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Second order density perturbations for dust cosmologies

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We present simple expressions for the relativistic first and second order fractional density perturbations for Friedmann-Lemaître cosmologies with dust, in four different gauges: the Poisson, uniform curvature, total matter and synchronous-comoving gauges. We include a cosmological constant and arbitrary spatial curvature in the background. A distinctive feature of our approach is our description of the spatial dependence of the perturbations using a canonical set of quadratic differential expressions involving an arbitrary spatial function that arises as a conserved quantity. This enables us to unify, simplify and extend previous seemingly disparate results. We use the primordial matter and metric perturbations that emerge at the end of the inflationary epoch to determine the additional arbitrary spatial function that arises when integrating the second order perturbation equations. This introduces a non-Gaussianity parameter into the expressions for the second order density perturbation. In the special case of zero spatial curvature we show that the time evolution simplifies significantly, and requires the use of only two nonelementary functions, the so-called growth suppression factor at the linear level, and one new function at the second order level. We expect that the results will be useful in applications, for example, studying the effects of primordial non-Gaussianity on the large scale structure of the Universe.

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I. INTRODUCTION

The increasingly accurate observations of the cosmic microwave background (CMB) and the large scale structure (LSS) of the Universe that are becoming available are stimulating theoretical developments in cosmology. The analysis of these observations is based on cosmological perturbation theory, with the bulk of the work to date using linear perturbations of Friedmann-Lemaître (FL) cosmologies. However, the study of possible deviations from linearity, for example, how primordial non-Gaussianity affects the anisotropy of the CMB and the LSS, necessitates the use of nonlinear perturbations (see for example [1–3]).

Much of the theoretical work on nonlinear perturbations has dealt with flat FL cosmologies with dust (a matter-dominated universe; see for example [4,5]) and more recently also with a cosmological constant (a ΛCDM universe; see for example [6–9]). One aspect of the work is to provide an expression for the second order fractional density perturbation ${}^{(2)}\boldsymbol{\delta}$ which is needed when analyzing observations of the LSS, and this is the focus of the present paper.

The analysis of the LSS is based on galaxy redshift surveys, which record the fluctuation of the number count of galaxies on the past light cone of the observer as a function of redshift and angular direction. On the other hand the fractional density perturbation, calculated using cosmological perturbation theory, is based on constant time slices in the Universe; i.e. it depends on a choice of temporal gauge. Thus in relating observations to theory one has to take into account both gauge effects and light cone effects. This has been done in detail in linear perturbation theory (see, for example, [10–14]), and more recently in second order theory [15]. These papers derive a formula that relates the observed fractional galaxy number overdensity as a function of redshift z and direction $\mathbf{n}$ to the fractional density perturbation as a function of spatial position and time in an appropriate gauge. In the present paper we focus on one aspect of this whole process by deriving simple explicit expressions for the relativistic first and second order fractional density perturbations for FL cosmologies with dust, and investigating how the choice of gauge affects the structure of the expressions.

In a recent paper [16], here referred to as UW, we derived a general expression for ${}^{(2)}\boldsymbol{\delta}$ using the Poisson gauge, also including the effects of spatial curvature, but subject to the restriction that the perturbation at linear order is purely scalar.[1] This expression for ${}^{(2)}\boldsymbol{\delta}$ contains the integral of a complicated quadratic source term involving two arbitrary spatial functions, which makes it difficult to obtain a clear physical understanding. If one assumes that the decaying mode of the scalar perturbation at the linear order is zero,

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[1]For the motivation for imposing this restriction, we refer to [3], p. 4.

however, then the scalar perturbation depends on only one such function. In this case we showed that the temporal and spatial dependence of ${}^{(2)}\boldsymbol{\delta}$ can be displayed explicitly, in a form that provides direct physical insight.

In the present paper we investigate what effect the choice of gauge has on the form of ${}^{(2)}\boldsymbol{\delta}$. Specifically, we first derive a new expression[2] for ${}^{(2)}\boldsymbol{\delta}$ in the Poisson gauge by solving the second order perturbation equations as given in UW. We then use the change of gauge formulas in Appendix B to calculate ${}^{(2)}\boldsymbol{\delta}$ in three other commonly used gauges: the uniform curvature, the total matter[3] and the synchronous-comoving gauges. Before continuing, we digress briefly to motivate our choice of gauges.[4] The two most commonly used gauges in cosmological perturbation theory are the synchronous-comoving gauge and the Poisson gauge. For linear perturbations of dust cosmologies the total matter gauge is the same as the synchronous-comoving gauge. For second order perturbations, however, this is no longer the case. Hence we chose the total matter gauge as one of the four gauges. Finally we decided to use the uniform curvature gauge because we noticed that for this gauge the second order fractional density perturbation has an interesting property; namely *the superhorizon part is a conserved quantity*.[5] We note that this result plays a key role in the process of simplifying the expressions for the second order fractional density perturbation that we give in this paper.

We will use the notation ${}^{(2)}\boldsymbol{\delta}_{\bullet}$ for the fractional density perturbation, where the bullet identifies the gauge. Our first main result is that ${}^{(2)}\boldsymbol{\delta}_{\bullet}$ has the following common structure for the Poisson, the uniform curvature and the total matter gauges:

$$
\begin{aligned}
{}^{(2)}\boldsymbol{\delta}_{\bullet} = {} & \underbrace{A_{1,\bullet}\zeta^2 + A_{2,\bullet}\mathcal{D}(\zeta)}_{\text{the superhorizon part}} \\
& + \underbrace{\frac{2}{3}m^{-2}xg[A_{3,\bullet}(\mathbf{D}\zeta)^2 + A_{4,\bullet}\mathbf{D}^2\mathcal{D}(\zeta) + A_{5,\bullet}\mathbf{D}^2\zeta^2]}_{\text{the post-Newtonian part}} \\
& + \underbrace{\frac{4}{9}m^{-4}x^2g^2[\mathcal{B}_3\mathbf{D}^2(\mathbf{D}\zeta)^2 + \mathcal{B}_4\mathbf{D}^4\mathcal{D}(\zeta)]}_{\text{the Newtonian part}}. \qquad (1)
\end{aligned}
$$

Here m is a constant and the coefficients $A_{i,\bullet}$, $i = 1, \ldots, 5$, $\mathcal{B}_3$ and $\mathcal{B}_4$ are functions of the scale factor x, while the spatial dependence is determined by the conserved quantity $\zeta(x^i)$, which appears in seven quadratic differential expressions:

$$
\begin{aligned}
&\zeta^2, \quad \mathcal{D}(\zeta), \quad (\mathbf{D}\zeta)^2, \quad \mathbf{D}^2\mathcal{D}(\zeta), \quad \mathbf{D}^2\zeta^2, \\
&\mathbf{D}^2(\mathbf{D}\zeta)^2, \quad \mathbf{D}^4\mathcal{D}(\zeta).
\end{aligned} \qquad (2)
$$

The spatial differential operators in (1) and (2) are defined in Appendix A. We will refer to the group of terms in (1) whose differential expressions have zero weight in the spatial differential operator $\mathbf{D}_i$ as *the superhorizon part*. The intermediate group of terms having weight two in $\mathbf{D}_i$ (coefficient m^{-2}) is referred to as *the post-Newtonian part*. Finally, we refer to the group of terms in (1) having weight four in $\mathbf{D}_i$ (coefficient m^{-4}) as *the Newtonian part*.

The common structure for ${}^{(2)}\boldsymbol{\delta}_{\bullet}$ in these three gauges is due to the fact that they all use the same spatial gauge. The differences thus depend on different temporal gauges, which affect the coefficients $A_{i,\bullet}$, $i = 1, \ldots, 5$, but not $\mathcal{B}_3$ and $\mathcal{B}_4$. On the other hand, the synchronous-comoving gauge uses a different spatial gauge, which has the effect of adding a term to the Newtonian part with spatial dependence given by $(\mathbf{D}^2\zeta)^2$, thereby adding a new quadratic differential expression to the set (2). For this reason we will treat this case separately in the paper.

The evolution of ${}^{(2)}\boldsymbol{\delta}$ in the Poisson gauge is determined by eight functions of time that are written as integrals, and these are the main source of the complexity of the expression. Our second main result is to show that if the background spatial curvature is zero, then seven of these integral functions can be written in an explicit form. This fact enables us to give simple expressions for ${}^{(2)}\boldsymbol{\delta}$ in all four gauges under consideration when the spatial geometry is flat.

The outline of the paper is as follows. In the next section we give a unified expression for the first order fractional density perturbation in the four gauges. In Sec. III we derive the corresponding second order results and address the issue of initial conditions. Section IV deals exclusively with the spatially flat case $K = 0$. We show that the time dependence simplifies significantly and then give a detailed comparison with previous work dealing with this case. Section V contains the concluding remarks. Finally, in Appendix A we define the various spatial differential operators and in Appendix B we derive the transformation laws that relate the density perturbations for the four gauges under consideration.

II. THE DENSITY PERTURBATION AT FIRST ORDER

The background cosmology is a FL universe with scale factor a, Hubble scalar H and curvature parameter K, containing dust with background matter density ${}^{(0)}\rho_m$ and a

[2]This expression differs from the one in UW referred to above in the way the time dependence is represented, which facilitates the subsequent analysis.

[3]The total matter gauge is referred to by Hwang and co-workers as the comoving gauge (see for example [17,18]). We also used this terminology in an earlier paper [19]. Here, however, we have chosen to follow the conventions and nomenclature of Malik and Wands [20]; see their Secs. 7.4 and 7.5.

[4]We give the definition of the gauges in Appendixes B 5 and B 7.

[5]This generalizes the known result for the linear fractional density perturbation in the uniform curvature gauge.

cosmological constant Λ. We introduce the usual density parameters[6]:

$$\Omega_m \coloneqq \frac{{}^{(0)}\rho_m}{3H^2}, \qquad \Omega_k \coloneqq -\frac{K}{\mathcal{H}^2}, \qquad \Omega_\Lambda \coloneqq \frac{\Lambda}{3H^2}, \tag{3}$$

which satisfy

$$\Omega_m + \Omega_k + \Omega_\Lambda = 1. \tag{4}$$

We use the dimensionless scale factor $x \coloneqq a/a_0$, normalized at some reference epoch t_0, as a time variable.[7] The conservation law shows that $a^{3(0)}\rho_m$ is constant, which we write in terms of Ω_m and the dimensionless Hubble scalar $\mathcal{H} \coloneqq aH$ as

$$\mathcal{H}^2 x \Omega_m = m^2. \tag{5}$$

Setting $x = 1$ shows that the constant m is given by $m^2 = \mathcal{H}_0^2\Omega_{m,0}$. It follows that

$$\mathcal{H}^2 = \mathcal{H}_0^2(\Omega_{\Lambda,0}x^2 + \Omega_{k,0} + \Omega_{m,0}x^{-1}). \tag{6}$$

Equations (3)–(6) determine $\mathcal{H}$, Ω_m, Ω_k and Ω_Λ explicitly in terms of x.

The gauge invariants[8] that describe the scalar linear perturbations of the metric and matter in the Poisson gauge are the Bardeen potentials ${}^{(1)}\Psi_{\rm p}$ and ${}^{(1)}\Phi_{\rm p}$, the velocity potential ${}^{(1)}\mathbf{v}_{\rm p}$ and the fractional density perturbation ${}^{(1)}\boldsymbol{\delta}_{\rm p}$. In the special case when the decaying mode of the scalar perturbation is set to zero we have the following expressions[9]:

$${}^{(1)}\Psi_{\rm p} = {}^{(1)}\Phi_{\rm p} = g(x)\zeta(x^i), \quad \mathcal{H}\,{}^{(1)}\mathbf{v}_{\rm p} = -\frac{2}{3}\Omega_m^{-1} f g\zeta, \tag{7a}$$

$${}^{(1)}\boldsymbol{\delta}_{\rm p} = -2\Omega_m^{-1}(f+\Omega_k)g\zeta + \frac{2}{3}m^{-2}xg\mathbf{D}^2\zeta, \tag{7b}$$

where[10]

$$g(x) \coloneqq \frac{3}{2}m^2\frac{\mathcal{H}}{x^2}\int_0^x \frac{d\bar{x}}{\mathcal{H}(\bar{x})^3}, \qquad f(x) \coloneqq 1 + g^{-1}x\partial_x g. \tag{8}$$

Here the arbitrary spatial function $\zeta(x^i)$ equals the conserved quantity that we introduced in [19], denoted by $\zeta_{\rm v}$, which can be written in the form[11]

$$\zeta_{\rm v} = \left(1 + \frac{2}{3}\Omega_k\Omega_m^{-1}\right){}^{(1)}\Psi_{\rm p} - \mathcal{H}\,{}^{(1)}\mathbf{v}_{\rm p}. \tag{9}$$

We emphasize that $\zeta_{\rm v}$ *is exactly conserved for dust* [see [21], Eq. (B2), and [19], Eqs. (69)–(A1)]. The fact that $\zeta = \zeta_{\rm v}$ was established in UW [see Eqs. (21)–(24)]. We note in passing that if the background spatial curvature is zero ($\Omega_k = 0$), then $\zeta_{\rm v}$ is the comoving curvature perturbation, often denoted by $\mathcal{R}$ [see [20], Eq. (7.46) in conjunction with (7.6) and (7.8)].

We can use (9) to simplify the expression for ${}^{(1)}\boldsymbol{\delta}_{\rm p}$, as follows. On substituting for ${}^{(1)}\Psi_{\rm p}$ and ${}^{(1)}\mathbf{v}_{\rm p}$ from (7a) into (9) the function ζ cancels as a common factor and we obtain the following algebraic constraint relating f to g:

$$(3\Omega_m + 2\Omega_k + 2f)g = 3\Omega_m. \tag{10}$$

Using this we can simplify the expression (7b) to obtain

$${}^{(1)}\boldsymbol{\delta}_{\rm p} = -3(1-g)\zeta + \frac{2}{3}m^{-2}xg\mathbf{D}^2\zeta. \tag{11}$$

By combining this equation with the transformation formulas (B28a), (B32a) and (B42) in Appendix B, we can give a unified expression for ${}^{(1)}\boldsymbol{\delta}_\bullet$, the gauge invariant associated with the first order density perturbation in the four gauges that we are considering in this paper:

$${}^{(1)}\boldsymbol{\delta}_\bullet = A_\bullet\zeta + \frac{2}{3}m^{-2}xg\mathbf{D}^2\zeta, \tag{12a}$$

where

$$A_{\rm p} = -3(1-g), \qquad A_{\rm c} = -3, \qquad A_{\rm v} = A_{\rm s} = -2\Omega_m^{-1}\Omega_k g. \tag{12b}$$

Here the subscripts p, c, v and s identify the Poisson, uniform curvature, total matter and synchronous-comoving gauges, respectively.

The first term of ${}^{(1)}\boldsymbol{\delta}_\bullet$ in (12), which has zero weight in the spatial differential operator $\mathbf{D}_i$, is referred to as the superhorizon term, while the second term, which has weight two, is referred to as the Newtonian term. We note that the superhorizon term is gauge dependent, while the Newtonian part is gauge independent for these four gauges.[12]

Finally the form of the superhorizon term in ${}^{(1)}\boldsymbol{\delta}_{\rm c}$, as given by (12), deserves comment: it is independent of time.

[6]We use units such that $8\pi G = 1 = c$.

[7]When t_0 is the present time, $x = (1+z)^{-1}$, where z is the redshift. We note that x is related to the conformal time η, which we shall sometimes use, according to $\partial_\eta = \mathcal{H}x\partial_x$.

