrm{d}}{\mathrm{d}t}F_t = \{H, F\}_t = \mathfrak{X}_H[F]_t \overset{\text{in general}}{\neq} 0. \tag{46}$$

The action (42) describes 12 harmonic oscillators coupled by six first-class constraints. The first three of them (43) impose the vanishing of the total "angular momentum" of the system, and the other three— these are the length matching conditions (28) on the faces— require that the spinors have equal "energy": $C_f = \|z_f\|^2 - \|w_f\|^2 = 0$. This "energy condition" resonates with recent developments of Frodden, Ghosh, and Perez [33] and Bianchi [34], who argued that in four dimensions the horizon area measures the local energy of a stationary observer at a short distance from the horizon. In three dimensions area becomes length, and indeed the length $\mathbf{L}[b] = \ell_\mathrm{P}\|w\|^2$ of the bones b linearly appear in our edge Hamiltonian (45). At the moment, this analogy is very vague and deserves a more profound investigation.

(iv) Gauge symmetries. What are the gauge symmetries of the system? First of all, there is the one-dimensional diffeomorphism invariance of the action. Replacing the t coordinate by $\tilde{t}(t)$ leaves the action invariant, provided we also change the Lagrange multipliers appropriately,

$$\tilde{A}^i_e(\tilde{t}) = \frac{\mathrm{d}t}{\mathrm{d}\tilde{t}}A^i_e(t), \quad \text{and} \quad \tilde{\varphi}(\tilde{t}) = \frac{\mathrm{d}t}{\mathrm{d}\tilde{t}}\varphi(t). \tag{47}$$

This gives us the first gauge symmetry. Then, there are those symmetries that are generated by the Hamiltonian vector field of the constraints of the system: In fact, the length matching constraint generates $U(1)$ gauge transformations,

$$\tilde{z}^A(t) = \mathrm{e}^{-\mathrm{i}\lambda(t)}z^A(t) = \exp(\lambda(t)\hbar\mathfrak{X}_C)z^A|_t, \tag{48a}$$

$$\tilde{w}^A(t) = \mathrm{e}^{-\mathrm{i}\lambda(t)}w^A(t) = \exp(\lambda(t)\hbar\mathfrak{X}_C)w^A|_t, \tag{48b}$$

where $\mathfrak{X}_C = \{C, \cdot\}$ is the Hamiltonian vector field of the constraint. These generators transform each (z, w) pair of spinors independently, so we rather have an $U(1)^3$ symmetry per edge. Equations (48) alone would not preserve the Lagrangian (42). The $U(1)$ gauge symmetry also shifts the Lagrange multipliers, which transform as $U(1)$ gauge potentials,

$$\tilde{\varphi}_f(t) = \varphi_f(t) + \dot{\lambda}_f(t). \tag{49}$$

The internal $SU(2)$ invariance gives us another local gauge symmetry. The Gauß constraint G_i generates, in fact, local $SU(2)$ gauge transformations $g(t) \in SU(2)$ and rigidly moves the spinors around,

$$\tilde{z}^A_f(t) = \exp(-\Lambda^i(t)\mathfrak{X}_{G_i})z^A_f|_t = g^{-1}(t)^A{}_B z^B_f(t),$$
$$\text{with} \quad g(t) = \exp(\Lambda^i\tau_i). \tag{50}$$

Just as for the $U(1)$ symmetry, we also have to change the gauge potential to keep the action invariant. The gauge potential $A^i_e(t)$ defines a $SU(2)$ connection on a line, and hence transforms inhomgenously under $SU(2)$,

$$\tilde{A}_e(t) = (\rho_\Lambda A)_e(t) := g^{-1}(t)\frac{\mathrm{d}}{\mathrm{d}t}g(t) + g^{-1}(t)A_e(t)g(t). \tag{51}$$

The local $SU(2)$ transformations (50), together with (51), clearly preserve the Lagrangian (42). In summary, the action (42) has three local gauge symmetries: First of all, there is the reparametrization invariance in t, next there are $U(1)$ phase transformations for each individual pair of (z_f, w_f) spinors, and then there are also $SU(2)$ transformations for the triple of spinors $(z_f)_{f=1,2,3}$ on an edge. These $SU(2)$ rotations move the dual triangle in internal space, but preserve its overall shape.

