Precision electroweak analysis after the Higgs boson discovery

James D. Wells and Zhengkang Zhang

Michigan Center for Theoretical Physics (MCTP), Department of Physics, University of Michigan, Ann Arbor, Michigan 48109, USA

(Received 7 July 2014; published 7 August 2014)

Until recently precision electroweak computations were fundamentally uncertain due to lack of knowledge about the existence of the Standard Model Higgs boson and its mass. For this reason substantial calculational machinery had to be carried along for each calculation that changed the Higgs boson mass and other parameters of the Standard Model. Now that the Higgs boson is discovered and its mass is known to within a percent, we are able to compute reliable semianalytic expansions of electroweak observables. We present results of those computations in the form of expansion formulas. In addition to the convenience of having these expressions, we show how the approach makes investigating new physics contributions to precision electroweak observables much easier.

DOI: 10.1103/PhysRevD.90.033006 PACS numbers: 12.15.-y, 13.66.Jn, 12.60.-i, 13.38.Dg

I. INTRODUCTION

Precision electroweak analysis has played an important role in testing the Standard Model (SM) and constraining new physics. Now this program has entered a new era with the discovery of the Higgs boson [1,2]. On one hand, the subpercentage-level determination of the Higgs boson mass [1–3] constitutes the last piece of a complete set of input observables. Electroweak observables can now be calculated to unprecedented accuracy, leading to unprecedented sensitivity to new physics beyond the SM. On the other hand, measurements of the Higgs observables, such as its decay widths and branching ratios, will push our understanding of elementary particle physics to more stringent tests. In this paper we focus on the former aspect. For the latter aspect, see e.g. [4].

The standard approach of precision electroweak analysis is to perform a χ^2 analysis, which involves varying the model parameters, or equivalently, a set of input observables to minimize the χ^2 function. In practice, this can be facilitated by an expansion about some reference values of the input, since we have a set of well-measured input observables that allows little variation. We present such an expansion formalism, and apply it to deriving constraints on new physics models. Most of the numerical results in this paper reflect state-of-the-art calculations of the electroweak observables, as implemented in the ZFITTER package [5,6].

Our paper is organized as follows. We first review the definition of the electroweak observables under consideration in Sec. II. Then in Sec. III we present the expansion formalism for calculating the SM and new physics contributions to the observables. The result will be that given the values of six input observables, and the new physics model, all observables can be easily calculated. The tools needed in this calculation, including the reference values of all observables, and the expansion coefficients, are presented. Next, we illustrate how to use the formalism by working out some new physics examples in Sec. IV. Finally, in Sec. V we summarize.

II. STANDARD MODEL PARAMETERS AND OBSERVABLES

The parameters of the SM include the gauge couplings g_3, g_2, g_1, the Yukawa couplings y_f, flavor angles, the Higgs vacuum expectation value v and self-coupling λ. For the purpose of precision electroweak analysis, with inconsequential errors we can treat all Yukawa couplings except that for the top quark as constants, and correspondingly set the lepton and light quark masses to their default values in ZFITTER (see [5]). Then there are six parameters[1] in the theory:

$$\{g_3, g_2, g_1, y_t, v, \lambda\}. \tag{1}$$

There are an infinite number of SM observables that can be defined. They correspond to well-defined quantities that are measured in experiments. The SM predicts each observable as a function of the parameters in Eq. (1). The success of the SM relies on the fact that the predictions for all observables agree with precision measurements, with suitable choices of the parameters. If some new physics beyond the SM were to exist, it could potentially destroy the agreement. Thus, precision analysis enables us to put stringent constraints on new physics models. In this paper we focus on the following list of observables, mostly relevant to precision tests of the electroweak theory.

(i) Pole mass of the particles: m_Z, m_W, m_t, m_H.

(ii) Observables associated with the strengths of the strong, weak, and electromagnetic interactions:

[1] We do not include flavor Cabibbo-Kobayashi-Maskawa angles in our calculations since all standard precision electroweak observables do not substantively depend on these angles.

$\alpha_s(m_Z)$, G_F, and $\alpha(m_Z)$. The Fermi constant G_F is defined via the muon lifetime [7]. $\alpha(m_Z)$ is related to the fine structure constant α_0 defined in the Thomson limit via

$$\alpha(m_Z) = \frac{\alpha_0}{1 - \Delta\alpha_\ell - \Delta\alpha_t - \Delta\alpha_{\rm had}^{(5)}}. \tag{2}$$

We treat $\alpha_0 = 1/137.035999074(44)$ [7,8] as a constant, since it is extraordinarily well measured. The contribution from leptons $\Delta\alpha_\ell$ and the top quark $\Delta\alpha_t$ are perturbatively calculable and known very accurately, so the uncertainty in $\alpha(m_Z)$ essentially comes from the incalculable light hadron contribution $\Delta\alpha_{\rm had}^{(5)}$, which is extracted from low energy $e^+e^- \to$ hadrons data via dispersion relations [7]. For simplicity, we will occasionally (especially in subscripts) drop the scale "(m_Z)" in $\alpha_s(m_Z)$ and $\alpha(m_Z)$, and write $\Delta\alpha_{\rm had}^{(5)}$ as $\Delta\alpha$ in the following.

(iii) Z boson decay observables: total width Γ_Z, and partial widths into fermions $\Gamma_f \equiv \Gamma(Z \to f\bar{f})$. Also we define and use the invisible and hadronic partial widths[2]:

$$\Gamma_{\rm inv} \equiv 3\Gamma_\nu,$$
$$\Gamma_{\rm had} \equiv \Gamma(Z \to \text{hadrons}) \simeq \Gamma_u + \Gamma_d + \Gamma_c + \Gamma_s + \Gamma_b. \tag{3}$$

The ratios of partial widths are defined and also included in our observables list:

$$R_\ell \equiv \frac{\Gamma_{\rm had}}{\Gamma_\ell}, \qquad R_q \equiv \frac{\Gamma_q}{\Gamma_{\rm had}}, \tag{4}$$

where ℓ and q denote any one of the lepton and quark species, respectively.

(iv) $e^+e^- \to$ hadrons cross section at the Z pole:

$$\sigma_{\rm had} = 12\pi \frac{\Gamma_e \Gamma_{\rm had}}{m_Z^2 \Gamma_Z^2}. \tag{5}$$

(v) Forward-backward asymmetries for $e^+e^- \to f\bar{f}$ at the Z pole:

$$A_{\rm FB}^f = \frac{\sigma_{\rm F} - \sigma_{\rm B}}{\sigma_{\rm F} + \sigma_{\rm B}} = \frac{3}{4}\mathcal{A}_e\mathcal{A}_f. \tag{6}$$

The asymmetry parameters $\mathcal{A}_f$ are related to the definition of the effective electroweak mixing angle $\sin^2\theta_{\rm eff}^f$ by

$$\mathcal{A}_f = \frac{2(1 - 4|Q_f|\sin^2\theta_{\rm eff}^f)}{1 + (1 - 4|Q_f|\sin^2\theta_{\rm eff}^f)^2}, \tag{7}$$

where Q_f is the electric charge of fermion f.

The experimental results for these observables are listed in Table I. For all the Z pole observables, we use the numbers presented in [10], which are combinations of various experimental results at LEP and SLC. Among these observables, lepton universality is assumed only for $\sin^2\theta_{\rm eff}^e$. For $\sin^2\theta_{\rm eff}^e$, we also list the PDG combination [7] of D0 [11] and CDF [12] results (the second number). m_W from [13] is the average of LEP2 [14] and Tevatron [13] results. m_H is the PDG average [7] of ATLAS [1] and CMS [3] results.

TABLE I. The list of observables, their experimental and reference values, and percent relative uncertainties. We set $\hat{O}_{i'}^{\rm ref} = \hat{O}_{i'}^{\rm expt}$ for the input observables, and calculate $\hat{O}_i^{\rm ref}$ for the other observables. The percent relative uncertainty $P[\hat{O}_i^{\rm ref}]$ is the maximum deviation of $\hat{O}_i$ from $\hat{O}_i^{\rm ref}$ in units of percentage when the input observables are varied within experimental errors; see Eq. (17) (e.g. m_W deviates from $[m_W]^{\rm ref}$ by at most 0.01%).

$\hat{O}_i$	$\hat{O}_i^{\rm expt}$	$\hat{O}_i^{\rm ref}$	$P[\hat{O}_i^{\rm ref}]$
m_Z [GeV]	91.1876(21) [10]	91.1876	
G_F [GeV^{-2}]	$1.1663787(6)\times10^{-5}$ [7]	1.1663787×10^{-5}	
$\Delta\alpha_{\rm had}^{(5)}$	0.02772(10) [7]	0.02772	
m_t [GeV]	173.20(87) [15]	173.20	
$\alpha_s(m_Z)$	0.1185(6) [7]	0.1185	
m_H [GeV]	125.9(4) [7]	125.9	
$\alpha(m_Z)$	$7.81592(86)\times10^{-3}$ [7]	7.75611×10^{-3}	0.01
m_W [GeV]	80.385(15) [13]	80.3614	0.01
Γ_e [MeV]	83.92(12) [10]	83.9818	0.02
Γ_μ [MeV]	83.99(18) [10]	83.9812	0.02
Γ_τ [MeV]	84.08(22) [10]	83.7916	0.02
Γ_b [MeV]	377.6(1.3) [10]	375.918	0.04
Γ_c [MeV]	300.5(5.3) [10]	299.969	0.06
$\Gamma_{\rm inv}$ [GeV]	0.4974(25) [10]	0.501627	0.02
$\Gamma_{\rm had}$ [GeV]	1.7458(27) [10]	1.74169	0.04
Γ_Z [GeV]	2.4952(23) [10]	2.49507	0.03
$\sigma_{\rm had}$ (nb)	41.541(37) [10]	41.4784	0.01
R_e	20.804(50) [10]	20.7389	0.03
R_μ	20.785(33) [10]	20.7391	0.03
R_τ	20.764(45) [10]	20.7860	0.03
R_b	0.21629(66) [10]	0.215835	0.02
R_c	0.1721(30) [10]	0.172229	0.01
$\sin^2\theta_{\rm eff}^e$	0.23153(16) [10]	0.231620	0.04
	0.23200(76) [7]		
$\sin^2\theta_{\rm eff}^b$	0.281(16) [10]	0.232958	0.03
$\sin^2\theta_{\rm eff}^c$	0.2355(59) [10]	0.231514	0.04
$\mathcal{A}_e$	0.1514(19) [10]	0.146249	0.44
$\mathcal{A}_b$	0.923(20) [10]	0.934602	0.00
$\mathcal{A}_c$	0.670(27) [10]	0.667530	0.04
$A_{\rm FB}^e$	0.0145(25) [10]	0.0160415	0.88
$A_{\rm FB}^b$	0.0992(16) [10]	0.102513	0.44
$A_{\rm FB}^c$	0.0707(35) [10]	0.0732191	0.48

[2] $\Gamma_{\rm had}$ is not quite the sum of all Γ_q, as there are $\mathcal{O}(\alpha_s^3)$ corrections that cannot be attributed to any Γ_q [9]. However, these corrections are small, and are neglected in ZFITTER. We will come back to this in Appendix B.