[8]These gauge invariants are introduced in Appendix B. Our strategy for working with gauge invariants is described in the first paragraph of that Appendix.

[9]See Eqs. (19b) and (30)–(32) in UW.

[10]We note in passing that the function $g(x)$, defined up to a constant factor, is sometimes referred to as the growth suppression factor. See for example [7], in the text following Eq. (2.3). Our function $f(x)$ equals their function $f(\eta)$ in Eq. (2.9), on noting that $g'(\eta) = \mathcal{H}x\partial_x g(x)$.

[11]Referring to UW, use Eqs. (11) and (12) to rewrite Eq. (21).

[12]Clearly the Newtonian part is not *universally* gauge independent since for the uniform density gauge it is zero.

This is to be expected since the quantity $\zeta_\rho := -\frac{1}{3}{}^{(1)}\boldsymbol{\delta}_c$ satisfies the "conservation law" $\partial_\eta \zeta_\rho = \frac{1}{2}\mathbf{D}^2\mathbf{v}_p$, which suggests that ${}^{(1)}\boldsymbol{\delta}_c$ will be constant in a regime in which spatial derivatives can be neglected.[13]

The expressions for the gauge invariants at first order given by Eqs. (7)–(9) provide the foundation for generalizing to second order perturbations. In particular the constraint (10) will be used frequently in simplifying the expressions for the second order fractional density perturbation.

III. THE DENSITY PERTURBATION AT SECOND ORDER

In this section we derive explicit expressions for the second order fractional density perturbation ${}^{(2)}\boldsymbol{\delta}_\bullet$ in the Poisson, uniform curvature, total matter and synchronous-comoving gauges, subject to the following restrictions: (i) the perturbations at linear order are purely scalar, and (ii) the decaying mode of the scalar perturbation is zero.

A. The Poisson gauge

The gauge invariants[14] that describe the scalar second order perturbations of the metric and matter in the Poisson gauge are the Bardeen potentials ${}^{(2)}\Psi_p$ and ${}^{(2)}\Phi_p$, the velocity potential ${}^{(2)}\mathbf{v}_p$ and the fractional density perturbation ${}^{(2)}\boldsymbol{\delta}_p$. The governing equations that determine these gauge invariants in the case of dust cosmologies are given in a concise form in Eqs. (14)–(16) in UW. A particular solution for ${}^{(2)}\Psi_p$ is given by Eq. (26) in UW:

$$ {}^{(2)}\Psi_p(x,x^i) = \frac{\mathcal{H}}{x^2}\int_0^x \frac{\bar{x}\Omega_m}{\mathcal{H}}\mathbb{S}(\bar{x},x^i)d\bar{x}, \tag{13a} $$

where

$$ \mathbb{S}(x,x^i) := m^{-2}\int_0^x \mathcal{S}(\tilde{x},x^i)d\tilde{x}, \tag{13b} $$

and the source term $\mathcal{S}$ is given by Eq. (70a) in UW.[15]

We can obtain a simple expression for the time derivative of ${}^{(2)}\Psi_p$ by differentiating (13a) with respect to x and using (34a) and (34b):

$$ \partial_x(x^{(2)}\Psi_p) = -\left(\frac{3}{2}\Omega_m + \Omega_k\right){}^{(2)}\Psi_p + \Omega_m\mathbb{S}. \tag{14} $$

On using this equation, the second order governing equations, given in UW as Eqs. (15) and (16),[16] lead directly to the following expressions for ${}^{(2)}\boldsymbol{\delta}_p$ and ${}^{(2)}\mathbf{v}_p$:

$$ {}^{(2)}\boldsymbol{\delta}_p = \frac{2}{3}m^{-2}\mathbf{D}^2(x^{(2)}\Psi_p) + 3{}^{(2)}\Psi_p - 2\mathbb{S} + \mathbf{S}_\delta, \tag{15a} $$

$$ \mathcal{H}{}^{(2)}\mathbf{v}_p = \left(1 + \frac{2}{3}\Omega_k\Omega_m^{-1}\right){}^{(2)}\Psi_p - \frac{2}{3}\mathbb{S} + \mathbf{S}_v, \tag{15b} $$

where

$$ \mathbf{S}_\delta = \mathcal{S}_{\mathbb{D}} + 3\mathcal{S}_{\mathbb{V}} - 2(\mathbf{D}^{(1)}\mathbf{v}_p)^2, \tag{16a} $$

$$ \mathbf{S}_v = \mathcal{S}_{\mathbb{V}} - 2\mathcal{S}^i[({}^{(1)}\boldsymbol{\delta}_p - {}^{(1)}\Psi_p)\mathbf{D}_i(\mathcal{H}^{(1)}\mathbf{v}_p)], \tag{16b} $$

and the source terms $\mathcal{S}_{\mathbb{D}}$ and $\mathcal{S}_{\mathbb{V}}$ are given by Eqs. (70b) and (70c) in UW. For the reader's convenience we quote these expressions[17]:

$$ \mathcal{S}_{\mathbb{D}} = \frac{2}{3}(\mathcal{H}\Omega_m)^{-1}[4(\mathbf{D}^2 + 3K)\Psi_p^2 - 5(\mathbf{D}\Psi_p)^2] + 6\mathcal{S}^i(\Psi_p\mathbf{D}_i\mathcal{H}\mathbf{v}_p) + \frac{9}{2}\Omega_m(\mathcal{H}\mathbf{v}_p)^2, \tag{17a} $$

$$ \mathcal{S}_{\mathbb{V}} = \frac{2}{3}\Omega_m{}^{-1}[\Psi_p^2 + 4\mathcal{D}(\Psi_p)] + 2[\mathcal{S}^i(\mathcal{H}\mathbf{v}_p\mathbf{D}_i\Psi_p) + 2\mathcal{D}(\mathcal{H}\mathbf{v}_p)]. \tag{17b} $$

We note that the mode extraction operator $\mathcal{S}^i$, which is defined by (A3), satisfies $\mathcal{S}^i\mathbf{D}_i f = f$, for any spatial function f; i.e. $\mathcal{S}^i$ is the inverse operator of $\mathbf{D}_i$. For brevity we have omitted the superscript (1) on the gauge invariants on the right side of Eqs. (17).

The next step is to obtain explicit expressions for ${}^{(2)}\Psi_p$, ${}^{(2)}\boldsymbol{\delta}_p$ and ${}^{(2)}\mathbf{v}_p$ by substituting the first order solution (7) into Eqs. (13) and (15). We begin by following UW in obtaining an explicit expression for the source term $\mathcal{S}(x,x^i)$ in (13b). The result is[18]

$$ \mathcal{S}(x,x^i) = m^2(T_1\zeta^2 + T_2\mathcal{D}(\zeta) + m^{-2}[T_3(\mathbf{D}\zeta)^2 + T_4\mathbf{D}^2\mathcal{D}(\zeta)]), \tag{18} $$

where

[13]See [19], Eqs. (65) and (66) specialized to a barotropic perfect fluid ($\bar{\Gamma} = 0$). See also [22], Eq. (18) in conjunction with (7)–(9).

[14]These gauge invariants are introduced in Appendix B.

[15]We have modified Eq. (26) in UW by using the constraint $x\mathcal{H}^2\Omega_m = m^2$ to replace $\mathcal{H}^2$ in the first integral and have rescaled $\mathbb{S}$ by a factor of m^{-2}. In addition we choose $x_{\text{initial}} = 0$, which is possible since we have set the decaying mode to zero.

[16]The gauge invariants with symbols Ψ, $\mathbf{v}$ and $\boldsymbol{\delta}$ all refer to the Poisson gauge, which we indicate here with a subscript p. Also note that $(\partial_\eta + \mathcal{H})^{(2)}\Psi = \mathcal{H}x\partial_x(x^{(2)}\Psi)$ on changing time derivative.

[17]We have replaced the scalar $\mathcal{A}$ by $\mathcal{A} = 3\mathcal{H}^2\Omega_m$, using Eq. (12) in UW.

[18]UW, Eqs. (A4) and (B1). We have rescaled the T_A by multiplying by m^{-2} and have used Eqs. (4) and (5) in this paper.

$$T_1(x) := (x\Omega_m)^{-1}g^2((f-1)^2 - 4\Omega_k), \tag{19a}$$

$$T_2(x) := -8(x\Omega_m)^{-1}g^2\left((f-1)^2 - \frac{1}{2}(1-\Omega_m) + \Omega_k\left(1+\frac{2}{3}\Omega_m^{-1}f^2\right)\right), \tag{19b}$$

$$T_3(x) := -\frac{1}{3}g^2\left(1-\frac{4}{3}\Omega_m^{-1}f^2\right), \tag{19c}$$

$$T_4(x) := \frac{4}{3}g^2\left(1+\frac{2}{3}\Omega_m^{-1}f^2\right). \tag{19d}$$

Substituting (18) in (13b) leads to

$$\mathbb{S}(x,x^i) = \hat{T}_1\zeta^2 + \hat{T}_2\mathcal{D}(\zeta) + m^{-2}[\hat{T}_3(\mathbf{D}\zeta)^2 + \hat{T}_4\mathbf{D}^2\mathcal{D}(\zeta)], \tag{20a}$$

where

$$\hat{T}_A(x) := \int_0^x T_A(\bar{x})d\bar{x}, \quad A = 1,\ldots,4. \tag{20b}$$

Next, on substituting (20a) in (13a) we obtain

$$^{(2)}\Psi_{\mathrm{p}} = g(B_1(x)\zeta^2 + B_2(x)\mathcal{D}(\zeta) + m^{-2}[B_3(x)(\mathbf{D}\zeta)^2 + B_4(x)\mathbf{D}^2\mathcal{D}(\zeta)]), \tag{21a}$$

where

$$B_A(x) := \frac{\mathcal{H}}{x^2 g}\int_0^x \frac{\bar{x}\Omega_m}{\mathcal{H}}\hat{T}_A(\bar{x})d\bar{x}, \quad A = 1,\ldots,4. \tag{21b}$$

At this stage it is convenient to rescale the coefficients and write Eqs. (21a) and (20a) in the form

$$^{(2)}\Psi_{\mathrm{p}} = g\left(\mathcal{B}_1\zeta^2 + \mathcal{B}_2\mathcal{D}(\zeta) + \frac{2}{3}m^{-2}xg[\mathcal{B}_3(\mathbf{D}\zeta)^2 + \mathcal{B}_4\mathbf{D}^2\mathcal{D}(\zeta)]\right), \tag{22a}$$

$$\mathbb{S} = \mathcal{T}_1\zeta^2 + \mathcal{T}_2\mathcal{D}(\zeta) + \frac{2}{3}m^{-2}xg(\mathcal{T}_3(\mathbf{D}\zeta)^2 + \mathcal{T}_4\mathbf{D}^2\mathcal{D}(\zeta)), \tag{22b}$$

where

$$\mathcal{B}_{1,2} := B_{1,2}, \qquad \mathcal{B}_{3,4} := \left(\frac{2}{3}xg\right)^{-1}B_{3,4}. \tag{23a}$$

$$\mathcal{T}_{1,2} := \hat{T}_{1,2}, \qquad \mathcal{T}_{3,4} := \left(\frac{2}{3}xg\right)^{-1}\hat{T}_{3,4}. \tag{23b}$$

We can now calculate $^{(2)}\mathbf{v}_{\mathrm{p}}$ and $^{(2)}\boldsymbol{\delta}_{\mathrm{p}}$ by substituting (22) into (15) and using the first order solution (7) to calculate the source terms (17) and (16).[19] The results are as follows. For $^{(2)}\mathbf{v}_{\mathrm{p}}$ we obtain

$$\mathcal{H}{}^{(2)}\mathbf{v}_{\mathrm{p}} = V_1\zeta^2 + V_2\mathcal{D}(\zeta) + \frac{2}{3}m^{-2}gx[V_3(\mathbf{D}\zeta)^2 + V_4\mathbf{D}^2\mathcal{D}(\zeta)], \tag{24a}$$

where

$$V_A(x) := -\frac{2}{3}\mathcal{T}_A + \left(1+\frac{2}{3}\Omega_k\Omega_m^{-1}\right)g\mathcal{B}_A + \frac{2}{3}\Omega_m^{-1}g\mathcal{V}_A, \tag{24b}$$

with

$$\mathcal{V}_1 = g(1-2\Omega_m^{-1}f(f+\Omega_m+\Omega_k)), \tag{24c}$$

$$\mathcal{V}_2 = 4g\left(1+\frac{2}{3}\Omega_m^{-1}f(f-\Omega_k)\right), \tag{24d}$$

$$\mathcal{V}_3 = -\frac{1}{3}f, \qquad \mathcal{V}_4 = \frac{4}{3}f. \tag{24e}$$

For the density perturbation $^{(2)}\boldsymbol{\delta}_{\mathrm{p}}$ we obtain (1) with • replaced by p, with the coefficients $A_{i,\mathrm{p}}$ having the following form:

$$A_{1,\mathrm{p}} = -2\mathcal{T}_1 + 3g\mathcal{B}_1 + 2\Omega_m^{-1}g^2((1-f)^2 - 4\Omega_k), \tag{25a}$$

$$A_{2,\mathrm{p}} = -2\mathcal{T}_2 + 3g\mathcal{B}_2 + 8\Omega_m^{-1}g^2\left(1+\frac{2}{3}\Omega_m^{-1}f^2\right), \tag{25b}$$

$$A_{3,\mathrm{p}} = -2\mathcal{T}_3 + 3g\mathcal{B}_3 - g\left(5+\frac{4}{3}\Omega_m^{-1}f^2\right), \tag{25c}$$

$$A_{4,\mathrm{p}} = -2\mathcal{T}_4 + 3g\mathcal{B}_4 + \mathcal{B}_2, \tag{25d}$$

$$A_{5,\mathrm{p}} = \mathcal{B}_1 + 4g. \tag{25e}$$

It should be noted that these coefficients give a *particular solution* for $^{(2)}\boldsymbol{\delta}_{\mathrm{p}}$ that corresponds to the particular solution (13) for $^{(2)}\Psi_{\mathrm{p}}$. We will give the general solution in Sec. III E. The Einstein–de Sitter background arises as a special case ($\Omega_m = 1$, $\Omega_\Lambda = 0$, $\Omega_k = 0$). It follows from (8), in conjunction with (5) and (6), that $g = \frac{3}{5}$, $f = 1$. The definition (23) then yields $\mathcal{B}_1 = \mathcal{B}_2 = 0$, $\mathcal{B}_3 = \frac{1}{21}$, $\mathcal{B}_4 = \frac{20}{21}$

[19] Use (3) to express K in terms of Ω_k and then use (5) to eliminate $\mathcal{H}^2$. In order to simplify the term in (16b) containing the mode extraction operator $\mathcal{S}^i$ it is necessary to use the identity (A4c).

and $\mathcal{T}_1 = \mathcal{T}_2 = 0$, $\mathcal{T}_3 = \frac{1}{10}$, $\mathcal{T}_4 = 2$. The expression for ${}^{(2)}\boldsymbol{\delta}_{\rm p}$ reduces to Eq. (44) in UW.

