

# Removing the gauge parameter dependence of the effective potential by a field redefinition

N. K. Nielsen\*

*Center of Cosmology and Particle Physics Phenomenology (CP3-Origins),  
University of Southern Denmark, DK 5230 Odense M, Denmark  
(Received 8 June 2014; published 27 August 2014)*

The gauge parameter dependence of the effective potential is determined by partial differential equations involving also the Higgs boson field expectation value. Solving these equations by the method of characteristics leads to elimination of the gauge parameter dependence of the effective potential. The construction is carried out in the case of the standard model of electroweak unification for the renormalization group improved effective potential up to the next-leading logarithmic order.

DOI: [10.1103/PhysRevD.90.036008](https://doi.org/10.1103/PhysRevD.90.036008)

PACS numbers: 11.15.-q, 12.15.-y, 12.15.Lk, 14.70.-e

## I. INTRODUCTION

With the discovery of the Higgs boson [1,2] the theoretical determination of its lower mass bound by means of the renormalization group method improved effective potential [3,4] has gained new prominence [5]. The stability analysis of [3–5] takes place in a specific gauge (the Landau gauge).

It was pointed out some time ago by Loinaz and Willey [6] that the gauge dependence of the effective potential could make the lower bound of the Higgs boson mass gauge dependent. They used an Abelian truncation of the standard model of electroweak unification, which in the bosonic sector is identical to the Abelian Higgs model, to make their point, working to one-loop and leading logarithmic order. A recent study by Andreassen [7] gives a very useful review of this difficulty. It is a problem of both phenomenological and conceptual importance, considering the fact that the observed Higgs mass is very close to the stability limit of the standard model of electroweak unification. It has been shown that the same problem arises in the context of Higgs inflation [8]. Quite recently the problem has been dealt with in connection with the so-called instability scale [9].

It will be shown in this paper that the gauge dependence of the effective potential can be removed by solving a set of homogeneous first order partial differential equation obeyed by the effective potential and involving the gauge parameters and the Higgs boson field expectation value. The validity of these equations was established many years ago by the author [10] and independently by Fukuda and Kugo [11] (see also [12–14]). Solving the differential equations leads to a field redefinition that eliminates the gauge dependence from the effective potential. The analysis is carried out in the context of the  $SU(2) \otimes U(1)$  standard model of electroweak unification for the renormalization

group improved effective potential at the leading and next-leading logarithmic order.

Suppressing in the effective potential  $V$  all variables except the Higgs field expectation value  $\phi$  and a gauge parameter  $\xi$ , the partial differential equations found in [10] and [11] have the form

$$\left( \xi \frac{\partial}{\partial \xi} + C(\phi, \xi) \frac{\partial}{\partial \phi} \right) V(\phi, \xi) = 0 \quad (1)$$

with  $C(\phi, \xi)$  a calculable function. Equation (1) can, as pointed out in [10] and [14], be solved by the method of characteristics. It is an obvious consequence of (1) that the potential  $V$  is unchanged if the gauge parameter  $\xi$  and the field  $\phi$  are subject to the following changes,

$$\xi \rightarrow \xi + \Delta\xi, \quad \phi \rightarrow \phi + C(\phi, \xi) \frac{\Delta\xi}{\xi} \quad (2)$$

with  $\Delta\xi$  infinitesimal. Taking this argument one step further, one can introduce a new field variable  $\Phi(\phi, \xi, \xi_0)$  that is a solution of the partial differential equation,

$$\left( \xi \frac{\partial}{\partial \xi} + C(\phi, \xi) \frac{\partial}{\partial \phi} \right) \Phi(\phi, \xi, \xi_0) = 0, \quad (3)$$

with the boundary condition,

$$\Phi(\phi, \xi, \xi_0)|_{\xi=\xi_0} = \phi, \quad (4)$$

with  $\xi_0$  a specific value of the gauge parameter  $\xi$ . Expressing the effective potential in terms of the new field variable one can now eliminate the dependence of the effective potential on the gauge parameter  $\xi$  altogether since  $\Phi(\phi, \xi, \xi_0)$  is invariant under (2) and thus is equal to the field variable at  $\xi = \xi_0$ , and (3) and (4) imply

$$V(\phi, \xi) = V(\Phi(\phi, \xi, \xi_0), \xi_0). \quad (5)$$

\*nkn@cp3-origins.net

What has been achieved by this construction is to show that the effective potential at an arbitrary value  $\xi$  of the gauge parameter is equal to the potential at a preferred value of the gauge parameter  $\xi_0$  by introducing a mapping through the solution of (3)–(4):

$$\phi \rightarrow \Phi(\phi, \xi, \xi_0). \quad (6)$$

The inverse mapping is obviously obtained by interchanging  $\xi$  and  $\xi_0$  in  $\Phi(\phi, \xi, \xi_0)$ . As mentioned above the stability analysis giving the Higgs boson mass lower bound [3–5] takes place in the Landau gauge, and the construction described above makes it possible to extend this analysis to arbitrary gauges; this is the problem raised in [6–9]. Our approach to this problem should be contrasted to that of [15] and [16], who suggested using instead a gauge-independent effective potential constructed either by a Hamiltonian method or by the Vilkovisky-DeWitt formalism [17], working in an Abelian model similar to the model used in [6] to leading logarithmic order.

The Higgs boson mass defined by the propagator pole is independent of the gauge parameters at all values of the quartic coupling  $\lambda$  entering the tree approximation potential term  $\frac{1}{4}\lambda\phi^4$ , provided one takes into account that the field value at the electroweak potential minimum is gauge parameter dependent; this statement was proven in [10] sec. IV and also verified in [12] (these proofs were carried out for the Abelian Higgs model but are easily extended to the standard model of electroweak unification). However, use of the gauge-dependent effective potential is necessary in order to determine the value of  $\lambda$  that leads to the lower bound on the Higgs mass such that the electroweak vacuum is stable. Extension of the analysis of this problem in [3–5] beyond the Landau gauge can be carried out by (6) or its inverse, which relate the field value at one set of values of the gauge parameters to the field value at another set of values of the gauge parameters, such that the value of the effective potential is unchanged.

In order to make (5) useful one has to provide a solution of (3)–(4). Usually the effective potential is found in the context of some approximation scheme, where it is obtained order by order. It is rather straightforward to find approximate solutions of (3)–(4) in the context of the loop expansion. In connection with the lower bound of the Higgs boson mass one is, however, dealing with the renormalization group improved effective potential, which involves a resummation of the effective potential at infinite loop order. Explicit solutions of (3)–(4) are, as mentioned above, obtained in this paper in the leading and next-leading logarithmic order for the standard model of electroweak unification, though there seems no reason why it should not be extended to arbitrary orders in this expansion. Only the bosonic part

of the standard model of electroweak unification is considered here since this is where complications involving gauge parameter dependence occurs; it is trivial to include also the fermionic part.

The outline of this paper is the following: Sec. II reviews ingredients necessary for the construction, where in Sec. II. A material on the standard model of electroweak unification is collected and in Sec. II. B the one-loop effective potential is briefly reviewed, while Sec. II. C deals with renormalization group improvement of the effective potential. Here the leading and next-leading approximations of the renormalization group solutions are defined, and it is noticed that different solutions of the renormalization group equation of the effective potential may have different orderings in this approximation scheme. In Sec. III the leading and next-leading logarithmic approximations of the effective potential are constructed, and (1) is verified and the relevant functions  $C(\phi, \xi)$  are determined. Finally in Sec. IV these functions are used to solve (3)–(4) and to check (5), such that the field redefinition alluded to in the title of the paper is achieved. An Appendix lists the renormalization group functions used.