Table I also contains the reference theory values around which we expand, and their percent relative uncertainties. These theory quantities will be introduced and discussed in detail in Sec. III B.

III. THE FORMALISM

A. Expansion about reference point

Let us denote the set of SM parameters by $\{p_{k'}\}$, and the set of SM observables by $\{\hat{O}_i\}$. The theoretical prediction for each observable can be calculated in the SM as a function of all parameters:

$$\hat{O}_i^{\rm th} = \hat{O}_i^{\rm SM}(\{p_{k'}\}). \tag{8}$$

The notation here is that primed roman indices run from 1 to N_p, the number of SM parameters, while unprimed ones run from 1 to N_O, the number of observables under consideration. Note that N_p is finite, while N_O can presumably be infinite (we must at least have $N_O > N_p$ in order to test any theory). The analysis in this paper is done with $N_p = 6$ and $N_O = 31$, with $\{p_{k'}\}$ given in Eq. (1) and $\{\hat{O}_i\}$ listed in Table I.

Next, suppose we want to study some new physics model beyond the SM, which contains a set of new parameters collectively denoted as $p^{\rm NP}$ ("NP" for "new physics"). Then at least some $\hat{O}_i^{\rm th}$ will receive new contribution. We expect such new contribution to be small, in the light of apparently good agreement between SM predictions and precision electroweak data. We can thus write

$$\hat{O}_i^{\rm th} = \hat{O}_i^{\rm SM}(\{p_{k'}\}) + \delta^{\rm NP}\hat{O}_i(\{p_{k'}\}, p^{\rm NP}). \tag{9}$$

We wish to decide whether the new physics model is compatible with precision electroweak data, i.e. whether the $\hat{O}_i^{\rm th}$ predicted by Eq. (9) are compatible with the experimentally measured values $\hat{O}_i^{\rm expt}$.

One common misconception in such analysis is that a new physics model would be ruled out if, for some very precisely measured observables, e.g. $G_F^{\rm expt} = 1.1663787(6) \times 10^{-5}\ {\rm GeV}^{-2}$, the new physics contribution $\delta^{\rm NP}\hat{O}_i$ exceeds the experimental error. The point is that the SM parameters $\{p_{k'}\}$ are not directly measured experimentally. Rather, in testing the SM, we adjust $\{p_{k'}\}$ and see that for some choice of all parameters $\{p_{k'}^{\rm ref}\}$, all $\hat{O}_i^{\rm SM}$ agree well with $\hat{O}_i^{\rm expt}$. In the presence of new physics, we should do the same thing, and will typically arrive at a different choice of $\{p_{k'}^{\rm ref}\}$, and hence different $\hat{O}_i^{\rm SM}$, which may allow the new physics model to survive (in some regions of parameter space spanned by $p^{\rm NP}$) despite a large $\delta^{\rm NP}\hat{O}_i$.

The statements above are made more precise by the χ^2 analysis, which is the standard way of doing precision electroweak analysis. With correlations among the observables ignored, and experimental errors assumed larger than theoretical errors, the χ^2 function is defined by

$$\chi^2(\{p_{k'}\}, p^{\rm NP}) = \sum_i \left[\frac{\hat{O}_i^{\rm th}(\{p_{k'}\}, p^{\rm NP}) - \hat{O}_i^{\rm expt}}{\Delta\hat{O}_i^{\rm expt}}\right]^2, \tag{10}$$

where $\Delta\hat{O}_i^{\rm expt}$ are the experimental uncertainties of the observables. To decide whether some $p^{\rm NP}$ in the new physics model parameter space survives precision tests, we vary $\{p_{k'}\}$ to minimize the χ^2 function to find the best fit to experimental data, and see if this minimum χ^2 is small enough. A good discussion of how to interpret the statistics of the χ^2 distribution can be found in [7].

In principle, one can calculate $\hat{O}_i^{\rm th}$ each time a different $\{p_{k'}\}$ is chosen in this minimization procedure. But in practice, we can do it once and for all by carrying out an expansion about some reference point in the SM parameter space $\{p_{k'}^{\rm ref}\}$. Such an expansion is useful because precision data do not allow much variation in each parameter. Thus, let us choose some $\{p_{k'}^{\rm ref}\}$ that lead to good agreement between $\hat{O}_i^{\rm SM}$ and $\hat{O}_i^{\rm expt}$, and write

$$\hat{O}_i^{\rm SM}(\{p_{k'}\}) = \hat{O}_i^{\rm ref} + \sum_{k'} \frac{\partial\hat{O}_i^{\rm SM}}{\partial p_{k'}}(p_{k'} - p_{k'}^{\rm ref}) + \cdots \tag{11}$$

where $\hat{O}_i^{\rm ref} \equiv \hat{O}_i^{\rm SM}(\{p_{k'}^{\rm ref}\})$, and the partial derivatives are taken at $p_{k'} = p_{k'}^{\rm ref}$ (this will be implicitly assumed in the following). Alternatively, define

$$\bar{\delta}^{\rm SM}\hat{O}_i(\{p_{k'}\}) \equiv \frac{\hat{O}_i^{\rm SM}(\{p_{k'}\}) - \hat{O}_i^{\rm ref}}{\hat{O}_i^{\rm ref}}, \quad \bar{\delta}p_{k'} \equiv \frac{p_{k'} - p_{k'}^{\rm ref}}{p_{k'}^{\rm ref}},$$
$$G_{ik'} \equiv \frac{p_{k'}^{\rm ref}}{\hat{O}_i^{\rm ref}} \frac{\partial\hat{O}_i^{\rm SM}}{\partial p_{k'}}. \tag{12}$$

Then we have a more concise expression for Eq. (11):

$$\bar{\delta}^{\rm SM}\hat{O}_i = \sum_{k'} G_{ik'}\bar{\delta}p_{k'} + \cdots \tag{13}$$

Here $\bar{\delta}$ means "fractional shift from the reference value," and the superscript on $\bar{\delta}^{\rm SM}\hat{O}_i$ indicates the shift comes from shifts in SM parameters. Ignoring higher-order terms in the expansion, the constant $G_{ik'}$ is the fractional change in $\hat{O}_i^{\rm SM}$ caused by the fractional change in $p_{k'}$, and hence characterizes the sensitivity of the ith SM observable (as calculated in the SM) to the k'th SM parameter.

In the presence of perturbative new physics contributions, let us define

$$\bar{\delta}\hat{O}_i^{\rm th}(\{p_{k'}\}, p^{\rm NP}) \equiv \frac{\hat{O}_i^{\rm th}(\{p_{k'}\}, p^{\rm NP}) - \hat{O}_i^{\rm ref}}{\hat{O}_i^{\rm ref}},$$
$$\xi_i(\{p_{k'}\}, p^{\rm NP}) \equiv \frac{\delta^{\rm NP}\hat{O}_i(\{p_{k'}\}, p^{\rm NP})}{\hat{O}_i^{\rm ref}}. \tag{14}$$

Then Eq. (9) can be expanded as, to first order,

$$\bar{\delta}\hat{O}_i^{\text{th}} = \bar{\delta}^{\text{SM}}\hat{O}_i + \xi_i = \sum_{k'} G_{ik'}\bar{\delta}p_{k'} + \xi_i. \tag{15}$$

The calculation of $\hat{O}_i^{\text{th}}$ and hence χ^2 is then facilitated if we have at hand the constants $p_{k'}^{\text{ref}}$, $\hat{O}_i^{\text{ref}}$ and $G_{ik'}$.

B. Recasting observables in terms of observables

The approach above is indirect, in the sense that the input of the analysis, the parameters $\{p_{k'}\}$, are not directly measurable—only $\{\hat{O}_i\}$ are well-defined observables. We can do better if we use N_p very well-measured observables $\{\hat{O}_{i'}\}$ as input. Note that primed indices, which run from 1 to N_p, are used for input observables. Inverting the functions $\hat{O}_{i'}^{\text{SM}}(\{p_{k'}\})$, we can express other observables as functions of these input observables. Then it is immediately clear from $\hat{O}_{i'}^{\text{expt}}$ and $\Delta\hat{O}_{i'}^{\text{expt}}$ what reference values for the input we should use, and by how much they are allowed to vary. In our analysis, $N_p = 6$, and a convenient choice for the six input observables is

$$\{\hat{O}_{i'}\} = \{m_Z, G_F, \Delta\alpha_{\text{had}}^{(5)}, m_t, \alpha_s(m_Z), m_H\}. \tag{16}$$

The reference values for these input observables are taken to be the central values experimentally measured; see Table I. All other observables are output observables, and their reference values $\hat{O}_i^{\text{ref}}$ are evaluated at $\hat{O}_{i'} = \hat{O}_{i'}^{\text{ref}}$ with the help of ZFITTER. See Appendix A for technical details.

We also show in Table I the "percent relative uncertainties" $P[\hat{O}_i^{\text{ref}}]$, defined as the maximum value of

$$100\left|\frac{\hat{O}_i^{\text{SM}}(\{\hat{O}_{i'}\}) - \hat{O}_i^{\text{ref}}}{\hat{O}_i^{\text{ref}}}\right| \tag{17}$$

when all $\{\hat{O}_{i'}\}$ are varied in their 1σ range around $\{\hat{O}_{i'}^{\text{expt}}\}$. We do not distinguish between positive and negative relative uncertainties because, as we have checked, the asymmetry in the uncertainties for all observables considered here is very small.

