Maximum strength of the magnetic field in the core of the most massive white dwarfs

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Massive white dwarfs endowed with ultrastrong magnetic fields have been recently proposed as the progenitors of overluminous type Ia supernovae. In our previous work, we have shown that such stars would become unstable against electron captures if the magnetic field is too strong. Using a more realistic model of dense matter taking into account electron-ion interactions and allowing for ionic mixtures, we estimate the strength of the magnetic field for the onset of electron captures in the core of the most massive white dwarfs. We have considered various compositions and lattice structures. The possibility of pycnonuclear fusion reactions and their impact on the stellar magnetic fields are also discussed. The strongest magnetic fields we find are considerably lower than those previously assumed in putative super-Chandrasekhar white dwarfs.

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I. INTRODUCTION

White dwarfs are the stellar remnants of low and intermediate mass stars [1] (i.e., with a mass $\lesssim 10 M_\odot$, $M_\odot$ being the mass of our Sun). Predicted in 1947 by Blackett [2], the existence of white dwarfs with strong magnetic fields $B \gtrsim 10^6$ G was confirmed in 1970 by Kemp [3] (for a brief historical review, see, e.g., Ref. [4]). Since then, several hundred magnetic white dwarfs already have been found [5]. Surface magnetic field strengths up to about 10^9 G have been inferred from Zeeman spectroscopy and polarimetry, as well as cyclotron spectroscopy (see, e.g., Ref. [6] for a review). About 10% of isolated white dwarfs have surface magnetic fields stronger than 10^6 G [7,8], and about 25% of cataclysmic variables are magnetic [6]. Among the latter, polars or AM Herculis systems have magnetic field strengths in the range 10^7–10^8 G while intermediate polars or DQ Herculis systems have weaker fields $< 10^7$ G. The population of strongly magnetized white dwarfs might also include some soft gamma-ray repeaters and anomalous x-ray pulsars [9–13], though it is widely believed that these objects are not white dwarfs but magnetars [14,15] (see e.g. Ref. [16] for a review).

The origin of strong magnetic fields in white dwarfs is still a matter of debate. According to the fossil field hypothesis, the magnetic field was generated at an earlier stage of the stellar evolution and was subsequently amplified during the formation of the white dwarf assuming that the magnetic flux was conserved (see, e.g., Ref. [17]). Chemically peculiar Ap and Bp stars have long been thought to be the progenitors of magnetic white dwarfs. However, this scenario has been challenged by population synthesis calculations [18]. Alternatively, the magnetic field in white dwarfs could be generated through dynamo action (see, e.g., Refs. [19,20] and references therein).

Although the strongest observed surface magnetic fields are of the order 10^9 G, much stronger fields may exist in the core of white dwarfs [21]. It has been proposed that very massive so called super-Chandrasekhar white dwarfs (with a mass $\gtrsim 2M_\odot$) endowed with ultrastrong magnetic fields $B \gg 10^{13}$ G could be the progenitors of overluminous type Ia supernovae [22–27]. Although some recent works have cast doubt on the existence of such stars [28–30], the question arises of how strong the magnetic field in white dwarfs could be. This question has been generally assessed using the virial theorem [31]: for a stellar configuration to be stable, the magnetic energy (Gaussian cgs units are used throughout this paper),

$$E_{\rm mag} = \frac{1}{8\pi}\int_0^M \frac{B^2}{\rho}\, dm, \tag{1}$$

has to be lower than the absolute value of the gravitational energy,

$$E_{\rm grav} = \frac{1}{2}\int_0^M \Phi dm, \qquad (2)$$

where Φ is the gravitational potential, ρ the average mass density, dm an infinitesimal mass element, and M the mass of the star. Considering a spherical star of radius R with a uniform magnetic field leads to

$$B < 2.2\times 10^8 \frac{M}{M_\odot}\left(\frac{R_\odot}{R}\right)^2 \text{ G}, \qquad (3)$$

where $R_\odot$ is the radius of the Sun. For typical white-dwarf masses and radii, the strongest magnetic field thus obtained is of the order $\sim 10^{13}$ G. However, this estimate should be taken with a grain of salt since a uniform magnetic field in a star is unstable [32]. Moreover, the global structure of the star may depend on the magnetic field configuration, which is not a priori known. Consequently, ultrastrong magnetic fields $B \gg 10^{13}$ G in white-dwarf cores cannot be ruled out: the global stability condition $E_{\rm mag} < |E_{\rm grav}|$ might still be fulfilled assuming that the magnetic field strength in the star decreases outwards. On the other hand, we have recently shown that the strength of the magnetic field in the most massive white dwarfs is limited by the onset of electron captures, independently of the global configuration of the star [28].

In this paper, we estimate the strength of the magnetic field above which the most massive white dwarfs would become unstable against electron captures using a more realistic model of dense matter, which takes into account electron-ion interactions and allows for mixtures of two different ionic species.

II. MODEL OF DENSE MATTER IN WHITE-DWARF CORES

The model we adopt here was initially developed for describing the outer crust of strongly magnetized neutron stars [33] (see also Ref. [34] for a recent development). The core of magnetic white dwarfs is assumed to be made of fully ionized atoms arranged in a regular crystal lattice. We shall consider both homogeneous crystalline structures made of only one type of ions, and heterogeneous crystalline structures made of an admixture of two ionic species (typically carbon and oxygen) following the work of Ref. [35] in the neutron-star context. In addition, we suppose that thermal effects are negligibly small, and we set the temperature to zero for simplicity.

The pressure P at the center of the star is thus given by the sum of the electron degeneracy pressure P_e and the lattice pressure P_L arising from the interactions between electrons and ions. According to the Bohr-van Leeuwen theorem [36], the lattice pressure is independent of the magnetic field apart from a small contribution due to quantum zero-point motion of ions [37], which we neglect. The lattice is composed of two types of ions: ${}^A_Z X$ with proton number Z and mass number A, and ${}^{A'}_{Z'}X'$ with proton number Z', and mass number A'. The crystal structures we shall consider are illustrated in Fig. 1. The fraction of the ion ${}^A_Z X$ will be denoted by ξ, so that the fraction of the ion ${}^{A'}_{Z'}X'$ will be given by $\xi' = 1-\xi$. The fractions corresponding to the different lattice types are indicated in the caption of Fig. 1. For point-like ions embedded in a uniform electron gas with number density n_e, the lattice pressure is simply given by

$$P_L = \frac{\mathcal{E}_L}{3}, \qquad (4)$$

where the lattice energy density $\mathcal{E}_L$ is given by [35]

$$\mathcal{E}_L = Ce^2 n_e^{4/3} f(Z,Z'), \qquad (5)$$

$$f(Z,Z') = \bar{Z}^{-4/3}[\eta Z^2 + \zeta Z'^2 + (1-\eta-\zeta)ZZ'], \qquad (6)$$

with the mean proton number $\bar{Z} = \xi Z + \xi' Z'$, e being the elementary electric charge. The lattice constants C, η and ζ are given in Table I for different crystal lattice types. Note that in the limiting case of homogeneous crystal structures, $f(Z,Z) = Z^{2/3}$ is independent of η and ζ.