## II. PRELIMINARIES

### A. $SU(2) \otimes U(1)$ electroweak theory

With  $\phi$  the expectation value of the Higgs boson field, the effective potential of the  $SU(2) \otimes U(1)$  standard model of electroweak unification is in the tree approximation:

$$V^{[0]}(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4, \quad \mu^2 < 0, \quad \lambda > 0. \quad (7)$$

The theory contains the gauge fields  $W_\mu^\pm$ ,  $W_\mu^3$  and  $B_\mu$  and in a renormalizable gauge the Goldstone boson fields  $\chi^\pm$  and  $\chi^3$ . A possible and rather general gauge fixing term of the Lagrangian is

$$\begin{aligned} \mathcal{L}_{\text{gf}} = & -\frac{1}{\xi} \left( \partial_\mu W^{+\mu} - \frac{gu}{2}\chi^+ \right) \left( \partial_\nu W^{-\nu} - \frac{gu}{2}\chi^- \right) \\ & - \frac{1}{2\xi} \left( \partial_\mu W^{3\mu} - \frac{gu}{2}\chi^3 \right)^2 - \frac{1}{2\xi'} \left( \partial_\mu B^\mu + \frac{g'u'}{2}\chi^3 \right)^2, \end{aligned} \quad (8)$$

with  $\xi, \xi', u$  and  $u'$  gauge parameters and where  $g$  and  $g'$  are the  $SU(2)$  and  $U(1)$  coupling constants. This leads to the following ghost Lagrangian in the bilinear approximation,

$$\begin{aligned} \mathcal{L}_{\text{FP}} = & -\bar{c}^+ \left( \partial - \frac{g^2 u \phi}{4} \right) c^+ - \bar{c}^- \left( \partial^2 - \frac{g^2 u \phi}{4} \right) c^- \\ & - \bar{c}^3 \partial^2 c^3 - \bar{c}' \partial^2 c' + \frac{\phi}{4} (gu\bar{c}^3 - g'u'\bar{c}') (gc^3 - g'c'), \end{aligned} \quad (9)$$

with  $c^+, \bar{c}^+, c^-, \bar{c}^-, c^3, \bar{c}^3, c'$  and  $\bar{c}'$  Faddeev-Popov ghost fields. The gauge parameters  $u$  and  $u'$  should be treated as independent variables in the context of the renormalization group since the gauge fixing Lagrangian is not renormalized [18] and the combinations  $gu\chi^\pm$  and  $gu\chi^3$  are thus renormalized by the same multiplicative renormalization as  $W_\mu^\pm$  and  $W_\mu^3$ , while  $g'u'\chi^3$  is renormalized like  $B_\mu$ . At general values of  $u$  and  $u'$  the vector and Goldstone boson fields mix in the bilinear approximation of the Lagrangian.

Consider first the charged vector fields. They are Stückelberg decomposed,

$$W_\mu^\pm = W_{\text{tr},\mu}^\pm + \frac{1}{\sqrt{\partial^2}} \partial_\mu \omega^\pm, \quad (10)$$

where

$$\partial_\mu W_{\text{tr}}^{\pm,\mu} = 0. \quad (11)$$

In momentum space with  $k$  a momentum variable this gives the following Lagrangian bilinear in the charged vector and Goldstone boson fields:

$$\begin{aligned} \mathcal{L} = & -W_{\text{tr}}^{+\mu}(-k) \left( k^2 - \frac{g^2 \phi^2}{4} \right) W_{\text{tr},\mu}^-(k) \\ & - (\chi^+(-k), \omega^+(-k)) \begin{pmatrix} a_{11}(k) & a_{12}(k) \\ a_{12}(k) & a_{22}(k) \end{pmatrix} \begin{pmatrix} \chi^- & (k) \\ \omega^- & (k) \end{pmatrix}, \end{aligned} \quad (12)$$

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \left( (W_{\text{tr},\mu}^3(-k), B_{\text{tr},\mu}(-k)) \begin{pmatrix} -k^2 + \frac{g^2 \phi^2}{4} & -\frac{gg' \phi^2}{4} \\ -\frac{gg' \phi^2}{4} & -k^2 + \frac{g'^2 \phi^2}{4} \end{pmatrix} \begin{pmatrix} W_{\text{tr}}^{3\mu}(k) \\ B_{\text{tr}}^\mu(k) \end{pmatrix} \right. \\ & \left. - (\chi^3(-k), \omega^3(-k), \omega'(-k)) \begin{pmatrix} b_{11}(k) & b_{12}(k) & b_{13}(k) \\ b_{12}(k) & b_{22}(k) & b_{23}(k) \\ b_{13}(k) & b_{23}(k) & b_{33}(k) \end{pmatrix} \begin{pmatrix} \chi^3(k) \\ \omega^3(k) \\ \omega'(k) \end{pmatrix} \right), \end{aligned} \quad (18)$$

with

$$b_{11}(k) = -k^2 + \mu^2 + \lambda \phi^2 + \frac{g^2 u^2}{4\xi} + \frac{g'^2 u'^2}{4\xi'} \quad (19)$$

$$b_{12}(k) = -\frac{g}{2} \left( \phi + \frac{u}{\xi} \right) \sqrt{-k^2}, \quad b_{13}(k) = \frac{g'}{2} \left( \phi + \frac{u'}{\xi'} \right) \sqrt{-k^2}, \quad (20)$$

$$b_{22}(k) = -\frac{k^2}{\xi} + \frac{g^2 \phi^2}{4}, \quad b_{33}(k) = -\frac{k^2}{\xi'} + \frac{g'^2 \phi^2}{4},$$

$$b_{23}(k) = -\frac{gg' \phi^2}{4} \quad (21)$$

where

$$\begin{aligned} a_{11}(k) &= -k^2 + \mu^2 + \lambda \phi^2 + \frac{g^2 u^2}{4\xi}, \\ a_{22}(k) &= -\frac{k^2}{\xi} + M_W^2, \quad a_{12}(k) = -\frac{g}{2} \left( \phi + \frac{u}{\xi} \right) \sqrt{-k^2} \end{aligned} \quad (13)$$

with  $M_W^2 = \frac{g^2 \phi^2}{4}$  and with the determinant

$$\text{Det} \begin{pmatrix} a_{11}(k) & a_{12}(k) \\ a_{12}(k) & a_{22}(k) \end{pmatrix} = \frac{1}{\xi} (k^2 - k_{+,W}^2)(k^2 - k_{-,W}^2), \quad (14)$$

where  $k_{\pm,W}$  are defined by

$$\begin{aligned} k_{\pm,W}^2 &= \frac{1}{2} \left( \mu^2 + \lambda \phi^2 - \frac{1}{2} g^2 u \phi \right) \\ &\quad \pm \frac{1}{2} \sqrt{(\mu^2 + \lambda \phi^2)(\mu^2 + \lambda \phi^2 - g^2 \phi(u + \xi \phi))}. \end{aligned} \quad (15)$$

The neutral vector fields are also Stückelberg decomposed,

$$W_\mu^3 = W_{\text{tr},\mu}^3 + \frac{1}{\sqrt{\partial^2}} \partial_\mu \omega^3, \quad B_\mu = B_{\text{tr},\mu} + \frac{1}{\sqrt{\partial^2}} \partial_\mu \omega', \quad (16)$$

where

$$\partial_\mu W_{\text{tr}}^{3,\mu} = \partial_\mu B_{\text{tr}}^\mu = 0, \quad (17)$$

leading to the quadratic Lagrangian in momentum space,

and

$$\begin{aligned} \text{Det} \begin{pmatrix} b_{11}(k) & b_{12}(k) & b_{13}(k) \\ b_{12}(k) & b_{22}(k) & b_{23}(k) \\ b_{13}(k) & b_{23}(k) & b_{33}(k) \end{pmatrix} \\ = -\frac{1}{\xi \xi'} k^2 (k^2 - k_{+,Z}^2)(k^2 - k_{-,Z}^2). \end{aligned} \quad (22)$$

Here was introduced

$$\begin{aligned} k_{\pm,Z}^2 &= \frac{1}{2} \left( \mu^2 + \lambda \phi^2 - \frac{1}{2} (g^2 + g'^2) u_Z \phi \right) \\ &\quad \pm \frac{1}{2} ((\mu^2 + \lambda \phi^2)(\mu^2 + \lambda \phi^2 - (g^2 + g'^2) \phi(u_Z + \xi_Z \phi)))^{\frac{1}{2}} \end{aligned} \quad (23)$$

cf. (15), with the definitions,

$$\xi_Z = \frac{\xi g^2 + \xi' g'^2}{g^2 + g'^2}, \quad u_Z = \frac{g^2 u + g'^2 u'}{g^2 + g'^2}. \quad (24)$$