To work out the expansion about the reference point, we assume the input observables $\{\hat{O}_{i'}\}$ are the first N_p observables in the list $\{\hat{O}_i\}$. Then we can simply invert the first N_p equations in Eq. (13). To first order,

$$\bar{\delta}^{\text{SM}}\hat{O}_{i'} = \sum_{k'} G_{i'k'}\bar{\delta}p_{k'} = \sum_{k'} \tilde{G}_{i'k'}\bar{\delta}p_{k'}$$
$$\Rightarrow \bar{\delta}p_{k'} = \sum_{i'} (\tilde{G}^{-1})_{k'i'}\bar{\delta}^{\text{SM}}\hat{O}_{i'}. \tag{18}$$

Note that G is a $N_O \times N_p$ matrix, while $\tilde{G}$ is the upper $N_p \times N_p$ block of G. Then Eq. (13) suggests

$$\bar{\delta}^{\text{SM}}\hat{O}_i = \sum_{k',i'} G_{ik'}(\tilde{G}^{-1})_{k'i'}\bar{\delta}^{\text{SM}}\hat{O}_{i'} \equiv \sum_{i'} c_{ii'}\bar{\delta}^{\text{SM}}\hat{O}_{i'}, \tag{19}$$

where we have defined

$$c_{ii'} \equiv \sum_{k'} G_{ik'}(\tilde{G}^{-1})_{k'i'} = \frac{\hat{O}_{i'}^{\text{ref}}}{\hat{O}_i^{\text{ref}}}\frac{\partial\hat{O}_i^{\text{SM}}}{\partial\hat{O}_{i'}^{\text{SM}}}. \tag{20}$$

Equation (19) expresses the shift in any observable in terms of shifts in the input observables, as calculated in the SM. Notably, the upper $N_p \times N_p$ block of the $N_O \times N_p$ matrix c is the identity matrix, i.e. $c_{j'i'} = \delta_{j'i'}$. For $i > N_p$, i.e. the output observables, the calculation of $c_{ii'}$ is nontrivial. We present in Table II the results for these expansion coefficients for the observables discussed in Sec. II, which we calculate using ZFITTER. These coefficients are useful not only because they facilitate the calculation of SM observables. They also give us information on the sensitivity of the calculated observables to each input observable.

In the presence of new physics, Eq. (15) becomes

$$\bar{\delta}\hat{O}_i^{\text{th}} = \sum_{i'} c_{ii'}\bar{\delta}^{\text{SM}}\hat{O}_{i'} + \xi_i = \sum_{i'} c_{ii'}(\bar{\delta}\hat{O}_{i'}^{\text{th}} - \xi_{i'}) + \xi_i$$
$$= \sum_{i'} c_{ii'}\bar{\delta}\hat{O}_{i'}^{\text{th}} + \bar{\delta}^{\text{NP}}\hat{O}_i, \tag{21}$$

where

$$\bar{\delta}^{\text{NP}}\hat{O}_i \equiv \xi_i - \sum_{i'} c_{ii'}\xi_{i'}$$
$$= \xi_i - c_{i,m_Z}\xi_{m_Z} - c_{i,G_F}\xi_{G_F} - c_{i,\Delta\alpha}\xi_{\Delta\alpha} - c_{i,m_t}\xi_{m_t} - c_{i,\alpha_s}\xi_{\alpha_s} - c_{i,m_H}\xi_{m_H}. \tag{22}$$

Equation (21) expresses the shift in any observable in terms of shifts in the input observables and new physics effects. Note that for the input observables, since $c_{j'i'} = \delta_{j'i'}$, Eq. (22) indicates $\bar{\delta}^{\text{NP}}\hat{O}_{i'} = 0$, and Eq. (21) trivially becomes $\bar{\delta}\hat{O}_{i'}^{\text{th}} = \bar{\delta}\hat{O}_{i'}^{\text{th}}$. This is forced to be true in our formalism, where $\hat{O}_{i'}^{\text{th}}$ are inputs of the analysis, independent of new physics. Of course, new physics does contribute $\xi_{i'}$ to the calculation of $\hat{O}_{i'}^{\text{th}}$, but as we decide to use some particular values for the input $\hat{O}_{i'}^{\text{th}}$ to be consistent with $\hat{O}_{i'}^{\text{expt}}$ (which are extraordinarily well measured), we find ourselves adjusting the SM parameters to compensate for $\xi_{i'}$. This adjustment gets propagated into the shift in $\hat{O}_i^{\text{th}}$ due to new physics for $i > N_p$. As a result, Eq. (22) shows that for the output observables, $\bar{\delta}^{\text{NP}}\hat{O}_i$ is not simply ξ_i, but is related to $\xi_{i'}$ for all input observables.

To close this subsection we remark on the calculation of ξ_i. In practice this is done at tree level or one-loop level, if we are only interested in constraining a new physics model at percentage level accuracy. Also, the definition of ξ_i, Eq. (14), instructs us to calculate them in terms of Lagrangian parameters, which can then be eliminated in

TABLE II. Expansion coefficients, as defined in Eq. (20), calculated in the basis of input observables containing $\Delta\alpha^{(5)}_{\rm had}$. These encode the dependence of the output observables on each input observable, and can be used to easily calculate the deviation of the theory prediction of the observables from their reference values via Eq. (21), including new physics contributions.

$\hat{O}_i$	c_{i,m_Z}	c_{i,G_F}	$c_{i,\Delta\alpha}$	c_{i,m_t}	c_{i,α_s}	c_{i,m_H}
m_Z	1	0	0	0	0	0
G_F	0	1	0	0	0	0
$\Delta\alpha^{(5)}_{\rm had}$	0	0	1	0	0	0
m_t	0	0	0	1	0	0
$\alpha_s(m_Z)$	0	0	0	0	1	0
m_H	0	0	0	0	0	1
$\alpha(m_Z)$	4.796×10^{-3}	0	0.02946	1.541×10^{-4}	-1.007×10^{-5}	0
m_W	1.427	0.2201	-6.345×10^{-3}	0.01322	-9.599×10^{-4}	-7.704×10^{-4}
Γ_e	3.377	1.198	-5.655×10^{-3}	0.01883	-1.253×10^{-3}	-7.924×10^{-4}
Γ_μ	3.377	1.198	-5.655×10^{-3}	0.01883	-1.253×10^{-3}	-7.924×10^{-4}
Γ_τ	3.383	1.198	-5.668×10^{-3}	0.01884	-1.254×10^{-3}	-7.931×10^{-4}
Γ_b	3.844	1.411	−0.01227	−0.01267	0.03672	-1.057×10^{-3}
Γ_c	4.151	1.590	−0.01721	0.02751	0.05046	-1.394×10^{-3}
$\Gamma_{\rm inv}$	2.996	1.006	5.635×10^{-5}	0.01567	-9.967×10^{-4}	-4.873×10^{-4}
$\Gamma_{\rm had}$	3.938	1.476	−0.01393	0.01578	0.03690	-1.204×10^{-3}
Γ_Z	3.692	1.353	−0.01028	0.01607	0.02543	-1.019×10^{-3}
$\sigma_{\rm had}$	−2.069	−0.03281	9.806×10^{-4}	2.476e-3	−0.01522	4.057×10^{-5}
R_e	0.5608	0.2780	-8.272×10^{-3}	-3.045×10^{-3}	0.03815	-4.120×10^{-4}
R_μ	0.5608	0.2780	-8.272×10^{-3}	-3.045×10^{-3}	0.03815	-4.120×10^{-4}
R_τ	0.5554	0.2776	-8.259×10^{-3}	-3.053×10^{-3}	0.03816	-4.113×10^{-4}
R_b	−0.09434	−0.06530	1.652×10^{-3}	−0.02845	-1.782×10^{-4}	1.477×10^{-4}
R_c	0.2133	0.1135	-3.284×10^{-3}	0.01173	0.01356	-1.898×10^{-4}
$\sin^2\theta^e_{\rm eff}$	−2.818	−1.423	0.04203	−0.02330	1.796×10^{-3}	2.195×10^{-3}
$\sin^2\theta^b_{\rm eff}$	−2.823	−1.417	0.04204	-6.914×10^{-3}	1.201×10^{-3}	2.116×10^{-3}
$\sin^2\theta^c_{\rm eff}$	−2.819	−1.423	0.04202	−0.02331	1.795×10^{-3}	2.194×10^{-3}
$\mathcal{A}_e$	35.13	17.74	−0.5239	0.2905	−0.02239	−0.02737
$\mathcal{A}_b$	0.4525	0.2271	-6.737×10^{-3}	1.108×10^{-3}	-1.924×10^{-4}	-3.390×10^{-4}
$\mathcal{A}_c$	3.386	1.710	−0.05048	0.02800	-2.156×10^{-3}	-2.636×10^{-3}
$A^e_{\rm FB}$	70.27	35.48	−1.048	0.5810	−0.04479	−0.05473
$A^b_{\rm FB}$	35.59	17.97	−0.5306	0.2916	−0.02259	−0.02771
$A^c_{\rm FB}$	38.52	19.45	−0.5744	0.3185	−0.02455	−0.03000

favor of input observables using the tree-level relations between the two. This does not conflict with the "precision" part of the analysis, since we are doing two different perturbative expansions in the calculation: the expansion in SM couplings, and the expansion in new physics effects. Since new physics makes tiny contributions to $\hat{O}^{\rm th}_i$, to discern them we have to calculate the SM part as precisely as possible, carrying out the expansion in SM couplings to as high order as possible. On the other hand, in most cases the new physics contributions ξ_i need not be calculated beyond leading order, since they are already very small. We will see explicitly how the reasoning above works out in specific examples in Section IV A.

C. Beyond first order

The above perturbative expansion carried out to first order is expected to be sufficient for the purpose of precision electroweak analysis, since we have chosen a very well-measured set of input observables, so that the expansion parameters $\bar{\delta}\hat{O}^{\rm th}_{i'}$ are tiny. The impact of higher-order terms in the expansion can be seen from the sensitivity of the expansion coefficients $c_{ii'}$ to the choice of reference values for the input observables $\hat{O}^{\rm ref}_{i'}$. In Table III we show the percent relative uncertainties for $c_{ii'}$, defined similarly to Eq. (17).

Alternatively, without varying $\hat{O}^{\rm ref}_{i'}$, we can explicitly write down the next order terms in the expansion:

$$\bar{\delta}^{\rm SM}\hat{O}_i = \sum_{i'} c_{ii'}\bar{\delta}^{\rm SM}\hat{O}_{i'} + \frac{1}{2!}\sum_{i'j'} c_{ii'j'}\bar{\delta}^{\rm SM}\hat{O}_{i'}\bar{\delta}^{\rm SM}\hat{O}_{j'} + \cdots$$
$$\equiv \sum_{i'}(c_{ii'} + \Delta c_{ii'})\bar{\delta}^{\rm SM}\hat{O}_{i'} + \cdots \quad (23)$$

where

$$c_{ii'j'} \equiv \frac{\hat{O}^{\rm ref}_{i'}\hat{O}^{\rm ref}_{j}}{\hat{O}^{\rm ref}_{i}}\frac{\partial^2\hat{O}^{\rm SM}_i}{\partial\hat{O}^{\rm SM}_{i'}\partial\hat{O}^{\rm SM}_{j'}}. \quad (24)$$

TABLE III. Percent relative uncertainties for the expansion coefficients $c_{ii'}$, with all input observables varied in their 1σ range.