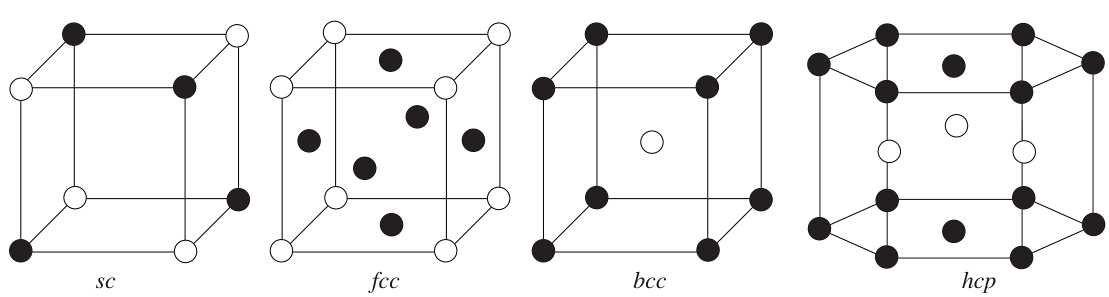

FIG. 1. Heterogeneous crystal structures with ion species ${}^A_Z X$ (black circles) and ${}^{A'}_{Z'}X'$ (white circles) : simple cubic lattice (sc), face-centered cubic lattice (fcc), body-centered cubic lattice (bcc), and hexagonal close-packed (hcp). The fractions of ions ${}^A_Z X$ and ${}^{A'}_{Z'}X'$ are equal to $\xi = 1/2$ and $\xi' = 1/2$, respectively, in all but the fcc lattice. In this latter case, these fractions are $\xi = 3/4$ and $\xi' = 1/4$.

TABLE I. Lattice constants for different types of heterogeneous crystal structures: simple cubic (sc), body-centered cubic (bcc), face-centered cubic (fcc), and hexagonal close-packed (hcp). Taken from Ref. [35].

Structure	C	η	ζ
sc	−1.418649	0.403981	0.403981
bcc	−1.444231	0.389821	0.389821
fcc	−1.444141	0.654710	0.154710
hcp	−1.444083	0.345284	0.345284

In the presence of a strong magnetic field, the electron motion perpendicular to the field is quantized into Landau levels (see, e.g., chapter 4 from Ref. [38]). Ignoring the small electron anomalous magnetic moment (see, e.g., Section 4.1.1 from Ref. [38] and references therein), and treating electrons as a relativistic Fermi gas, the energies of Landau levels are given by

$$\epsilon_\nu = \sqrt{c^2 p_z^2 + m_e^2 c^4 (1 + 2\nu B_\star)} \tag{7}$$

$$\nu = n_L + \frac{1}{2} + \sigma, \tag{8}$$

where m_e is the electron mass, c is the speed of light, n_L is any non-negative integer, $\sigma = \pm 1/2$ is the spin, p_z is the component of the momentum along the field, and $B_\star = B/B_{\rm crit}$ with the critical magnetic field $B_{\rm crit}$ defined by

$$B_{\rm crit} = \frac{m_e^2 c^3}{e\hbar} \approx 4.4 \times 10^{13}\ \mathrm{G}. \tag{9}$$

For a given magnetic field strength $B_\star$, the number of occupied Landau levels is determined by the electron number density n_e,

$$n_e = \frac{2B_\star}{(2\pi)^2 \lambda_e^3} \sum_{\nu=0}^{\nu_{\rm max}} g_\nu x_e(\nu), \tag{10}$$

$$x_e(\nu) = \sqrt{\gamma_e^2 - 1 - 2\nu B_\star}, \tag{11}$$

where $\lambda_e = \hbar/m_e c$ is the electron Compton wavelength, γ_e is the electron chemical potential in units of the electron rest mass energy, that is,

$$\gamma_e = \frac{\mu_e}{m_e c^2}, \tag{12}$$

with $\mu_e = d\mathcal{E}_e/dn_e$ ($\mathcal{E}_e$ being the electron energy density), while the degeneracy g_ν is $g_\nu = 1$ for $\nu = 0$ and $g_\nu = 2$ for $\nu \geq 1$. The electric charge neutrality condition implies that the mass density can be expressed as $\rho = n_e m / y_e$ where n_e is the electron number density, $y_e = \bar{Z}/\bar{A}$ the mean electron fraction, $\bar{A} = \xi A + \xi' A'$ the mean mass number, and m the average mass per nucleon (approximated here by the unified atomic mass unit).

The electron energy density $\mathcal{E}_e$ and corresponding electron pressure P_e are given by

$$\mathcal{E}_e = \frac{B_\star m_e c^2}{(2\pi)^2 \lambda_e^3} \sum_{\nu=0}^{\nu_{\rm max}} g_\nu (1 + 2\nu B_\star) \psi_+ \left[\frac{x_e(\nu)}{\sqrt{1 + 2\nu B_\star}}\right], \tag{13}$$

and

$$P_e = \frac{B_\star m_e c^2}{(2\pi)^2 \lambda_e^3} \sum_{\nu=0}^{\nu_{\rm max}} g_\nu (1 + 2\nu B_\star) \psi_- \left[\frac{x_e(\nu)}{\sqrt{1 + 2\nu B_\star}}\right], \tag{14}$$

respectively, where

$$\psi_\pm(x) = x\sqrt{1 + x^2} \pm \ln\left(x + \sqrt{1 + x^2}\right). \tag{15}$$

A magnetic field is strongly quantizing if only the lowest level $\nu = 0$ is filled, or equivalently whenever $n_e < n_{e\mathrm{B}}$, where (see Appendix A)

$$n_{e\mathrm{B}} = \frac{B_\star^{3/2}}{\sqrt{2}\pi^2 \lambda_e^3}. \tag{16}$$

To the electron density (16) corresponds the mass density

$$\rho_{\rm B} = \frac{m}{y_e \lambda_e^3} \frac{B_\star^{3/2}}{\sqrt{2}\pi^2}. \tag{17}$$

Conversely, for a given mass density ρ the strongly quantizing regime corresponds to magnetic field strengths

$$B_\star > \left(\frac{\rho y_e \lambda_e^3 \sqrt{2}\pi^2}{m}\right)^{2/3} \approx 180(2 y_e \rho_{10})^{2/3}, \tag{18}$$

where $\rho_{10} = \rho/(10^{10}\ \mathrm{g\,cm^{-3}})$.