IV. PATH-INTEGRAL QUANTIZATION

We are now ready to study the resulting quantum theory and define the vacuum to vacuum amplitude $\langle\Omega|\Omega\rangle = Z_M$ for the discretized manifold M as the path integral over the exponential of the spinorial action (34), and hence study the following expression:

$$Z_M = \int_{\substack{\text{all spinors be}\\ \text{periodic in}\,\partial f}} \prod_{f:\,\text{faces}} \mathcal{D}[z_f]\mathcal{D}[w_f]\mathcal{D}[\varphi_f]\Delta^{\psi}_{\mathrm{FP}}[\varphi_f]\delta(\psi[\varphi_f]) \times \prod_{e:\,\text{edges}} \mathcal{D}[A_e]\Delta^{\Psi}_{\mathrm{FP}}[A_e]\delta(\Psi[A_e])\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_M[\underline{z},\underline{w},\underline{\varphi},\underline{A}]}. \tag{52}$$

The underlying manifold M shall be closed, and the amplitude Z_M is therefore a pure number, not depending on any boundary data. Insertions of gauge invariant observables give the n-point functions of the theory. All fields are supported only on the one-dimensional edges of the discretization, and fulfill periodic boundary conditions once we go around a spin-foam face. Furthermore, $\prod_f \mathcal{D}[z_f]$ with $\mathcal{D}[z] = \prod_t \frac{d^4 z(t)}{\pi^2}$ denotes the flat integration measure in the infinite dimensional space of spinor-valued functions over the edges of the discretization.[8] The spinorial action S_M (34) has local $U(1)$ and $SU(2)$ gauge symmetries (48)–(51) on the edges. This necessitates a gauge fixing, for we cannot integrate over the gauge orbits, because this generically yields an infinity. We thus take the integral only over a gauge fixing surface, which intersects every gauge orbit exactly once. The gauge fixing functions $\Psi[A]$ and $\psi[\varphi]$ define such a gauge section, while the corresponding Faddeev-Popov determinants, $\Delta^{\Psi}_{\mathrm{FP}}[A]$ and $\Delta^{\psi}_{\mathrm{FP}}[\varphi]$, are needed to end up with a gauge invariant integration measure. This also guarantees the invariance of the resulting amplitude under small deformations of the gauge fixing surface.

A. Step 0: Bargmann's quantization of the harmonic oscillator.

Before we go on and actually calculate this integral, let us first study the kinematical structure of the resulting quantum theory, its Hilbert space, and the operators [6,11]. We start from the space of analytic functions $\mathcal{H} \ni f\colon \mathbb{C}^2 \to \mathbb{C}, z \mapsto f(z)$, which carry a natural representation of the classical commutation relations $\{z^\dagger_A, z^B\} = \frac{\mathrm{i}}{\hbar}\delta^B_A$. Following Bargmann's analytic quantization of the harmonic oscillator, the z^A spinor acts by multiplication, while $z^\dagger_A$ turns into a derivative,

$$(\hat{z}^A f)(z) = z^A f(z), \qquad (\hat{z}^\dagger_A f)(z) = \frac{\partial}{\partial z^A} f(z). \tag{53}$$

The reality conditions $\bar{z}^{\bar{A}} = \delta^{A\bar{A}} z^\dagger_A$ uniquely determine the inner product as the Gaußian integral,

$$\langle f, f'\rangle = \frac{1}{\pi^2}\int_{\mathbb{C}^2} d^4 z \mathrm{e}^{-\delta_{A\bar{A}} z^A \bar{z}^{\bar{A}}} \overline{f(z)} f'(z), \tag{54}$$