### B. One-loop effective potential

In one-loop order the effective potential corresponding to the gauge fixing (8) can be seen from [4–9], using also (9), (12), (14), (18) and (22). It has in  $D = 4 - \epsilon$  dimensions the following contribution from the Higgs field,

$$V_H^{[1]} = -\frac{i}{2} \int \frac{d^D k}{(2\pi)^D} \log \frac{k^2 - \mu^2 - 3\lambda\phi^2 + i\epsilon}{k^2 + i\epsilon} \\ \simeq -\frac{9\lambda^2\phi^4}{64\pi^2} \left( \frac{2}{\epsilon} - \log \frac{3\lambda\phi^2}{M^2} + \frac{3}{2} \right), \quad (25)$$

where in the last step the asymptotic part at large values of  $\phi$  was kept. The contribution from the  $W^\pm$  field with associated Goldstone bosons and ghosts is by means of (12) and (14),

$$V_{W^\pm}^{[1]} = -i \int \frac{d^D k}{(2\pi)^D} \left( (D-1) \log \frac{k^2 - M_W^2 + i\epsilon}{k^2 + i\epsilon} + \log \frac{(k^2 - k_{+,W}^2 + i\epsilon)(k^2 - k_{-,W}^2 + i\epsilon)}{(k^2 + i\epsilon)^2} - 2 \log \frac{k^2 + \frac{g^2 u \phi}{4} + i\epsilon}{k^2 + i\epsilon} \right) \\ \simeq -\frac{3M_W^4}{32\pi^2} \left( \frac{2}{\epsilon} - \log \frac{M_W^2}{M^2} + \frac{5}{6} \right) - \frac{\lambda(\lambda - \frac{1}{2}\xi g^2)\phi^4}{32\pi^2} \left( \frac{2}{\epsilon} - \frac{1}{2} \log \frac{\frac{1}{4}\xi g^2 \lambda \phi^4}{M^4} + \frac{3}{2} \right) + \frac{\lambda\phi^4}{64\pi^2} \sqrt{\lambda(\lambda - \xi g^2)} \log \frac{\lambda + \sqrt{\lambda(\lambda - \xi g^2)}}{\lambda - \sqrt{\lambda(\lambda - \xi g^2)}}, \quad (26)$$

where in the last step again only the asymptotic part at large values of  $\phi$  was kept, with the gauge parameter  $u$  kept fixed. Also we get from the  $A^-$  and  $Z^0$  -fields with associated Goldstone boson and ghosts by (18) and (22),

$$V_{Z^0}^{[1]} = -\frac{i}{2} \int \frac{d^D k}{(2\pi)^D} \left( (D-1) \log \frac{k^2 - M_Z^2 + i\epsilon}{k^2 + i\epsilon} + \log \frac{(k^2 - k_{+,Z}^2 + i\epsilon)(k^2 - k_{-,Z}^2 + i\epsilon)}{(k^2 + i\epsilon)^2} - 2 \log \frac{k^2 + \frac{(g^2 + g'^2)u_Z \phi}{4} + i\epsilon}{k^2 + i\epsilon} \right) \\ \simeq -\frac{3M_Z^4}{64\pi^2} \left( \frac{2}{\epsilon} - \log \frac{M_Z^2}{M^2} + \frac{5}{6} \right) - \frac{\lambda(\lambda - \frac{1}{2}\xi_Z(g^2 + g'^2)\phi^4)}{64\pi^2} \left( \frac{2}{\epsilon} - \frac{1}{2} \log \frac{\frac{1}{4}\xi_Z(g^2 + g'^2)\lambda\phi^4}{M^4} + \frac{3}{2} \right) \\ + \frac{\lambda\phi^4}{128\pi^2} \sqrt{\lambda(\lambda - \xi_Z(g^2 + g'^2))} \log \frac{\lambda + \sqrt{\lambda(\lambda - \xi_Z(g^2 + g'^2))}}{\lambda - \sqrt{\lambda(\lambda - \xi_Z(g^2 + g'^2))}}, \quad (27)$$

where again only the asymptotic part, with  $\phi \gg u, u'$ , was kept, and with  $M_Z^2 = \frac{(g^2 + g'^2)\phi^2}{4}$ .

The infinite parts of (25), (26) and (27) are canceled by the standard one-loop counterterms, and collecting the asymptotic part one gets, with  $\beta_\lambda^{[1]}$  and  $\gamma_\phi^{[1]}$  one-loop renormalization group functions [see (A3)],

$$V_{\text{ren}}^{[1]} \simeq \frac{1}{4} (\beta_\lambda^{[1]} - 4\lambda\gamma_\phi^{[1]}) \phi^4 \log \frac{\phi}{M} + \frac{1}{4} \Delta\lambda\phi^4, \quad (28)$$

where

$$\Delta\lambda \simeq \frac{1}{16\pi^2} \left( \frac{3}{8} g^4 \left( \log \left( \frac{g^2}{4} \right) - \frac{5}{6} \right) + \frac{3}{16} (g^2 + g'^2)^2 \left( \log \left( \frac{g^2 + g'^2}{4} \right) - \frac{5}{6} \right) + 9\lambda^2 \left( \log(3\lambda) - \frac{3}{2} \right) \right) \\ + 2\lambda \left( \lambda - \frac{1}{2}\xi g^2 \right) \left( \frac{1}{2} \log \left( \frac{1}{4}\xi g^2 \lambda \right) - \frac{3}{2} \right) + \lambda \left( \lambda - \frac{1}{2}\xi_Z(g^2 + g'^2) \right) \left( \frac{1}{2} \log \left( \frac{1}{4}\xi_Z(g^2 + g'^2)\lambda \right) - \frac{3}{2} \right) \\ + \lambda \sqrt{\lambda(\lambda - \xi g^2)} \log \frac{\lambda + \sqrt{\lambda(\lambda - \xi g^2)}}{\lambda - \sqrt{\lambda(\lambda - \xi g^2)}} + \frac{\lambda}{2} \sqrt{\lambda(\lambda - \xi_Z(g^2 + g'^2))} \log \frac{\lambda + \sqrt{\lambda(\lambda - \xi_Z(g^2 + g'^2))}}{\lambda - \sqrt{\lambda(\lambda - \xi_Z(g^2 + g'^2))}}. \quad (29)$$

The important point concerning this expression is that it has a nontrivial dependence on the gauge parameters  $\xi$  and  $\xi'$ . Its value at  $\xi = \xi_0$  and  $\xi' = \xi'_0$ , with  $\xi_0$  and  $\xi'_0$  specific values of the gauge parameters, is denoted  $\Delta\lambda_0$ .

### C. Renormalization group improved effective potential

The effective potential is a solution of the renormalization group equation,

$$\left( M \frac{\partial}{\partial M} + \sum_i \beta_{g_i} \frac{\partial}{\partial g_i} - \gamma_\phi \phi \frac{\partial}{\partial \phi} \right) V(\phi, M, g_i) = 0, \quad (30)$$

where  $M$  denotes the renormalization scale and  $g_i$  are coupling constants and dimensionless gauge fixing parameters (we only consider the renormalization group asymptotically such that the dependence of the effective potential on  $\mu^2$  and the gauge parameters  $u, u'$  can be disregarded).

Ford, Jones, Stephenson and Einhorn [19] have found the following solution of this equation,

$$V(\phi, M, g_i) = V(\phi(t), M(t), g_i(t)), \quad (31)$$

with

$$\begin{aligned} \phi(t) &= \phi\eta(t), & \eta(t) &= \exp\left(-\int_0^t dt' \gamma_\phi(g_i(t'))\right), \\ M(t) &= e^t M, \end{aligned} \quad (32)$$

and

$$\frac{dg_i(t)}{dt} = \beta_i(g_j(t)). \quad (33)$$

In (31) one conveniently chooses  $\phi(t) = M(t)$ , which by (32) is the same as

$$t = \log \frac{\phi}{M} - \int_0^t dt' \gamma_\phi(t'). \quad (34)$$

This choice of  $t$  makes terms involving logarithmic factors,

$$\log \frac{\phi(t)}{M(t)},$$

disappear on the right-hand side of (31).

Another solution of (30) was obtained by Coleman and Weinberg [20]. From dimensional considerations it follows that

$$\left(M \frac{\partial}{\partial M} + \phi \frac{\partial}{\partial \phi}\right) V = 4V \quad (35)$$

which is combined with (30) and becomes by elimination of  $\phi$ ,

$$\left(M \frac{\partial}{\partial M} + \bar{\beta}_i \frac{\partial}{\partial g_i} - 4\bar{\gamma}_\phi\right) V(\phi, M, g_i) = 0, \quad (36)$$

with  $\bar{\beta}_i = \frac{\beta_i}{1+\gamma_\phi}$ ,  $\bar{\gamma}_\phi = \frac{\gamma_\phi}{1+\gamma_\phi}$  and with solution

$$V(\phi, M, g_i) = \bar{\eta}(\bar{t})^4 V(\phi, M(\bar{t}), \bar{g}_i(\bar{t})), \quad (37)$$

where  $\bar{\eta}(\bar{t})$  and  $\bar{g}_i(\bar{t})$  are defined by (32) and (33) with the  $\bar{\gamma}_\phi$  and  $\bar{\beta}_i$ , and with a new running variable  $\bar{t}$ . Here we take

$$\bar{t} = \log \frac{\phi}{M} \quad (38)$$

and thus  $M(\bar{t}) = \phi$ . This choice of  $\bar{t}$  makes terms involving logarithmic factors,

$$\log \frac{\phi}{M(\bar{t})},$$

disappear on the right-hand side of (37). The two solutions (31) and (37) of (30) are equivalent.