III. MAGNETIC FIELD STRENGTH IN WHITE DWARFS

The equilibrium configuration of a white dwarf with a strongly magnetized core is determined from the magneto-hydrostatic equation (see, e.g., Ref. [1])

$$\nabla P = -\rho \nabla \Phi + \frac{1}{4\pi} \nabla \times \boldsymbol{B} \times \boldsymbol{B}, \tag{19}$$

where the gravitational potential Φ obeys Poisson's equation

$$\nabla^2 \Phi = 4\pi G \rho, \tag{20}$$

G being the gravitational constant. Since the pressure depends not only on the density ρ but also on magnetic field strength B, Eq. (19) can be written as

$$\left.\frac{\partial P}{\partial \rho}\right|_B \nabla\rho + \left.\frac{\partial P}{\partial B}\right|_\rho \nabla B = -\rho\nabla\Phi + \frac{1}{4\pi}\nabla\times\boldsymbol{B}\times\boldsymbol{B}. \quad (21)$$

Let us consider that the magnetic field is *locally* uniform, as in Refs. [22–27]. The magnetohydrostatic equation (21) thus reduces to

$$\left.\frac{\partial P}{\partial \rho}\right|_B \nabla\rho = -\rho\nabla\Phi. \quad (22)$$

Therefore, the star is mechanically unstable against gravitational collapse whenever $\partial P/\partial\rho = 0$: in such case, the matter pressure force could not resist the huge gravitational pull. As shown in Appendix A, this situation occurs at the density $\rho_{\rm B}$. The corresponding pressure is given by

$$P_{\rm B} = P_e(n_{e{\rm B}}) + P_L(n_{e{\rm B}}, Z, Z'). \quad (23)$$

The most massive magnetic white dwarfs are thus expected to have central densities close to $\rho_{\rm B}$.

As shown in our previous work [28], the stability of such stars is further limited by the onset of electron captures in their core, whereby the nucleus ${}^A_Z X$ (similarly for the nucleus ${}^{A'}_{Z'}X'$) transforms into a nucleus ${}^{A}_{Z-1}Y$ with proton number $Z-1$ and mass number A with the emission of an electron neutrino ν_e:

$${}^A_Z X + e^- \rightarrow {}^{A}_{Z-1}Y + \nu_e. \quad (24)$$

This reaction is generally almost immediately followed by a second electron capture on the daughter nucleus Y. During this process, the pressure does not change and since $P \approx P_e(n_e)$ the electron density remains approximately constant. On the contrary, the mass density increases from $\rho \approx n_e m\bar{A}/\bar{Z}$ to $\rho \approx n_e m\bar{A}/(\bar{Z}-2\xi)$. As a result, electron captures soften the equation of state and make the star unstable. The nuclei ${}^A_Z X$ and ${}^{A'}_{Z'}X'$ will be stable against electron captures in the dense matter of the stellar core at pressure P if the Gibbs free energy per nucleon g is lower than those of their daughter nuclei. The Gibbs free energy per nucleon is defined by

$$g = \frac{\mathcal{E}+P}{n}, \quad (25)$$

where n is the average nucleon number density, and $\mathcal{E}$ is the average energy density given by

$$\mathcal{E} = n_X M(Z,A)c^2 + n_{X'}M(Z',A')c^2 + \mathcal{E}_e + \mathcal{E}_L - n_e m_e c^2, \quad (26)$$

where n_X and $n_{X'}$ are the number densities of nuclei ${}^A_Z X$ and ${}^{A'}_{Z'}X'$, respectively, $M(Z,A)$ and $M(Z',A')$ are their mass (including the rest mass of nucleons and Z electrons). The average nucleon density and the fraction ξ are given by

$$n = An_X + A'n_{X'}, \quad (27)$$

$$\xi = \frac{n_X}{n_X + n_{X'}}. \quad (28)$$

The number densities of nuclei can thus be conveniently expressed as $n_X = \xi n/\bar{A}$ and $n_{X'} = \xi' n/\bar{A}$. From the electric charge neutrality, we obtain

$$n_e = Zn_X + Z'n_{X'}. \quad (29)$$

Using the thermodynamic identity $\mathcal{E}_e + P_e = n_e\mu_e$ and Eq. (4), the Gibbs free energy per nucleon can be written as

$$g = mc^2 + \xi\frac{\mathcal{E}(A,Z)}{\bar{A}} + \xi'\frac{\mathcal{E}(A',Z')}{\bar{A}} + y_e\left[\mu_e - m_e c^2 + \frac{4}{3}\frac{\mathcal{E}_L}{n_e}\right], \quad (30)$$

$\mathcal{E}(A,Z) = M(A,Z)c^2 - Amc^2$ and $\mathcal{E}(A',Z') = M(A',Z')c^2 - A'mc^2$ being the mass excesses of the nuclei ${}^A_Z X$ and ${}^{A'}_{Z'}X'$, respectively. Note that the nucleon number is conserved during electron captures so that neither A nor A' change. Likewise, the fractions ξ and ξ' remain constant since they are completely determined by the lattice structure. On the contrary, the electron density does change. The electron density before and after the captures will be denoted by n_e^- and n_e^+, respectively. These densities are not independent because the pressure P has to remain constant. Before the capture, the pressure can be written as

$$P = P_e(n_e^-) + P_L(n_e^-, Z, Z'). \quad (31)$$

For clarity, we have indicated explicitly the dependence on the electron densities and proton numbers. After the capture, the pressure can be expressed as

$$P = P_e(n_e^+) + P_L(n_e^+, Z-\Delta Z, Z'-\Delta Z'), \quad (32)$$

where ΔZ and $\Delta Z'$ denote the changes in Z and Z' (e.g., $\Delta Z = 1$ and $\Delta Z' = 0$ if the nuclei ${}^A_Z X$ capture electrons).

Note that $n_e^- \approx n_e^+$ since $P_L \ll P_e$. Let us write $n_e^\pm = n_e + \delta n_e^\pm$ where n_e is the electron density corresponding to the pressure P such that

$$P \equiv P_e(n_e) + P_L(n_e, Z, Z'). \quad (33)$$

Solving Eqs. (31) and (32) to first order in $\delta n_e^\pm$ yields $\delta n_e^- = 0$ and

$$\delta n_e^+ = [P_L(n_e,Z,Z') - P_L(n_e, Z-\Delta Z, Z'-\Delta Z')]\left(\frac{\partial P_e}{\partial n_e}\right)^{-1}. \quad (34)$$

In the strongly quantizing regime with $x_e \gg 1$, $P_e \approx m_e c^2 n_e^2 \pi^2 \lambda_e^3/B_\star$ so that Eq. (34) can be expressed as

$$\delta n_e^+ \approx n_{e\mathrm{B}} \frac{C\alpha}{3(4\pi^2)^{1/3}} \left(\frac{n_e}{n_{e\mathrm{B}}}\right)^{1/3} \Delta f, \tag{35}$$

where $\alpha = e^2/(\hbar c)$ is the fine structure constant and

$$\Delta f \equiv f(Z, Z') - f(Z - \Delta Z, Z' - \Delta Z'). \tag{36}$$

As previously discussed, we expect $n_e \approx n_{e\mathrm{B}}$ in the core of the most massive magnetic white dwarfs. Since $\alpha/(3(4\pi^2)^{1/3}) \simeq 1/1400$, we thus find that $\delta n_e^{\pm} \ll n_{e\mathrm{B}}$.