$\hat{O}_i$	$P[c_{i,m_Z}]$	$P[c_{i,G_F}]$	$P[c_{i,\Delta\alpha}]$	$P[c_{i,m_t}]$	$P[c_{i,\alpha_s}]$	$P[c_{i,m_H}]$
$\alpha(m_Z)$	0.05	0.00	0.37	1.19	1.64	0.00
m_W	0.02	0.05	0.44	0.87	1.20	0.23
Γ_e	0.04	0.07	0.42	1.09	1.53	0.60
Γ_μ	0.04	0.07	0.42	1.09	1.53	0.60
Γ_τ	0.04	0.07	0.42	1.09	1.53	0.60
Γ_b	0.01	0.02	0.43	0.96	0.41	0.27
Γ_c	0.01	0.01	0.39	0.88	0.64	0.33
$\Gamma_{\rm inv}$	0.00	0.01	0.63	1.04	1.51	0.74
$\Gamma_{\rm had}$	0.01	0.01	0.41	1.10	0.50	0.35
Γ_Z	0.00	0.01	0.39	1.07	0.52	0.39
$\sigma_{\rm had}$	0.06	2.08	2.41	1.31	0.50	2.81
R_e	0.31	0.32	0.69	1.40	0.47	0.36
R_μ	0.31	0.32	0.69	1.40	0.47	0.36
R_τ	0.32	0.33	0.69	1.40	0.47	0.36
R_b	0.13	0.28	0.41	0.92	22.06	0.88
R_c	0.12	0.14	0.41	0.87	1.26	0.35
$\sin^2\theta^e_{\rm eff}$	0.02	0.01	0.39	0.97	1.26	0.12
$\sin^2\theta^b_{\rm eff}$	0.02	0.02	0.39	0.75	1.16	0.05
$\sin^2\theta^c_{\rm eff}$	0.02	0.01	0.39	0.97	1.26	0.12
$\mathcal{A}_e$	0.51	0.50	0.88	1.10	1.42	0.46
$\mathcal{A}_b$	0.09	0.09	0.46	0.80	1.21	0.11
$\mathcal{A}_c$	0.14	0.14	0.52	1.00	1.30	0.16
$A^e_{\rm FB}$	0.51	0.50	0.88	1.10	1.42	0.46
$A^b_{\rm FB}$	0.50	0.49	0.88	1.10	1.42	0.46
$A^c_{\rm FB}$	0.48	0.47	0.85	1.09	1.41	0.43

TABLE IV. The $r_{ii'}$'s defined in Eq. (25), characterizing the ratios of second-order vs first-order terms in the expansion (in units of percentage).

$\hat{O}_i$	r_{i,m_Z}	r_{i,G_F}	$r_{i,\Delta\alpha}$	r_{i,m_t}	r_{i,α_s}	r_{i,m_H}
$\alpha(m_Z)$	0.03	0.00	0.01	0.85	0.66	0.00
m_W	0.01	0.03	0.03	0.18	0.35	0.18
Γ_e	0.03	0.04	0.20	0.30	0.52	0.18
Γ_μ	0.03	0.04	0.20	0.30	0.52	0.18
Γ_τ	0.03	0.04	0.20	0.30	0.52	0.18
Γ_b	0.02	0.02	0.04	0.24	0.10	0.07
Γ_c	0.02	0.03	0.02	0.21	0.09	0.16
$\Gamma_{\rm inv}$	0.01	0.01	0.12	0.27	0.51	0.21
$\Gamma_{\rm had}$	0.02	0.02	0.02	0.29	0.04	0.14
Γ_Z	0.02	0.02	0.02	0.29	0.05	0.13
$\sigma_{\rm had}$	0.03	1.04	1.02	0.39	0.02	1.49
R_e	0.17	0.17	0.17	0.46	0.02	0.31
R_μ	0.17	0.17	0.17	0.46	0.02	0.31
R_τ	0.17	0.17	0.17	0.46	0.02	0.31
R_b	0.05	0.13	0.05	0.20	10.69	0.59
R_c	0.06	0.07	0.05	0.19	0.38	0.31
$\sin^2\theta^e_{\rm eff}$	0.03	0.02	0.03	0.24	0.38	0.19
$\sin^2\theta^b_{\rm eff}$	0.03	0.02	0.03	0.13	0.34	0.17
$\sin^2\theta^c_{\rm eff}$	0.03	0.02	0.03	0.24	0.38	0.19
$\mathcal{A}_e$	0.04	0.03	0.04	0.24	0.38	0.20
$\mathcal{A}_b$	0.04	0.04	0.05	0.14	0.35	0.18
$\mathcal{A}_c$	0.05	0.05	0.06	0.24	0.39	0.20
$A^e_{\rm FB}$	0.18	0.19	0.18	0.42	0.55	0.37
$A^b_{\rm FB}$	0.03	0.03	0.04	0.24	0.38	0.19
$A^c_{\rm FB}$	0.00	0.01	0.01	0.23	0.37	0.19

Then the size of second-order terms in Eq. (23) compared with the first-order term is characterized by the ratio

$$\left|\frac{\Delta c_{ii'}}{c_{ii'}}\right| = \left|\frac{\sum_{j'} c_{ii'j'}\bar{\delta}^{\rm SM}\hat{O}_{j'}}{2c_{ii'}}\right| \leq \frac{\sum_{j'}|c_{ii'j'}||\bar{\delta}^{\rm SM}\hat{O}_{j'}|}{2|c_{ii'}|} \equiv 0.01 r_{ii'}. \tag{25}$$

We show in Table IV the $r_{ii'}$ calculated with $\bar{\delta}^{\rm SM}\hat{O}_{j'} = \Delta\hat{O}^{\rm expt}_{j'}/\hat{O}^{\rm ref}_{j'}$. The results follow a similar pattern as in Table III.

Tables III and IV both show that the uncertainties on the observables calculations are negligible due to uncertainty in the first-order expansion coefficient $c_{ii'}$'s. Most entries manifestly demonstrate this with values of less than 1% corrections to the first-order coefficients that are already governing less than 1% shifts in the observables due to the small uncertainties of the input observables to the calculation (see Table I). Only in a couple of places does the uncertainty reach more than 1%, but the final uncertainty on the observables themselves is of course significantly lower than that. To illustrate this, let us consider the largest $P[c_{ii'}]$ in Table III, $P[c_{R_b,\alpha_s}]$, which is the uncertainty in the expansion coefficient of $\alpha_s - \alpha_s^{\rm ref}$ in the computation for R_b. It yields an uncertainty on R_b of

$$\Delta R_b \simeq R_b^{\rm ref}|22\% \times c_{R_b,\alpha_s} \times \bar{\delta}\alpha_s| \simeq 0.216(0.22 \times 0.0002 \times 0.005) \simeq 5 \times 10^{-8}, \tag{26}$$

which is much smaller than the experimental uncertainty of 7×10^{-4}. Therefore, in practice this 22% uncertainty does not concern us, and we can be confident that the first-order expansion expressions are sufficient for any precision electroweak analysis given the current uncertainties in observables.

However, this large uncertainty in c_{R_b,α_s}, plus the intuitively unexpected large difference in c_{Γ_q,α_s} among different quarks (see Table VIII in Appendix B), inspire us to examine closely the calculation of the QCD corrections to Z decay. We will address this issue and explain these features in Appendix B.

D. Change of basis

Our choice of input observables as in Eq. (16) is convenient for the calculation of expansion coefficients in ZFITTER. In principle, any set of $N_p = 6$ independent observables can serve as input, though we should better choose those most precisely measured observables to minimize the uncertainty due to higher-order terms in the expansion. In this respect, an equally good choice as Eq. (16) could be

$$\{\hat{O}_{i'}\} = \{m_Z, G_F, \alpha(m_Z), m_t, \alpha_s(m_Z), m_H\}, \quad (27)$$

since essentially all the uncertainty in $\alpha(m_Z)$ comes from $\Delta\alpha^{(5)}_{\rm had}$. This basis may be preferable in practice, since it is often more convenient to do calculations with $\alpha(m_Z)$, rather than $\Delta\alpha^{(5)}_{\rm had}$, as input. In this subsection we derive the rules for translating the expansion coefficients $c_{ii'}$, which are calculated in the basis Eq. (16), into those for the basis Eq. (27). To avoid confusion, denote the latter by $d_{ii'}$. Also, superscripts "SM" will be dropped for simplicity in this subsection.

First, consider $d_{i,\alpha}$. We need to determine the shift in $\hat{O}_i$ caused by $\bar{\delta}\alpha(m_Z)$, with the other five input observables held fixed. If we work in the basis Eq. (16), this shift in $\alpha(m_Z)$ is an outcome of the following shift in $\Delta\alpha^{(5)}_{\rm had}$ (with other input observables fixed):

$$\bar{\delta}\Delta\alpha^{(5)}_{\rm had} = [c_{\alpha,\Delta\alpha}]^{-1}\bar{\delta}\alpha(m_Z). \quad (28)$$

And the shift in $\hat{O}_i$ is

$$\bar{\delta}\hat{O}_i = c_{i,\Delta\alpha}\bar{\delta}\Delta\alpha^{(5)}_{\rm had} = c_{i,\Delta\alpha}[c_{\alpha,\Delta\alpha}]^{-1}\bar{\delta}\alpha(m_Z). \quad (29)$$

Thus,

$$d_{i,\alpha} = \frac{\bar{\delta}\hat{O}_i}{\bar{\delta}\alpha(m_Z)} = c_{i,\Delta\alpha}[c_{\alpha,\Delta\alpha}]^{-1}. \quad (30)$$

Next, consider $d_{ii'}$ for $i' \neq \alpha(m_Z)$. Take d_{i,m_Z} as an example. We need to shift m_Z while keeping other observables in Eq. (27), including $\alpha(m_Z)$, fixed, and find the resulting shift in $\hat{O}_i$. Working in the basis Eq. (16), we can do this in two steps. First, shift m_Z by $\bar{\delta}m_Z$. As a result,

$$\bar{\delta}\hat{O}_i = c_{i,m_Z}\bar{\delta}m_Z, \qquad \bar{\delta}\alpha(m_Z) = c_{\alpha,m_Z}\bar{\delta}m_Z. \quad (31)$$

Second, shift $\Delta\alpha^{(5)}_{\rm had}$ by

TABLE V. Expansion coefficients calculated in the basis of input observables containing $\alpha(m_Z)$, which are derived from the numbers in Table II by a change of basis described in Sec. III D. These encode the dependence of the output observables on each input observable, and can be used to easily calculate the deviation of the theory prediction of the observables from their reference values via Eq. (38), including new physics contributions.