B. The uniform curvature and total matter gauges

The second order density perturbations ${}^{(2)}\boldsymbol{\delta}_{\rm c}$ and ${}^{(2)}\boldsymbol{\delta}_{\rm v}$ in the uniform curvature and total matter gauges are related to ${}^{(2)}\boldsymbol{\delta}_{\rm p}$ according to Eqs. (B28b) and (B32b) in Appendix B, which we for the reader's convenience repeat here:

$$ {}^{(2)}\boldsymbol{\delta}_{\rm c} = {}^{(2)}\boldsymbol{\delta}_{\rm p} - 3\,{}^{(2)}\Psi_{\rm p} + \mathbb{S}_\delta[{}^{(1)}\mathbf{Z}_{\rm c,p}], \tag{26a} $$

$$ {}^{(2)}\boldsymbol{\delta}_{\rm v} = {}^{(2)}\boldsymbol{\delta}_{\rm p} - 3\mathcal{H}\,{}^{(2)}\mathbf{v}_{\rm p} + \mathbb{S}_\delta[{}^{(1)}\mathbf{Z}_{\rm c,p}]. \tag{26b} $$

The source terms $\mathbb{S}_\delta[{}^{(1)}\mathbf{Z}_{\rm c,p}]$ and $\mathbb{S}_\delta[{}^{(1)}\mathbf{Z}_{\rm c,p}]$ are given by (B30) and (B34) in terms of ${}^{(1)}\Psi_{\rm p}$, ${}^{(1)}\mathbf{v}_{\rm p}$ and ${}^{(1)}\boldsymbol{\delta}_{\rm p}$, which in the case of zero decaying mode are given by (7). It follows that the source terms are a linear combination of the expressions for ζ in (2) of weights zero and two in $\mathbf{D}_i$, as are ${}^{(2)}\Psi_{\rm p}$ and ${}^{(2)}\mathbf{v}_{\rm p}$ which are given by (22a) and (24). Equations (26) thus imply that ${}^{(2)}\boldsymbol{\delta}_{\rm c}$ and ${}^{(2)}\boldsymbol{\delta}_{\rm v}$ *are of the canonical form* (1), with the Newtonian terms being unaffected by the change of gauge. The coefficients $A_{i,\rm c}$ and $A_{i,\rm v}$ are obtained by appropriately collecting terms on the right side of Eqs. (26), leading to

$$ A_{1,\rm c} = -2\mathcal{T}_1 + 2\Omega_m^{-1} g^2\left(1 + 4f + f^2 + 2\Omega_k + \frac{3}{2}\Omega_m\right), \tag{27a} $$

$$ A_{2,\rm c} = -2\mathcal{T}_2 + 8\Omega_m^{-1} g^2\left(1 + \frac{2}{3}\Omega_m^{-1} f^2\right), \tag{27b} $$

$$ A_{3,\rm c} = -2\mathcal{T}_3 + g\left(1 - 2f - \frac{3}{2}\Omega_m - \frac{4}{3}\Omega_m^{-1} f^2\right), \tag{27c} $$

$$ A_{4,\rm c} = -2\mathcal{T}_4 + \mathcal{B}_2 + \frac{3}{2} g\Omega_m, \tag{27d} $$

$$ A_{5,\rm c} = \mathcal{B}_1 + g(1 + f), \tag{27e} $$

and

$$ A_{1,\rm v} = 2\kappa x g\left[\mathcal{B}_1 - \frac{2}{3}\Omega_m^{-1} g(f^2 - 3f - 6\Omega_m)\right], \tag{28a} $$

$$ A_{2,\rm v} = 2\kappa x g\left[\mathcal{B}_2 - \frac{8}{3}\Omega_m^{-1} g f\right], \tag{28b} $$

$$ A_{3,\rm v} = 2\kappa x g\mathcal{B}_3 - \frac{5}{3}\Omega_m^{-1} g(2f + 3\Omega_m), \tag{28c} $$

$$ A_{4,\rm v} = 2\kappa x g\mathcal{B}_4 + \mathcal{B}_2 - \frac{8}{3}\Omega_m^{-1} g f, \tag{28d} $$

$$ A_{5,\rm v} = \mathcal{B}_1 - \frac{2}{3}\Omega_m^{-1} g(f^2 - 3f - 6\Omega_m). \tag{28e} $$

Here we have used the fact that

$$ \Omega_k\Omega_m^{-1} = -\kappa x, \tag{29a} $$

where the constant κ is given by

$$ \kappa := \frac{K}{m^2}. \tag{29b} $$

C. The synchronous-comoving gauge

The second order density perturbation ${}^{(2)}\boldsymbol{\delta}_{\rm s}$ in *the synchronous-comoving gauge* is related to ${}^{(2)}\boldsymbol{\delta}_{\rm v}$ according to Eq. (B42) in Appendix B, which we repeat here:

$$ {}^{(2)}\boldsymbol{\delta}_{\rm s} = {}^{(2)}\boldsymbol{\delta}_{\rm v} - \frac{4}{3} x m^{-2}(\mathbf{D}^i\boldsymbol{\delta}_{\rm v})(\mathbf{D}_i\Psi_{\rm p}). \tag{30} $$

On evaluating the source term using (7a), (12) and the identity (A4c) in Appendix A, and noting that ${}^{(2)}\boldsymbol{\delta}_{\rm v}$ has the general form (1), we obtain

$$ \begin{aligned} {}^{(2)}\boldsymbol{\delta}_{\rm s} = {} & A_{1,\rm v}\zeta^2 + A_{2,\rm v}\mathcal{D}(\zeta) \\ & + \frac{2}{3} m^{-2} x g[A_{3,\rm s}(\mathbf{D}\zeta)^2 + A_{4,\rm s}\mathbf{D}^2\mathcal{D}(\zeta) + A_{5,\rm v}\mathbf{D}^2\zeta^2] \\ & + \frac{4}{9} m^{-4} x^2 g^2\left[\left(\mathcal{B}_3 + \frac{1}{3}\right)\mathbf{D}^2(\mathbf{D}\zeta)^2 + \left(\mathcal{B}_4 - \frac{4}{3}\right)\mathbf{D}^4\mathcal{D}(\zeta) + 2(\mathbf{D}^2\zeta)^2\right], \end{aligned} \tag{31a} $$

where

$$ A_{3,\rm s} = A_{3,\rm v} - 4\kappa x g, \tag{31b} $$

$$ A_{4,\rm s} = A_{4,\rm v} - \frac{8}{3}\kappa x g, \tag{31c} $$

and the $A_{i,\rm v}$ coefficients are given by (28). Note the appearance of the additional quadratic differential expression $(\mathbf{D}^2\zeta)^2$ in the Newtonian part. Equations (12) and (31) highlight important similarities and differences between the total matter gauge and the synchronous-comoving gauge for dust perturbations; namely, at first order the fractional density perturbations are the same while at second order only the superhorizon parts coincide.

D. A second order conserved quantity ${}^{(2)}\boldsymbol{\delta}_{\mathrm{c}}$

We have shown earlier that the superhorizon term in ${}^{(1)}\boldsymbol{\delta}_{\mathrm{c}}$ is independent of time. This reflects the fact that $\zeta_\rho := -\frac{1}{3}{}^{(1)}\boldsymbol{\delta}_{\mathrm{c}}$ satisfies a conservation law that reduces to $\partial_\eta \zeta_\rho = 0$ when spatial derivative terms can be neglected. At second order, we conjecture that ${}^{(2)}\zeta_\rho := -\frac{1}{3}{}^{(2)}\boldsymbol{\delta}_{\mathrm{c}}$ has a similar property, in other words that the superhorizon terms in ${}^{(2)}\boldsymbol{\delta}_{\mathrm{c}}$, namely $A_{1,\mathrm{c}}$ and $A_{2,\mathrm{c}}$, as given by (27a) and (27b), are independent of time. We can confirm this by showing directly that

$$\partial_x A_{1,\mathrm{c}} = 0, \qquad \partial_x A_{2,\mathrm{c}} = 0. \tag{32}$$

This calculation uses the fact that $\partial_x \mathcal{T}_{1,2} = T_{1,2}$, the constraint (10) and the following derivatives[20]:

$$x\partial_x \Omega_m = (2q-1)\Omega_m, \tag{33a}$$

$$x\partial_x g = g(f-1), \tag{33b}$$

$$x\partial_x f = (1+f)(q-2-f) + 2f + 3 - \Omega_k, \tag{33c}$$

where the deceleration parameter q is defined by[21]

$$x\partial_x \mathcal{H} = -q\mathcal{H}. \tag{34a}$$

It follows from (B25) that for dust

$$q = \frac{3}{2}\Omega_m + \Omega_k - 1. \tag{34b}$$

We can then determine the constant values of $A_{1,\mathrm{c}}$ and $A_{2,\mathrm{c}}$ by evaluating the limit of the expressions (27a) and (27b) as $x \to 0$, leading to

$$A_{1,\mathrm{c}} = \frac{27}{5}, \qquad A_{2,\mathrm{c}} = \frac{24}{5}. \tag{35}$$

When these values are substituted into (27a) and (27b) we obtain

$$\mathcal{T}_1 = \Omega_m^{-1} g^2 \left(1 + 4f + f^2 + 2\Omega_k + \frac{3}{2}\Omega_m\right) - \frac{27}{10}, \tag{36a}$$

$$\mathcal{T}_2 = 4\Omega_m^{-1} g^2 \left(1 + \frac{2}{3}\Omega_m^{-1} f^2\right) - \frac{12}{5}. \tag{36b}$$

We have thus shown that ${}^{(1)}\boldsymbol{\delta}_{\mathrm{c}}$ and ${}^{(2)}\boldsymbol{\delta}_{\mathrm{c}}$ are conserved quantities in the sense that the superhorizon part is constant in time. These conserved quantities are in fact closely related to the quantities introduced by Malik and Wands[22]:

$${}^{(1)}\zeta_{\mathrm{mw}} := -{}^{(1)}\Psi_\rho, \qquad {}^{(2)}\zeta_{\mathrm{mw}} := -{}^{(2)}\Psi_\rho, \tag{37}$$

which we shall use in the following section. At first order we have the simple relation ${}^{(1)}\Psi_\rho = -\frac{1}{3}{}^{(1)}\boldsymbol{\delta}_{\mathrm{c}}$, as follows from (12) and (42a). At second order these conserved quantities are related through their superhorizon terms:

$$({}^{(2)}\Psi_\rho - {}^{(1)}\Psi_\rho^2)|_{\mathrm{superhorizon}} = -\frac{1}{3}{}^{(2)}\boldsymbol{\delta}_{\mathrm{c}}|_{\mathrm{superhorizon}}, \tag{38}$$

as follows from (35), (40) and (42b).

E. Initial conditions

The solution (13) for ${}^{(2)}\Psi_{\mathrm{p}}$ is a particular solution that satisfies $\lim_{x\to 0}{}^{(2)}\Psi_{\mathrm{p}} = 0$. The general solution for ${}^{(2)}\Psi_{\mathrm{p}}$ for a zero decaying scalar mode at the linear level is given by

$${}^{(2)}\Psi_{\mathrm{p}}|_{\mathrm{gen}} = {}^{(2)}\Psi_{\mathrm{p}} + C(x^i) g(x), \tag{39}$$

where $C(x^i)$ is an arbitrary function. Note that the second term on the right side of (39) is the general solution of the homogeneous equation for ${}^{(2)}\Psi_{\mathrm{p}}$ [see UW, Eq. (37)]. The corresponding general expression for ${}^{(2)}\boldsymbol{\delta}_\bullet$ for the Poisson, the uniform curvature, the total matter and the synchronous-comoving gauges is given by

$${}^{(2)}\boldsymbol{\delta}_\bullet|_{\mathrm{gen}} = {}^{(2)}\boldsymbol{\delta}_\bullet + A_\bullet C + \frac{2}{3} x g m^{-2} \mathbf{D}^2 C, \tag{40}$$

where $A_\bullet$ is the coefficient in the expression (12) for ${}^{(1)}\boldsymbol{\delta}_\bullet$. Note that the extra terms on the right side of (40) take the same form as the first order density perturbation, but with $\zeta(x^i)$ replaced by the arbitrary function $C(x^i)$.

In applications the arbitrary function $C(x^i)$ is usually determined by using the metric and matter perturbations at the end of inflation as initial conditions. Various theories of inflation predict that these perturbations will not be purely Gaussian; i.e. there will be a certain level of primordial non-Gaussianity. It is convenient to use the first and second order conserved quantities given by (37) to parameterize this primordial non-Gaussianity on superhorizon scales. Specifically, it is assumed that

$${}^{(2)}\zeta_{\mathrm{mw}} = 2a_{\mathrm{nl}}({}^{(1)}\zeta_{\mathrm{mw}})^2, \tag{41}$$

[20]Equation (8) gives (33b), apply ∂_x to (5) and use (34a) to get (33a), and finally apply ∂_x to (10) to get (33c).

[21]This is equivalent to $q = -\frac{\ddot{a}a}{(\dot{a})^2}$.

[22]See [20], Eqs. (7.61) and (7.71). The subscript ρ stands for the uniform density gauge, and the subscript mw stands for Malik-Wands. The gauge invariant ${}^{(2)}\zeta_{\mathrm{mw}}$ was first defined in [23] [see Eq. (4.18)]. We note that Langlois and Vernizzi [24] derived an analogous conserved quantity at second order using the $1+3$ approach to perturbations [see Eq. (50)].

where $a_{\rm nl}$ is a parameter that depends on the physics of the model of inflation.[23] It has been shown that primordial non-Gaussianity in the CMB temperature anisotropy at second order is represented by the quantity ${}^{(2)}\zeta_{\rm mw} - 2{}^{(1)}\zeta_{\rm mw}^2$ [[25], Eq. (8)]. It follows that the absence of primordial non-Gaussianity corresponds to $a_{\rm nl} = 1$.

It follows from Eqs. (B36) in Appendix B in conjunction with (7a) and (12) that the gauge invariants ${}^{(1)}\Psi_\rho$ and ${}^{(2)}\Psi_\rho$ in (37) are given by

$$ {}^{(1)}\Psi_\rho = \zeta - \frac{2}{9} x g m^{-2} \mathbf{D}^2 \zeta, \tag{42a} $$

$$ {}^{(2)}\Psi_\rho = -\frac{1}{5}[4\zeta^2 + 8\mathcal{D}(\zeta)] + C(x^i) + (\mathbf{D}_i \text{ terms up to order } 6). \tag{42b} $$

These equations and the restriction (41) determine the arbitrary function $C(x^i)$ in terms of ζ and $\mathcal{D}(\zeta)$. The resulting function is denoted by $C_{\rm nl}$:

$$ C_{\rm nl}(x^i) = \frac{4}{5}\left[\left(1 - \frac{5}{2}a_{\rm nl}\right)\zeta^2 + 2\mathcal{D}(\zeta)\right]. \tag{43} $$

We will denote the density perturbation ${}^{(2)}\boldsymbol{\delta}_{\bullet}$ corresponding to this choice of initial condition, which is determined by substituting the expression (43) into (40), by ${}^{(2)}\boldsymbol{\delta}_{\bullet}|_{\rm nl}$:

$$ {}^{(2)}\boldsymbol{\delta}_{\bullet}|_{\rm nl} = {}^{(2)}\boldsymbol{\delta}_{\bullet} + A_{\bullet} C_{\rm nl} + \frac{2}{3} x g m^{-2} \mathbf{D}^2 C_{\rm nl}. \tag{44} $$

It follows from (1), (44) and (43) that the coefficients $A_{i,\bullet}|_{\rm nl}$ are given by

$$ A_{1,\bullet}|_{\rm nl} = A_{1,\bullet} + \frac{4}{5}\left(1 - \frac{5}{2}a_{\rm nl}\right)A_{\bullet}, \tag{45a} $$

$$ A_{2,\bullet}|_{\rm nl} = A_{2,\bullet} + \frac{8}{5}A_{\bullet}, \tag{45b} $$

$$ A_{3,\bullet}|_{\rm nl} = A_{3,\bullet}, \tag{45c} $$

$$ A_{4,\bullet}|_{\rm nl} = A_{4,\bullet} + \frac{8}{5}, \tag{45d} $$

$$ A_{5,\bullet}|_{\rm nl} = A_{5,\bullet} + \frac{4}{5}\left(1 - \frac{5}{2}a_{\rm nl}\right), \tag{45e} $$

where $A_{\bullet}$ is given by (12).