The stability of the stellar core against electron captures is embedded in the inequality

$$g(n_e^-, Z, Z') < g(n_e^+, Z - \Delta Z, Z' - \Delta Z'). \tag{37}$$

Since $n_e^- \approx n_e^+ \approx n_e$, the inequality (37) can be approximately expressed as

$$\mu_e \Delta\bar{Z} + \frac{4}{3} C e^2 n_e^{1/3} \Delta(\bar{Z} f(Z, Z')) < \bar{\mu}_e^\beta, \tag{38}$$

where

$$\Delta\bar{Z} \equiv \xi \Delta Z + \xi' \Delta Z', \tag{39}$$

$$\bar{\mu}_e^\beta \equiv \xi \mu_e^\beta(A, Z) + \xi' \mu_e^\beta(A', Z'), \tag{40}$$

$$\mu_e^\beta(A, Z) \equiv \mathcal{E}(A, Z - \Delta Z) - \mathcal{E}(A, Z) + \Delta Z m_e c^2, \tag{41}$$

$$\mu_e^\beta(A', Z') \equiv \mathcal{E}(A', Z' - \Delta Z') - \mathcal{E}(A', Z') + \Delta Z' m_e c^2, \tag{42}$$

and $\Delta(\bar{Z} f(Z, Z'))$ denotes the difference in $\bar{Z} f(Z, Z')$ before and after the captures, i.e.

$$\Delta(\bar{Z} f(Z, Z')) = \bar{Z} f(Z, Z') - (\bar{Z} - \Delta\bar{Z}) f(Z - \Delta Z, Z' - \Delta Z'). \tag{43}$$

The left-hand side of the inequality (38) increases with electron density hence with pressure, and will therefore reach the threshold value given by $\bar{\mu}_e^\beta$ at some pressure P_β. However, as previously discussed, the pressure in the core of super-Chandrasekhar magnetic white dwarfs will be approximately limited by P_B. The core will thus become unstable against electron captures whenever $P_\beta < P_\mathrm{B}$. This situation occurs if the magnetic field strength exceeds some threshold value, denoted by $B_\star^\beta$. Indeed, setting $n_e = n_{e\mathrm{B}} \propto B_\star^{3/2}$ and using $\mu_e \approx 2\pi^2 m_e c^2 \lambda_e^3 n_{e\mathrm{B}}/B_\star \propto \sqrt{B_\star}$, the inequality (38) can be expressed as $B_\star < B_\star^\beta$, where

$$B_\star^\beta \approx \frac{1}{2} \left(\frac{\bar{\mu}_e^\beta}{m_e c^2 \Delta\bar{Z}}\right)^2 \left[1 + \left(\frac{4}{\pi}\right)^{2/3} \frac{C\alpha}{3} \frac{\Delta(\bar{Z} f(Z, Z'))}{\Delta\bar{Z}}\right]^{-2}. \tag{44}$$

At sufficiently high pressures, the quantum-zero point fluctuations of nuclei about their equilibrium position may become large enough to trigger pycnonuclear fusion reactions

$${}^A_Z X + {}^A_Z X \to {}^{2A}_{2Z} Y, \tag{45}$$

and similarly for the nuclei ${}^{A'}_{Z'} X'$. The threshold pressure $P_\beta(2A, 2Z)$ for the onset of electron capture by the daughter nucleus ${}^{2A}_{2Z} Y$ is generally lower than the corresponding pressure $P_\beta(A, Z)$ for the original nucleus ${}^A_Z X$. For this reason, pycnonuclear fusion reactions, if they occur at a pressure $P_\mathrm{pyc} < P_\beta(2A, 2Z)$, further reduce the maximum strength of the magnetic field in the core of white dwarfs. The corresponding value of the magnetic field strength can be easily obtained from Eq. (44).

IV. RESULTS AND DISCUSSION

The core of a white dwarf is expected to contain mainly carbon and oxygen, the primary ashes of helium burning. However, the cores of some white dwarfs may be composed of other elements like helium [39–41], neon and magnesium [42], or even much heavier elements like iron. Iron white dwarfs could be formed from the explosive ignition of electron degenerate oxygen-neon-magnesium cores [43], or from failed detonation supernovae [44]. One possible candidate of iron white dwarfs is WD0433 + 270 [45].

The masses of these nuclei have all been measured in the laboratory. Using the latest experimental data from the Atomic Mass Evaluation 2012 [46], we have solved numerically the set of equations (31), (32) and (37) considering different matter composition and crystal structures. The nuclear masses can be obtained from the tabulated atomic masses after subtracting out the binding energy of the atomic electrons; however the differences are very small [47] and have been ignored here. We have used the values of the various constants from CODATA 2010 [48]. We have considered the following three different processes: (i) $\Delta Z = 1$, $\Delta Z' = 0$, (ii) $\Delta Z = 0$, $\Delta Z' = 1$, and (iii) $\Delta Z = 1$, $\Delta Z' = 1$. For the peculiar case of helium white dwarfs, we have also examined electron captures accompanied by neutron emission, i.e. ${}^4\mathrm{He} + e^- \to {}^3\mathrm{H} + n + \nu_e$, since this process generally occurs at a lower density (see, e.g., Ref. [1]). The formalism for treating this case is developed in Appendix B.

The lowest values of $B_\star^\beta$ and the corresponding mass density are indicated in Tables II and III. The threshold magnetic field strength $B_\star^\beta$ is found to be highly dependent on the composition of the white-dwarf core, ranging from 867 for ^{4}He down to 6.5 for ^{40}Ca. As a consequence, the occurrence of pycnonuclear fusion reactions would drastically reduce the magnetic field strength in the most massive white dwarfs, e.g., from 383 to 74 for the fusion of ^{12}C into ^{24}Mg, from 240 to 9.7 for the fusion ^{16}O in ^{32}S, or from 115 to 6.5 for the fusion of ^{20}Ne into ^{40}Ca. However, the rates at which pycnonuclear fusion reactions

TABLE II. Magnetic field strength $B_\star^\beta$ above which the core of the most massive white dwarfs becomes unstable against electron captures for different compositions. The corresponding average mass density is given by ρ. The core is assumed to be made of homogeneous crystalline structures. The values of $B_\star^\beta$ are given for bcc, fcc and hcp lattices since the results are the same for the adopted accuracy. Values in parentheses are for a sc lattice.

${}^A_ZX$	$B_\star^\beta$	ρ (g cm^{-3})
^{4}He	867.0 (866.8)	1.05×10^{11}
^{12}C	383.2 (382.9)	3.10×10^{10}
^{16}O	239.7 (239.5)	1.53×10^{10}
^{22}Ne	259.6 (259.3)	1.90×10^{10}
^{21}Ne	77.3 (77.2)	2.95×10^9 (2.94×10^9)
^{20}Ne	114.6 (114.5)	5.07×10^9 (5.06×10^9)
^{23}Na	48.1 (48.0)	1.44×10^9
^{24}Mg	73.7 (73.6)	2.61×10^9
^{25}Mg	38.1 (38.0)	1.01×10^9 (1.00×10^9)
^{26}Mg	198.2 (198.00)	1.25×10^{10}
^{32}S	9.71 (9.69)	1.25×10^8
^{40}Ca	6.45 (6.43)	6.76×10^7 (6.74×10^7)
^{44}Ca	80.2 (80.0)	3.26×10^9 (3.25×10^9)
^{56}Fe	37.3 (37.2)	1.01×10^9

occur still remain very uncertain (see, e.g., Ref. [49]). The dependence of $B_\star^\beta$ on the lattice structure is very small, as can be easily understood from Eq. (44). Table III also shows that the magnetic field strength $B_\star^\beta$ in heterogeneous structures is almost completely determined by the most unstable ion species regardless of the proportion. As shown in Tables IV and V, the error of Eq. (44) lies below 1% for ^{12}C and ^{16}O, the primary constituents of white dwarfs, and amounts at most to ~9% for ^{40}Ca.