The renormalization group equations (33) are solved in the leading and next-leading logarithmic approximation, where for  $g_i^2 = (\lambda, g^2, g'^2)$ ,  $i = 1, 2, 3$ ,

$$(g_i^2)^{\{0\}}(t) = g_i^2 + \sum_{m_1+m_2+m_3=n>1} a_{m_1, m_2, m_3} g_1^{2m_1} g_2^{2m_2} g_3^{2m_3} t^{n-1}, \quad (39)$$

where the coefficients  $a_{m_1, m_2, m_3}$  are determined by (33) and the leading logarithmic approximation is indicated by a superscript  $\{0\}$ . In next-leading logarithmic order one gets, with the next-leading logarithmic approximation indicated by a superscript  $\{1\}$ , instead of (39),

$$(g_i^2)^{\{1\}}(t) = \sum_{m_1+m_2+m_3=n>1} b_{m_1, m_2, m_3} g_1^{2m_1} g_2^{2m_2} g_3^{2m_3} t^{n-2}, \quad (40)$$

with new coefficients  $b_{m_1, m_2, m_3}$ .

For the gauge parameter  $\xi$  one also has a renormalization group equation with a solution  $\xi(t)$  that also depends on  $\xi$ , and  $\xi(t)$  also has a leading and next-leading logarithmic approximation given by a power series like (39) and (40) (more details are given in the Appendix). In an expansion involving leading and next-leading logarithms one also has the running Higgs boson field expectation value,

$$\begin{aligned} \phi^{\{0\}}(t) &\simeq \phi \exp\left(-\int_0^t dt' \gamma_\phi^{\{0\}}(t')\right), \\ \phi^{\{1\}}(t) &\simeq -\int_0^t dt' \gamma_\phi^{\{1\}}(t') \phi^{\{0\}}(t), \end{aligned} \quad (41)$$

where  $\gamma_\phi$  is the anomalous dimension of the scalar field, and details on  $\gamma_\phi^{\{0\}}(t)$  and  $\gamma_\phi^{\{1\}}(t)$  can be found in the Appendix.

Equation (34) reduces to

$$t \simeq \log \frac{\phi}{M} \quad (42)$$

in the leading logarithmic approximation, and in the next-leading logarithmic approximation one gets

$$t \simeq \log \frac{\phi}{M} - \int_0^{\log \frac{\phi}{M}} dt' \gamma_\phi^{\{0\}}(t'). \quad (43)$$

Inserting here (38) one gets (43) in the form

$$t \simeq \bar{t} - \int_0^{\bar{t}} dt' \gamma_\phi^{\{0\}}(t'), \quad dt \simeq d\bar{t} (1 - \gamma^{\{0\}}(\bar{t})). \quad (44)$$

Combining (41) with (43) one finds in the next-leading logarithmic order

$$\begin{aligned} \phi^{\{0\}}(t) + \phi^{\{1\}}(t) &\simeq \left(1 - \int_0^{\log \frac{\phi}{M}} dt' \gamma_{\phi}^{\{1\}}(t') + \gamma^{\{0\}} \left(\log \frac{\phi}{M}\right)\right) \\ &\times \int_0^{\log \frac{\phi}{M}} dt' \gamma_{\phi}^{\{0\}}(t') \phi^{\{0\}} \left(\log \frac{\phi}{M}\right). \end{aligned} \quad (45)$$

This equation should be compared with the next-leading logarithmic approximation of the quantity  $\bar{\eta}(\bar{t})$  obtained from Coleman and Weinberg's solution (37):

$$\begin{aligned} \bar{\eta}^{\{0\}}(\bar{t}) + \bar{\eta}^{\{1\}}(\bar{t}) &\simeq \left(1 - \int_0^{\bar{t}} d\bar{t}' (\gamma^{\{1\}}(\bar{t}') - (\gamma^{\{0\}}(\bar{t}'))^2)\right) \\ &\times \exp\left(-\int_0^{\bar{t}} d\bar{t}' \gamma_{\phi}^{\{0\}}(\bar{t}')\right). \end{aligned} \quad (46)$$

The discrepancy between (45) and (46) is caused by the fact that the two integration variables  $t$  and  $\bar{t}$  are related by the transformation (44), which mixes different orders in the expansion in leading and next-leading logarithms, and it is removed by carrying out in (41) a change of integration variable by (44).

Similarly one gets the quartic coupling constant  $\lambda$  in the leading and next-leading logarithmic order,

$$\begin{aligned} \lambda^{\{0\}}(t) + \lambda^{\{1\}}(t) - \lambda &\simeq \int_0^t dt' (\beta_{\lambda}^{\{0\}}(t') + \beta_{\lambda}^{\{1\}}(t')) \\ &\simeq \int_0^{\log \frac{\phi}{M}} dt' (\beta_{\lambda}^{\{0\}}(t') + \beta_{\lambda}^{\{1\}}(t')) \\ &\quad - \beta_{\lambda}^{\{0\}} \left(\log \frac{\phi}{M}\right) \int_0^{\log \frac{\phi}{M}} dt' \gamma_{\phi}^{\{0\}}(t'), \end{aligned} \quad (47)$$

and

$$\begin{aligned} \bar{\lambda}^{\{0\}}(\bar{t}) + \bar{\lambda}^{\{1\}}(\bar{t}) - \lambda &\simeq \int_0^{\bar{t}} d\bar{t}' (\beta_{\lambda}^{\{0\}}(\bar{t}') + \beta_{\lambda}^{\{1\}}(\bar{t}')) \\ &\quad - \gamma_{\phi}^{\{0\}}(\bar{t}') \beta_{\lambda}^{\{0\}}(\bar{t}'), \end{aligned} \quad (48)$$

where (47) is converted into (48) through the transformation (44).

### III. LEADING AND NEXT-LEADING LOGARITHMIC APPROXIMATION OF THE EFFECTIVE POTENTIAL

The renormalization group improved potential is in the leading logarithmic approximation by (7),

$$V^{\{0\}} \simeq \frac{1}{4} \lambda^{\{0\}}(t) \phi^{\{0\}}(t)^4 \Big|_{t=\log \frac{\phi}{M}}, \quad (49)$$

where  $\lambda^{\{0\}}(t)$  and  $\phi^{\{0\}}(t)$  only include leading logarithms. Equation (49) follows immediately from both the renormalization group equation solutions (31) and (37). Since the

anomalous dimension, in contrast to the running coupling constant, is gauge parameter dependent, one gets from (49)

$$\begin{aligned} \xi \frac{\partial}{\partial \xi} V^{\{0\}} &\simeq -4 \int_0^t dt' \xi \frac{\partial \gamma_{\phi}^{\{0\}}(t')}{\partial \xi} \Big|_{t=\log \frac{\phi}{M}} V^{\{0\}}, \\ \xi' \frac{\partial}{\partial \xi'} V^{\{0\}} &\simeq -4 \int_0^t dt' \xi' \frac{\partial \gamma_{\phi}^{\{0\}}(t')}{\partial \xi'} \Big|_{t=\log \frac{\phi}{M}} V^{\{0\}}. \end{aligned} \quad (50)$$

These relations are consistent with the partial differential equations (1) in the context of the leading logarithms,

$$\xi \frac{\partial}{\partial \xi} V^{\{0\}} \simeq -C^{\{0\}} \frac{\partial}{\partial \phi} V^{\{0\}}, \quad \xi' \frac{\partial}{\partial \xi'} V^{\{0\}} \simeq -C'^{\{0\}} \frac{\partial}{\partial \phi} V^{\{0\}}, \quad (51)$$

with

$$\begin{aligned} C^{\{0\}} &= \phi \xi \frac{\partial}{\partial \xi} \int_0^{\log \frac{\phi}{M}} dt' \gamma_{\phi}^{\{0\}}(t'), \\ C'^{\{0\}} &= \phi \xi' \frac{\partial}{\partial \xi'} \int_0^{\log \frac{\phi}{M}} dt' \gamma_{\phi}^{\{0\}}(t') \end{aligned} \quad (52)$$

valid since the differentiation of the logarithms in (49) convert leading logarithmic terms into next-leading logarithmic terms that are neglected in this approximation.

Including next-leading logarithms one gets from (7) and (28) combined with (31) and using (43) the asymptotic effective potential

$$\begin{aligned} V^{\{0\}} + V^{\{1\}} &\simeq \left(\frac{1}{4} \lambda^{\{0\}}(t) \phi^{\{0\}}(t)^4 + \frac{1}{4} \lambda^{\{1\}}(t) \phi^{\{0\}}(t)^4\right. \\ &\quad + \lambda^{\{0\}}(t) \phi^{\{0\}}(t)^3 \phi^{\{1\}}(t) \\ &\quad \left. + \frac{1}{4} \Delta \lambda^{\{0\}}(t) \phi^{\{0\}}(t)^4\right) \Big|_{t=\log \frac{\phi}{M}} \int_0^{\log \frac{\phi}{M}} dt' \gamma_{\phi}^{\{0\}}(t'). \end{aligned} \quad (53)$$

Here the coupling constants as well as the field are taken only to the leading logarithmic approximation in the last term, but in the first three terms both leading and next-leading logarithms are included. This is because the expression  $\Delta \lambda$  as seen from (29) has two extra powers of  $g$  or  $g'$  or one extra power of  $\lambda$  in front and hence the leading logarithmic approximation of this term matches the next-leading logarithmic approximation of the renormalization group improved tree potential.