[23] See for example [25], p. 4, and [8], Eq. (9), and references given in these papers. When making comparisons with CMB observations it is customary to use a nonlinearity parameter $f_{\rm nl}$, which takes into account that the nonlinear gravitational dynamics after inflation contributes to the non-Gaussianity. This parameter has the form $f_{\rm nl} = \frac{5}{3}(a_{\rm nl} - 1) + \cdots$, where $+\cdots$ refers to terms that describe the effect of the postinflation nonlinear gravitational dynamics on the primordial non-Gaussianity. See for example [25], Eq. (9), [8], Eq. (31) and [1], Sec. 8.4.2.

IV. THE SPECIALIZATION TO A FLAT BACKGROUND

In the previous section we showed that the time dependence of ${}^{(2)}\boldsymbol{\delta}_{\bullet}$ is determined by the linear perturbation function $g(x)$ and the background functions Ω_m and $\mathcal{H}$, either algebraically, or as the integrals $\mathcal{B}_A$ given by (21b) and (23a), and $\mathcal{T}_A$ given by (20b) and (23b), with $A = 1, \ldots, 4$. Subsequently, we showed that $\mathcal{T}_1$ and $\mathcal{T}_2$ could be written algebraically, as in (36). In this section we show that if the spatial curvature is zero, a significant simplification occurs: *only one integral function is required.*

A. The flatness conditions

We here show that if the background is flat, then $\mathcal{T}_{3,4}$ and $\mathcal{B}_{1,2}$ are algebraic expressions in g and Ω_m, and in addition $\mathcal{B}_3 + \mathcal{B}_4 = 1$. For convenience we define

$$ \mathbf{T}_3 \coloneqq \mathcal{T}_3 - \frac{1}{2}g + \frac{3}{4}\Omega_m g^{-1}(1-g)^2, \tag{46a} $$

$$ \mathbf{T}_4 \coloneqq \mathcal{T}_4 - 3 + 2g - \frac{3}{4}\Omega_m g^{-1}(1-g)^2, \tag{46b} $$

$$ \mathbf{B}_1 \coloneqq \mathcal{B}_1 - \frac{1}{5} + g - \frac{3}{2}\Omega_m g^{-1}(1-g)^2, \tag{46c} $$

$$ \mathbf{B}_2 \coloneqq \mathcal{B}_2 - \frac{12}{5} + 4g, \tag{46d} $$

$$ \mathbf{B}_{3+4} \coloneqq xg(\mathcal{B}_3 + \mathcal{B}_4 - 1). \tag{46e} $$

Then the result can be expressed as follows: if $\Omega_k = 0$, then $\mathbf{T}_{3,4} = 0$, $\mathbf{B}_{1,2} = 0$, and $\mathbf{B}_{3+4} = 0$.

These results can be proved by differentiation, as follows. First we show that if $K = 0$, then $\partial_x(xg\mathbf{T}_{3,4}) = 0$. This calculation requires $\partial_x(xg\mathcal{T}_{3,4}) = \frac{3}{2}T_{3,4}$, as follows from (20b) and (23b), and also Eqs. (33b) and (33a). It follows that $xg\mathbf{T}_{3,4} = C_{3,4}$, a constant. Since $T_{3,4}$ is bounded as $x \to 0$ we conclude that $C_{3,4} = 0$, which gives the desired result.

Second we show that if $K = 0$, then the quantities $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_{3+4}$ satisfy

$$ x\partial_x \mathbf{B}_{\bullet} = -\frac{3}{2}\Omega_m g^{-1}\mathbf{B}_{\bullet}, \qquad \lim_{x\to 0}\mathbf{B}_{\bullet} = 0, \tag{47} $$

Since $g > 0$ it follows from (47) that $(\mathbf{B}_{\bullet})^2$ is monotone decreasing or identically zero. The limit condition then implies that $\mathbf{B}_{\bullet} \equiv 0$, which gives the desired relations $\mathbf{B}_{1,2} = 0$, and $\mathbf{B}_{3+4} = 0$. This calculation requires

$$ \partial_x(x^2 g\mathcal{H}^{-1}B_A) = x\Omega_m\mathcal{H}^{-1}\hat{T}_A, \\ \partial_x(x^2 g\mathcal{H}^{-1}) = \frac{3}{2}x\Omega_m\mathcal{H}^{-1}. \tag{48} $$

The first of these follows from (21b), together with the definitions (23a) and (23b). The second follows from (33a), (34a) and (10). Note that the constraint (10) can be written in the form $2g(f+q+1)=3\Omega_m$, using (34b).

B. Alternate expressions for ${}^{(2)}\boldsymbol{\delta}_{\bullet}$

We now cast the expressions for ${}^{(2)}\boldsymbol{\delta}_{\bullet}$ into a form in which the role played by the spatial curvature becomes clear. We use (36) to eliminate $\mathcal{T}_{1,2}$ in the expressions for ${}^{(2)}\boldsymbol{\delta}_{\bullet}$ and use (46) to express $\mathcal{T}_{3,4}$ and $\mathcal{B}_{1,2}$ in terms of $\mathbf{T}_{3,4}$ and $\mathbf{B}_{1,2}$. We also use the constraint (10) to eliminate f in favor of g, and use (45) to introduce the non-Gaussianity initial condition. The coefficients $A_{i,\bullet}$ in Eqs. (25), (27) and (28) assume the form

$$A_{1,\mathrm{p}}=3(1-g)\left(1+2a_{\mathrm{nl}}-4g+\frac{3}{2}\Omega_m(1-g)\right)+3g\mathbf{B}_1, \tag{49a}$$

$$A_{2,\mathrm{p}}=12g(1-g)+3g\mathbf{B}_2, \tag{49b}$$

$$A_{3,\mathrm{p}}=3g(\mathcal{B}_3-2)-\frac{3}{2}\Omega_m g^{-1}(1-g)^2+4\Omega_k\left(1-g+\frac{1}{3}\kappa xg\right)-2\mathbf{T}_3, \tag{49c}$$

$$A_{4,\mathrm{p}}=-2+3g\mathcal{B}_4-\frac{3}{2}\Omega_m g^{-1}(1-g)^2-2\mathbf{T}_4+\mathbf{B}_2, \tag{49d}$$

$$A_{5,\mathrm{p}}=1-2a_{\mathrm{nl}}+3g+\frac{3}{2}\Omega_m g^{-1}(1-g)^2+\mathbf{B}_1, \tag{49e}$$

$$A_{1,\mathrm{c}}=3(1+2a_{\mathrm{nl}}), \tag{50a}$$

$$A_{2,\mathrm{c}}=0, \tag{50b}$$

$$A_{3,\mathrm{c}}=-\frac{3}{2}\Omega_m g^{-1}+2\Omega_k\left(2-g+\frac{2}{3}\kappa xg\right)-2\mathbf{T}_3, \tag{50c}$$

$$A_{4,\mathrm{c}}=-2-\frac{3}{2}\Omega_m g^{-1}(1-2g)-2\mathbf{T}_4+\mathbf{B}_2, \tag{50d}$$

$$A_{5,\mathrm{c}}=1-2a_{\mathrm{nl}}+\frac{3}{2}\Omega_m g^{-1}(1-g)-\Omega_k g+\mathbf{B}_1, \tag{50e}$$

$$A_{1,\mathrm{v}}=2\kappa xg\left[\mathbf{B}_1+\frac{16}{5}+2\kappa xg\left(1+\frac{1}{3}\Omega_k\right)+2\Omega_k(1-g))\right], \tag{51a}$$

$$A_{2,\mathrm{v}}=2\kappa xg\left[\mathbf{B}_2-\frac{8}{5}-\frac{8}{3}\kappa xg\right], \tag{51b}$$

$$A_{3,\mathrm{v}}=2\kappa xg\left(\mathcal{B}_3-\frac{5}{3}\right)-5, \tag{51c}$$

$$A_{4,\mathrm{v}}=2\kappa xg\left(\mathcal{B}_4-\frac{4}{3}\right)+\mathbf{B}_2, \tag{51d}$$

$$A_{5,\mathrm{v}}=2(2-a_{nl})+2\kappa xg\left(1+\frac{1}{3}\Omega_k\right)+2\Omega_k(1-g)+\mathbf{B}_1, \tag{51e}$$

where the constant κ is defined by (29). For the synchronous-comoving gauge it follows from (31) that

$$(A_1,A_2,A_3,A_4,A_5)_{\mathrm{s}}=(A_1,A_2,A_3,A_4,A_5)_{\mathrm{v}}-4\kappa xg\left(0,0,1,\frac{2}{3},0\right), \tag{52}$$

with the Newtonian part unchanged.

C. Zero spatial curvature

In the case of zero spatial curvature we have $\Omega_k=0$, $\kappa=0$, $\mathbf{T}_{3,4}=0$, $\mathbf{B}_{1,2}=0$ and $\mathcal{B}_3+\mathcal{B}_4=1$. We write $\mathcal{B}_3=\mathcal{B}$ and $\mathcal{B}_4=1-\mathcal{B}$ and can express the scalar $\mathcal{B}$ as a standard integral involving g, Ω_m and $\mathcal{H}$:

$$\mathcal{B}(x)=\frac{\mathcal{H}(x)}{2x^3g(x)^2}\int_0^x\frac{\bar{x}^2\Omega_m}{\mathcal{H}}\left(g^2-\frac{3}{2}\Omega_m(1-g)^2\right)d\bar{x}, \tag{53}$$

where g, Ω_m and $\mathcal{H}$ inside the integral are functions of $\bar{x}$. This result follows from Eqs. (21b) and (23) with $A=3$, when one uses the expression for $\mathcal{T}_3$ given by (23b) and (46a) with $\mathbf{T}_3=0$.

With these simplifications the expressions (49)–(51) for the coefficients in ${}^{(2)}\boldsymbol{\delta}_{\bullet}$ for the Poisson, uniform curvature and total matter gauges reduce to those in [26], where the present results for zero curvature are summarized. The full expression is given by (1), with the Newtonian part given by

$${}^{(2)}\boldsymbol{\delta}_{\bullet}|_{\mathrm{Newtonian}}=\frac{4}{9}m^{-4}x^2g^2[\mathcal{B}\mathbf{D}^2(\mathbf{D}\zeta)^2+(1-\mathcal{B})\mathbf{D}^4\mathcal{D}(\zeta)]. \tag{54}$$

For the reader's convenience, we give the coefficients $A_{i,\mathrm{p}}$ in the Poisson gauge when $K=0$, obtained by specializing (49):

$$A_{1,\mathrm{p}}=3(1-g)\left(1+2a_{\mathrm{nl}}-4g+\frac{3}{2}\Omega_m(1-g)\right), \tag{55a}$$

$$A_{2,\mathrm{p}}=12g(1-g), \tag{55b}$$

$$A_{3,\mathrm{p}}=3g(\mathcal{B}-2)-\frac{3}{2}\Omega_m g^{-1}(1-g)^2, \tag{55c}$$

$$A_{4,\mathrm{p}} = -2 + 3g(1-\mathcal{B}) - \frac{3}{2}\Omega_m g^{-1}(1-g)^2, \tag{55d}$$

$$A_{5,\mathrm{p}} = 1 - 2a_{\mathrm{nl}} + 3g + \frac{3}{2}\Omega_m g^{-1}(1-g)^2. \tag{55e}$$

We also give the full expression for the synchronous-comoving gauge:

$$\begin{aligned}{}^{(2)}\boldsymbol{\delta}_{\mathrm{s}} = {}& \frac{2}{3}m^{-2}xg[-5(\mathbf{D}\zeta)^2 + 2(2-a_{\mathrm{nl}})\mathbf{D}^2\zeta^2] \\ &+ \frac{4}{9}m^{-4}x^2g^2\left[\left(\mathcal{B}+\frac{1}{3}\right)(\mathbf{D}^2(\mathbf{D}\zeta)^2 \right. \\ &\left. - \mathbf{D}^4\mathcal{D}(\zeta)) + 2(\mathbf{D}^2\zeta)^2\right],\end{aligned} \tag{56}$$

as follows from (51) and (52).

D. Relation with the literature

In the course of doing the research reported in this paper we made a detailed comparison of our expressions for ${}^{(2)}\boldsymbol{\delta}_{\bullet}$ with those in the literature, which deal solely with the case where *the background spatial curvature is zero*. In addition the expressions for ${}^{(2)}\boldsymbol{\delta}_{\bullet}$ with $\Lambda > 0$ are *restricted to the synchronous-comoving and Poisson gauges*. In [26] we gave a brief overview of the results in the literature. In this section we describe in detail the relation between our results and the papers in the literature, focusing in particular on the work of Tomita [6] and Bartolo and collaborators [8].

Comparing the different results is not straightforward since there are many different ways of representing the spatial dependence in the expression for ${}^{(2)}\boldsymbol{\delta}_{\bullet}$, which involves an arbitrary spatial function and the spatial differential operator $\mathbf{D}_i$. We thus begin by describing the various quadratic differential expressions that have been used in the literature and showing how they are related to our canonical set (2).

1. Spatial quadratic differential expressions

In discussing our canonical set of quadratic differential expressions (2) we note that the operator $\mathcal{D}(A)$, as defined by (A2) and (A3), plays a key role. We begin with the zero order derivative expressions ζ^2 and $\mathcal{D}(\zeta)$ that determine the superhorizon terms in ${}^{(2)}\boldsymbol{\delta}_{\bullet}$. Two of the three second order derivative expressions that determine the post-Newtonian terms are obtained by acting with $\mathbf{D}^2$ on the zero order expressions, while the third, $(\mathbf{D}\zeta)^2 \equiv \mathbf{D}^i\zeta\mathbf{D}_i\zeta$, is a new expression. Finally, the two fourth order derivative expressions that determine the Newtonian terms are obtained by acting with $\mathbf{D}^2$ on two of the second order expressions. Before continuing we mention that the appearance of $\mathcal{D}(\zeta)$ in the second order density perturbation has its origin in the quadratic source term in the evolution equation for the second order Bardeen potential ${}^{(2)}\Psi_{\mathrm{p}}$ [see Eqs. (61b) and (61f) in [27]], through the use of the mode extraction operator $\mathcal{S}^{ij}$, as defined by (A3).

We now list the various other spatial quadratic differential expressions that have appeared in the literature:

$$A\mathbf{D}^2A, \qquad \mathbf{D}^i\mathbf{D}^j(\mathbf{D}_iA\mathbf{D}_jA), \qquad (\mathbf{D}^i\mathbf{D}^jA)(\mathbf{D}_i\mathbf{D}_jA), \tag{57a}$$

$$(\mathbf{D}^iA)(\mathbf{D}_i\mathbf{D}^2A), \qquad \mathbf{D}^i(\mathbf{D}_iA\mathbf{D}^2A), \qquad \mathbf{D}^i\mathbf{D}^j(A\mathbf{D}_i\mathbf{D}_jA), \tag{57b}$$

sometimes with $\mathbf{D}^{-2}$ acting on the left. Each of these expressions can be written as a linear combination of our canonical set (2) augmented by the terms $(\mathbf{D}^2A)^2$ and $\mathbf{D}^4A^2$ using the identities (A4) in the Appendix A. Here we use the generic symbol $A = A(x^i)$ to denote the arbitrary spatial function. Although there is no consensus for this function the various choices differ only by an overall numerical factor.