$\hat{O}_i$	d_{i,m_Z}	d_{i,G_F}	$d_{i,\alpha}$	d_{i,m_t}	d_{i,α_s}	d_{i,m_H}
m_Z	1	0	0	0	0	0
G_F	0	1	0	0	0	0
$\Delta\alpha^{(5)}_{\rm had}$	0	0	1	0	0	0
m_t	0	0	0	1	0	0
$\alpha_s(m_Z)$	0	0	0	0	1	0
m_H	0	0	0	0	0	1
$\Delta\alpha^{(5)}_{\rm had}$	−0.1628	0	33.94	-5.232×10^{-3}	3.417×10^{-4}	0
m_W	1.428	0.2201	−0.2154	0.01325	-9.621×10^{-4}	-7.704×10^{-4}
Γ_e	3.378	1.198	−0.1920	0.01886	-1.255×10^{-3}	-7.924×10^{-4}
Γ_μ	3.378	1.198	−0.1920	0.01886	-1.255×10^{-3}	-7.924×10^{-4}
Γ_τ	3.384	1.198	−0.1924	0.01887	-1.256×10^{-3}	-7.931×10^{-4}
Γ_b	3.846	1.411	−0.4166	−0.01260	0.03672	-1.057×10^{-3}
Γ_c	4.154	1.590	−0.5842	0.02760	0.05045	-1.394×10^{-3}
$\Gamma_{\rm inv}$	2.996	1.006	1.913×10^{-3}	0.01567	-9.967×10^{-4}	-4.873×10^{-4}
$\Gamma_{\rm had}$	3.940	1.476	−0.4727	0.01586	0.03690	-1.204×10^{-3}
Γ_Z	3.694	1.353	−0.3490	0.01612	0.02543	-1.019×10^{-3}
$\sigma_{\rm had}$	−2.070	−0.03281	0.03328	2.471×10^{-3}	−0.01522	4.057×10^{-5}
R_e	0.5622	0.2780	−0.2807	-3.002×10^{-3}	0.03815	-4.120×10^{-4}
R_μ	0.5622	0.2780	−0.2807	-3.002×10^{-3}	0.03815	-4.120×10^{-4}
R_τ	0.5568	0.2776	−0.2803	-3.009×10^{-3}	0.03815	-4.113×10^{-4}
R_b	−0.09461	−0.06530	0.05608	−0.02846	-1.777×10^{-4}	1.477×10^{-4}
R_c	0.2138	0.1135	−0.1115	0.01174	0.01356	-1.898×10^{-4}
$\sin^2\theta^e_{\rm eff}$	−2.825	−1.423	1.426	−0.02352	1.811×10^{-3}	2.195×10^{-3}
$\sin^2\theta^b_{\rm eff}$	−2.830	−1.417	1.427	-7.134×10^{-3}	1.215×10^{-3}	2.116×10^{-3}
$\sin^2\theta^c_{\rm eff}$	−2.826	−1.423	1.426	−0.02353	1.809×10^{-3}	2.194×10^{-3}
$\mathcal{A}_e$	35.22	17.74	−17.78	0.2932	−0.02257	−0.02737
$\mathcal{A}_b$	0.4536	0.2271	−0.2287	1.143×10^{-4}	-1.947×10^{-4}	-3.390×10^{-4}
$\mathcal{A}_c$	3.395	1.710	−1.713	0.02827	-2.174×10^{-3}	-2.636×10^{-3}
$A^e_{\rm FB}$	70.44	35.48	−35.56	0.5865	−0.04515	−0.05473
$A^b_{\rm FB}$	35.67	17.97	−18.01	0.2944	−0.02277	−0.02771
$A^c_{\rm FB}$	38.61	19.45	−19.50	0.3215	−0.02475	−0.03000

$$\bar{\delta}\Delta\alpha_{\rm had}^{(5)} = -[c_{\alpha,\Delta\alpha}]^{-1} c_{\alpha,m_Z} \bar{\delta} m_Z. \tag{32}$$

As a result,

$$\bar{\delta}\hat{O}_i = c_{i,\Delta\alpha}\bar{\delta}\Delta\alpha_{\rm had}^{(5)} = -c_{i,\Delta\alpha}[c_{\alpha,\Delta\alpha}]^{-1} c_{\alpha,m_Z}\bar{\delta} m_Z, \tag{33}$$

$$\bar{\delta}\alpha(m_Z) = c_{\alpha,\Delta\alpha}\bar{\delta}\Delta\alpha_{\rm had}^{(5)} = -c_{\alpha,m_Z}\bar{\delta} m_Z. \tag{34}$$

The effect of both steps is to hold all observables in Eq. (27) other than m_Z, in particular $\alpha(m_Z)$, fixed. And we get the desired result

$$d_{i,m_Z} = \frac{\bar{\delta}\hat{O}_i}{\bar{\delta} m_Z} = c_{i,m_Z} - c_{i,\Delta\alpha}[c_{\alpha,\Delta\alpha}]^{-1} c_{\alpha,m_Z}. \tag{35}$$

As a special case, Eqs. (30) and (35) also hold for $i = \Delta\alpha_{\rm had}^{(5)}$:

$$d_{\Delta\alpha,\alpha} = [c_{\alpha,\Delta\alpha}]^{-1}, \tag{36}$$

$$d_{\Delta\alpha,m_Z} = -[c_{\alpha,\Delta\alpha}]^{-1} c_{\alpha,m_Z}, \tag{37}$$

where we have used $c_{\Delta\alpha,\Delta\alpha} = 1$, $c_{\Delta\alpha,m_Z} = 0$.

In the basis Eq. (27), the theory predictions for the observables (with respect to the reference values) are calculated from

$$\bar{\delta}\hat{O}_i^{\rm th} = \sum_{i'} d_{ii'}\bar{\delta}\hat{O}_{i'}^{\rm th} + \bar{\delta}^{\rm NP}\hat{O}_i, \tag{38}$$

where

$$\begin{aligned}\bar{\delta}^{\rm NP}\hat{O}_i &\equiv \xi_i - \sum_{i'} d_{ii'}\xi_{i'} \\ &= \xi_i - d_{i,m_Z}\xi_{m_Z} - d_{i,G_F}\xi_{G_F} - d_{i,\alpha}\xi_\alpha - d_{i,m_t}\xi_{m_t} \\ &\quad - d_{i,\alpha_s}\xi_{\alpha_s} - d_{i,m_H}\xi_{m_H}.\end{aligned} \tag{39}$$

We list the expansion coefficients $d_{ii'}$, as calculated from Eqs. (30) and (35), in Table V.

IV. NEW PHYSICS EXAMPLES

In this section we present some examples of calculating new physics contributions to electroweak observables, using the formalism developed in Sec. III. We work in the basis Eq. (27), with $\alpha(m_Z)$ as an input observable.

A. Dimension-6 effective operators

The SM, when viewed as an effective field theory below some cutoff scale Λ, can be supplemented by higher dimensional operators suppressed by powers of Λ [16,17], which presumably come from new physics at or above Λ. Two examples at dimension 6 are

$$\mathfrak{O}_L = \frac{1}{2\Lambda_L^2}(\bar{L}\gamma_\mu\sigma^a L)^2, \qquad \mathfrak{O}_H = \frac{1}{\Lambda_H^2}|H^\dagger D_\mu H|^2, \tag{40}$$

where L and H are the lepton and Higgs $SU(2)_L$ doublets, respectively, and σ^a ($a = 1, 2, 3$) are the Pauli matrices. In this subsection we consider these two operators separately, and illustrate how to use the formalism developed in this paper to work out the precision electroweak constraints on Λ_L, Λ_H.

First consider $\mathfrak{O}_L$. At tree level the only nonzero ξ_i at $\mathcal{O}(\frac{1}{\Lambda_L^2})$ is

$$\xi_{G_F} = \frac{v^2}{\Lambda_L^2} = \frac{1}{\sqrt{2}G_F\Lambda_L^2} \quad \text{(tree level)}. \tag{41}$$

This computation should not be compared with the experimental uncertainty in G_F measurement to get limits on Λ_L^2. Rather, we should calculate

$$\bar{\delta}^{\rm NP}\hat{O}_i = \xi_i - d_{i,G_F}\xi_{G_F} \simeq \xi_i - d_{i,G_F}\left(\frac{246\ \text{GeV}}{\Lambda_L}\right)^2 \tag{42}$$

for all observables using the d_{i,G_F} listed in Table V, and perform a χ^2 analysis. Indeed, Eq. (42) gives $\bar{\delta}^{\rm NP}G_F = 0$, which is an essential check to the formalism since G_F is an input observable that is by definition set to whatever value we wish it to have. In other words, if new physics does appear to want to shift G_F, the parameters in the theory adjust themselves such that the total shift is zero. That is the nature of being a fixed input observable to precision electroweak computations.

Because of the rearrangement of SM parameters due to accommodating the contribution to G_F from new physics, every output observable will feel a shift. For example,

$$\begin{aligned}\bar{\delta}^{\rm NP} m_W &\simeq -d_{m_W,G_F}\left(\frac{246\ \text{GeV}}{\Lambda_L}\right)^2 \\ &\simeq -0.220\left(\frac{246\ \text{GeV}}{\Lambda_L}\right)^2,\end{aligned} \tag{43}$$

$$\begin{aligned}\bar{\delta}^{\rm NP}\mathcal{A}_e &\simeq -d_{\mathcal{A}_e,G_F}\left(\frac{246\ \text{GeV}}{\Lambda_L}\right)^2 \\ &\simeq -17.7\left(\frac{246\ \text{GeV}}{\Lambda_L}\right)^2.\end{aligned} \tag{44}$$

Similar expressions exist for all SM precision electroweak observables. To find limits on Λ_L a global χ^2 analysis must be performed, or at least a semiglobal χ^2 analysis using the most sensitive observables, such as Γ_e, m_W and $\sin^2\theta_{\rm eff}^e$ [18].