where $d^4 z = -\frac{1}{4} \mathrm{d}z^0 \mathrm{d}\bar{z}^{\bar{0}} \mathrm{d}z^1 \mathrm{d}\bar{z}^{\bar{1}}$ is the flat integration measure, and both f and f' are analytic in $\mathbb{C}^2$. A short moment of reflection reveals the Gaußian measure $\propto d^4 z \exp(-\|z\|^2)$ truly respects the reality conditions: $\forall f, f' \in \mathcal{H}\colon \langle f, \hat{z}^A f'\rangle = \langle \hat{\bar{z}}^{\bar{A}} f, f'\rangle$. Next, we need a complete orthonormal basis. A convenient choice is given by the following family of polynomials:

$$\langle z|j,m\rangle = \frac{1}{\sqrt{(j-m)!(j+m)!}}(z^0)^{j-m}(z^1)^{j+m}, \quad \text{and}$$
$$\langle j,m|j',m'\rangle = \delta_{jj'}\delta_{mm'}. \tag{55}$$

Then there are the operators. The quantization of the fluxes (7) yields the generators of angular momentum,

$$L_i = \mathrm{i}\tau_i^{AB}\hat{z}_A\hat{z}^\dagger_B = -\ell_{\mathrm{P}}{}^{-1}\hat{\ell}_i, \tag{56}$$

which satisfy the usual commutation relations $[L_i, L_j] = \mathrm{i}\epsilon_{ij}{}^m L_m$ of $\mathfrak{su}(2)$. Another important operator is the spinor's norm. Any homogenous function diagonalizes this operator, choosing a normal ordering we find, in fact,

$$:\|\hat{z}\|^2: = \frac{1}{2}(\hat{z}^A\hat{z}^\dagger_A + \hat{z}^\dagger_A\hat{z}^A) = z^A\frac{\partial}{\partial z^A} + 1,$$
$$\text{thus } :\|\hat{z}\|^2:|j,m\rangle = (2j+1)|j,m\rangle. \tag{57}$$

This gives us the spectrum of the length operator: Classically, each bone has a physical length given by $\boldsymbol{L}[b] = \sqrt{\ell_i[b]\ell^i[b]}$, but now the z spinors parametrize the fluxes (11), and their squared norm measures the length of b. Choosing a normal ordering and looking back at (57), we thus get the spectrum of the length operator

$$\mathrm{spec}(\hat{\boldsymbol{L}}) = \left\{\ell_{\mathrm{P}}\left(j + \frac{1}{2}\right)\right\}_{2j\in\mathbb{N}_0}. \tag{58}$$

[8] Equally for $\mathcal{D}[A_e]$ and $\mathcal{D}[\varphi_f]$: They are formal Lebesgue measures in the space of $\mathfrak{su}(2)$-valued functions $A_e\colon e \to \mathfrak{su}(2)$ and real-valued functions $\varphi_f\colon \partial f \to \mathbb{R}$, respectively.

B. Step 1: Integration over the spinors

The integral over the spinors factorizes into a product over the individual spin-foam faces. We take the contribution from a single face, integrate over the spinors, and are hence left with a functional that we just call $Z_f[\varphi, A]$. This functional can depend only on the $SU(2)$ connection on the edges, and the Lagrange multiplier φ imposing the length matching condition; all other variables have been integrated out,

$$Z_f[A,\varphi] \coloneqq \int_{\substack{z^A(0)=z^A(1)\\ w^A(0)=w^A(1)}} \mathcal{D}[z]\mathcal{D}[w]\mathrm{e}^{\int_{\partial f}\mathrm{d}t(z^\dagger_A\frac{D}{\mathrm{d}t}z^A - w^\dagger_A\frac{\mathrm{d}}{\mathrm{d}t}w^A - \mathrm{i}\varphi(\|z\|^2-\|w\|^2))}. \tag{59}$$

If we now want to recover the path integral for the full discretized manifold, we just take the product over all face amplitudes (59) and integrate over all remaining configuration variables,

$$Z_M = \int \prod_{f:\ \text{faces}} \mathcal{D}[\varphi_f]\Delta^{\psi}_{\mathrm{FP}}[\varphi_f]\delta(\psi[\varphi_f]) \times \prod_{e:\ \text{edges}} \mathcal{D}[A_e]\Delta^{\Psi}_{\mathrm{FP}}[A_e]\delta(\Psi[A_e])Z_f[\varphi_f, A]. \tag{60}$$