TABLE III. Same as Table II for heterogeneous crystalline structures.

% ${}^A_ZX$	% ${}^{A'}_{Z'}X'$	lattice	$B_\star^\beta$	ρ (g cm^{-3})
50% ^{12}C	50% ^{16}O	sc	239.4	1.53×10^{10}
50% ^{12}C	50% ^{16}O	bcc	239.5	1.53×10^{10}
50% ^{12}C	50% ^{16}O	hcp	239.3	1.53×10^{10}
75% ^{12}C	25% ^{16}O	fcc	239.3	1.53×10^{10}
25% ^{12}C	75% ^{16}O	fcc	239.5	1.55×10^{10}
50% ^{20}Ne	50% ^{16}O	sc	114.4	5.06×10^9
50% ^{20}Ne	50% ^{16}O	bcc	114.5	5.06×10^9
50% ^{20}Ne	50% ^{16}O	hcp	114.4	5.06×10^9
75% ^{20}Ne	25% ^{16}O	fcc	114.5	5.06×10^9
25% ^{20}Ne	75% ^{16}O	fcc	114.4	5.06×10^9
50% ^{16}O	50% ^{24}Mg	sc	73.5	2.61×10^9
50% ^{16}O	50% ^{24}Mg	bcc	73.6	2.61×10^9
50% ^{16}O	50% ^{24}Mg	hcp	73.5	2.60×10^9
75% ^{16}O	25% ^{24}Mg	fcc	73.5	2.60×10^9
25% ^{16}O	75% ^{24}Mg	fcc	73.6	2.61×10^9

TABLE IV. Relative error of Eq. (44) as compared to numerical results shown in Table II. The stellar core is assumed to be made of homogeneous crystalline structures with a bcc lattice.

${}^A_ZX$	Error (%)
^{12}C	0.5
^{16}O	0.7
^{22}Ne	0.8
^{21}Ne	1.2
^{20}Ne	1.0
^{23}Na	1.7
^{24}Mg	1.4
^{25}Mg	1.9
^{26}Mg	1.0
^{32}S	6.3
^{40}Ca	9.2
^{44}Ca	1.6
^{56}Fe	2.7

The presence of ultrastrong magnetic fields may change nuclear masses [50–52], an effect which we haven't taken into account. In order to assess the corresponding error in $B_\star^\beta$, we have estimated the change $\delta\mathcal{E} \equiv \mathcal{E}(B_\star^\beta) - \mathcal{E}(0)$ in the mass excess induced by a magnetic field for various nuclei expected to be found in the core of white dwarfs. For this purpose, we have carried out fully self-consistent relativistic mean-field calculations, as described in Ref. [52] except that we have now considered density-dependent meson-nucleon couplings using the DD-ME2 parameter set [53]. More specifically, for each nucleus ${}^A_ZX$ and its daughter nucleus ${}^{A}_{Z-1}Y$, we have computed $\delta\mathcal{E}(A,Z)$ and $\delta\mathcal{E}(A,Z-1)$ for the same magnetic field strength $B_\star^\beta({}^A_ZX)$, as given in Table II. Using Eq. (44) we have estimated the relative error in $B_\star^\beta$ as

$$\frac{\delta B_\star^\beta}{B_\star^\beta} = \frac{2\delta\bar{\mu}_e^\beta(A,Z)}{\bar{\mu}_e^\beta(A,Z)}, \tag{46}$$

where

$$\delta\bar{\mu}_e^\beta \equiv \xi\delta\mu_e^\beta(A,Z) + \xi'\delta\mu_e^\beta(A',Z'), \tag{47}$$

$$\delta\mu_e^\beta(A,Z) \equiv \delta\mathcal{E}(A,Z-\Delta Z) - \delta\mathcal{E}(A,Z). \tag{48}$$

TABLE V. Relative error of Eq. (44) as compared to numerical results shown in Table III. The stellar core is assumed to be made of heterogeneous crystalline structures with a bcc lattice.

% ${}^A_ZX$	% ${}^{A'}_{Z'}X'$	Error (%)
50% ^{12}C	50% ^{16}O	0.8
50% ^{20}Ne	50% ^{16}O	1.1
50% ^{24}Mg	50% ^{16}O	1.5

TABLE VI. Change in mass excess of nuclei ${}^A_ZX$ due to the presence of a strong magnetic field and estimated error on the magnetic field strength $B_\star^\beta$ above which the most massive white dwarfs become unstable against electron captures. Values in parentheses are the changes in the mass excess of the daughter nuclei ${}^A_{Z-1}Y$.

${}^A_ZX$	$\delta\mathcal{E}$ (keV)	$\delta B_\star^\beta/B_\star^\beta$ (%)
^{12}C	−5 (697)	10.1
^{16}O	0 (−57)	−0.8
^{22}Ne	−1 (151)	2.7
^{21}Ne	−19 (117)	4.4
^{20}Ne	0 (169)	4.5
^{23}Na	358 (−13)	−15.2
^{24}Mg	0 (621)	20.6
^{25}Mg	−11 (337)	16.0
^{26}Mg	−2 (493)	10.0
^{32}S	0 (489)	44.0
^{40}Ca	0 (346)	38.0
^{44}Ca	0 (469)	15.1
^{56}Fe	0 (52)	2.5

$$\delta\mu_e^\beta(A',Z') \equiv \delta\mathcal{E}(A',Z'-\Delta Z') - \delta\mathcal{E}(A',Z'). \qquad (49)$$

Results are summarized in Table VI. We have not considered ^{4}He since a mean-field approach is questionable for such a light nucleus. The errors $\delta\mathcal{E}$ in the mass excess of the daughter nuclei ${}^A_{Z-1}Y$ are found to be larger than for the nuclei ${}^A_ZX$, the notable exception being ^{23}Na. The reason lies in the fact that for a given magnetic field strength, the structure of odd nuclei is generally more affected than that of even-even nuclei. The corresponding errors in $B_\star^\beta$ vary appreciably from one nucleus to another. Although $\delta B_\star^\beta/B_\star^\beta \lesssim 10\%$, for ^{12}C, ^{16}O and Ne isotopes, much larger deviations are observed for the corresponding end-products of pycnonuclear fusion reactions, namely ^{24}Mg, ^{32}S and Ca isotopes, respectively. The errors on the average mass density in the core of super-Chandrasekhar white dwarfs can be estimated as

$$\frac{\delta\rho}{\rho} = \frac{3}{2}\frac{\delta B_\star^\beta}{B_\star^\beta}, \qquad (50)$$

where we have used Eq. (17).