Next consider $\mathfrak{O}_H$. In the unitary gauge,

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h \end{pmatrix}$$
$$\Rightarrow \mathfrak{O}_H = \frac{v^2}{2\Lambda_H^2}\left[\frac{1}{2}(\partial_\mu h)^2\left(1+\frac{h}{v}\right)^2 + \frac{1}{4}(g_2^2+g_1^2)v^2 Z_\mu Z^\mu\left(1+\frac{h}{v}\right)^4\right]. \quad (45)$$

Noting that $m_Z = \frac{1}{2}\sqrt{g_2^2+g_1^2}\,v$ at tree level, we have

$$\xi_{m_Z} = -1 + \left(1+\frac{v^2}{2\Lambda_H^2}\right)^{1/2} \simeq \frac{v^2}{4\Lambda_H^2},$$
$$\xi_{m_H} = -1 + \left(1+\frac{v^2}{2\Lambda_H^2}\right)^{-1/2} \simeq -\frac{v^2}{4\Lambda_H^2} \quad \text{(tree level)}. \quad (46)$$

The shift in m_H comes from rescaling the field h such that its kinetic term is canonically normalized, as necessitated by the first term in Eq. (45). To derive constraints on Λ_H, a χ^2 analysis has to be done, which can be facilitated by the expansion

$$\bar{\delta}^{\mathrm{NP}}\hat{O}_i = \xi_i - d_{i,m_Z}\xi_{m_Z} - d_{i,m_H}\xi_{m_H}$$
$$\simeq \xi_i - (d_{i,m_Z} - d_{i,m_H})\left(\frac{123\ \mathrm{GeV}}{\Lambda_H}\right)^2. \quad (47)$$

Among the output observables in Table I, only those related to Z boson decay have nonzero ξ_i at tree level due to the shift in m_Z:

$$\xi_{\Gamma_f} = \xi_{\Gamma_{\mathrm{inv}}} = \xi_{\Gamma_{\mathrm{had}}} = \xi_{\Gamma_Z} = \xi_{m_Z} = \frac{v^2}{4\Lambda_H^2}, \quad (48)$$

$$\xi_{\sigma_{\mathrm{had}}} = -2\xi_{m_Z} = -\frac{v^2}{2\Lambda_H^2}. \quad (49)$$

Thus, for example,

$$\bar{\delta}^{\mathrm{NP}}\Gamma_Z \simeq (1 - d_{\Gamma_Z,m_Z} + d_{\Gamma_Z,m_H})\left(\frac{123\ \mathrm{GeV}}{\Lambda_H}\right)^2$$
$$\simeq -2.70\left(\frac{123\ \mathrm{GeV}}{\Lambda_H}\right)^2, \quad (50)$$

$$\bar{\delta}^{\mathrm{NP}}R_b \simeq -(d_{R_b,m_Z} - d_{R_b,m_H})\left(\frac{123\ \mathrm{GeV}}{\Lambda_H}\right)^2$$
$$\simeq 0.0948\left(\frac{123\ \mathrm{GeV}}{\Lambda_H}\right)^2. \quad (51)$$

For both operators considered above, the new physics contribution is on the order $\frac{v^2}{\Lambda^2}$. If we were to calculate $\bar{\delta}^{\mathrm{NP}}\hat{O}_i$ to higher order, we would have

$$\bar{\delta}^{\mathrm{NP}}\hat{O}_i \sim \mathcal{O}\left(\frac{v^2}{\Lambda^2}\right)\left[1+\mathcal{O}\left(\frac{v^2}{\Lambda^2}\right)\right]\left[1+\mathcal{O}\left(\frac{\alpha_s}{4\pi}\right)\right]. \quad (52)$$

Neglecting these higher-order corrections will result in errors in the derived constraints on Λ, typically at the percentage level. However, much effort has been devoted to calculating observables within the SM to a much higher accuracy, and such accuracy is reflected in $\hat{O}_i^{\mathrm{ref}}$ and $d_{ii'}$ presented in this paper. There is no contradiction here, because [recall Eq. (38)]

$$\hat{O}_i^{\mathrm{th}} = \hat{O}_i^{\mathrm{ref}}(1+\bar{\delta}\hat{O}_i^{\mathrm{th}}) = \hat{O}_i^{\mathrm{ref}}\left(1+\sum_{i'} d_{ii'}\bar{\delta}\hat{O}_{i'}^{\mathrm{th}} + \bar{\delta}^{\mathrm{NP}}\hat{O}_i\right). \quad (53)$$

To discern new physics contributions of order $\frac{v^2}{\Lambda^2}$, we must calculate $\hat{O}_i^{\mathrm{ref}}$ and $d_{ii'}$ to a better accuracy, hence the need for higher loop–order calculations. The higher-order calculation of ξ_i, on the other hand, usually does not contribute as much to $\hat{O}_i^{\mathrm{th}}$, because $\bar{\delta}^{\mathrm{NP}}\hat{O}_i$ is $\mathcal{O}(\frac{v^2}{\Lambda^2})$ anyway. In a word, if we only calculate ξ_i (and hence $\bar{\delta}^{\mathrm{NP}}\hat{O}_i$) at tree level, we will constrain new physics models with a few percent uncertainty; but if we did not calculate $\hat{O}_i^{\mathrm{ref}}$ and $d_{ii'}$ to multiloop level, we would not be able to constrain them at all.

B. Shifts in $Zb\bar{b}$ couplings

Suppose some new physics model shifts the Z boson couplings to left- and right-handed b quarks [19]

$$c_L^b \to c_L^b(1+\varepsilon_L), \qquad c_R^b \to c_R^b(1+\varepsilon_R). \quad (54)$$

None of the input observables is affected at tree level. Thus, the impact of the shifts of these couplings can be calculated straightforwardly from observables that directly depend on c_L^b and c_R^b. The set of observables directly affected include Γ_b, Γ_{had}, $R_{e,\mu,\tau}$, $R_{c,b}$, Γ_Z, σ_{had}, $\mathcal{A}_b$, A_{FB}^b, and $\sin^2\theta_{\mathrm{eff}}^b$. Their shifts from this new physics contribution can be expressed as

$$\bar{\delta}^{\mathrm{NP}}\hat{O}_i = \xi_i. \quad (55)$$

Let us begin by computing the shift in Γ_b. At tree level, $\Gamma_b \propto [(c_L^b)^2 + (c_R^b)^2]$, which when expanded leads to the shift $\bar{\delta}^{\mathrm{NP}}\Gamma_b = \xi_{\Gamma_b}$, where

$$\xi_{\Gamma_b} = \frac{2(c_L^b)^2}{(c_L^b)^2+(c_R^b)^2}\varepsilon_L + \frac{2(c_R^b)^2}{(c_L^b)^2+(c_R^b)^2}\varepsilon_R$$
$$\simeq 1.94\varepsilon_L + 0.0645\varepsilon_R. \quad (56)$$

Knowing this shift in Γ_b enables us to simply compute the shift of other observables that depend on Γ_b in terms of ξ_{Γ_b}:

$$\bar{\delta}^{\rm NP}\Gamma_{\rm had} = \bar{\delta}^{\rm NP}R_e = \bar{\delta}^{\rm NP}R_\mu = \bar{\delta}^{\rm NP}R_\tau = -\bar{\delta}^{\rm NP}R_c = R_b\xi_{\Gamma_b} \simeq 0.216\xi_{\Gamma_b}, \tag{57}$$

$$\bar{\delta}^{\rm NP}R_b = \bar{\delta}^{\rm NP}\Gamma_b - \bar{\delta}^{\rm NP}\Gamma_{\rm had} = (1-R_b)\xi_{\Gamma_b} \simeq 0.784\xi_{\Gamma_b}, \tag{58}$$

$$\bar{\delta}^{\rm NP}\Gamma_Z = B_b\xi_{\Gamma_b} \simeq 0.151\xi_{\Gamma_b}, \tag{59}$$

$$\bar{\delta}^{\rm NP}\sigma_{\rm had} = \bar{\delta}^{\rm NP}\Gamma_{\rm had} - 2\bar{\delta}^{\rm NP}\Gamma_Z = (R_b - 2B_b)\xi_{\Gamma_b} \simeq -0.0855\xi_{\Gamma_b}, \tag{60}$$

where $B_b = \Gamma_b/\Gamma_Z$ is the branching ratio of $Z \to b\bar{b}$.

The asymmetry observables are also affected due to the shift in $\mathcal{A}_b$. At tree level,

$$\mathcal{A}_b = \frac{(c_L^b)^2 - (c_R^b)^2}{(c_L^b)^2 + (c_R^b)^2}, \tag{61}$$

which leads to a shift $\bar{\delta}^{\rm NP}\mathcal{A}_b = \xi_{\mathcal{A}_b}$, where

$$\xi_{\mathcal{A}_b} = \frac{4(c_L^b)^2(c_R^b)^2}{(c_L^b)^4 - (c_R^b)^4}(\epsilon_L - \epsilon_R) \simeq 0.134(\epsilon_L - \epsilon_R). \tag{62}$$

We can then straightforwardly compute $\bar{\delta}^{\rm NP}A_{\rm FB}^b$ and $\bar{\delta}^{\rm NP}\sin^2\theta_{\rm eff}^b$ in terms of $\xi_{\mathcal{A}_b}$:

$$\bar{\delta}^{\rm NP}A_{\rm FB}^b = \xi_{\mathcal{A}_b}, \tag{63}$$

and

$$\bar{\delta}^{\rm NP}\sin^2\theta_{\rm eff}^b = \left[\frac{\sin^2\theta_{\rm eff}^b}{\mathcal{A}_b}\frac{\partial\mathcal{A}_b}{\partial\sin^2\theta_{\rm eff}^b}\right]^{-1}\xi_{\mathcal{A}_b} = \frac{(1-\frac{4}{3}\sin^2\theta_{\rm eff}^b)[1+(1-\frac{4}{3}\sin^2\theta_{\rm eff}^b)^2]}{-\frac{4}{3}\sin^2\theta_{\rm eff}^b[1-(1-\frac{4}{3}\sin^2\theta_{\rm eff}^b)^2]}\xi_{\mathcal{A}_b} \simeq -6.24\xi_{\mathcal{A}_b}. \tag{64}$$

Thus, $\bar{\delta}^{\rm NP}\hat{O}_i$ for all observables are expressed in terms of ξ_{Γ_b} or $\xi_{\mathcal{A}_b}$, which are simply related to ϵ_L, ϵ_R via Eqs. (56) and (62).