The Lagrangian (33) for the face action $S_f[z,w,\varphi,A]$ is quadratic in the spinors. This considerably simplifies the evaluation of the path integral $Z_f[\varphi, A]$. We only have to calculate an infinite product of Gaußian integrals. This eventually yields a trace over the underlying Hilbert space,

$$Z_f[A,\varphi] = \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}}\left[\mathrm{Pexp}\left(-\int_{\partial f}\mathrm{d}t(A^i(t)\tau^{AB}{}_i\hat{z}_A\hat{z}^\dagger_B + \mathrm{i}\varphi(t)(:\|\hat{z}\|^2: - :\|\hat{w}\|^2:))\right)\right]. \tag{61}$$

The trace goes over an orthonormal basis in the Hilbert space $\mathcal{H}\otimes\mathcal{H}\ni f(z,w)$ of analytic functions in the z and w spinors, square integrable with respect to the inner product (54). In terms of the spin (j,m)-basis (55) this trace turns into an infinite sum,

$$Z_f[A,\varphi] = \sum_{2j=0}^{\infty}\sum_{m=-j}^{j}\sum_{2l=0}^{\infty}(2l+1)\langle j,m|\mathrm{Pexp}\times\left(\mathrm{i}\int_{\partial f}\mathrm{d}tA^i(t)L_i\right)|j,m\rangle \mathrm{e}^{-\mathrm{i}\int_{\partial f}\mathrm{d}t\varphi(t)(2j-2l)}. \tag{62}$$

C. Step 2: Integration over the $U(1)$ gauge potential

The next step is to perform the integrals over the gauge potentials. Let us first do the integral over φ. This requires a gauge fixing, and we choose the following:

$$\psi[\varphi](t) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = 0. \tag{63}$$

The variation $\delta_\lambda = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0}$ of the gauge fixing condition (63) under an infinitesimal $U(1)$ gauge transformation $\varphi_{\varepsilon\lambda} = \varphi + \varepsilon\dot{\lambda}$ determines the Faddeev-Popov determinant $\Delta^{\psi}_{\mathrm{FP}}[\varphi]$ as the functional determinant of the following differential operator $\hat{m}$:

$$\hat{m}[\lambda] \coloneqq \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\psi[\varphi_{\varepsilon\lambda}](t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\lambda(t). \tag{64}$$

The eigenvectors of $\hat{m}$ are clearly independent of φ, and so is the Faddeev-Popov determinant $\Delta^{\psi}_{\mathrm{FP}}[\varphi]$, which can therefore only affect the overall normalization of the measure. The gauge potential φ determines a $U(1)$ angle, periodic in 2π. We require that the integration measure is normalized, which in turn implies $\Delta^{\psi}_{\mathrm{FP}}[\varphi] = (2\pi)^{-1}$ once we restrict the integral over just one period of φ. The resulting integral gives the Dirac distribution[9] of the holonomy around the spin-foam face,

$$\begin{aligned}Z_f[A] &= \int \mathcal{D}[\varphi]\Delta^{\psi}_{\mathrm{FP}}[\varphi]\delta(\psi[\varphi])Z_f[A,\varphi]\\ &= \frac{1}{2\pi}\sum_{2l=0}^{\infty}\sum_{2j=0}^{\infty}\sum_{m=-j}^{j}\int_0^{2\pi}\mathrm{d}\varphi\mathrm{e}^{-\varphi(2j-2l)}(2l+1)\langle j,m|\mathrm{Pexp}\left(\mathrm{i}\int_{\partial f}\mathrm{d}tA^i(t)L_i\right)|j,m\rangle\\ &= \sum_{2j=0}^{\infty}\sum_{m=-j}^{j}(2j+1)\langle j,m|\mathrm{Pexp}\left(\mathrm{i}\int_{\partial f}\mathrm{d}tA^i(t)L_i\right)|j,m\rangle = \delta_{SU(2)}\left(\mathrm{Pexp}\left(-\int_{\partial f}\mathrm{d}tA^i(t)\tau_i\right)\right),\end{aligned} \tag{65}$$

where the last equality follows from the Peter-Weyl theorem.