A quantity closely related to our $\mathcal{D}(A)$ in (A2) has been defined by several authors as follows. Let

$$\Psi_0 := \frac{1}{2}\lambda\mathbf{D}^{-2}(\mathbf{D}^i\mathbf{D}^jA\mathbf{D}_i\mathbf{D}_jA - (\mathbf{D}^2A)^2), \tag{58a}$$

$$\Theta_0 := \mathbf{D}^{-2}\left(\Psi_0 - \frac{1}{3}\lambda(\mathbf{D}A)^2\right), \tag{58b}$$

where λ is a numerical factor that we have introduced to accommodate different scalings. It follows from (A4d) that

$$\Psi_0 := -\frac{1}{3}\lambda(\mathbf{D}^2\mathcal{D}(A) - (\mathbf{D}A)^2), \qquad \Theta_0 = -\frac{1}{3}\lambda\mathcal{D}(A). \tag{59}$$

This makes clear that Θ_0 corresponds to our $\mathcal{D}(A)$, while Ψ_0 is closely related to our $\mathbf{D}^2\mathcal{D}(A)$. These quantities were used in [4,5] with $A = \varphi = \frac{3}{5}\zeta$ and $\lambda = 1$ [see Eqs. (4.36) and (6.6) in [4], and following Eq. (9) in [5]]. They were also used in [6] with $A = F = -2\zeta$ and $\lambda = \frac{9}{100}$ [see Eq. (4.11), noting that a factor of $\mathbf{D}^{-2}$ is missing in the first equation]. Furthermore, Bartolo *et al.* [8] come close to introducing our $\mathcal{D}(A)$. They define [see their Eq. (18)]

$$\alpha_0 := \mathbf{D}^{-2}(\mathbf{D}\varphi_0)^2 - 3\mathbf{D}^{-4}\mathbf{D}^i\mathbf{D}^j(\mathbf{D}_i\varphi_0\mathbf{D}_j\varphi_0), \tag{60}$$

where $\varphi_0 = \frac{3}{5}\zeta$. It follows from (A4b) that

$$\alpha_0 = -2\mathcal{D}(\varphi_0). \tag{61}$$

2. Synchronous-comoving gauge with $K = 0$, $\Lambda \geq 0$

The first expression for ${}^{(2)}\boldsymbol{\delta}_{\mathrm{s}}$ with $\Lambda > 0$ was given by Tomita [6] [see his Eq. (2.22)]. The time dependence in his expression is described by two functions $P(\eta)$ and $Q(\eta)$ that satisfy second order differential equations[24]:

[24] Here η denotes conformal time. Note that $\partial_\eta = \mathcal{H}x\partial_x$.

$$(\partial_\eta^2 + 2\mathcal{H}\partial_\eta)P = 1, \qquad (\partial_\eta^2 + 2\mathcal{H}\partial_\eta)Q = P - \frac{5}{2}(\partial_\eta P)^2, \tag{62}$$

and the spatial dependence is described by a function $F(x^i)$ and its first and second partial derivatives, including the following quadratic differential expressions:

$$F\mathbf{D}^2 F, \qquad \mathbf{D}_i\mathbf{D}_j F\mathbf{D}^i\mathbf{D}^j F, \tag{63}$$

in addition to the ones in our canonical list (2). We use the identities (A4a) and (A4d) to relate these expressions to our canonical expressions. To match the density perturbations requires

$$g = 1 - \mathcal{H}\partial_\eta P, \qquad F(x^i) = -2\zeta(x^i), \tag{64a}$$

at linear order and

$$P = \frac{2}{3}m^{-2}xg, \tag{64b}$$

$$\partial_\eta Q = \frac{1}{3}m^{-2}\frac{x}{\mathcal{H}}\left[21g^2\mathcal{B} - g(9g-2) + \frac{21}{2}\Omega_m(1-g)^2\right], \tag{64c}$$

at second order. With these equations it follows that Tomita's expression (2.22) is transformed into our expression for ${}^{(2)}\boldsymbol{\delta}_{\rm s}$ given by (56), but with $a_{\rm nl} = 0$.

3. Poisson gauge with $K = 0$, $\Lambda \geq 0$

The first expression for ${}^{(2)}\boldsymbol{\delta}_{\rm p}$ with $\Lambda > 0$ was given by Tomita [6] [see his Eq. (4.16)]. As with ${}^{(2)}\boldsymbol{\delta}_{\rm s}$ the time dependence is described by P and Q and the spatial dependence is described by the quadratic differential expressions in our canonical list (2) together with the expressions (63) and

$$(\mathbf{D}^i F)\mathbf{D}^2\mathbf{D}_i F. \tag{65}$$

In particular the combinations Ψ_0 and Θ_0, as defined by (58a) and (58b), are used with $F = -2\zeta$ and $\lambda = \frac{9}{100}$. We write these combinations in the form (59), and use the identity (A4c) for the expression in (65). Using Eqs. (64) we can now show that Tomita's expression (4.16) with a few minor typos corrected[25] is transformed into our expression for ${}^{(2)}\boldsymbol{\delta}_{\rm p}$ given by (1), (54) and (55), but with $a_{\rm nl} = 0$.

An expression for ${}^{(2)}\boldsymbol{\delta}_{\rm p}$ with $\Lambda > 0$ has more recently been derived by Bartolo and collaborators [8] [see their Eq. (29)]. In order to make a comparison with our expression which has the general form (1), we first consider their linear perturbation. The time dependence of the linear perturbation is described by a function g, which we denote by $g_{\rm b}$ to distinguish it from our g, and the spatial dependence is described by a function $\varphi(x^i)$ which is a constant multiple of our ζ. Since $g_{\rm b} = 1$ and $g = \frac{3}{5}$ when $\Lambda = 0$ it follows by comparing their Eq. (11) with our (7a) that

$$g_{\rm b} = \frac{5}{3}g, \qquad \varphi_0 = \frac{3}{5}\zeta. \tag{66}$$

We next consider the Bardeen potential ${}^{(2)}\Psi_{\rm p}$ [Eq. (20) in [8]] whose time dependence is described by $g_{\rm b}$ and four functions $\mathbb{B}_A$, $A = 1, 2, 3, 4$. Our expression for ${}^{(2)}\Psi_{\rm p}$, including the non-Gaussianity initial condition, is given by Eqs. (22a), (39) and (43). In order to match the spatial dependence terms we note that their α_0 is given by (61). We also need to use the identities (A4a) and (A4b). Comparing the two expressions for ${}^{(2)}\Psi_{\rm p}$ leads to the following relation between our B_A and the quantities $\mathbb{B}_A$ in [8]:

$$(B_1, B_2, B_3, B_4) = \frac{9}{25}g^{-1}\left(\mathbb{B}_1, -2\mathbb{B}_2, m^2\left(\frac{1}{3}\mathbb{B}_3 + \mathbb{B}_4\right), \frac{2}{3}m^2\mathbb{B}_3\right). \tag{67}$$

To establish consistency we need to show that the definition of the $\mathbb{B}_A$ in [8] [see Eqs. (22)–(26)] translates into our definition of the B_A in (21b) under the transformation (67). The definitions of the $\mathbb{B}_A$ in [8] can be collectively written in the form[26]

$$\frac{9}{25}g^{-1}\mathbb{B}_A = \left(\frac{5}{3}g_0\right)\frac{\mathcal{H}}{x^2 g}\int_0^x (I(x) - I(\bar{x}))g(\bar{x})^2\mathbb{T}_A(\bar{x})d\bar{x}, \tag{68}$$

where $I(x)$ is expressed in terms of $g(x)$ by (8). The functions $\mathbb{T}_A$ are related to our functions T_A according to

$$m^2(T_1, T_2, T_3, T_4) = g^2\left(\mathbb{T}_1, -2\mathbb{T}_2, m^2\left(\frac{1}{3}\mathbb{T}_3 + \mathbb{T}_4\right), \frac{2}{3}m^2\mathbb{T}_3\right), \tag{69}$$

where we note that our variable f coincides with the f in [8]. On the other hand our Eq. (21b) expressing B_A in terms of T_A can be written in the equivalent form[27]

[25] In line 2 in Tomita's Eq. (4.16), $-\frac{2a'}{a}PP'$ should be $-\frac{a'}{2a}PP'$, in line 3, $-\frac{1}{2}P'$ should be $-P'$, and in line 4, $-\frac{1}{2}P$ should be $-P$. In addition the sign of Q, which appears in lines 1 and 4, should be reversed.

[26] Here we have used (10) and $m^2 = \Omega_{m,0}\mathcal{H}_0^2$ to write the equation $\tilde{B}_A = \mathcal{H}_0^2(f_0 + \frac{3}{2}\Omega_{m,0})B_A$ in [8] in the form $\tilde{B}_A = \frac{3}{2}m^2 g_0^{-1}B_A$.

[27] Equation (21b), together with (20b), is an iterated integral. Use $x\Omega_m\mathcal{H}^2 = m^2$ in the integrand, reverse the order of integration and make use of the definition of $I(x)$.

$$B_A = \frac{\mathcal{H}}{x^2 g}\int_0^x (I(x) - I(\bar{x}))m^2 T_A(\bar{x})d\bar{x},$$
$$I(x) := \int_0^x \frac{d\bar{x}}{\mathcal{H}(\bar{x})^3}. \tag{70}$$

The common structure of (67) and (69) ensures that (68) translates into Eq. (70), provided that $g_0 = \frac{3}{5}$.

We are now in a position to show that the expression for ${}^{(2)}\boldsymbol{\delta}_{\rm p}$ in [8] can be transformed into our expression. We first convert the quadratic differential expressions in [8] in φ_0 into our canonical expressions (2) in ζ using (A4a), (A4b), (61) and (66). We then express $g_{\rm b}$ and $\mathbb{B}_A$ in the time-dependent coefficients in terms of g and B_A, using (66) and (67). It is necessary to use (14) in order to eliminate the derivatives $\partial_x \mathbb{B}_A$. We find agreement except with the coefficient of ζ^2, which corresponds to the coefficient of φ^2 in Eq. (29) in [8]. We conclude that the term $(f-1)^2 - 1$ in this coefficient should be replaced by $2(f-1)^2$. The φ^2 term then correctly specializes to $-\frac{8}{3}(1 - \frac{2}{5}a_{\rm nl})\varphi^2$ when one restricts consideration to the Einstein–de Sitter universe ($\Lambda = 0$), in agreement with Eq. (8) in [5].

V. CONCLUDING REMARKS

The results in this paper fall under three headings. First, we have presented exact expressions for the second order fractional density perturbation for dust, a cosmological constant and spatial curvature in a simple and physically transparent form in four popular gauges: the Poisson, the uniform curvature, the total matter and the synchronous-comoving gauges. Our results unify and generalize all the known results in the literature, which are confined to the case of zero spatial curvature and, when $\Lambda > 0$, to the Poisson and synchronous-comoving gauges. Our approach has two novel features. We have introduced a canonical way of representing the spatial dependence of the perturbations at second order which makes clear how the choice of gauge affects the form of the expressions. In addition we have formulated the time dependence in such a way that the dynamics of the perturbations and the effect of spatial curvature can be read off by inspection. In particular, in the special case of zero spatial curvature we have shown that the time evolution simplifies dramatically and requires the use of only two nonelementary functions, the so-called growth suppression factor g that arises at the linear level, and one new function $\mathcal{B}$ at the second order level. We emphasize that the assumption of zero decaying mode underlies the simple expressions for ${}^{(2)}\boldsymbol{\delta}_{\bullet}$ that we have presented. This assumption is usually made in cosmological perturbation theory, presumably on the grounds that the decaying mode will become negligible. However, if $\Lambda > 0$, the name "decaying mode" is a misnomer since this mode, after decaying in the matter-dominated epoch ($\Omega_m \approx 1$), increases when Ω_Λ becomes significant and contributes to the density perturbation on an equal footing with the growing mode in the de Sitter regime. This is made clear by the asymptotic expressions given in UW [see Eq. (66a)]. Into the past the decaying mode grows without bound on approach to the initial singularity. On the other hand, if the decaying mode is set to zero, the perturbations remain finite into the past and one is essentially considering perturbations in a universe with an isotropic singularity [28].

Second, we have made a detailed comparison of our results with the known expressions for ${}^{(2)}\boldsymbol{\delta}$ in different gauges when the background spatial curvature is zero. Our canonical representation of the spatial dependence has enabled us to unify seemingly disparate results, while at the same time revealing a number of errors in the expressions in the literature. For example, two expressions for ${}^{(2)}\boldsymbol{\delta}_{\rm p}$ with $\Lambda > 0$ have been given. The first, by Tomita [6], was derived by solving the perturbation equations at second order in the synchronous-comoving gauge and then transforming to the Poisson gauge. The second, by Bartolo *et al.* [8], was derived by solving the perturbation equations directly in the Poisson gauge. The two expressions appear to be completely different. However, by simplifying the $\mathbb{B}$ functions of Bartolo and introducing our canonical representation of the spatial dependence we have been able to show, after correcting some typos, that both of these expressions can be written in our canonical form for ${}^{(2)}\boldsymbol{\delta}_{\rm p}$, which is given by (1), (54) and (55).

Third, we have given a systematic procedure for performing a change of gauge for second order perturbed quantities. The derivation of our expressions for ${}^{(2)}\boldsymbol{\delta}_{\bullet}$ relied on solving the perturbation equations in the Poisson gauge as done in UW and then using our change of gauge procedure to calculate ${}^{(2)}\boldsymbol{\delta}_{\bullet}$ in the other gauges. The procedure is easy to implement in this application since the change of gauge induces a simple change in the time-dependent coefficients $A_{i,\bullet}$ in (1) while preserving the overall structure of ${}^{(2)}\boldsymbol{\delta}_{\bullet}$. However, we anticipate that the generality of our procedure will make it useful in other contexts.

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APPENDIX A: SPATIAL DIFFERENTIAL OPERATORS

The definitions of the spatial differential operators that we use are as follows. First, the second order spatial differential operators are defined by

$$\mathbf{D}^2 := \gamma^{ij}\mathbf{D}_i\mathbf{D}_j, \qquad \mathbf{D}_{ij} := \mathbf{D}_{(i}\mathbf{D}_{j)} - \frac{1}{3}\gamma_{ij}\mathbf{D}^2, \tag{A1}$$

where $\mathbf{D}_i$ denotes covariant differentiation with respect to the conformal background spatial metric γ_{ij}. Second, we use the shorthand notation

$$(\mathbf{D}A)^2 := (\mathbf{D}^k A)(\mathbf{D}_k A), \quad \mathcal{D}(A) := \mathcal{S}^{ij}(\mathbf{D}_i A)(\mathbf{D}_j A), \quad \text{(A2)}$$

where A is a scalar field and $\mathcal{S}^{ij}$ is defined in (A3). Finally, we define the *mode extraction operators* [see [27], Eq. (B11)]:

$$\mathcal{S}^i = \mathbf{D}^{-2}\mathbf{D}^i, \qquad \mathcal{S}^{ij} = \frac{3}{2}\mathbf{D}^{-2}(\mathbf{D}^2 + 3K)^{-1}\mathbf{D}^{ij}, \quad \text{(A3a)}$$

$$\mathcal{V}_i^{\,j} = \delta_i^j - \mathbf{D}_i\mathcal{S}^j, \qquad \mathcal{V}_i^{\,jk} = (\mathbf{D}^2 + 2K)^{-1}\mathcal{V}_i^{\langle j}\mathbf{D}^{k\rangle}, \quad \text{(A3b)}$$

$$\mathcal{T}_{ij}{}^{km} = \delta_i^{\langle k}\delta_j^{m\rangle} - \mathbf{D}_{(i}\mathcal{V}_{j)}{}^{km} - \mathbf{D}_{ij}\mathcal{S}^{km}. \quad \text{(A3c)}$$

If some expression $L(\mathbf{D}_i)$ involving $\mathbf{D}_i$ scales as $L(\lambda\mathbf{D}_i) = \lambda^p L(\mathbf{D}_i)$ under a rescaling of coordinates $x^i \to \lambda^{-1}x^i$, $\eta \to \lambda^{-1}\eta$, we say that $L(\mathbf{D}_i)$ *has weight* p *in* $\mathbf{D}_i$. It follows that the canonical differential expressions in (2) have the following weights[28]: $\zeta^2, \mathcal{D}(\zeta)$ are of weight zero, $(\mathbf{D}\zeta)^2, \mathbf{D}^2\mathcal{D}(\zeta), \mathbf{D}^2\zeta^2$ are of weight two and $\mathbf{D}^2(\mathbf{D}\zeta)^2, \mathbf{D}^4\mathcal{D}(\zeta)$ are of weight four.