Carrying out an expansion of the first term on the right-hand side of (53), one obtains in this approximation

$$\begin{aligned} V^{\{1\}} &\simeq \left(\frac{1}{4} \lambda^{\{1\}}(t) \phi^{\{0\}}(t)^4 + \lambda^{\{0\}}(t) \phi^{\{0\}}(t)^3 \phi^{\{1\}}(t)\right. \\ &\quad - \frac{1}{4} (\beta_{\lambda}^{\{0\}}(t) - 4 \lambda^{\{0\}}(t) \gamma_{\phi}^{\{0\}}(t)) \phi^{\{0\}}(t)^4 \int_0^t dt' \gamma_{\phi}^{\{0\}}(t') \\ &\quad \left. + \frac{1}{4} \Delta \lambda^{\{0\}}(t) \phi^{\{0\}}(t)^4\right) \Big|_{t=\log \frac{\phi}{M}}. \end{aligned} \quad (54)$$

Using the solution (37) of the renormalization group equation one obtains a different result:

$$V^{\{1\}} \simeq \left( \frac{1}{4} \lambda^{\{1\}}(t) \phi^{\{0\}}(t)^4 + \lambda^{\{0\}}(t) \phi^{\{0\}}(t)^3 \phi^{\{1\}}(t) - \frac{1}{4} \int_0^t dt' (\beta_\lambda^{\{0\}}(t') - 4\lambda^{\{0\}}(t') \gamma_\phi^{\{0\}}(t')) \gamma_\phi^{\{0\}}(t') \phi^{\{0\}}(t)^4 + \frac{1}{4} \Delta \lambda^{\{0\}}(t) \phi^{\{0\}}(t)^4 \right) \Big|_{t=\log \frac{\phi}{M}}. \quad (55)$$

From (54) one gets, using (41),

$$\begin{aligned} \xi \frac{\partial}{\partial \xi} V^{\{1\}} &\simeq -4 \int_0^t dt' \xi \frac{\partial \gamma_\phi^{\{0\}}(t')}{\partial \xi} \Big|_{t=\log \frac{\phi}{M}} V^{\{1\}} \\ &+ \left( -4 \int_0^t dt' \xi \frac{\partial \gamma_\phi^{\{1\}}(t')}{\partial \xi} \right. \\ &+ 4\xi \frac{\partial \gamma_\phi^{\{0\}}(t)}{\partial \xi} \int_0^t dt' \gamma_\phi^{\{0\}}(t') \\ &- \int_0^t dt' \xi \frac{\partial \gamma_\phi^{\{0\}}(t')}{\partial \xi} \left( \beta_\lambda^{\{0\}}(t) - 4\gamma_\phi^{\{0\}}(t) \right) \\ &\left. + \xi \frac{\partial \Delta \lambda^{\{0\}}(t)}{\partial \xi} \lambda^{\{0\}}(t) \right) \Big|_{t=\log \frac{\phi}{M}} V^{\{0\}}. \end{aligned} \quad (56)$$

Equations (50) and (56) are combined with

$$\phi \frac{\partial}{\partial \phi} V^{\{0\}} \simeq 4V^{\{0\}} + \left( \frac{\beta_\lambda^{\{0\}}(t)}{\lambda^{\{0\}}(t)} - 4\gamma_\phi^{\{0\}}(t) \right) \Big|_{t=\log \frac{\phi}{M}} V^{\{0\}}, \quad (57)$$

correct to the next-leading logarithmic approximation. Thus (50) implies in this approximation

$$\begin{aligned} \xi \frac{\partial}{\partial \xi} V^{\{0\}} &\simeq -C^{\{0\}} \frac{\partial}{\partial \phi} V^{\{0\}} \\ &+ \int_0^t dt' \xi \frac{\partial \gamma_\phi^{\{0\}}(t')}{\partial \xi} \left( \frac{\beta_\lambda^{\{0\}}(t)}{\lambda^{\{0\}}(t)} - 4\gamma_\phi^{\{0\}}(t) \right) \Big|_{t=\log \frac{\phi}{M}} V^{\{0\}}, \end{aligned} \quad (58)$$

and adding (56) and (58), using again (57), one obtains

$$\begin{aligned} \xi \frac{\partial}{\partial \xi} (V^{\{0\}} + V^{\{1\}}) &\simeq -C^{\{0\}} \frac{\partial}{\partial \phi} (V^{\{0\}} + V^{\{1\}}) \\ &- C^{\{1\}} \frac{\partial}{\partial \phi} V^{\{0\}}, \end{aligned} \quad (59)$$

correct to the next-leading logarithmic approximation, with

$$\begin{aligned} C^{\{1\}} &= \phi \left( \int_0^t dt' \xi \frac{\partial \gamma_\phi^{\{1\}}(t')}{\partial \xi} - \xi \frac{\partial \gamma_\phi^{\{0\}}(t)}{\partial \xi} \int_0^t dt' \gamma_\phi^{\{0\}}(t') \right. \\ &\left. - \frac{1}{4} \xi \frac{\partial \Delta \lambda^{\{0\}}(t)}{\partial \xi} \lambda^{\{0\}}(t) \right) \Big|_{t=\log \frac{\phi}{M}}. \end{aligned} \quad (60)$$

A similar construction leads to

$$\begin{aligned} C'^{\{1\}} &= -\phi \left( \xi' \frac{\partial \gamma_\phi^{\{0\}}(t)}{\partial \xi'} \int_0^t dt' \gamma_\phi^{\{0\}}(t') \right. \\ &\left. + \frac{1}{4} \xi' \frac{\partial \Delta \lambda^{\{0\}}(t)}{\partial \xi'} \lambda^{\{0\}}(t) \right) \Big|_{t=\log \frac{\phi}{M}}, \end{aligned} \quad (61)$$

where it should be kept in mind that  $\gamma_\phi^{\{1\}}(t)$  does not depend on  $\xi'$  since the anomalous dimension is independent of  $\xi'$  at two-loop order and at one-loop order it only depends on  $\xi'$  in the renormalization group invariant combination  $\xi' g^2$  which only contributes to  $\gamma_\phi^{\{0\}}(t)$ .

Using (55) as the next-leading logarithmic approximation of the effective potential one gets instead of (59)

$$\begin{aligned} \xi \frac{\partial}{\partial \xi} (V^{\{0\}} + V^{\{1\}}) &\simeq -C^{\{0\}} \frac{\partial}{\partial \phi} (V^{\{0\}} + V^{\{1\}}) \\ &- (C^{\{1\}} + \Delta C^{\{1\}}) \frac{\partial}{\partial \phi} V^{\{0\}}, \end{aligned} \quad (62)$$

where

$$\begin{aligned} \Delta C^{\{1\}} &= \phi \xi \frac{\partial}{\partial \xi} \left( \int_0^t dt' \gamma_\phi^{\{0\}}(t') \left( \frac{1}{4} \frac{\beta_\lambda^{\{0\}}(t') - \beta_\lambda^{\{0\}}(t)}{\lambda^{\{0\}}(t)} \right. \right. \\ &\left. \left. - (\gamma_\phi^{\{0\}}(t') - \gamma_\phi^{\{0\}}(t)) \right) \right) \Big|_{t=\log \frac{\phi}{M}}, \end{aligned} \quad (63)$$

with similar equations involving the gauge parameter  $\xi'$ .

#### IV. FIELD REDEFINITION

All ingredients are now available to solve (3)–(4) in the context of the renormalization group improved effective potential in the leading and next-leading logarithmic approximation.

The leading logarithmic approximation of the effective potential is given by (49). Here a solution of (3)–(4) is

$$\Phi^{\{0\}} \simeq \phi \exp \left( - \int_0^{\log \frac{\phi}{M}} dt' (\gamma_\phi^{\{0\}}(t') - \gamma_{\phi,0}^{\{0\}}(t')) \right) \quad (64)$$

with

$$\xi \frac{\partial \Phi^{\{0\}}}{\partial \xi} \simeq -C^{\{0\}} \frac{\Phi^{\{0\}}}{\phi} \simeq -C^{\{0\}} \frac{\partial \Phi^{\{0\}}}{\partial \phi}, \quad (65)$$

where  $C^{\{0\}}$  is given in (52), and with a similar equation for  $\xi'$ , and where  $\gamma_{\phi,0}$  denotes the value of the anomalous dimension  $\gamma_\phi$  at the specific values  $\xi_0$  and  $\xi'_0$  of the gauge parameters. Here it was used that differentiation of the exponential of (42) with respect to  $\phi$  only gives a nonvanishing contribution in the next-leading logarithmic approximation.