D. Step 3: Integral over the $SU(2)$ gauge potential on the edges

We are now left to perform the integral over the $SU(2)$ connection. Our strategy is to solve this path integral on each edge separately. The calculation can be seen as a one-dimensional analogue of what has been found in [35,36]. In fact, Bianchi's

[9] The Dirac distribution evaluates any $f: SU(2)\to\mathbb{C}$ at the identity 1: $\int_{SU(2)} d\mu_{\mathrm{Haar}}(U)f(U)\delta_{SU(2)}(U) = f(\mathbb{1})$, where $d\mu_{\mathrm{Haar}}(U)$ is the normalized Haar measure on the group.

conjecture [35] of equivalence between the Ashtekar-Lewandowski measure [16,18] on a fixed graph and the canonical measure in the moduli space of flat connections was one of the key motivating ideas behind this work.

Simplifying our notation, let us first parametrize each edge $e, e', \ldots$, by a t coordinate running from 0 to 1. Every edge carries its own $SU(2)$ gauge potential $A_e(t) = A_\mu(e(t))\dot{e}^\mu(t)$, defined as in (35). We now choose our gauge condition and require that for every edge e the gauge potential be constant in t,

$$\forall e\colon\ \Psi^i[A_e](t) = \frac{\mathrm{d}}{\mathrm{d}t}A^i_e(t) = 0. \tag{66}$$

Notice that this is only a partial gauge fixing.[10] For a single edge e, residual gauge transformations can shift the connection $A_e \equiv A$ to any other constant value $\tilde{A}$. The proof is immediate; consider the gauge element,

$$g(t) = \mathrm{e}^{-At}g_o\mathrm{e}^{\tilde{A}t}, \qquad g_o \in SU(2), \quad \text{and} \quad A, \tilde{A} \in \mathfrak{su}(2). \tag{67}$$

The gauge transformed connection yields the $\mathfrak{su}(2)$ element $\tilde{A}$, which is again constant in t,

$$\tilde{A} = g^{-1}(t)\dot{g}(t) + g^{-1}(t)Ag(t). \tag{68}$$

A typical gauge invariant observable is the Wilson loop—the trace of the holonomy around the boundary of the spin-foam face. If the gauge condition (66) holds on all edges, we have, in fact,

$$\mathrm{Tr}\left(\mathrm{Pexp}\left(-\int_{\partial f} A\right)\right) = \mathrm{Tr}\left(\mathrm{P}\prod_{e\in\partial f}\mathrm{e}^{-A_e}\right), \tag{69}$$

where $\mathrm{P}\prod$ denotes the path ordered product.[11]

We can impose the gauge fixing condition (66) globally, on each individual edge of the discretization. This can be seen as follows. Start with some generic gauge potential, not subject to (66). We now need a gauge transformation $g(t)$ mapping $A^i(t)$ into an element $\tilde{A}^i$ of the constraint hypersurface: $\Psi^i[\tilde{A}](t) = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}^i(t) = 0$. We compute the parallel transport $U(t) = \mathrm{Pexp}(-\int_0^t \mathrm{d}tA(t))$ along the edge, and define the $SU(2)$ angle ϕ^i as the logarithm of the holonomy along the entire edge: $U(1) = \exp(-\phi^i\tau_i)$. The gauge transformation

$$g(t) = U(t)\mathrm{e}^{t\phi^i\tau_i} \tag{70}$$

fulfills our requirement, for it turns the connection into a constant $\mathfrak{su}(2)$ element,

$$\tilde{A}(t) = g^{-1}(t)\dot{g}(t) + g^{-1}(t)A(t)g(t) = \phi^i\tau_i. \tag{71}$$

This enables us to solve $\Psi^i[A] = 0$ all along the edge. In fact, we can achieve (66) on all edges $e, e', \ldots$, at the same time, simply because the gauge transformation (70) vanishes at the edge's source and target points: $g(0) = g(1) = \mathbb{1}$.