Although our calculations of $B_\star^\beta$ have been restricted to $T=0$, the results we have obtained still remain valid at finite temperatures $T>0$ provided the following conditions are fulfilled: (i) the core of white dwarfs has crystallized, (ii) the electron gas is highly degenerate, and (iii) the magnetic field is strongly quantizing. The first condition reads $T<T_m$, where T_m is the crystallization temperature, defined as (see, e.g., Ref. [38])

$$T_m = \frac{e^2}{a_e k_\mathrm{B}\Gamma_m}\overline{Z^{5/3}}, \qquad (51)$$

where $a_e = (3/(4\pi n_e))^{1/3}$ is the electron-sphere radius, k_B is the Boltzmann's constant, and Γ_m is the Coulomb coupling parameter at melting. Remarking that in the stellar core $n_e = n_{e\mathrm{B}}$, and using Eq. (16) with $B_\star = B_\star^\beta$, the crystallization temperature can be expressed as

$$T_m = \frac{e^2}{\lambda_e k_\mathrm{B}}\frac{\overline{Z^{5/3}}}{\Gamma_m}\left(\frac{2\sqrt{2}}{3\pi}\right)^{1/3}\sqrt{B_\star^\beta}. \qquad (52)$$

For a classical one-component plasma $\Gamma_m \approx 175$ [38]. Adopting this value yields

$$T_m \approx 1.66\times 10^5\overline{Z^{5/3}}\sqrt{B_\star^\beta}\ \mathrm{K}. \qquad (53)$$

The second condition is satisfied if the temperature is lower than the Fermi temperature defined by

$$T_\mathrm{F} = \frac{\mu_e - m_e c^2}{k_\mathrm{B}}. \qquad (54)$$

In the stellar core, the electron chemical potential is given by $\mu_e = m_e c^2\sqrt{1+2B_\star^\beta}$ (see Appendix A), so that

$$T_\mathrm{F} = \frac{m_e c^2}{k_\mathrm{B}}\left(\sqrt{1+2B_\star^\beta}-1\right) \approx 5.93\times 10^9\left(\sqrt{1+2B_\star^\beta}-1\right)\ \mathrm{K}. \qquad (55)$$

Finally, the last condition requires $n_e < n_{e\mathrm{B}}$ and $T<T_\mathrm{B}$ with (see, e.g., Ref. [38])

$$T_\mathrm{B} = \frac{\hbar\omega_c}{k_\mathrm{B}}, \qquad (56)$$

$\omega_c = eB/(m_e c)$ being the electron cyclotron frequency. Using Eq. (9) with $B_\star = B_\star^\beta$ leads to

$$T_\mathrm{B} = \frac{m_e c^2}{k_\mathrm{B}}B_\star^\beta \approx 5.93\times 10^9 B_\star^\beta\ \mathrm{K}. \qquad (57)$$

The temperatures T_m, T_F and T_B are indicated in Tables VII and VIII for different core compositions. The lowest temperature is found to be T_m, with values comparable to those prevailing in white dwarfs. However, these values of T_m are probably underestimated since strongly magnetized Coulomb crystals tend to melt at a higher temperature [54]. In addition, we have shown that the magnetic field strength $B_\star^\beta$ is almost independent of the spatial arrangement of ions. We therefore anticipate that our calculations provide fairly accurate estimates of $B_\star^\beta$, even at temperatures $T>T_m$.

So far we have implicitly assumed that electron captures occur on a very short timescale as compared to typical stellar evolutionary timescales so that white dwarfs with central densities $\sim\rho_\mathrm{B}$ and magnetic field strength $B_\star > B_\star^\beta$

TABLE VII. Characteristic temperatures (in K) in the core of a strongly magnetized white dwarf for different compositions. The core is assumed to be made of homogeneous crystalline structures with a bcc lattice.

${}^A_Z X$	T_F (K)	T_B (K)	T_m (K)
^{4}He	2.4×10^{11}	5.1×10^{12}	1.6×10^7
^{12}C	1.6×10^{11}	2.3×10^{12}	6.4×10^7
^{16}O	1.2×10^{11}	1.4×10^{12}	8.2×10^7
^{22}Ne	1.3×10^{11}	1.5×10^{12}	1.2×10^8
^{21}Ne	6.8×10^{10}	4.6×10^{11}	6.8×10^7
^{20}Ne	8.4×10^{10}	6.8×10^{11}	8.2×10^7
^{23}Na	5.3×10^{10}	2.9×10^{11}	6.3×10^7
^{24}Mg	6.6×10^{10}	4.4×10^{11}	9.0×10^7
^{32}S	2.1×10^{10}	5.8×10^{10}	5.3×10^7
^{40}Ca	1.6×10^{10}	3.8×10^{10}	6.2×10^7
^{44}Ca	6.9×10^{10}	4.8×10^{11}	2.2×10^8

would almost immediately collapse. We have tested this assumption by estimating electron capture rates. Although the magnetic field can impact the rate of electron captures, the effects of the magnetic field is negligible at densities $\rho \approx \rho_B$ [33] (the rates only depend on $B_\star$ indirectly through the actual value of ρ_B). Therefore, we have calculated electron captures as in the absence of magnetic fields at the density $\rho = \rho_B$, using the nuclear model described in Refs. [55,56]. In this model, the single-nucleon basis and the occupation factors in the target nucleus are calculated using the finite-temperature Skyrme Hartree-Fock (HF) method. The $J^\pi = 0^\pm, 1^\pm, 2^\pm$ charge-exchange transitions are determined in the finite-temperature random-phase approximation (RPA). This scheme is self-consistent; i.e., both the HF and the RPA equations are based on the same Skyrme functional. The functional employed in this work is the Brussels-Montreal Skyrme functional BSk17 [57]. The dependence of the rates on the magnetic field only arises from the electron chemical potential, which we set equal to $\mu_e = \hbar c(3\pi^2 n_{e\mathrm{B}})^{1/3}$ with $n_{e\mathrm{B}}$ given by Eq. (16). Results are summarized in Table IX. We find that strongly magnetized white dwarfs with central densities $\sim\rho_\mathrm{B}$ would be highly unstable since electrons in their core would be captured at a rate of the order 10^2–10^5 per second.

TABLE VIII. Same as Table VII for heterogeneous crystalline structures.

% ${}^A_Z X$	% ${}^{A'}_{Z'} X'$	lattice	T_F (K)	T_B (K)	T_m (K)
50% ^{12}C	50% ^{16}O	bcc	1.2×10^{11}	1.4×10^{12}	6.7×10^7
75% ^{12}C	25% ^{16}O	fcc	1.2×10^{11}	1.4×10^{12}	5.9×10^7
25% ^{12}C	75% ^{16}O	fcc	1.1×10^{11}	1.1×10^{12}	6.5×10^7
50% ^{20}Ne	50% ^{16}O	bcc	8.4×10^{10}	6.8×10^{11}	7.0×10^7
75% ^{20}Ne	25% ^{16}O	fcc	8.4×10^{10}	6.8×10^{11}	7.6×10^7
25% ^{20}Ne	75% ^{16}O	fcc	8.4×10^{10}	6.8×10^{11}	6.3×10^7
50% ^{16}O	50% ^{24}Mg	bcc	6.6×10^{10}	4.4×10^{11}	6.8×10^7
75% ^{16}O	25% ^{24}Mg	fcc	6.6×10^{10}	4.4×10^{11}	5.7×10^7
25% ^{16}O	75% ^{24}Mg	fcc	6.6×10^{10}	4.4×10^{11}	7.9×10^7