C. Shifts in vector boson self-energies

In many new physics scenarios, there exist exotic states that do not couple directly to SM fermions but have charges under the SM gauge groups. These states affect electroweak observables via shifts in vector boson self-energies [20]. At one-loop level, the dependence of various observables on vector boson self-energies is as follows [21]:

$$m_Z^2 = [m_Z^2]^{(0)}(1+\pi_{zz}), \tag{65}$$

$$m_W^2 = [m_W^2]^{(0)}(1+\pi_{ww}), \tag{66}$$

$$G_F = [G_F]^{(0)}(1-\pi_{ww}^0), \tag{67}$$

$$\alpha(m_Z) = [\alpha(m_Z)]^{(0)}(1+\pi'_{\gamma\gamma}), \tag{68}$$

$$\sin^2\theta_{\rm eff}^f = s^2\left(1-\frac{c}{s}\pi_{\gamma z}\right), \tag{69}$$

$$\Gamma_f = [\Gamma_f]^{(0)}\left(1+\pi'_{zz}+\frac{1}{2}\pi_{zz}+a_f\pi_{\gamma z}\right), \tag{70}$$

where superscripts "(0)" denote tree-level values, and $s = \frac{g_1}{\sqrt{g_1^2+g_2^2}}$, $c = \frac{g_2}{\sqrt{g_1^2+g_2^2}}$. We have also defined

$$\pi_{zz} \equiv \frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2}, \tag{71}$$

$$\pi'_{zz} \equiv \lim_{q^2\to m_Z^2}\frac{\Pi_{ZZ}(q^2)-\Pi_{ZZ}(m_Z^2)}{q^2-m_Z^2}, \tag{72}$$

$$\pi_{\gamma z} \equiv \frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2}, \tag{73}$$

$$\pi'_{\gamma\gamma} \equiv \lim_{q^2\to 0}\frac{\Pi_{\gamma\gamma}(q^2)-\Pi_{\gamma\gamma}(0)}{q^2}, \tag{74}$$

$$\pi_{ww} \equiv \frac{\Pi_{WW}(m_W^2)}{m_W^2}, \tag{75}$$

$$\pi_{ww}^0 \equiv \frac{\Pi_{WW}(0)}{m_W^2}. \tag{76}$$

The a_f in Eq. (70) can be derived from

$$\Gamma_f = [\Gamma_f]^{(0)}(1+\pi'_{zz}+\pi_{zz})\frac{1+(1-4|Q_f|\sin^2\theta_{\rm eff}^f)^2}{1+(1-4|Q_f|s^2)^2} \tag{77}$$

and Eq. (69). The result is

$$a_f = \frac{8sc|Q_f|(1-4|Q_f|s^2)}{1+(1-4|Q_f|s^2)^2} = 4sc|Q_f|[\mathcal{A}_f]^{(0)}. \tag{78}$$

With $s^2 \simeq \sin^2\theta_{\rm eff}^e = 0.231620$, which is good at tree level, we have

$$a_\nu = 0, \quad a_\ell = 0.2468, \quad a_u = 0.7505, \quad a_d = 0.5262. \tag{79}$$

With Eqs. (65)–(70), it is straightforward to calculate contributions from new physics. Denote the shifts in vector boson self-energies by $\delta^{\rm NP}\pi_{zz}$, etc.; i.e.

$$\pi_{zz} \to \pi_{zz} + \delta^{\rm NP}\pi_{zz}, \quad \text{etc.} \tag{80}$$

Note the absence of "bar" on δ, since this is the absolute shift, not the fractional shift. Then for the input observables,

$$\xi_{m_Z} = \frac{1}{2}\delta^{\rm NP}\pi_{zz}, \qquad \xi_{G_F} = -\delta^{\rm NP}\pi^0_{ww}, \qquad \xi_\alpha = \delta^{\rm NP}\pi'_{\gamma\gamma},$$
$$\xi_{m_t} = \xi_{\alpha_s} = \xi_{m_H} = 0. \tag{81}$$

These shifts propagate into shifts in the output observables, while leaving the input observables unchanged due to new physics (i.e. $\bar\delta^{\rm NP}\hat O_{i'} = 0$). The new physics contribution to the output observables can be conveniently expressed as

$$\bar\delta^{\rm NP}\hat O_i = \xi_i - \sum_{i'} d_{ii'}\xi_{i'}$$
$$\equiv b_{i,zz}\delta^{\rm NP}\pi_{zz} + b'_{i,zz}\delta^{\rm NP}\pi'_{zz} + b_{i,\gamma z}\delta^{\rm NP}\pi_{\gamma z} + b'_{i,\gamma\gamma}\delta^{\rm NP}\pi'_{\gamma\gamma} + b_{i,ww}\delta^{\rm NP}\pi_{ww} + b^0_{i,ww}\delta^{\rm NP}\pi^0_{ww}. \tag{82}$$

In the following we discuss the calculation of these b coefficients.

(i) $b'_{i,zz}$, $b_{i,ww}$ are the simplest, since they vanish for most of the observables. In particular, $b'_{i,zz}$, which comes from wave function renormalization, is nonzero only for Z boson decay widths:

$$b'_{\Gamma_f,zz} = b'_{\Gamma_{\rm inv},zz} = b'_{\Gamma_{\rm had},zz} = b'_{\Gamma_Z,zz} = 1. \tag{83}$$

Note that wave function renormalization cancels out in $\sigma_{\rm had}$, and ratios of decay widths. $b_{i,ww}$ is related to the shift in the W boson mass, so is nonzero only for

$$b_{m_W,ww} = \frac{1}{2}. \tag{84}$$

(ii) $b_{i,zz}$, $b'_{i,\gamma\gamma}$, $b^0_{i,ww}$ are simply related to d_{i,m_Z}, $d_{i,\alpha}$, d_{i,G_F}, respectively. Since $\pi'_{\gamma\gamma}$, π^0_{ww} only enter $\alpha(m_Z)$, G_F, respectively, we have

$$b'_{i,\gamma\gamma} = -d_{i,\alpha}, \qquad b^0_{i,ww} = d_{i,G_F} \tag{85}$$

for all $\hat O_i$. Similarly,

$$b_{i,zz} = -\frac{1}{2}d_{i,m_Z} \tag{86}$$

except for those observables having direct dependence on the Z boson mass:

$$b_{i,zz} = \frac{1}{2}(1 - d_{i,m_Z}) \quad \text{for } i = \Gamma_f, \Gamma_{\rm inv}, \Gamma_{\rm had}, \Gamma_Z, \tag{87}$$

$$b_{\sigma_{\rm had},zz} = -\frac{1}{2}(2 + d_{i,m_Z}). \tag{88}$$

(iii) Finally, $b_{i,\gamma z}$ should be derived from the dependence on $\sin^2\theta^f_{\rm eff}$. For the Z partial widths, it can be read off from Eq. (70):

TABLE VI. The b coefficients defined in Eq. (82), characterizing the shift in the output observables due to new physics that shifts vector boson self-energies.

$\hat O_i$	$b_{i,zz}$	$b'_{i,zz}$	$b_{i,\gamma z}$	$b'_{i,\gamma\gamma}$	$b_{i,ww}$	$b^0_{i,ww}$
m_W	−0.7140	0	0	0.2154	0.5	0.2201
Γ_e	−1.189	1	0.2468	0.1920	0	1.198
Γ_μ	−1.189	1	0.2468	0.1920	0	1.198
Γ_τ	−1.192	1	0.2468	0.1924	0	1.198
Γ_b	−1.423	1	0.5262	0.4166	0	1.411
Γ_c	−1.577	1	0.7505	0.5842	0	1.590
$\Gamma_{\rm inv}$	−0.9982	1	0	-1.913×10^{-3}	0	1.006
$\Gamma_{\rm had}$	−1.470	1	0.6027	0.4727	0	1.476
Γ_Z	−1.347	1	0.4420	0.3490	0	1.353
$\sigma_{\rm had}$	0.03475	0	−0.03460	−0.03328	0	−0.03281
R_e	−0.2811	0	0.3559	0.2807	0	0.2780
R_μ	−0.2811	0	0.3559	0.2807	0	0.2780
R_τ	−0.2784	0	0.3559	0.2803	0	0.2776
R_b	0.04731	0	−0.07647	−0.05608	0	−0.06530
R_c	−0.1069	0	0.1479	0.1115	0	0.1135
$\sin^2\theta^e_{\rm eff}$	1.413	0	−1.821	−1.426	0	−1.423
$\sin^2\theta^b_{\rm eff}$	1.415	0	−1.821	−1.427	0	−1.417
$\sin^2\theta^c_{\rm eff}$	1.413	0	−1.821	−1.426	0	−1.423
$\mathcal{A}_e$	−17.61	0	22.71	17.78	0	17.74
$\mathcal{A}_b$	−0.2268	0	0.2876	0.2287	0	0.2271
$\mathcal{A}_c$	−1.697	0	2.192	1.713	0	1.710
$A^e_{\rm FB}$	−35.22	0	45.41	35.56	0	35.48
$A^b_{\rm FB}$	−17.84	0	22.99	18.01	0	17.97
$A^c_{\rm FB}$	−19.31	0	24.90	19.50	0	19.45

$$b_{\Gamma_f,\gamma z} = a_f, \qquad b_{\Gamma_{\rm inv},\gamma z} = 3a_\nu = 0, \tag{89}$$

with a_f given in Eqs. (78) and (79). For $i = \Gamma_{\rm had}, \Gamma_Z$, $b_{i,\gamma z}$ is a weighted sum. At leading order,

$$b_{\Gamma_{\rm had},\gamma z} = \sum_{f\in {\rm had}} \frac{\Gamma_f}{\Gamma_{\rm had}} b_{\Gamma_f,\gamma z} = \frac{\sum_{f\in{\rm had}}[1+(1-4|Q_f|s^2)^2] b_{\Gamma_f,\gamma z}}{\sum_{f\in{\rm had}}[1+(1-4|Q_f|s^2)^2]}, \tag{90}$$

$$b_{\Gamma_Z,\gamma z} = \sum_f \frac{\Gamma_f}{\Gamma_Z} b_{\Gamma_f,\gamma z} = \frac{\sum_f [1+(1-4|Q_f|s^2)^2] b_{\Gamma_f,\gamma z}}{\sum_f [1+(1-4|Q_f|s^2)^2]}. \tag{91}$$

For the ratios of partial widths, and the Z-pole cross section,

$$b_{R_\ell,\gamma z} = b_{\Gamma_{\rm had},\gamma z} - b_{\Gamma_\ell,\gamma z}, \qquad b_{R_q,\gamma z} = b_{\Gamma_q,\gamma z} - b_{\Gamma_{\rm had},\gamma z},$$
$$b_{\sigma_{\rm had},\gamma z} = b_{\Gamma_e,\gamma z} + b_{\Gamma_{\rm had},\gamma z} - 2b_{\Gamma_Z,\gamma z}. \tag{92}$$

For the asymmetry observables, we can read off from Eq. (69):

$$b_{\sin^2\theta^f_{\rm eff},\gamma z} = -\frac{c}{s}. \tag{93}$$

And hence, at leading order,

$$b_{\mathcal{A}_f,\gamma z} = \frac{s^2}{[\mathcal{A}_f]^{(0)}} \frac{\partial[\mathcal{A}_f]^{(0)}}{\partial(s^2)} b_{\sin^2\theta^f_{\rm eff},\gamma z} = \frac{4|Q_f|sc[1-(1-4|Q_f|s^2)^2]}{(1-4|Q_f|s^2)[1-(1+4|Q_f|s^2)^2]}, \tag{94}$$

$$b_{A^f_{\rm FB},\gamma z} = b_{\mathcal{A}_e,\gamma z} + b_{\mathcal{A}_f,\gamma z}. \tag{95}$$

The numerical values for these b coefficients are listed in Table VI. The calculation is done with $s^2 = 0.231620$, and the sign conventions for the gauge couplings are $g_1 > 0$, $g_2 > 0$ (hence $s > 0$).