The residual gauge transformations (67) preserve the gauge fixing condition $\Psi^i[A] = 0$. The Faddeev-Popov procedure does not remove these "horizontal" transformations, it only deals with transversal gauge transformations that take us out of the gauge fixing surface. Transversal gauge transformations vanish at the two boundary points $t = 0, 1$ of the edge $e(t)$, but not in between. We can now split any gauge element $\Lambda^i(t)\colon [0,1]\mapsto\mathfrak{su}(2)$ into its horizontal and transversal components: $\Lambda^i(t) = \Lambda^i_\parallel(t) + \Lambda^i_\perp(t)$, where $\Lambda^i_\parallel(t)$ maps the gauge fixing surface into itself (67), while $\Lambda^i_\perp(t)$ deforms it nontrivially.

Since any transversal gauge element $\Lambda^i(t)_\perp \equiv \Lambda^i(t)$ vanishes at the boundary, it implicitly defines a periodic function $\Lambda^i(t+n) = \Lambda^i(t)$ on the real line. We can thus introduce Fourier modes $\mathrm{e}^{2\pi \mathrm{i}nt}$ and write

$$\Lambda^i(t) = \sum_{n=1}^{\infty}\Lambda^i_n\mathrm{e}^{2\pi\mathrm{i}nt} + \text{c.c.}, \qquad \Lambda^i(0) = 0, \qquad \Lambda^i_n \in \mathbb{C}^3, \tag{72}$$

where c.c. denotes the complex conjugate of all preceding terms. Notice the absence of the $n = 0$ mode, which would generate residual gauge transformations preserving the gauge fixing surface $\Psi^i[A] = 0$ (66). We exclude this constant gauge element because only the transversal modes that map the gauge fixed connection out of the constraint hypersurface $\Psi^i[A] = 0$ can contribute to the Faddeev-Popov determinant.

We now need the Faddeev-Popov operator $\hat{M}$. Infinitesimal gauge transformations (51) of the gauge fixing condition (66) define this operator as

$$\begin{aligned}\hat{M}^i{}_j\Lambda^j(t) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}(\rho_{\varepsilon\Lambda}A)^i(t)\\ &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Lambda^i(t) + \epsilon^i{}_{jk}A^j\frac{\mathrm{d}}{\mathrm{d}t}\Lambda^k(t).\end{aligned} \tag{73}$$

Its eigenvalues determine the Faddeev-Popov determinant in the space of transversal gauge elements (72). Let us do

[10] A complete gauge fixing condition is inconvenient, because it would depend on the topological details of how the edges bound another. The gauge fixing (66), on the other hand, is more general. We can simultaneously impose it on every single edge, no matter how the edges glue together. The proof follows in a minute.

[11] Let ∂f consist of edges $e_i\colon[0,1]\to\partial M$. Their orientation agrees with the induced orientation of ∂f, and they also are already appropriately ordered: $\forall i\colon\ e_{i+1}(0) = e_i(1)$, $e_{N+1} \equiv e_1$. We can now define the path ordered product simply as $\mathrm{P}\prod_{i=1}^N U_{e_i} \coloneqq U_{e_N}U_{e_{N-1}}\cdots U_{e_1}$, where each edge carries the holonomy $\mathrm{e}^{-A_{e_i}} = U_{e_i} \in SU(2)$.

the calculation for only one direction of A^i. Setting, without loss of generality, $A^i = A\delta^i_3$, we are thus led to the following eigenvalue equation:

$$\begin{pmatrix} \frac{d^2}{dt^2}\Lambda^1 - A\frac{d}{dt}\Lambda^2 = E\Lambda^1 \\ \frac{d^2}{dt^2}\Lambda^2 + A\frac{d}{dt}\Lambda^1 = E\Lambda^2 \\ \frac{d^2}{dt^2}\Lambda^3 = E\Lambda^3 \end{pmatrix}. \tag{74}$$