We now give identities involving the spatial differential operators that we use to relate results in the literature to our results:

$$A\mathbf{D}^2A = \frac{1}{2}\mathbf{D}^2A^2 - (\mathbf{D}A)^2, \quad \text{(A4a)}$$

$$\mathbf{D}^i\mathbf{D}^j(\mathbf{D}_iA\mathbf{D}_jA) = \frac{1}{3}\mathbf{D}^2[(\mathbf{D}A)^2 + 2(\mathbf{D}^2 + 3K)\mathcal{D}(A)], \quad \text{(A4b)}$$

$$\mathbf{D}^i(\mathbf{D}_iA\mathbf{D}^2A) = -\frac{1}{6}\mathbf{D}^2[(\mathbf{D}A)^2 - 4(\mathbf{D}^2 + 3K)\mathcal{D}(A)], \quad \text{(A4c)}$$

$$(\mathbf{D}^i\mathbf{D}^jA)(\mathbf{D}_i\mathbf{D}_jA) = \frac{2}{3}[(\mathbf{D}^2 - 3K)(\mathbf{D}A)^2 - (\mathbf{D}^2 + 3K)\mathbf{D}^2\mathcal{D}(A)] + (\mathbf{D}^2A)^2, \quad \text{(A4d)}$$

$$\mathbf{D}^i\mathbf{D}^j(A\mathbf{D}_i\mathbf{D}_jA) = -\frac{1}{6}\mathbf{D}^2[2(\mathbf{D}A)^2 + 4(\mathbf{D}^2 + 3K)\mathcal{D}(A) - 3(\mathbf{D}^2 + 2K)A^2]. \quad \text{(A4e)}$$

APPENDIX B: TRANSFORMATION LAWS FOR GAUGE INVARIANTS

The first purpose of this Appendix is to define the gauge invariants that are associated with the perturbed metric and matter distribution. We do not, however, write out the expressions for the gauge invariants in terms of gauge-variant quantities since our strategy is to work solely with gauge invariants. First we require the governing equations that determine the gauge invariants in the Poisson gauge, and these are given in UW. Second we require a framework for determining how gauge invariants transform under a change of gauge at second order. For example, given ${}^{(2)}\boldsymbol{\delta}_{\rm p}$ (Poisson gauge) how can one calculate ${}^{(2)}\boldsymbol{\delta}_{\rm c}$ (uniform curvature gauge) efficiently? The framework that we present in this Appendix is based on the transformation law for the perturbations of a given tensor field up to second order under a gauge transformation, first given by Bruni *et al.* [29]. This transformation law has been used for this purpose in a number of specific cases (for example, synchronous-comoving to Poisson [4,6], Poisson to uniform curvature [30], synchronous-comoving to total matter [9] and Poisson to total matter [31].) Our goal is to give a general framework that is valid for a specific gauge invariant and two chosen gauges. In the body of the paper we consider pressure-free matter, but in this Appendix we assume that the matter content is a perfect fluid with equation of state $p = w\rho$, $w = \text{const}$, in order to increase the applicability of the results.

1. Gauge invariants associated with an arbitrary tensor field

In cosmological perturbation theory a second order gauge transformation can be represented in coordinates as follows:

$$\tilde{x}^a = x^a + \epsilon^{(1)}\xi^a + \frac{1}{2}\epsilon^2({}^{(2)}\xi^a + {}^{(1)}\xi^a{}_{,b}{}^{(1)}\xi^b), \quad \text{(B1)}$$

where ${}^{(1)}\xi^a$ and ${}^{(2)}\xi^a$ are independent dimensionless background vector fields. We consider a one-parameter family of tensor fields $A(\epsilon)$, which we assume can be expanded in powers of ϵ, i.e. as a Taylor series:

$$\mathrm{A}(\epsilon) = {}^{(0)}\mathrm{A} + \epsilon^{(1)}\mathrm{A} + \frac{1}{2}\epsilon^{2(2)}\mathrm{A} + \cdots, \quad \text{(B2)}$$

where ${}^{(0)}A$ is called the *unperturbed value*, ${}^{(1)}$A is called the *first order (linear) perturbation* and ${}^{(2)}$A is called the *second order perturbation* of $A(\epsilon)$. Such a transformation induces a change in the first and second order perturbations of $A(\epsilon)$ according to

$${}^{(1)}A[\xi] = {}^{(1)}A + \pounds_{{}^{(1)}\xi}{}^{(0)}A, \quad \text{(B3a)}$$

$${}^{(2)}A[\xi] = {}^{(2)}A + \pounds_{{}^{(2)}\xi}{}^{(0)}A + \pounds_{{}^{(1)}\xi}(2^{(1)}A + \pounds_{{}^{(1)}\xi}{}^{(0)}A), \quad \text{(B3b)}$$

where $\pounds$ is the Lie derivative [see [29], Eqs. (1.1)–(1.3)]. One fixes a gauge by requiring some components of the perturbations of some tensor fields ${}^{(r)}A[\xi]$, ${}^{(r)}B[\xi]$, etc., with $r = 1, 2$, to be zero, thereby determining unique values for

[28]Note that $\mathcal{H} \to \lambda\mathcal{H}$ and $K \to \lambda^2 K$.

${}^{(1)}\xi^a$ and ${}^{(2)}\xi^a$ which we denote by ${}^{(1)}\xi^a_\bullet$ and ${}^{(2)}\xi^a_\bullet$. Since there is no remaining gauge freedom, the nonzero components ${}^{(r)}\tilde{A}[\xi_\bullet]$, obtained by replacing ξ by $\xi_\bullet$ in (B3), are gauge invariants. We refer to Malik and Wands [20] (see pp. 18–20) for an illustration of this process using the Poisson gauge. When uniquely determined, the vector fields ${}^{(1)}\xi^a_\bullet$ and ${}^{(2)}\xi^a_\bullet$ will be referred to as *gauge fields*.[29]

In order to derive a transformation law for gauge invariants under a change of gauge we consider two gauge fields ${}^{(r)}\xi^a_\bullet$ and ${}^{(r)}\xi^a_\mathrm{o}$ and define

$$ {}^{(1)}\mathbf{Z}^a[\xi_\bullet,\xi_\mathrm{o}] := {}^{(1)}\xi^a_\bullet - {}^{(1)}\xi^a_\mathrm{o}, \tag{B4a}$$

$$ {}^{(2)}\mathbf{Z}^a[\xi_\bullet,\xi_\mathrm{o}] := {}^{(2)}\xi^a_\bullet - {}^{(2)}\xi^a_\mathrm{o} + [{}^{(1)}\xi_\bullet, {}^{(1)}\xi_\mathrm{o}]^a. \tag{B4b}$$

We now consider (B3) with $\xi = \xi_\bullet$ and $\xi = \xi_\mathrm{o}$ and form the difference of the two sets of equations. This leads to the following transformation law relating the gauge invariants ${}^{(r)}A[\xi_\bullet]$ and ${}^{(r)}A[\xi_\mathrm{o}]$:

$$ {}^{(1)}A[\xi_\bullet] = {}^{(1)}A[\xi_\mathrm{o}] + \pounds_{{}^{(1)}\mathbf{Z}}{}^{(0)}A, \tag{B5a}$$

$$ {}^{(2)}A[\xi_\bullet] = {}^{(2)}A[\xi_\mathrm{o}] + \pounds_{{}^{(2)}\mathbf{Z}}{}^{(0)}A + \pounds_{{}^{(1)}\mathbf{Z}}(2{}^{(1)}A[\xi_\mathrm{o}] + \pounds_{{}^{(1)}\mathbf{Z}}{}^{(0)}A), \tag{B5b}$$

where ${}^{(r)}\mathbf{Z} \equiv {}^{(r)}\mathbf{Z}^a[\xi_\bullet,\xi_\mathrm{o}]$. We shall refer to the functions ${}^{(r)}\mathbf{Z}^a[\xi_\bullet,\xi_\mathrm{o}]$, which are gauge invariants, as the *transition functions*. They are determined by the conditions that specify the gauge fields ${}^{(r)}\xi^a_\bullet$ and ${}^{(r)}\xi^a_\mathrm{o}$. We note the formal similarity between (B3) and (B5). In going from (B3) to (B5) one replaces gauge-variant quantities by gauge-invariant quantities: ${}^{(r)}A[\xi]$ by ${}^{(r)}A[\xi_\bullet] = {}^{(r)}A_\bullet$, ${}^{(r)}A$ by ${}^{(r)}A[\xi_\mathrm{o}] = {}^{(r)}A_\mathrm{o}$ and ${}^{(r)}\xi$ by ${}^{(r)}\mathbf{Z}$.

a. Shorthand notation for gauge invariants and transition functions

The full notation for the first and second order gauge invariants associated with a tensor $A(\epsilon)$ is ${}^{(r)}A[\xi_\bullet]$, $r = 1, 2$, where $\xi_\bullet$ is a gauge field. If there is no danger of confusion, we will use a subscript notation:

$$ {}^{(r)}A_\bullet \equiv {}^{(r)}A[\xi_\bullet]. \tag{B6a}$$

The full notation for the transition functions is ${}^{(r)}\mathbf{Z}^a[\xi_\bullet,\xi_\mathrm{o}]$, where $\xi_\bullet$ and ξ_o are the two gauge fields. In general we will use the kernel $\mathbf{Z}$ as shorthand for $\mathbf{Z}[\xi_\bullet,\xi_\mathrm{o}]$. If specific gauge fields are used, for example, $\xi_\bullet = \xi_\mathrm{c}$ and $\xi_\mathrm{o} = \xi_\mathrm{p}$, we will use subscripts:

$$ \mathbf{Z} \equiv \mathbf{Z}[\xi_\bullet,\xi_\mathrm{o}], \qquad \mathbf{Z}_{\mathrm{c,p}} \equiv \mathbf{Z}[\xi_\mathrm{c},\xi_\mathrm{p}]. \tag{B6b}$$

The source terms in the transformation laws, which have the general form $\mathcal{F}[{}^{(1)}\mathbf{Z}]$ or $\mathbb{S}[{}^{(1)}\mathbf{Z}]$, are quadratic in first order gauge invariants. Specific source terms of the form $\mathcal{F}[{}^{(1)}\mathbf{Z}_{\bullet,\mathrm{p}}]$ or $\mathbb{S}[{}^{(1)}\mathbf{Z}_{\bullet,\mathrm{p}}]$ are quadratic in the first order gauge invariants ${}^{(1)}\Psi_\mathrm{p}$, ${}^{(1)}\boldsymbol{\delta}_\mathrm{p}$ and ${}^{(1)}\mathbf{v}_\mathrm{p}$. We will omit the superscript (1) and the subscript p when there is no danger of confusion.

2. The metric gauge invariants

The gauge invariants ${}^{(r)}g_{ab}[\xi]$ associated with the metric g_{ab} are given by (B3) with the arbitrary tensor A chosen to be g_{ab}. Since $a^{-2}g_{ab}$ is dimensionless we can define dimensionless gauge invariants by

$$ {}^{(r)}\mathbf{f}_{ab}[\xi] := a^{-2}{}^{(r)}g_{ab}[\xi]. \tag{B7}$$

We choose the tensor A in Eq. (B5) to be g_{ab} and use (B7) to obtain the following transformation law for ${}^{(r)}\mathbf{f}_{ab}[\xi]$:

$$ {}^{(1)}\mathbf{f}_{ab}[\xi_\bullet] = {}^{(1)}\mathbf{f}_{ab}[\xi_\mathrm{o}] + a^{-2}\pounds_{{}^{(1)}\mathbf{Z}}(a^2\gamma_{ab}), \tag{B8a}$$

$$ {}^{(2)}\mathbf{f}_{ab}[\xi_\bullet] = {}^{(2)}\mathbf{f}_{ab}[\xi_\mathrm{o}] + a^{-2}\pounds_{{}^{(2)}\mathbf{Z}}(a^2\gamma_{ab}) + \mathcal{F}_{ab}[{}^{(1)}\mathbf{Z}], \tag{B8b}$$

where

$$ \mathcal{F}_{ab}[{}^{(1)}\mathbf{Z}] := a^{-2}\pounds_{{}^{(1)}\mathbf{Z}}(2a^{2}{}^{(1)}\mathbf{f}_{ab}[\xi_\mathrm{o}] + \pounds_{{}^{(1)}\mathbf{Z}}(a^2\gamma_{ab})). \tag{B8c}$$

Here γ_{ab} is the conformally related background metric, given by ${}^{(0)}g_{ab} = a^2\gamma_{ab}$.

We now perform a mode decomposition of ${}^{(r)}\mathbf{f}_{ab}[\xi]$ as follows[30]:

$$ {}^{(r)}\mathbf{f}_{00}[\xi] = -2{}^{(r)}\Phi[\xi], \tag{B9a}$$

$$ {}^{(r)}\mathbf{f}_{0i}[\xi] = \mathbf{D}_i{}^{(r)}\mathbf{B}[\xi] + {}^{(r)}\mathbf{B}_i[\xi], \tag{B9b}$$

$$ {}^{(r)}\mathbf{f}_{ij}[\xi] = -2{}^{(r)}\Psi[\xi]\gamma_{ij} + 2\mathbf{D}_i\mathbf{D}_j{}^{(r)}\mathbf{C}[\xi] + 2\mathbf{D}_{(i}{}^{(r)}\mathbf{C}_{j)}[\xi] + 2{}^{(r)}\mathbf{C}_{ij}[\xi]. \tag{B9c}$$

We can apply the mode extraction operators defined in Eqs. (A3) in Appendix A to (B8) to obtain the transformation laws for the individual gauge invariants on the right side of (B9), obtaining[31]

[29]In previous papers [19,27,32] and UW, influenced by the approach of Nakamura [33,34] to cosmological perturbations, we used the kernel $-X$ to denote a gauge field. Here we use the kernel ξ but with a subscript, to indicate that the arbitrary vector field ξ^a has been uniquely determined, thereby fixing a gauge.

[30]We use notation that is compatible with the notation in [27]. See Eqs. (24) and (B14).