The new running variable is

$$\tau \simeq \log \frac{\Phi^{\{0\}}}{M} \simeq \log \frac{\phi}{M} - \int_0^{\log \frac{\phi}{M}} dt' (\gamma_\phi^{\{0\}}(t') - \gamma_{\phi,0}^{\{0\}}(t')) \quad (66)$$

according to (42), using also (64). The second term of (66) is of next-leading logarithmic order, and thus

$$\tau \simeq t \quad (67)$$

at leading logarithmic order. Using the anomalous dimension  $\gamma_{\phi,0}$  corresponding to the gauge parameters  $\xi_0$  and  $\xi'_0$ , one next gets the new running field variable  $\Phi^{\{0\}}(t)$  from (41),

$$\Phi^{\{0\}}(t) = \Phi^{\{0\}} \exp\left(-\int_0^t dt' \gamma_{\phi,0}^{\{0\}}(t')\right) = \phi^{\{0\}}(t), \quad (68)$$

with  $t \simeq \log \frac{\phi}{M}$ , and with  $\phi^{\{0\}}(t)$  given in (41). Eliminating  $\phi$  and introducing instead  $\Phi^{\{0\}}$  in (49) one thus gets, correct to the leading logarithmic approximation,

$$\frac{1}{4} \lambda^{\{0\}}(t) \phi^{\{0\}}(t)^4 \Big|_{t=\log \frac{\phi}{M}} \simeq \frac{1}{4} \lambda^{\{0\}}(t) \Phi^{\{0\}}(t)^4 \Big|_{t=\log \frac{\Phi^{\{0\}}}{M}}, \quad (69)$$

and this establishes the invariance of the effective potential under a change of the gauge parameters at leading logarithmic order in agreement with (5).

At next-leading logarithmic order the solution of (3)–(4) is instead of (64) for the renormalization group solution (31),

$$\begin{aligned} \Phi^{\{0\}} + \Phi^{\{1\}} &\simeq \Phi^{\{0\}} \left( 1 + (\gamma_\phi^{\{0\}}(t) - \gamma_{\phi,0}^{\{0\}}(t)) \int_0^t dt' \gamma_\phi^{\{0\}}(t') \right. \\ &\quad \left. - \int_0^t dt' (\gamma_\phi^{\{1\}}(t') - \gamma_{\phi,0}^{\{1\}}(t')) \right. \\ &\quad \left. + \frac{1}{4} \frac{\Delta \lambda^{\{0\}}(t) - \Delta \lambda_0^{\{0\}}(t)}{\lambda^{\{0\}}(t)} \right) \Big|_{t=\log \frac{\phi}{M}}, \end{aligned} \quad (70)$$

since

$$\xi \frac{\partial}{\partial \xi} (\Phi^{\{0\}} + \Phi^{\{1\}}) \simeq -C^{\{0\}} \frac{\partial}{\partial \phi} (\Phi^{\{0\}} + \Phi^{\{1\}}) - C^{\{1\}} \frac{\partial}{\partial \phi} \Phi^{\{0\}}, \quad (71)$$

by (52) and (60), with a similar equation for  $\xi'$ , and with

$$\phi \frac{\partial}{\partial \phi} (\Phi^{\{0\}} + \Phi^{\{1\}}) \simeq (1 - (\gamma_\phi^{\{0\}}(t) - \gamma_{\phi,0}^{\{0\}}(t))) \Phi^{\{0\}} + \Phi^{\{1\}} \quad (72)$$

by (64) and (70), correct at next-leading logarithmic order.

It is next verified that the effective potential in the next-leading logarithmic approximation (54) is obtained also from the new field (70). First it is shown that the new running variable still obeys (67). From (64) and (70) it follows that

$$\begin{aligned} \log \frac{\Phi^{\{0\}} + \Phi^{\{1\}}}{M} &\simeq \log \frac{\phi}{M} + \left( (\gamma_\phi^{\{0\}}(t) - \gamma_{\phi,0}^{\{0\}}(t)) \int_0^t dt' \gamma_\phi^{\{0\}}(t') \right. \\ &\quad \left. - \int_0^t dt' (\gamma_\phi^{\{0\}}(t') + \gamma_\phi^{\{1\}}(t') \right. \\ &\quad \left. - \gamma_{\phi,0}^{\{0\}}(t') - \gamma_{\phi,0}^{\{1\}}(t')) \right. \\ &\quad \left. + \frac{1}{4} \frac{\Delta \lambda^{\{0\}}(t) - \Delta \lambda_0^{\{0\}}(t)}{\lambda^{\{0\}}(t)} \right) \Big|_{t=\log \frac{\phi}{M}}, \end{aligned} \quad (73)$$

correct in the next-leading logarithmic approximation, and thus the new running variable  $\tau$  replacing  $t$  is by (43),

$$\begin{aligned} \tau &\simeq \log \frac{\Phi^{\{0\}} + \Phi^{\{1\}}}{M} - \int_0^{\log \frac{\Phi^{\{0\}} + \Phi^{\{1\}}}{M}} dt' \gamma_{\phi,0}^{\{0\}}(t') \\ &\simeq t - \int_{\log \frac{\phi}{M}}^{\log \frac{\Phi^{\{0\}} + \Phi^{\{1\}}}{M}} dt' \gamma_{\phi,0}^{\{0\}}(t') \\ &\quad + \left( (\gamma_\phi^{\{0\}}(t) - \gamma_{\phi,0}^{\{0\}}(t)) \int_0^t dt' \gamma_\phi^{\{0\}}(t') \right. \\ &\quad \left. - \int_0^t dt' (\gamma_\phi^{\{1\}}(t') - \gamma_{\phi,0}^{\{1\}}(t')) \right. \\ &\quad \left. + \frac{1}{4} \frac{\Delta \lambda^{\{0\}}(t) - \Delta \lambda_0^{\{0\}}(t)}{\lambda^{\{0\}}(t)} \right) \Big|_{t=\log \frac{\phi}{M}}, \end{aligned} \quad (74)$$

with the quantity  $t$  given by (43), and the other terms on the right-hand side are only nonvanishing at next-next-leading logarithmic order, and so (67) holds true also at next-leading logarithmic order.

The running field variable is thus instead of (68), keeping in mind that the anomalous dimension  $\gamma_{\phi,0}$  should be used,

$$\begin{aligned} (\Phi^{\{0\}} + \Phi^{\{1\}})(t) &= (\Phi^{\{0\}} + \Phi^{\{1\}}) \\ &\quad \times \exp\left(-\int_0^t dt' (\gamma_{\phi,0}^{\{0\}}(t') + \gamma_{\phi,0}^{\{1\}}(t'))\right), \end{aligned} \quad (75)$$

where  $t$  is given by (43), with



$$\exp\left(-\int_0^t dt' (\gamma_{\phi,0}^{\{0\}}(t') + \gamma_{\phi,0}^{\{1\}}(t'))\right)\Big|_{t=\log\frac{\phi}{M}}^{\int_0^{\log\frac{\phi}{M}} dt' \gamma_{\phi}^{\{0\}}(t')} \simeq \exp\left(-\int_0^t dt' \gamma_{\phi,0}^{\{0\}}(t')\right) \left(1 + \gamma_{\phi,0}^{\{0\}}(t) \int_0^t dt' \gamma_{\phi}^{\{0\}}(t') - \int_0^t dt' \gamma_{\phi,0}^{\{1\}}(t')\right)\Big|_{t=\log\frac{\phi}{M}}. \quad (76)$$

Inserting (64), (70) and (76) into (75) one then finds

$$(\Phi^{\{0\}} + \Phi^{\{1\}})(t) \simeq \phi^{\{0\}}(t) \left(1 + \frac{1}{4} \frac{\Delta\lambda^{\{0\}}(t) - \Delta\lambda_0^{\{0\}}(t)}{\lambda^{\{0\}}(t)}\right) + \phi^{\{1\}}(t), \quad (77)$$

with  $t$  again given by (43), and using (77), (53) can be tested for invariance under a change of the gauge parameters,

$$\frac{1}{4}(\lambda^{\{0\}} + \lambda^{\{1\}})(t)(\Phi^{\{0\}} + \Phi^{\{1\}})(t)^4 + \frac{1}{4}\Delta\lambda_0^{\{0\}}(t)\Phi^{\{0\}}(t)^4 \simeq \frac{1}{4}(\lambda^{\{0\}} + \lambda^{\{1\}})(t)(\phi^{\{0\}} + \phi^{\{1\}})(t)^4 + \frac{1}{4}\Delta\lambda^{\{0\}}(t)\phi^{\{0\}}(t)^4, \quad (78)$$

which completes the proof that the renormalization group improved effective potential is gauge parameter independent in the next-leading logarithmic approximation in the context of the renormalization group solution (31), again in agreement with (5).