The eigenvectors are $\Lambda^{(n)}_\pm$ and $\Lambda^{(n)}_z$ with corresponding eigenvalues $E^{(n)}_\pm$ and $E^{(n)}_z$, respectively,

$$\Lambda^{(n)}_\pm(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} e^{2\pi i n t}, \quad \text{with :}$$
$$E^{(n)}_\pm = -(2\pi n)^2\left(1 \mp \frac{A}{2\pi n}\right), \quad n \in \mathbb{Z} - \{0\}, \tag{75}$$

$$\Lambda^{(n)}_z(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2\pi i n t}, \quad \text{with :}$$
$$E^{(n)}_z = -(2\pi n)^2, \quad n \in \mathbb{Z} - \{0\}. \tag{76}$$

Only transversal gauge transformations can contribute to the Faddeev-Popov determinant, and therefore the $n = 0$ modes do not appear in this list. The functional determinant of $\hat{M}$ is the product over all eigenvalues. This badly diverges, but we can easily remove this infinity by looking at a regularized expression,

$$\Delta^\Psi_{FP}[A] := \frac{\det \hat{M}|_A}{\det \hat{M}|_{A=0}} = \prod_{n\in\mathbb{Z}-\{0\}}\left(1 - \frac{|A|}{2\pi n}\right)^2 = \frac{4\sin^2(\frac{|A|}{2})}{|A|^2}. \tag{77}$$

The square sinc function combines with the flat integration measure d^3A to the Haar measure of $SU(2)$. Taking the parametrization $U = \exp(-A^i\tau_i)$ of $U \in SU(2)$ and setting $|A| = \sqrt{\delta_{ij}A^iA^j}$ we find, in fact,

$$d^3A\frac{4\sin^2(\frac{|A|}{2})}{|A|^2} = \frac{4}{3}\mathrm{Tr}(U^{-1}dU\wedge U^{-1}dU\wedge U^{-1}dU)|_{U=\exp(-A^i\tau_i)}$$
$$= 32\pi^2 d\mu_{\mathrm{Haar}}(U). \tag{78}$$

Our gauge condition (66) has thus turned the functional integral into an ordinary integral over the group. Absorbing the overall normalization $32\pi^2$ into the definition of the flat integration measure $\mathcal{D}[A] \propto \prod_{t\in[0,1]} d^3A(t)$ we get the following rule:

$$\int \mathcal{D}[A]\Delta^\Psi_{FP}[A]\delta(\Psi[A])f\left(\mathrm{Pexp}(-\int_e A)\right)$$
$$= \int_{SU(2)} d\mu_{\mathrm{Haar}}(U)f(U), \tag{79}$$

where $f: SU(2) \to \mathbb{C}$ denotes some integrable function on the group.

We arrive at our final result once we repeat the calculation for all edges in the discretization. Combining the integration formula (79) with our final expression (79) for the face amplitude $Z_f[A]$, we see that the path integral Z_M over the spinorial action (52) eventually assumes a very neat form,

$$Z_M = \prod_{e:\ \text{edges}} \int_{SU(2)} d\mu_{\mathrm{Haar}}(U_e) \prod_{f:\ \text{faces}} \delta_{SU(2)}\left(\mathrm{P}\prod_{e\in\partial f} U_e\right), \tag{80}$$

where $\mathrm{P}\prod$ again denotes the path ordered product (see footnote 11). Equation (80) reproduces the Ponzano-Regge model for the simplicial manifold M. This is our final result. It proves the equivalence between our one-dimensional spinorial path integral (52) and the discrete spin-foam approach [3,24,37].

V. CONCLUSION

A. Summary

This article developed two results: First of all, we wrote the discretized Palatini action as a one-dimensional line integral. We then took that action and used it to define the path integral. The resulting amplitudes reproduced the Ponzano-Regge model.