[31]Here η is conformal time, and $\partial_\eta = \mathcal{H}x\partial_x$.

$$ {}^{(r)}\Phi[\xi_\bullet] = {}^{(r)}\Phi[\xi_\mathrm{o}] + (\partial_\eta + \mathcal{H}){}^{(r)}\mathbf{Z}^0 - \frac{1}{2}\mathcal{F}_{00}[\mathbf{Z}], \tag{B10a} $$

$$ {}^{(r)}\mathbf{B}[\xi_\bullet] = {}^{(r)}\mathbf{B}[\xi_\mathrm{o}] - {}^{(r)}\mathbf{Z}^0 + \partial_\eta{}^{(r)}\mathbf{Z} + \mathcal{S}^i\mathcal{F}_{0i}[\mathbf{Z}], \tag{B10b} $$

$$ {}^{(r)}\Psi[\xi_\bullet] = {}^{(r)}\Psi[\xi_\mathrm{o}] - \mathcal{H}{}^{(r)}\mathbf{Z}^0 - \frac{1}{6}(\mathcal{F}^k{}_k - \mathbf{D}^2\mathcal{S}^{ij}\hat{\mathcal{F}}_{ij})[\mathbf{Z}], \tag{B10c} $$

$$ {}^{(r)}\mathbf{B}_i[\xi_\bullet] = {}^{(r)}\mathbf{B}_i[\xi_\mathrm{o}] + \partial_\eta{}^{(r)}\tilde{\mathbf{Z}}_i + \mathcal{V}_i^j\mathcal{F}_{0j}[\mathbf{Z}], \tag{B10d} $$

$$ {}^{(r)}\mathbf{C}[\xi_\bullet] = {}^{(r)}\mathbf{C}[\xi_\mathrm{o}] + {}^{(r)}\mathbf{Z} + \frac{1}{2}\mathcal{S}^{ij}\hat{\mathcal{F}}_{ij}[\mathbf{Z}], \tag{B10e} $$

$$ {}^{(r)}\mathbf{C}_i[\xi_\bullet] = {}^{(r)}\mathbf{C}_i[\xi_\mathrm{o}] + {}^{(r)}\tilde{\mathbf{Z}}_i + \mathcal{V}_i^{jk}\hat{\mathcal{F}}_{jk}[\mathbf{Z}], \tag{B10f} $$

$$ {}^{(r)}\mathbf{C}_{ij}[\xi_\bullet] = {}^{(r)}\mathbf{C}_{ij}[\xi_\mathrm{o}] + \frac{1}{2}\mathcal{T}_{ij}{}^{km}\hat{\mathcal{F}}_{km}[\mathbf{Z}], \tag{B10g} $$

where $r = 1, 2$. The source terms $\mathcal{F}_{ab}[\mathbf{Z}]$, which are given by (B8c), do not appear when $r = 1$. We will give explicit expressions for them later. Here we have decomposed the transition functions ${}^{(r)}\mathbf{Z}^a \equiv {}^{(r)}\mathbf{Z}^a[\xi_\bullet, \xi_\mathrm{o}]$ according to

$$ {}^{(r)}\mathbf{Z}^a = ({}^{(r)}\mathbf{Z}^0, {}^{(r)}\mathbf{Z}^i), \qquad {}^{(r)}\mathbf{Z}^i = \mathbf{D}^{i(r)}\mathbf{Z} + {}^{(r)}\tilde{\mathbf{Z}}^i, \qquad \mathbf{D}_i{}^{(r)}\tilde{\mathbf{Z}}^i = 0. \tag{B11} $$

3. Density gauge invariants

We choose $A = \rho$, the matter density scalar in Eq. (B5). On evaluating the Lie derivatives we obtain

$$ {}^{(1)}\rho[\xi_\bullet] = {}^{(1)}\rho[\xi_\mathrm{o}] + {}^{(1)}\mathbf{Z}^{0(0)}\rho', \tag{B12a} $$

$$ {}^{(2)}\rho[\xi_\bullet] = {}^{(2)}\rho[\xi_\mathrm{o}] + {}^{(2)}\mathbf{Z}^{0(0)}\rho' + ({}^{(1)}\mathbf{Z}^0\partial_\eta + {}^{(1)}\mathbf{Z}^i\mathbf{D}_i)(2{}^{(1)}\rho[\xi_\mathrm{o}] + {}^{(1)}\mathbf{Z}^{0(0)}\rho'), \tag{B12b} $$

where $'$ denotes differentiation with respect to conformal time η. Here and in the rest of this section the kernel $\mathbf{Z}$ is shorthand for $\mathbf{Z}[\xi_\bullet, \xi_\mathrm{o}]$. We introduce dimensionless gauge invariants by normalizing with the inertial mass density:

$$ {}^{(r)}\boldsymbol{\delta}[\xi] := \frac{{}^{(r)}\rho[\xi]}{{}^{(0)}\rho + {}^{(0)}p}, \tag{B13} $$

which in the case of dust is just the usual fractional density perturbation. Then (B12) leads to the following transformation law for the density gauge invariants:

$$ {}^{(1)}\boldsymbol{\delta}_\bullet = {}^{(1)}\boldsymbol{\delta}_\mathrm{o} - 3\mathcal{H}{}^{(1)}\mathbf{Z}^0, \tag{B14a} $$

$$ {}^{(2)}\boldsymbol{\delta}_\bullet = {}^{(2)}\boldsymbol{\delta}_\mathrm{o} - 3\mathcal{H}{}^{(2)}\mathbf{Z}^0 + \mathcal{F}_\delta[{}^{(1)}\mathbf{Z}], \tag{B14b} $$

where

$$ \mathcal{F}_\delta[{}^{(1)}\mathbf{Z}] := (\mathbf{Z}^0(\partial_\eta - 3(1+w)\mathcal{H}) + \mathbf{Z}^i\mathbf{D}_i)(2\boldsymbol{\delta}[\xi_\mathrm{o}] - 3\mathcal{H}\mathbf{Z}^0), \tag{B14c} $$

and we are using the shorthand notation (B6). Here we have dropped the superscript (1) on the first order quantities on the right-hand side of this equation. In deriving Eqs. (B14) we used the following background equations for a perfect fluid:

$$ {}^{(0)}\rho' = -3\mathcal{H}({}^{(0)}\rho + {}^{(0)}p), \qquad {}^{(0)}p' = w{}^{(0)}\rho', \tag{B15} $$

where w is the constant equation of state parameter.

4. Velocity gauge invariants

The gauge invariants ${}^{(r)}u_a[\xi]$ associated with the covariant unit vector field u_a are given by (B3) with the arbitrary tensor A chosen to be u_a. Since $a^{-1}u_a$ is dimensionless we can define dimensionless gauge invariants by

$$ {}^{(r)}\mathbf{v}_a[\xi] := a^{-1(r)}u_a[\xi]. \tag{B16} $$

Equation (B5), with the tensor A chosen to be u_a, in conjunction with (B16), then leads to the following transformation law for ${}^{(r)}\mathbf{v}_a[\xi_\mathrm{o}]$:

$$ {}^{(1)}\mathbf{v}_a[\xi_\bullet] := {}^{(1)}\mathbf{v}_a[\xi_\mathrm{o}] + a^{-1}\pounds_{{}^{(1)}\mathbf{Z}}(a^{(0)}v_a), \tag{B17a} $$

$$ {}^{(2)}\mathbf{v}_a[\xi_\bullet] := {}^{(2)}\mathbf{v}_a[\xi_\mathrm{o}] + a^{-1}\pounds_{{}^{(2)}\mathbf{Z}}(a^{(0)}v_a) + (\mathcal{F}_v)_a[{}^{(1)}\mathbf{Z}], \tag{B17b} $$

where

$$ (\mathcal{F}_v)_a[{}^{(1)}\mathbf{Z}] := a^{-1}\pounds_{{}^{(1)}\mathbf{Z}}(2a{}^{(1)}\mathbf{v}_a[\xi_\mathrm{o}] + \pounds_{{}^{(1)}\mathbf{Z}}(a^{(0)}v_a)), \tag{B17c} $$

and $a^{(0)}v_a \equiv {}^{(0)}u_a$. Evaluating the Lie derivatives and restricting to the spatial components yields the following:

$$ {}^{(1)}\mathbf{v}_i[\xi_\bullet] = {}^{(1)}\mathbf{v}_i[\xi_\mathrm{o}] - \mathbf{D}_i{}^{(1)}\mathbf{Z}^0, \tag{B18a} $$

$$ {}^{(2)}\mathbf{v}_i[\xi_\bullet] = {}^{(2)}\mathbf{v}_i[\xi_\mathrm{o}] - \mathbf{D}_i{}^{(2)}\mathbf{Z}^0 + (\mathcal{F}_\mathrm{v})_i[{}^{(1)}\mathbf{Z}], \tag{B18b} $$

where

$$ (\mathcal{F}_\mathrm{v})_i[{}^{(1)}\mathbf{Z}] := 2\mathbf{Z}^0(\partial_\eta + \mathcal{H})\mathbf{v}_i[\xi_\mathrm{o}] - 2\Phi[\xi_\mathrm{o}]\mathbf{D}_i\mathbf{Z}^0 - \frac{1}{2}\mathbf{D}_i(\partial_\eta + 2\mathcal{H})(\mathbf{Z}^0)^2 - \mathbf{D}_i(\mathbf{Z}^j\mathbf{D}_j\mathbf{Z}^0) + 2(\mathbf{Z}^j\mathbf{D}_j\mathbf{v}_i[\xi_\mathrm{o}] + (\mathbf{D}_i\mathbf{Z}^j)\mathbf{v}_j[\xi_\mathrm{o}]). \tag{B18c} $$

We now mode decompose ${}^{(r)}\mathbf{v}_i[\xi]$ into a scalar and vector part according to $\mathbf{v}_i = \mathbf{D}_i\mathbf{v} + \tilde{\mathbf{v}}_i$, $\mathbf{D}^i\tilde{\mathbf{v}}_i = 0$. On restricting to the purely scalar case at linear order (i.e. $\mathbf{v}_i = \mathbf{D}_i\mathbf{v}$, $\mathbf{Z}^i = \mathbf{D}^i\mathbf{Z}$), (B18) reduces to[32]

$$ {}^{(1)}\mathbf{v}_{\bullet} = {}^{(1)}\mathbf{v}_{\mathrm{o}} - {}^{(1)}\mathbf{Z}^0, \tag{B19a}$$

$$ {}^{(2)}\mathbf{v}_{\bullet} = {}^{(2)}\mathbf{v}_{\mathrm{o}} - {}^{(2)}\mathbf{Z}^0 + \mathcal{F}_{\mathrm{v}}[{}^{(1)}\mathbf{Z}], \tag{B19b}$$

where

$$\mathcal{F}_{\mathrm{v}}[{}^{(1)}\mathbf{Z}] = 2\mathcal{S}^i(\mathbf{Z}^0(\partial_\eta + \mathcal{H})\mathbf{D}_i\mathbf{v}[\xi_{\mathrm{o}}] - \Phi[\xi_{\mathrm{o}}]\mathbf{D}_i\mathbf{Z}^0) - \frac{1}{2}(\partial_\eta + 2\mathcal{H})(\mathbf{Z}^0)^2 - (\mathbf{D}^j\mathbf{Z})\mathbf{D}_j(\mathbf{Z}^0 - 2\mathbf{v}[\xi_{\mathrm{o}}]), \tag{B19c}$$

and we are using the shorthand notation (B6).

5. Transformation laws between the Poisson, the uniform curvature and the total matter gauges

The Poisson, uniform curvature and total matter gauges all satisfy the following conditions on the metric gauge invariants:

$$ {}^{(r)}\mathbf{C}[\xi] = 0, \qquad {}^{(r)}\mathbf{C}_i[\xi] = 0, \tag{B20}$$

for $r = 1, 2$, where ξ is any of the gauge fields ξ_{p}, ξ_{c} and ξ_{v}. It follows from (B10e) and (B10f) with $r = 1$ that the spatial part of the first order transition function ${}^{(1)}\mathbf{Z}^a$ relating these three gauges will be zero:

$$ {}^{(1)}\mathbf{Z}[\xi_{\bullet}, \xi] = 0, \qquad {}^{(1)}\tilde{\mathbf{Z}}_i[\xi_{\bullet}, \xi] = 0, \tag{B21}$$

where $\xi_{\bullet}$ and ξ can be chosen to be any two of the gauge fields ξ_{p}, ξ_{c} and ξ_{v}. On the other hand, these three gauges are distinguished by the specification of the temporal gauge, as follows:

$$ {}^{(r)}\mathbf{B}[\xi_{\mathrm{p}}] = 0, \qquad {}^{(r)}\Psi[\xi_{\mathrm{c}}] = 0, \qquad {}^{(r)}\mathbf{v}[\xi_{\mathrm{v}}] = 0, \tag{B22}$$

for $r = 1, 2$, respectively.

We now give the components of the source terms $\mathcal{F}_{ab}[\mathbf{Z}]$ in the transformation laws (B10), assuming that the linear metric perturbation is purely scalar and that the conditions (B20) and hence (B21) are satisfied. We calculate the Lie derivatives in (B8c), making use of (B9), (B20) and (B21), which leads to

$$\mathcal{F}_{00}[{}^{(1)}\mathbf{Z}] = -2[\mathbf{Z}^0(\partial_\eta + 2\mathcal{H}) + 2(\partial_\eta\mathbf{Z}^0)][(\partial_\eta + \mathcal{H})\mathbf{Z}^0 + 2\Phi[\xi]], \tag{B23a}$$

$$\mathcal{F}_{0i}[{}^{(1)}\mathbf{Z}] = -[\mathbf{Z}^0(\partial_\eta + 2\mathcal{H}) + (\partial_\eta\mathbf{Z}^0)]\mathbf{D}_i(\mathbf{Z}^0 - 2\mathbf{B}[\xi]) - 2(\mathbf{D}_i\mathbf{Z}^0)((\partial_\eta + \mathcal{H})\mathbf{Z}^0 + 2\Phi[\xi]), \tag{B23b}$$

$$\mathcal{F}^k{}_k[{}^{(1)}\mathbf{Z}] = 6\mathbf{Z}^0(\partial_\eta + 2\mathcal{H})(\mathcal{H}\mathbf{Z}^0 - 2\Psi[\xi]) - 2(\mathbf{D}^k\mathbf{Z}^0)\mathbf{D}_k(\mathbf{Z}^0 - 2\mathbf{B}[\xi]), \tag{B23c}$$

$$\hat{\mathcal{F}}_{ij}[{}^{(1)}\mathbf{Z}] = -2\mathbf{D}_{\langle i}(\mathbf{Z}^0 - 2\mathbf{B}[\xi])(\mathbf{D}_{j\rangle}\mathbf{Z}^0), \tag{B23d}$$

where $\mathbf{Z}^0 \equiv \mathbf{Z}^0[\xi_{\bullet}, \xi]$, and $\xi_{\bullet}$ and ξ can be chosen to be any two of the gauge fields ξ_{p}, ξ_{c} and ξ_{v}.

When evaluating the source terms (B14c), (B19c) and (B23) in the following sections it is convenient to eliminate the temporal derivatives of the first order gauge invariants $\boldsymbol{\delta}$, $\mathbf{v}$ and Ψ in the Poisson gauge. To do this we use the linearized conservation equations for a perfect fluid in the following form[33]:

$$ x\partial_x(\boldsymbol{\delta}_{\mathrm{p}} - 3\Psi_{\mathrm{p}}) + \mathcal{H}^{-2}\mathbf{D}^2(\mathcal{H}\mathbf{v}_{\mathrm{p}}) = 0, \tag{B24a}$$

$$ \mathcal{H}(x\partial_x + 1)\mathbf{v}_{\mathrm{p}} + \Psi_{\mathrm{p}} + w(\boldsymbol{\delta}_{\mathrm{p}} - 3\mathcal{H}\mathbf{v}_{\mathrm{p}}) = 0, \tag{B24b}$$

and the velocity equation in the form

$$ (x\partial_x + 1)\Psi_{\mathrm{p}} = -\frac{\mathcal{A}}{2\mathcal{H}^2}(\mathcal{H}\mathbf{v}_{\mathrm{p}}) \tag{B24c}$$

[see Eqs. (15b) and (16b) in UW]. Here the scalar $\mathcal{A}$ is given by

$$\mathcal{A} = 2(-\partial_\eta\mathcal{H} + \mathcal{H}^2 + K) = 3(1 + w)\mathcal{H}^2\Omega_m, \tag{B25}$$

the second equality holding for a perfect fluid with linear equation of state.[34] In addition (B24b) and (B25) lead to

$$ x\partial_x(\mathcal{H}\mathbf{v}_{\mathrm{p}}) = -\frac{\mathcal{A} - 2K}{2\mathcal{H}^2}(\mathcal{H}\mathbf{v}_{\mathrm{p}}) - \Psi_{\mathrm{p}} - w(\boldsymbol{\delta}_{\mathrm{p}} - 3\mathcal{H}\mathbf{v}_{\mathrm{p}}), \tag{B26}$$

on noting that $\partial_\eta = \mathcal{H}x\partial_x$.