Using (55) as the next-leading approximation of the effective potential the solution of (3)–(4) contains at next-leading logarithmic order in addition to (70), as seen from (63),

$$\begin{aligned} \Delta\Phi^{\{1\}} \simeq & -\Phi^{\{0\}} \left( \int_0^t dt' \gamma_{\phi}^{\{0\}}(t') \left( \frac{1}{4} \frac{\beta_{\lambda}^{\{0\}}(t') - \beta_{\lambda}^{\{0\}}(t)}{\lambda^{\{0\}}(t)} - (\gamma_{\phi}^{\{0\}}(t') - \gamma_{\phi}^{\{0\}}(t)) \right) \right. \\ & \left. - \int_0^t dt' \gamma_{\phi,0}^{\{0\}}(t') \left( \frac{1}{4} \frac{\beta_{\lambda}^{\{0\}}(t') - \beta_{\lambda}^{\{0\}}(t)}{\lambda^{\{0\}}(t)} - (\gamma_{\phi,0}^{\{0\}}(t') - \gamma_{\phi,0}^{\{0\}}(t)) \right) \right) \Big|_{t=\log\frac{\phi}{M}} \end{aligned} \quad (79)$$

with

$$\xi \frac{\partial}{\partial \xi} (\Phi^{\{0\}} + \Phi^{\{1\}} + \Delta\Phi^{\{1\}}) \simeq -C^{\{0\}} \frac{\partial}{\partial \phi} (\Phi^{\{0\}} + \Phi^{\{1\}} + \Delta\Phi^{\{1\}}) - (C^{\{1\}} + \Delta C^{\{1\}}) \frac{\partial}{\partial \phi} \Phi^{\{0\}}, \quad (80)$$

using again (72) and again with a similar equation for  $\xi'$ .

The running variable is in this case by (38),

$$\bar{\tau} = \log \frac{\Phi^{\{0\}} + \Phi^{\{1\}} + \Delta\Phi^{\{1\}}}{M}, \quad (81)$$

which in the next-leading logarithmic approximation reduces to (66), and thus

$$\lambda^{\{0\}}(\bar{\tau}) \simeq \left( \lambda^{\{0\}}(t) - \beta_{\lambda}^{\{0\}}(t) \int_0^t dt' (\gamma_{\phi}^{\{0\}}(t') - \gamma_{\phi,0}^{\{0\}}(t')) \right) \Big|_{t=\log\frac{\phi}{M}}. \quad (82)$$

For  $\Phi^{\{1\}}(\bar{\tau})$  and  $\Delta\Phi^{\{1\}}(\bar{\tau})$  it is sufficient to take  $\bar{\tau} \simeq \log\frac{\phi}{M}$  at next-leading logarithmic order. One thus finds by (41), (66), (68), (70) and (79) the running field variable in this case,

$$\begin{aligned} (\Phi^{\{0\}} + \Phi^{\{1\}} + \Delta\Phi^{\{1\}})(\bar{\tau}) &= (\Phi^{\{0\}} + \Phi^{\{1\}} + \Delta\Phi^{\{1\}}) \exp\left(-\int_0^{\bar{\tau}} dt' (\gamma_{\phi,0}^{\{0\}}(t') + \gamma_{\phi,0}^{\{1\}}(t'))\right) \\ &\simeq \left( \phi^{\{1\}}(t) + \phi^{\{0\}}(t) \left(1 + \int_0^t dt' ((\gamma_{\phi}^{\{0\}}(t'))^2 - (\gamma_{\phi,0}^{\{0\}}(t'))^2) + \frac{1}{4} \frac{\Delta\lambda^{\{0\}}(t) - \Delta\lambda_0^{\{0\}}(t)}{\lambda^{\{0\}}(t)} \right. \right. \\ &\quad \left. \left. - \frac{1}{4} \int_0^t dt' (\gamma_{\phi}^{\{0\}}(t') - \gamma_{\phi,0}^{\{0\}}(t')) \frac{\beta_{\lambda}^{\{0\}}(t') - \beta_{\lambda}^{\{0\}}(t)}{\lambda^{\{0\}}(t)} \right) \right) \Big|_{t=\log\frac{\phi}{M}}, \end{aligned} \quad (83)$$

and therefore, using again (68),

$$\begin{aligned}
& \frac{1}{4}\lambda^{\{0\}}(\bar{t})\Phi^{\{0\}}(\bar{t})^4 + \frac{1}{4}\lambda^{\{1\}}(t)\Phi^{\{0\}}(t)^4 \\
& + \lambda^{\{0\}}(t)\Phi^{\{0\}}(t)^3(\Phi^{\{1\}} + \Delta\Phi^{\{1\}})(t) \\
& - \frac{1}{4}\int_0^t dt'(\beta_\lambda^{\{0\}}(t') - 4\lambda^{\{0\}}(t)\gamma_{\phi,0}^{\{0\}}(t'))\gamma_{\phi,0}^{\{0\}}(t')\Phi^{\{0\}}(t)^4 + \frac{1}{4}\Delta\lambda_0^{\{0\}}(t)\Phi^{\{0\}}(t)^4 \\
& \simeq \frac{1}{4}\lambda^{\{0\}}(t)\phi^{\{0\}}(t)^4 + \frac{1}{4}\lambda^{\{1\}}(t)\phi^{\{0\}}(t)^4 + \lambda^{\{0\}}(t)\phi^{\{0\}}(t)^3\phi^{\{1\}}(t) \\
& - \frac{1}{4}\int_0^t dt'(\beta_\lambda^{\{0\}}(t') - 4\lambda^{\{0\}}(t)\gamma_\phi^{\{0\}}(t'))\gamma_\phi^{\{0\}}(t')\phi^{\{0\}}(t)^4 + \frac{1}{4}\Delta\lambda^{\{0\}}(t)\phi^{\{0\}}(t)^4, \tag{84}
\end{aligned}$$

valid in next-leading logarithmic order, with  $t = \log \frac{\phi}{M}$  and  $\bar{t}$  given by (81). Eq. (84) verifies (5) for this case also. This proves according to (55) that the effective potential is unchanged under gauge parameter changes for the renormalization group solution (37) in the next-leading logarithmic order as well.

## V. CONCLUSION AND COMMENTS

It has been demonstrated above that Eq. (1) allows a redefinition of the field variable of the effective potential in the  $SU(2) \otimes U(1)$  model of electroweak unification in the leading and next-leading logarithmic approximation for both of the renormalization group equation solutions (31) and (37). As a result of the redefinition, the arbitrary gauge parameters  $(\xi, \xi')$  are eliminated in terms of fixed gauge parameters  $(\xi_0, \xi'_0)$  of some preferred gauge such as the Landau gauge. The redefinition is achieved by solving (3)–(4) to the required accuracy, and the solutions are (64) in the leading logarithmic approximation and (70) and (79) in the next-leading logarithmic approximation. Remarkably, the construction in the leading logarithmic approximation only involves the anomalous dimension of the Higgs boson field, while in the next-leading logarithmic approximation the quantity  $\Delta\lambda$  defined in (29) plays a crucial role.

As mentioned in the Introduction, (64), (70) and (79) define mappings of the field variable  $\phi$ , and the inverse mappings are obtained by interchanging  $(\xi, \xi')$  and  $(\xi_0, \xi'_0)$  in (64), (70) and (79); this is also easily shown directly and represents a consistency check on the solutions of (3) and (4) represented by (64), (70) and (79). This observation makes it possible to obtain the field variable in an arbitrary gauge from the field variable in the Landau gauge as used in [3–5] such that the effective potential is invariant: one substitutes instead of the field variable  $\phi$  the solutions (64), (70) and (79) determined above, with the replacements  $(\xi, \xi') \rightarrow (0, 0)$  and  $(\xi_0, \xi'_0) \rightarrow (\xi, \xi')$ , since one is now dealing with the inverse mapping of the field variable. The gauge parameter dependence of the instability scale studied numerically in [9] can be found analytically this way.

The conditions (34) and (38) were imposed on the renormalization group running variables in order to obtain well-defined approximation schemes. They may not be the

optimal choices of running variables from the point of view of numerical accuracy. Also it was pointed out that the two solutions (31) and (37), though equivalent, have different expansions in this approximation scheme, and thus one may give more precise estimates than the other one. Similarly, different gauge choices may lead to different degrees of accuracy, though they are formally equivalent.

## ACKNOWLEDGMENTS

I wish to thank Dr. A. Fettouhi and Dr. K. Boye for taking part in the early stages of the investigation presented here. Also I am grateful to Professor Peter van Nieuwenhuizen at the C. N. Yang Institute of Theoretical Physics, State University of New York at Stony Brook, where this work was initiated some years ago, for his hospitality. Finally I would like to express my gratitude to Professor J.-M. Frère, Professor P. M. Lavrov and Professor O. M. Del Cima for drawing my attention to their work and to the referee for his questions and comments.