Section III gave the first result. To discretize the first-order action, we introduced a simplicial decomposition of the three-dimensional manifold. The equations of motion for a discretized field theory normally give a tangled system of difference equations. This would make it hard to speak about the symplectic structure, the Hamiltonian, the time evolution, and the constraint equations of the theory— all of which are crucial elements for the quantization program.[12] Our partial continuum limit circumvents these difficulties. On each spin-foam face we performed a continuum limit in the t variable parametrizing the boundary of the spin-foam face. The resulting action (34) is a one-dimensional line integral over the one-skeleton of the underlying simplicial manifold. The action is a functional of the loop gravity spinors [5], but depends also on connection variables. There is a $SU(2)$ connection on each edge, and a $U(1)$ connection φ for each spin-foam face. All fields are continuous, but have support only on the

[12]Despite this difficulty, Hoehn and Dittrich [30–32] have recently achieved impressive progress toward this goal.

one-dimensional edges of the simplicial complex. The action variation gave us the equations of motion, and we proved agreement with the continuum theory: The Gauß law (43) is the discrete analogue of the torsionless equation (2a), while the evolution equations (36) imply that the loop holonomy transports the spinors into themselves (41). This represents the flatness constraint (2b) in the discrete theory. We closed the classical part with two more comments: First of all, the equations of motion (36) admit a Hamiltonian formulation (46). Then we also discussed the local gauge symmetries of the theory. The action has a diffeomorphism symmetry, since it does not depend on the actual parametrization of the edges, but there is also a $U(1)$ symmetry for each face, and an internal $SU(2)$ gauge symmetry.

This was our first complex of results. The second part considered the quantization of the theory, as developed in Sec. IV. We started with a short review showing how to recover the loop gravity Hilbert space from the spinorial representation [6]. The remaining part developed the path integral for the simplicial manifold. The integral over the spinors is easy to solve: The action (33) is quadratic in the spinors, and the path integral reduces to an infinte product of Gaußian integrals. Then there are the local gauge symmetries. We removed the redundant integrals according to the usual Faddeev-Popov procedure. The result reduced the functional integral to an ordinary integral, with the emergence of the canonical Haar measure of $SU(2)$. Our final expression (80) agrees with the Ponzano-Regge state sum model [3,24,37–39] of three-dimensional quantum gravity.

B. Prospects of the formalism

Our analysis encourages further questions: What is the physical interpretation of the $U(1)$ winding number n_f as defined in (39)? Does the edge Hamiltonian (45) introduce a local notion of energy? Is there a way to add a cosmological constant to the spinorial action (along the lines of e.g. [40–42])? Can we generalize the formalism to the Lorentzian signature, thus replacing $SU(2)$ by $SU(1,1)$? Can we use the one-dimensional action to bring causal sets [43] and spin foams closer together, an idea first studied in [44,45]?

The most important question is, however, rather simple: What does all this machinery actually tell us for the four-dimensional Lorentzian case? So far, we only have a partial answer: In 3+1 dimensions there is a spinorial (or rather twistorial) formulation of $SL(2,\mathbb{C})$ BF theory [13]. Once again the action is a *one-dimensional* integral over the edges of the discretization. This action defines a topological theory. Additional constraints break the topological symmetries and bring us back to general relativity [46,47]. How these simplicity constraints translate into conditions on the spinors has been shown in [10,12]. These results can lead us to a version of first-order Regge calculus in four spacetime dimensions with spinors as the fundamental configuration variables. Generalizing our derivation of the Ponzano-Regge model to the four-dimensional case would then give us a neat definition of the transition amplitudes: We would start with the one-dimensional spinorial path integral for $SL(2,\mathbb{C})$ BF theory, add the simplicity constraints, and evaluate the integral for a given simplicial manifold. My hope is that this will improve our understanding of the mathematical structure of loop quantum gravity, its causal structure, and the continuum limit of the theory.

ACKNOWLEDGMENTS

I thank Eugenio Bianchi and Marc Geiller for helpful comments and discussions on a first draft of this article. I also gratefully acknowledge support from the Institute for Gravitation and the Cosmos, the National Science Foundation through Grant No. PHY-12-05388, and the Eberly research funds of The Pennsylvania State University.

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