[32] Apply the mode extraction operator $\mathcal{S}^i$ to the second of Eqs. (B18) to get the second of Eqs. (B19). We introduce the shorthand notation $\mathcal{F}_{\mathrm{v}} \equiv \mathcal{S}^i(\mathcal{F}_{\mathrm{v}})_i$.

[33] Choose $\xi = \xi_{\mathrm{p}}$ in Eq. (43) in [19], and specialize to a perfect fluid by setting $\bar{\Gamma} = 0$, $\bar{\Xi} = 0$ and noting that $\Phi_{\mathrm{p}} = \Psi_{\mathrm{p}}$. Also note that $\mathbb{V}_{\mathrm{p}} = \mathbf{v}_{\mathrm{p}}$ and $\mathbb{D} \equiv \boldsymbol{\delta}_{\mathrm{v}} = \boldsymbol{\delta}_{\mathrm{p}} - 3\mathcal{H}\mathbf{v}_{\mathrm{p}}$.

[34] See Eqs. (16) and (36) in [19], noting that the background Einstein equations imply $\mathcal{A}_G = \mathcal{A}_T$, or Eq. (6) in UW, but note the typo: the signs on $\mathcal{H}'$ and $\mathcal{H}^2$ are reversed.

a. Transforming from the Poisson to the uniform curvature gauge

The transition quantities ${}^{(r)}\mathbf{Z}^0_{\mathrm{c,p}} \equiv {}^{(r)}\mathbf{Z}^0[\xi_\mathrm{c}, \xi_\mathrm{p}]$ are obtained by choosing $\xi_\bullet = \xi_\mathrm{c}$ and $\xi_\mathrm{o} = \xi_\mathrm{p}$ in (B10c) and using the second of Eqs. (B22). This leads to

$$\mathcal{H}{}^{(1)}\mathbf{Z}^0_{\mathrm{c,p}} = {}^{(1)}\Psi_\mathrm{p}, \tag{B27a}$$

$$\mathcal{H}{}^{(2)}\mathbf{Z}^0_{\mathrm{c,p}} = {}^{(2)}\Psi_\mathrm{p} - \frac{1}{6}(\mathcal{F}^k{}_k - \mathbf{D}^2\mathcal{S}^{ij}\hat{\mathcal{F}}_{ij})[{}^{(1)}\mathbf{Z}_{\mathrm{c,p}}]. \tag{B27b}$$

Next, we substitute (B27) in (B14) with • and o replaced by c and p, respectively, and obtain

$${}^{(1)}\boldsymbol{\delta}_\mathrm{c} = {}^{(1)}\boldsymbol{\delta}_\mathrm{p} - 3{}^{(1)}\Psi_\mathrm{p}, \tag{B28a}$$

$${}^{(2)}\boldsymbol{\delta}_\mathrm{c} = {}^{(2)}\boldsymbol{\delta}_\mathrm{p} - 3{}^{(2)}\Psi_\mathrm{p} + \mathbb{S}_\delta[{}^{(1)}\mathbf{Z}_{\mathrm{c,p}}], \tag{B28b}$$

where

$$\mathbb{S}_\delta[{}^{(1)}\mathbf{Z}_{\mathrm{c,p}}] := \frac{1}{2}(2\mathcal{F}_\delta + \mathcal{F}^k k - \mathbf{D}^2\mathcal{S}^{ij}\hat{\mathcal{F}}_{ij})[{}^{(1)}\mathbf{Z}_{\mathrm{c,p}}]. \tag{B28c}$$

We now use (B14c), (B23), the first of Eqs. (B22), and (B27a) to show that

$$\mathbb{S}_\delta[{}^{(1)}\mathbf{Z}_{\mathrm{c,p}}] = 3\Psi[(1+3w)\Psi - 2(1+w)\boldsymbol{\delta}] + 2\Psi x\partial_x(\boldsymbol{\delta} - 3\Psi) - \mathcal{H}^{-2}[(\mathbf{D}\Psi)^2 - \mathbf{D}^2\mathcal{D}(\Psi)], \tag{B29}$$

where we for brevity drop the subscript p on the first order quantities in the source terms. Finally we use (B24a) to eliminate the temporal derivative, obtaining

$$\mathbb{S}_\delta[{}^{(1)}\mathbf{Z}_{\mathrm{c,p}}] = 3\Psi[(1+3w)\Psi - 2(1+w)\boldsymbol{\delta}] - \mathcal{H}^{-2}[2\Psi\mathbf{D}^2(\mathcal{H}\mathbf{v}) + (\mathbf{D}\Psi)^2 - \mathbf{D}^2\mathcal{D}(\Psi)]. \tag{B30}$$

In summary, Eq. (B28b), with the source term given by (B30), is the transformation law that relates ${}^{(2)}\boldsymbol{\delta}_\mathrm{c}$ to ${}^{(2)}\boldsymbol{\delta}_\mathrm{p}$.

b. Transforming from the Poisson to the total matter gauge

The transition quantities ${}^{(r)}\mathbf{Z}^0_{\mathrm{v,p}} \equiv {}^{(r)}\mathbf{Z}^0[\xi_\mathrm{v}, \xi_\mathrm{p}]$ are obtained by replacing • and o with v and p, respectively, in (B19) and using the third of Eqs. (B22). This leads to

$${}^{(1)}\mathbf{Z}^0_{\mathrm{v,p}} = {}^{(1)}\mathbf{v}_\mathrm{p}, \tag{B31a}$$

$${}^{(2)}\mathbf{Z}^0_{\mathrm{v,p}} = {}^{(2)}\mathbf{v}_\mathrm{p} + \mathcal{F}_\mathrm{v}[{}^{(1)}\mathbf{Z}_{\mathrm{c,p}}]. \tag{B31b}$$

It follows from (B14) that ${}^{(r)}\boldsymbol{\delta}_\mathrm{v}$ is related to ${}^{(r)}\boldsymbol{\delta}_\mathrm{p}$ according to

$${}^{(1)}\boldsymbol{\delta}_\mathrm{v} = {}^{(1)}\boldsymbol{\delta}_\mathrm{p} - 3\mathcal{H}{}^{(1)}\mathbf{v}_\mathrm{p}, \tag{B32a}$$

$${}^{(2)}\boldsymbol{\delta}_\mathrm{v} = {}^{(2)}\boldsymbol{\delta}_\mathrm{p} - 3\mathcal{H}{}^{(2)}\mathbf{v}_\mathrm{p} + \mathbb{S}_\delta[{}^{(1)}\mathbf{Z}_{\mathrm{v,p}}], \tag{B32b}$$

where

$$\mathbb{S}_\delta[{}^{(1)}\mathbf{Z}_{\mathrm{v,p}}] := (\mathcal{F}_\delta - 3\mathcal{H}\mathcal{F}_\mathrm{v})[{}^{(1)}\mathbf{Z}_{\mathrm{v,p}}]. \tag{B32c}$$

Then we use (B14c), (B19c), the first of Eqs. (B22), and (B31a) to calculate an explicit expression for the source term:

$$\mathcal{F}_\delta[{}^{(1)}\mathbf{Z}_{\mathrm{v,p}}] = \mathcal{H}\mathbf{v}(x\partial_x - 3(1+w))(2\boldsymbol{\delta} - 3\mathcal{H}\mathbf{v}), \tag{B33a}$$

$$\mathcal{F}_\mathrm{v}[{}^{(1)}\mathbf{Z}_{\mathrm{v,p}}] = 2\mathcal{S}^i[\mathcal{H}\mathbf{v}\mathbf{D}_i(x\partial_x\mathbf{v}) - \Phi\mathbf{D}_i\mathbf{v}] - \mathcal{H}\mathbf{v}x\partial_x\mathbf{v}. \tag{B33b}$$

Eliminating the time derivatives using Eqs. (B24) and (B26) and making use of (3) and (B25) we obtain

$$\mathbb{S}_\delta[{}^{(1)}\mathbf{Z}_{\mathrm{v,p}}] = \mathcal{H}\mathbf{v}\left[-6\boldsymbol{\delta} + 3\left(3 - \frac{3}{2}(1+w)\Omega_m + \Omega_k\right)\mathcal{H}\mathbf{v} - 2\mathcal{H}^{-2}\mathbf{D}^2(\mathcal{H}\mathbf{v})\right] - 6w\mathcal{S}^i(\boldsymbol{\delta}\mathbf{D}_i(\mathcal{H}\mathbf{v})). \tag{B34}$$

In summary, Eq. (B32b), with the source term given by (B24), is the transformation law that relates ${}^{(2)}\boldsymbol{\delta}_\mathrm{v}$ to ${}^{(2)}\boldsymbol{\delta}_\mathrm{p}$.

6. Transforming from the Poisson to the uniform density gauge

The uniform density gauge is defined by

$${}^{(r)}\boldsymbol{\delta}[\xi_\rho] = 0, \tag{B35}$$

with the spatial part of the gauge field fixed as for the Poisson gauge in (B20). We specialize (B10c) by choosing $\xi_\bullet = \xi_\rho$ and $\xi = \xi_\mathrm{p}$, which relates ${}^{(r)}\Psi_\rho$ to ${}^{(r)}\Psi_\mathrm{p}$. Substituting Eq. (B35) in (B14) with • and o replaced by ρ and p, respectively, yields expressions for the required transition quantities ${}^{(r)}\mathbf{Z}^0_{\rho,\mathrm{p}} \equiv {}^{(r)}\mathbf{Z}^0[\xi_\rho, \xi_\mathrm{p}]$, which when substituted in (B10c) lead to

$${}^{(1)}\Psi_\rho = {}^{(1)}\Psi_\mathrm{p} - \frac{1}{3}{}^{(1)}\boldsymbol{\delta}_\mathrm{p}, \tag{B36a}$$

$${}^{(2)}\Psi_\rho = {}^{(2)}\Psi_\mathrm{p} - \frac{1}{3}{}^{(2)}\boldsymbol{\delta}_\mathrm{p} + \mathbb{S}_\Psi[\mathbf{Z}_{\rho,\mathrm{p}}], \tag{B36b}$$

where

$$\mathbb{S}_\Psi[\mathbf{Z}_{\rho,\mathrm{p}}] := -\frac{1}{6}(2\mathcal{F}_\delta + \mathcal{F}^k{}_k - \mathbf{D}^2\mathcal{S}^{ij}\hat{\mathcal{F}}_{ij})[\mathbf{Z}_{\rho,\mathrm{p}}]. \tag{B36c}$$

Finally it follows from (B14c), (B23) and (B24a) that

$$\mathbb{S}_\Psi[\mathbf{Z}_{\rho,\mathrm{p}}] = \frac{1}{9}\boldsymbol{\delta}((1+3w)\boldsymbol{\delta}+12\Psi) + \frac{1}{27}\mathcal{H}^{-2}((\mathbf{D}\boldsymbol{\delta})^2 - \mathbf{D}^2\mathcal{D}(\boldsymbol{\delta}) + 6\boldsymbol{\delta}\mathbf{D}^2\mathcal{H}\mathbf{v}). \tag{B36d}$$

7. Transforming from the total matter to the synchronous-comoving gauge

In cosmological perturbation theory when considering perturbations of FL universes containing pressure-free matter that is *irrotational* one can specialize to the synchronous-comoving gauge.[35]

In our notation this gauge is defined by the following conditions:

$$ {}^{(r)}\mathbf{B}[\xi_\mathrm{s}] = 0, \quad {}^{(r)}\Phi[\xi_\mathrm{s}] = 0, \quad {}^{(r)}\mathbf{v}[\xi_\mathrm{s}] = 0, \quad r = 1, 2, \tag{B37}$$

where the subscript s stands for synchronous-comoving. We have to determine the transition quantities ${}^{(r)}\mathbf{Z}^0_{\mathrm{s,v}} \equiv {}^{(r)}\mathbf{Z}^0[\xi_\mathrm{s}, \xi_\mathrm{v}]$ and ${}^{(1)}\mathbf{Z}_{\mathrm{s,v}} \equiv {}^{(1)}\mathbf{Z}[\xi_\mathrm{s}, \xi_\mathrm{v}]$, where the subscript v refers to the total matter gauge. First, since ${}^{(r)}\mathbf{v}[\xi_\mathrm{v}] = 0$, it follows from (B37) and (B19) with $\xi_\bullet = \xi_\mathrm{s}$ and $\xi = \xi_\mathrm{v}$ that the temporal parts are zero:

$$ {}^{(1)}\mathbf{Z}^0_{\mathrm{s,v}} = 0, \qquad {}^{(2)}\mathbf{Z}^0_{\mathrm{s,v}} = 0, \tag{B38}$$

unlike in the previous cases. The spatial part is determined as follows. Since ${}^{(1)}\mathbf{B}[\xi_\mathrm{s}] = 0$ Eq. (B10b) with $r = 1$ leads to

$$ x\mathcal{H}\partial_x({}^{(1)}\mathbf{Z}_{\mathrm{s,v}}) = -{}^{(1)}\mathbf{B}_\mathrm{v} = {}^{(1)}\mathbf{v}_\mathrm{p}. \tag{B39}$$

Noting that in the case of dust we have $\mathcal{A}x = 3m^2$, which follows from (5) and (B25), we can write (B24c) in the form

$$ {}^{(1)}\mathbf{v}_\mathrm{p} = -x\mathcal{H}\left(\frac{2}{3}m^{-2}\partial_x(x\,{}^{(1)}\Psi_\mathrm{p})\right). \tag{B40}$$

A comparison with (B39) leads to an exact temporal differential, which when integrated yields a spatial function. Setting this function to zero fixes the residual gauge freedom in the synchronous-comoving gauge and leads to a one-to-one gauge-invariant relationship with the total matter gauge determined by

$$ {}^{(1)}\mathbf{Z}_{\mathrm{s,v}} = -\frac{2}{3}m^{-2}x\,{}^{(1)}\Psi_\mathrm{p}. \tag{B41}$$

Using (B38) and (B41) it follows from (B14) that

$$ {}^{(1)}\boldsymbol{\delta}_\mathrm{s} = {}^{(1)}\boldsymbol{\delta}_\mathrm{v}, \qquad {}^{(2)}\boldsymbol{\delta}_\mathrm{s} = {}^{(2)}\boldsymbol{\delta}_\mathrm{v} - \frac{4}{3}xm^{-2}(\mathbf{D}^i\boldsymbol{\delta}_\mathrm{v})(\mathbf{D}_i\Psi_\mathrm{p}). \tag{B42}$$

[35]See for example [12,13,17]. The last reference discusses the relation between the synchronous-comoving gauge and the traditional synchronous gauge.

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