## APPENDIX: RENORMALIZATION GROUP FUNCTIONS

$g_i$  represents the couplings of the theory (including the dimensionless gauge fixing parameters  $\xi, \xi'$ ) that obey the renormalization group equations (33). Solving these equations with the one-loop  $\beta$  functions leads to the leading logarithmic approximation of the running coupling constants, and including also two-loop  $\beta$  functions one obtains the next-leading logarithmic approximation. The  $\beta$  functions for coupling constants are gauge parameter independent, and up to two-loop order they can be seen from [4,5,7].

Anomalous dimensions are, in contrast to coupling constant  $\beta$  functions, dependent on the dimensionless gauge fixing parameters. The anomalous dimension of the  $W$  field in the electroweak theory is at one-loop order, keeping only bosonic contributions,

$$\gamma_W^{[1]} \simeq -\frac{1}{16\pi^2} \left( \frac{25}{6} - \xi \right) g^2, \tag{A1}$$

and the function  $\beta_\xi$  for the gauge parameter  $\xi$  is in general given by

$$\beta_\xi = -2\xi\gamma_W. \quad (\text{A2})$$

Solving (33) for the gauge parameter  $\xi$  with the  $\beta$  function (A2) one gets, using the one-loop value of  $\beta_\xi$ , the running gauge parameter in the leading logarithmic approximation. The quantity  $\xi'g^2$  is renormalization group invariant. The anomalous dimension of the scalar field is at one-loop order, disregarding fermions,

$$\gamma_\phi^{[1]} \simeq -\frac{1}{16\pi^2} \left( \frac{3}{4}(3g^2 + g'^2) - \frac{3}{4}\xi g^2 - \frac{1}{4}\xi' g^2 \right). \quad (\text{A3})$$

Inserting here the leading logarithmic approximations of the coupling constants and gauge fixing parameters one obtains the leading logarithmic approximation of the anomalous dimension, the function  $\gamma_\phi^{\{0\}}(t)$  used in the text.

The anomalous dimension of the  $W$  field is at two-loop order, keeping only terms involving the coupling constant  $g$  (gauge parameter dependence only occurs in these terms),

$$\gamma_W^{[2]} \simeq \left( \frac{1}{16\pi^2} \right)^2 \left( -\frac{231}{8} + \frac{11}{2}\xi + \xi^2 \right) g^4, \quad (\text{A4})$$

where the value in pure  $SU(2)$  Yang-Mills theory is found from [21] and the scalar field contribution from [22].  $\beta_\xi$  is determined by (A2) also at two-loop order, and solving (33) in this case one obtains the running gauge parameter in the next-leading logarithmic approximation. The two-loop anomalous dimension of the scalar field  $\gamma_\phi^{[2]}$  is also found from [22]. The part proportional to  $g^4$  is, neglecting again the fermion contribution,

$$\gamma_\phi^{[2]} \simeq \left( \frac{1}{16\pi^2} \right)^2 \left( -\frac{511}{32} + 3\xi + \frac{3}{8}\xi^2 \right) g^4. \quad (\text{A5})$$

$\gamma_\phi^{[2]}$  has no dependence on  $\xi'$ . Inserting  $g^2$  and  $\xi$  in leading logarithmic order into (A5) and in next-leading logarithmic order into (A3), the sum is the function  $\gamma_\phi^{\{1\}}(t)$  (the result is only complete in the gauge parameter dependent part).

- 
- [1] ATLAS Collaboration, G. Aad *et al.*, *Phys. Lett. B* **716**, 1 (2012).
- [2] CMS Collaboration, S. Chatrchyan *et al.*, *Phys. Lett. B* **716**, 30 (2012).
- [3] N. Cabibbo, L. Maiani, G. Parisi, and R. Petronzio, *Nucl. Phys.* **B158**, 295(1979).
- [4] M. Sher, *Phys. Rep.* **179**, 273 (1989); M. Lindner, M. Sher, and H. W. Zaglauer, *Phys. Lett. B* **228**, 139 (1989); P. B. Arnold, *Phys. Rev. D* **40**, 613 (1989); M. Sher, *Phys. Lett. B* **317**, 159 (1993); J. A. Casas, J. R. Espinosa, and M. Quirós, *Phys. Lett. B* **342**, 171 (1995); **382**, 374 (1996); *Nucl. Phys.* **B436**, 3 (1995).
- [5] M. Holthausen, K. S. Lim, and M. Lindner, *J. High Energy Phys.* **02** (2012) 037; J. Elias-Miró, J. R. Espinosa, G. F. Giudice, G. Isidori, A. Riotto, and A. Strumia, *Phys. Lett. B* **709**, 222 (2012); F. Bezrukov, M. Y. Kalmykov, B. A. Kniehl, and M. Shaposhnikov, *J. High Energy Phys.* **10** (2012) 140; G. Degrassi, S. Di Vita, J. Elias-Miró, J. R. Espinosa, G. F. Giudice, G. Isidori, and A. Strumia, *J. High Energy Phys.* **08** (2012) 098; S. Alekhin, A. Djouadi, and S. Moch, *Phys. Lett. B* **716**, 214 (2012); I. Masina, *Phys. Rev. D* **87**, 053001 (2013); O. Antipin, M. Gillioz, J. Krog, E. Mølgaard, and F. Sannino, *J. High Energy Phys.* **08** (2013) 034; D. Buttazzo, G. Degrassi, P. P. Giardino, G. F. Giudice, F. Sala, A. Salvio, and A. Strumia, *J. High Energy Phys.* **12** (2013) 089; V. Branchina and E. Messina, *Phys. Rev. Lett.* **111**, 241801 (2013).
- [6] W. Loinaz and R. S. Willey, *Phys. Rev. D* **56**, 7416 (1997).
- [7] A. J. Andreassen, master's thesis, NTNU-Trondheim, 2013.
- [8] J. L. Cook, L. M. Krauss, A. J. Long, and S. Sabharwal, *Phys. Rev. D* **89**, 103525 (2014).
- [9] L. Di Luzio and L. Mihaila, *J. High Energy Phys.* **06** (2014) 079.
- [10] N. K. Nielsen, *Nucl. Phys.* **B101**, 173 (1975).
- [11] R. Fukuda and T. Kugo, *Phys. Rev. D* **13**, 3469 (1976).
- [12] I. J. R. Aitchison and C. M. Fraser, *Ann. Phys. (N.Y.)* **156**, 1 (1984); D. Johnston, *Nucl. Phys.* **B253**, 687 (1985).
- [13] W. Fischler and R. Brout, *Phys. Rev. D* **11**, 905 (1975); J.-M. Frère and P. Nicoletopoulos, *Phys. Rev. D* **11**, 2332 (1975); H. Kluberg-Stern and J. B. Zuber, *Phys. Rev. D* **12**, 467 (1975); **12**, 482 (1975); W. Konetschny and W. Kummer, *Nucl. Phys.* **B100**, 106 (1975); O. Piguet and K. Sibold, *Nucl. Phys.* **B253**, 517 (1985); O. M. Del Cima, *Phys. Lett. B* **457**, 307 (1999); A. Lewandowski, arXiv:1307.4055.
- [14] P. M. Lavrov and I. V. Tyutin, *Sov. J. Nucl. Phys.* **34**, 156 (1981); **34**, 474 (1981); O. M. Del Cima, D. H. T. Franco, and O. Piguet, *Nucl. Phys.* **B551**, 813 (1999).
- [15] D. Boyanovski, W. Loinaz, and R. S. Willey, *Phys. Rev. D* **57**, 100 (1998).
- [16] G.-L. Lin and T.-K. Chyi, *Phys. Rev. D* **60**, 016002 (1999).
- [17] G. Vilkovisky, *Nucl. Phys.* **B234**, 125 (1984); B. S. DeWitt, in *Quantum Field Theory and Quantum Statistics: Essays in Honour of the 60th Birthday of E. S. Fradkin*, edited by I. A. Batalin, C. J. Isham, and G. A. Vilkovisky (Hilger, Bristol, 1987), p. 191.
- [18] J. Zinn-Justin, in *Trends in Elementary Particle Theory, International Summer Institute on Theoretical Physics in Bonn 1974*, Lecture Notes in Physics (Springer-Verlag, Berlin, 1975), Vol. 37.
- [19] C. Ford, D. R. T. Jones, P. W. Stephenson, and M. Einhorn, *Nucl. Phys.* **B395**, 17 (1993).
- [20] S. Coleman and E. Weinberg, *Phys. Rev. D* **7**, 1888 (1973).
- [21] O. V. Tarasov and A. A. Vladimirov, *Sov. J. Nucl. Phys.* **25**, 585 (1977).
- [22] M. E. Machacek and M. T. Vaughn, *Nucl. Phys.* **B222**, 83 (1983).