TABLE IX. Electron capture rates for some nuclei expected to be found in the core of a super-Chandrasekhar magnetic white dwarf with a central density given by ρ_B, and for two different magnetic field strengths $B_\star$.

	rate (s^{-1})	
Species	$B_\star = 2 \times 10^3$	$B_\star = 2 \times 10^4$
^{12}C	3.5×10^3	6.2×10^4
^{16}O	4.4×10^2	1.3×10^4
^{20}Ne	1.3×10^4	1.1×10^5
^{22}Ne	2.8×10^3	4.5×10^4
^{24}Mg	3.6×10^4	2.6×10^5
^{32}S	1.2×10^5	6.8×10^5
^{40}Ca	1.7×10^4	2.2×10^5
^{44}Ca	4.7×10^3	8.7×10^4
^{56}Fe	1.3×10^5	7.9×10^5

V. CONCLUSION

The onset of electron captures sets an upper limit on the strongest possible magnetic fields in the core of the most massive white dwarfs. Results are summarized in Tables II and III. The limiting magnetic field strength $B_\star^\beta$ is approximately given by Eq. (44), from which the central mass density can be inferred using Eq. (17). Although $B_\star^\beta$ is almost independent of the spatial arrangement of ions, it is extremely sensitive to the core composition. The strongest possible magnetic fields are predicted to be found in pure helium white dwarfs. However, the presence of heavier elements in the stellar core considerably lowers the value of $B_\star^\beta$. More importantly, the value of $B_\star^\beta$ is dramatically reduced if pycnonuclear fusion reactions are allowed. All in all, our estimates of the magnetic field strengths are significantly lower than those expected to be found in the core of the putative super-Chandrasekhar white dwarfs proposed in Refs. [22–27]. This confirms the results we obtained in our previous analysis [28]. Although stars endowed with strong central magnetic fields such that $B_\star \geq 1$ and $B_\star < B_\star^\beta$ would be stable against electron captures, their global stability is not guaranteed and needs to be further studied using realistic equations of state of strongly magnetized matter, such as those obtained in this work.

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APPENDIX A: SOFTENING OF THE EQUATION OF STATE OF A RELATIVISTIC ELECTRON GAS IN A STRONGLY QUANTIZING MAGNETIC FIELD

It has been known for a long time that the presence of a strongly quantizing magnetic field considerably softens the equation of state of cold dense matter [58] (see also chapter 4 from Ref. [38] and references therein). In this appendix, we briefly review the calculation of the pressure P and its partial derivative $\partial P/\partial\rho$.

In the strongly quantizing regime, only the lowest level $\nu = 0$ is filled so that $\nu_{\rm max} = 0$. This situation occurs whenever the electron chemical potential μ_e is lower than the energy $\epsilon_1(p_z = 0) = m_e c^2\sqrt{1 + 2B_\star}$, i.e. whenever $x_e \leq \sqrt{2B_\star}$ or equivalently $n_e \leq n_{e\rm B} = B_\star^{3/2}/(\sqrt{2}\pi^2\lambda_e^3)$. In this limiting case, the electron pressure is given by

$$P_e = \frac{B_\star m_e c^2}{(2\pi)^2\lambda_e^3}\psi_-(x_e), \qquad (A1)$$

with

$$x_e = \frac{2\pi^2\lambda_e^3 n_e}{B_\star}. \qquad (A2)$$

The electron chemical potential can be obtained from

$$\gamma_e = \sqrt{1 + x_e^2}. \qquad (A3)$$

The stiffness of the equation of state can be characterized by the adiabatic index defined by

$$\Gamma_e \equiv \frac{n_e}{P_e}\frac{\partial P_e}{\partial n_e}\bigg|_{B_\star}, \qquad (A4)$$

where the partial derivative is taken at fixed value of the magnetic field strength. In the limit $x_e \ll 1$, the electron pressure is approximately given by

$$P_e \approx \frac{1}{3} m_e c^2 n_e^3 \left[\frac{2\pi^2\lambda_e^3}{B_\star}\right]^2, \qquad (A5)$$

and therefore $\Gamma_e \approx 3$. In the other limit $x_e \gg 1$ and $x_e \leq \sqrt{2B_\star}$, the electron pressure is approximately given by

$$P \simeq P_e \approx m_e c^2 n_e^2 \frac{\pi^2\lambda_e^3}{B_\star}, \qquad (A6)$$

and $\Gamma_e \approx 2$.

Let us now consider that the levels $\nu = 0$ and $\nu = 1$ are filled so that $\nu_{\rm max} = 1$. Equation (10) thus reads

$$n_e = \frac{2B_\star}{(2\pi)^2\lambda_e^3}\left[\sqrt{\gamma_e^2 - 1} + 2\sqrt{\gamma_e^2 - 1 - 2B_\star}\right]. \qquad (A7)$$

Solving this equation for γ_e yields

$$\gamma_e = \frac{1}{3}\sqrt{9 + 24B_\star + 5x_e^2 - 4x_e\sqrt{6B_\star + x_e^2}}, \qquad (A8)$$

where $x_e = 2\pi^2\lambda_e^3 n_e/B_\star$. Equation (A8) is the only solution lying in the range

$$\epsilon_1(p_z = 0) \leq \gamma_e m_e c^2 \leq \epsilon_2(p_z = 0), \qquad (A9)$$

the lower and upper bounds corresponding to the complete filling of the levels $\nu = 0$ and $\nu = 1$, respectively. Using Eqs. (7) and (A7), this range of values for the chemical potential can be equivalently expressed as

$$\sqrt{2B_\star} \leq x_e \leq 2(1 + \sqrt{2})\sqrt{B_\star}, \qquad (A10)$$

$$n_{e\rm B} \leq n_e \leq n_{e\rm B2}, \qquad (A11)$$

where $n_{e\rm B2} = \sqrt{2}(1 + \sqrt{2})n_{e\rm B}$.

The partial derivative of γ_e with respect to x_e at constant $B_\star$ is given by

$$\frac{\partial\gamma_e}{\partial x_e}\bigg|_{B_\star} = \frac{5x_e\sqrt{6B_\star + x_e^2} - 4(3B_\star + x_e^2)}{9\gamma_e\sqrt{6B_\star + x_e^2}}. \qquad (A12)$$

It is readily seen that this derivative vanishes for $x_e = \sqrt{2B_\star}$, i.e., when the level $\nu = 0$ is completely filled. On the other hand, we find from Eq. (14) and $\gamma_e m_e c^2 = e_1(p_z = 0) = \sqrt{1 + 2B_\star}$ that the partial derivative of the pressure with respect to γ_e is given by

$$\frac{\partial P_e}{\partial\gamma_e}\bigg|_{B_\star} = \frac{m_e c^2}{\lambda_e^3}\frac{2B_\star^{5/2}}{\pi^2}\sqrt{\frac{1 + 2B_\star}{1 + 4B_\star}}. \qquad (A13)$$

Consequently, combining Eqs. (A12) and (A13) using the chain rule shows that the partial derivative of the pressure with respect to the electron density vanishes whenever the level $\nu = 1$ starts to be filled,

$$\frac{\partial P_e}{\partial n_e}\bigg|_{B_\star} = \frac{(2\pi)^2\lambda_e^3}{2B_\star}\frac{\partial P_e}{\partial\gamma_e}\bigg|_{B_\star}\frac{\partial\gamma_e}{\partial x_e}\bigg|_{B_\star} = 0; \qquad (A14)$$

therefore, $\Gamma_e = 0$. Note that the adiabatic index varies discontinuously at $n_e = n_{e\rm B}$, dropping from $\Gamma_e = 2$ to $\Gamma_e = 0$. The variation of Γ_e with the electron density is shown in Fig. 2. Using Eqs. (A6), (4), (5), and (16), we find that the lattice pressure is negligible for $n_e \approx n_{e\rm B}$:

$$\frac{P_L}{P_e} \approx \frac{C\alpha}{3}\left(\frac{\sqrt{2}}{\pi}\right)^{2/3} f(Z, Z') \ll 1. \qquad (A15)$$

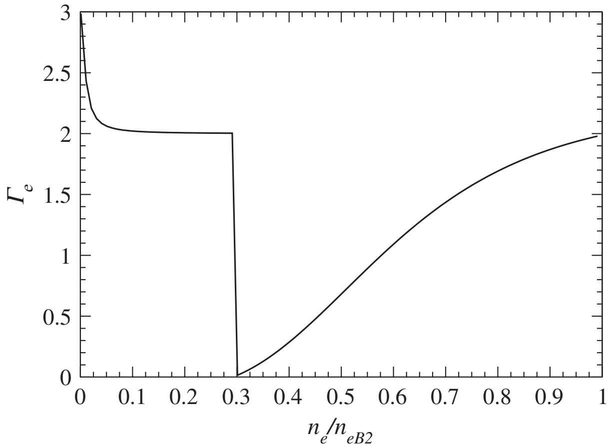

FIG. 2. Adiabatic index Γ_e of a relativistic electron gas in a strong magnetic field with $B_\star = 1000$, as a function of the electron density n_e in units of the electron density n_{eB2} at which electrons start to populate the level $\nu = 2$. Note that in this case $n_{eB} \approx 0.29 n_{eB2}$.

Since $P_L \ll P_e$ and $\rho = n_e m / y_e$, we obtain $\partial P/\partial \rho \approx 0$ for $\rho = \rho_B$: Landau quantization thus leads to a strong softening of the equation of state.

APPENDIX B: STABILITY OF DENSE MATTER AGAINST NEUTRON EMISSION

At high pressures, the nucleus ${}^A_Z X$ (similarly for the nucleus ${}^{A'}_{Z'} X'$) may become unstable against the capture of electrons accompanied by the emission of free neutrons. The nucleus ${}^A_Z X$ will thus transform into a nucleus ${}^{A-1}_{Z-1} Y$ (with proton number $Z-1$ and mass number $A-1$) with the emission of a neutron n and an electron neutrino ν_e:

$$ {}^A_Z X + e^- \to {}^{A-1}_{Z-1} Y + n + \nu_e. \tag{B1} $$

In order to examine the occurrence of such a process, we have to generalize the expression of the Gibbs free energy per nucleon so as to include the contribution of free neutrons. In the following, we shall consider homogeneous crystalline structures for simplicity. However, the generalization to heterogeneous structures is straightforward.

From the general definition (25) and neglecting the effect of free neutrons on nuclei, it can be easily seen that the neutron contribution to g will be simply given by $n_n \mu_n / n$, where μ_n is the neutron chemical potential. Let us determine the expression of the Gibbs free energy per nucleon of nuclei coexisting with free neutrons. The average baryon number density is now given by

$$ n = (A-1) n_X + n_n. \tag{B2} $$

Note that one neutron is associated with each nucleus so that the neutron density is given by

TABLE X. Electron chemical potentials for the onset of electron captures with and without neutron emission in the core of the most massive magnetic white dwarfs. See text for details.

${}^A_Z X$	$\mu_e^{\beta n}$ (MeV)	μ_e^{β} (MeV)
^{4}He	21.107	22.707
^{12}C	17.25	13.88
^{16}O	13.420	10.932
^{20}Ne	14.137	7.536
^{21}Ne	14.297	6.195
^{22}Ne	16.559	11.329
^{56}Fe	11.476	4.206
^{23}Na	10.087	4.887
^{24}Mg	12.986	6.027
^{32}S	10.157	2.222
^{40}Ca	9.621	1.822
^{44}Ca	13.476	6.199

$$ n_n = n_X. \tag{B3} $$

Therefore, substituting Eq. (B3) in (B2) shows that Eq. (27) still holds. The electric charge neutrality leads to

$$ n_e = (Z-1) n_X. \tag{B4} $$

Approximating the neutron chemical potential by the neutron rest mass energy $\mu_n \approx m_n c^2$, we finally obtain

$$ g_n = \left(1 - \frac{1}{A}\right) mc^2 + \frac{m_n c^2}{A} + \frac{\mathcal{E}(A-1, Z-1)}{A} + \frac{Z-1}{A}\left[\mu_e - m_e c^2 + \frac{4}{3}\frac{\mathcal{E}_L (Z-1)}{n_e}\right], \tag{B5} $$

where the subscript n is to remind that the electron capture is accompanied by neutron emission. The core of white dwarfs will be stable against the process (B1) whenever

$$ g(n_e^-, Z) < g_n(n_e^+, Z-1). \tag{B6} $$

Neglecting the contribution of free neutrons to the pressure, the electron densities $n_e^\pm$ will still be given by Eqs. (31) and (32). Since $n_e^- \approx n_e^+ \approx n_e$, the inequality (B6) can be approximately expressed as

$$ \mu_e + \frac{4}{3} C e^2 n_e^{1/3} (Z^{5/3} - (Z-1)^{5/3}) < \mu_e^{\beta n}, \tag{B7} $$

where

$$ \mu_e^{\beta n}(A, Z) \equiv \mathcal{E}(A-1, Z-1) - \mathcal{E}(A, Z) + Q, \tag{B8} $$

with $Q \equiv (m_n + m_e - m)c^2$. Following the same line of arguments as those leading to Eq. (44), it can be easily seen that the core of the most massive white dwarfs will be unstable against electron captures accompanied by neutron emission if the magnetic field strength exceeds some threshold value, approximately given by

$$B_\star^{\beta n} \approx B_\star^{\beta} \left(\frac{\mu_e^{\beta n}}{\mu_e^{\beta}}\right)^2. \tag{B9}$$

This shows that the stability of the star will be limited by neutron emission rather than electron capture alone whenever $\mu_e^{\beta n} < \mu_e^{\beta}$. As shown in Table X, this situation only occurs for helium white dwarfs. In this case, $B_\star^{\beta n} \approx 867$ ($\rho \approx 1.1 \times 10^{11}$ g cm^{-3}) whereas $B_\star^{\beta} \approx 1003$ ($\rho \approx 1.3 \times 10^{11}$ g cm^{-3}).

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