V. CONCLUSION

In this paper we presented an expansion formalism that facilitates precision electroweak analysis. By recasting all observables in terms of six very well-measured input observables, we can calculate each of them easily by expanding about the reference values of the input observables, chosen in accord with experimental measurements. Also, the formalism developed here can be applied in a simple manner to calculate new physics corrections to electroweak observables and derive constraints on new physics models. Some examples were worked out for illustration.

For numerical results we calculated the reference values and expansion coefficients using the ZFITTER package. Most, though not all, of these results reflect state-of-the-art calculations in the literature. Various higher-order calculations of electroweak observables have been done since the release of ZFITTER 6.42 in 2005, but their impact on precision analysis is not significant at present because the power of the precision program is limited by experimental errors. However, improvements of our results to better accuracy with the inclusion of these and future calculations may be necessary in the future, if experimental priorities of next-generation facilities involve Giga-Z or Tera-Z options [22,23]. With 10^9 or 10^{12} Z bosons produced at a future collider, unprecedented levels of reliable theoretical calculations will be needed to meet the unprecedented levels of experimental accuracy. We hope that the formalism presented here, with improving numerical results, will continue to be helpful for efficient and reliable calculations of SM results and beyond the SM corrections in the precision electroweak program.

ACKNOWLEDGMENTS

This work was supported in part by the Department of Energy. We wish to thank T. Riemann for helpful communications regarding ZFITTER, and A. Freitas for pointing out a mistake in our implementation of higher-order QCD corrections in an earlier version.

APPENDIX A: TECHNICAL DETAILS OF ZFITTER

We rely on ZFITTER 6.42 for all numerical calculations of observables, and obtain the expansion coefficients $c_{ii'}$, $c_{ii'j'}$ by numerical differentiation. Some calculational details are presented in this appendix.

We use the DIZET package in ZFITTER, modified slightly to allow for G_F as input. The flags are set to default listed in [6], with the following exceptions:

(i) NPAR(7) = IALEM = 2 (default = 3) to allow for $\Delta\alpha^{(5)}_{\rm had}$ as input.

(ii) NPAR(20) = IGFER = 3 (default = 2) to allow for G_F as input. Note that the only available options for this flag in ZFITTER are 0, 1, 2, and none of them allows us to treat G_F as input (since it is extraordinarily well measured), but we added a new option 3 to be consistent with the modification of the codes mentioned above.

In principle, alternative choices for the flags are possible. But to be consistent with our formalism, the following flags should not be changed from default:

(i) NPAR(2) = IAMT4 (default = 4): 4 is the only option consistent with treating G_F as input.

(ii) NPAR(4) = IMOMS (default = 1): 1 treats m_Z as input and m_W as output, not otherwise.

TABLE VII. The h chosen for each input observable in numerical differentiation. See Eq. (A1).

$\hat{O}_{i'}$	m_Z	G_F	$\Delta\alpha^{(5)}_{\rm had}$	m_t	$\alpha_s(m_Z)$	m_H
h	10^{-6}	10^{-5}	10^{-4}	10^{-4}	10^{-4}	10^{-4}

The derivatives appearing in $c_{ii'}$ [Eq. (20)] are carried out numerically via [24]

$$\frac{\partial \hat{O}_i^{\rm SM}}{\partial \hat{O}_{i'}^{\rm SM}} \simeq \frac{\hat{O}_i^{\rm SM}|_{(1+h)\hat{O}_{i'}^{\rm ref}} - \hat{O}_i^{\rm SM}|_{(1-h)\hat{O}_{i'}^{\rm ref}}}{2h\hat{O}_{i'}^{\rm ref}}, \tag{A1}$$

where h is chosen differently for different input observables; see Table VII. The choices are made empirically, and are expected to be optimal in reducing the combination of truncation and roundoff errors.[3] We found that the numerical errors typically occur at the seventh or eighth digit, and thus do not affect the digits presented in the tables earlier in this paper.

For calculating $c_{ii'j'}$ [Eq. (24)], on the other hand, we make use of the fact that

$$c_{ii'j'} = \left[\hat{O}_{j'}^{\rm SM}\frac{\partial c_{ii'}}{\partial \hat{O}_{j'}^{\rm SM}} + c_{ii'}c_{ij'} - \delta_{i'j'}c_{ii'}\right]\Bigg|_{\hat{O}_{i'}=\hat{O}_{i'}^{\rm ref}}, \tag{A2}$$

and evaluate the derivatives with the same h mentioned above.

APPENDIX B: QCD CORRECTIONS TO Z DECAY

In this appendix we discuss the calculation of Γ_q. As was mentioned in Sec. III C, this discussion is motivated by two features in our numerical results. First, the uncertainty in c_{R_b,α_s} is much larger than that in all other expansion coefficients. Second, c_{Γ_q,α_s}, which characterize the sensitivity of $Z \to q\bar{q}$ partial widths to the strong coupling constant, are very different for different quarks (see Table VIII), though at leading order QCD corrections are flavor universal. This second feature led us to investigate and confirm the reliability of our numerical calculation. Both features are related to $\mathcal{O}(\alpha_s^2)$ corrections, as we will explain in the following.

Following the notations in ZFITTER [5], we write the formula that calculates the partial width of the Z boson to $q\bar{q}$ as follows:

[3] We calculated the derivatives with h varied within a wide range, and recognized the regime where the results fluctuate (roundoff error dominates) and the regime where the results vary monotonically (truncation error dominates). The optimal h is in between these two regimes. In principle, the optimal h can be determined from the machine precision and the algorithm for evaluating the functions. But in practice, this is difficult due to the complexity of calculations in ZFITTER, so we took this empirical approach.

TABLE VIII. Numerical values of c_{Γ_q,α_s} and $c_{\Gamma_{\rm had},\alpha_s}$. The difference among these numbers is explained in the text. Note that $\Gamma_{u,d,s}$ are not in our observables list, since they are practically unmeasurable.

q	u	c	d	s	b	had
c_{Γ_q,α_s}	0.04892	0.05046	0.02697	0.02697	0.03672	0.03690

$$\Gamma_q = 3\Gamma_0|\rho_Z^q|(|g_Z^q|^2R_V^q + R_A^q) + \Delta_{\rm EW/QCD}, \tag{B1}$$

where

$$\Gamma_0 = \frac{G_F m_Z^3}{24\sqrt{2}\pi} \simeq 83 \text{ MeV}. \tag{B2}$$

ρ_Z^q and g_Z^q are effective couplings that incorporate electroweak loop corrections to the Z decay; in particular, g_Z^q is the ratio of effective vector and axial couplings. R_V^q and R_A^q are vector and axial radiator functions, which deal with final state QCD and QED radiation. There is also an additive mixed EW/QCD correction term $\Delta_{\rm EW/QCD}$ that does not factorize.

The radiator functions R_V^q and R_A^q actually depend on the energy scale. In Eq. (B1) it is implicit that they are evaluated at the Z mass. Explicitly, the vector radiator function is given by

$$R_V^q = 1 + \frac{3}{4}Q_q^2\frac{\alpha}{\pi} + \frac{\alpha_s}{\pi} - \frac{1}{4}Q_q^2\frac{\alpha}{\pi}\frac{\alpha_s}{\pi} + \left[C_{02} + C_2^t\left(\frac{m_Z^2}{m_t^2}\right)\right]\left(\frac{\alpha_s}{\pi}\right)^2 + C_{03}\left(\frac{\alpha_s}{\pi}\right)^3 + \mathcal{O}(\alpha^2), \mathcal{O}(\alpha_s^4), \mathcal{O}(m_q^2), \tag{B3}$$

where

$$C_{02} = \frac{365}{24} - 11\zeta(3) + \left[-\frac{11}{12} + \frac{2}{3}\zeta(3)\right]n_q, \tag{B4}$$

$$C_2^t(x) = x\left(\frac{44}{675} - \frac{2}{135}\ln x\right) + \mathcal{O}(x^2), \tag{B5}$$

$$C_{03} = \frac{87029}{288} - \frac{121}{8}\zeta(2) - \frac{1103}{4}\zeta(3) + \frac{275}{6}\zeta(5) + \left[-\frac{7847}{216} + \frac{11}{6}\zeta(2) + \frac{262}{9}\zeta(3) - \frac{25}{9}\zeta(5)\right]n_q + \left[\frac{151}{162} - \frac{1}{18}\zeta(2) - \frac{19}{27}\zeta(3)\right]n_q^2. \tag{B6}$$

ζ is the Riemann zeta function. At the Z pole the number of light quark flavors $n_q = 5$.

To the order shown in Eq. (B3), R_A^q receives additional contributions at $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_s^3)$:

$$R_A^q = R_V^q - 2T_q^{(3)}\left[I^{(2)}\left(\frac{m_Z^2}{m_t^2}\right)\left(\frac{\alpha_s}{\pi}\right)^2 + I^{(3)}\left(\frac{m_Z^2}{m_t^2}\right)\left(\frac{\alpha_s}{\pi}\right)^3\right] + \mathcal{O}(\alpha^2), \mathcal{O}(\alpha_s^4), \mathcal{O}(m_q^2), \tag{B7}$$

where $T_q^{(3)} = +\frac{1}{2}\,(-\frac{1}{2})$ for up (down) type quarks, and

$$I^{(2)}(x) = -\frac{37}{12} + \ln x + \frac{7}{81}x + \frac{79}{6000}x^2 + \mathcal{O}(x^3), \tag{B8}$$

$$I^{(3)}(x) = -\frac{5075}{216} + \frac{23}{6}\zeta(2) + \zeta(3) + \frac{67}{18}\ln x + \frac{23}{12}\ln^2 x + \mathcal{O}(x). \tag{B9}$$

These terms are called singlet axial corrections. $I^{(2)}$ was first calculated in [25,26]. There the focus was on the total hadronic width, and the singlet axial corrections (approximately) cancel among the "light" quarks u, d, c, s. However, these terms are visible in each partial width, and are numerically comparable to the $\mathcal{O}(\alpha_s)$ terms. Being negative, they make c_{Γ_u,α_s}, c_{Γ_c,α_s} larger than c_{Γ_d,α_s}, c_{Γ_s,α_s}.

We might expect c_{Γ_b,α_s} to be close to c_{Γ_d,α_s}, c_{Γ_s,α_s}, but in Table VIII it is seen to be larger. This is due to a positive contribution from the $\mathcal{O}(m_q^2)$ terms, which are significant only for the b quark. To be precise, m_q in these terms should be taken as the running masses at the Z pole, obtained by solving renormalization group (RG) equations. For the b quark, the dependence of these RG equations on α_s is strong enough to overcome the $\frac{m_b^2}{m_Z^2}$ suppression, and the contribution to c_{Γ_b,α_s} turns out to be positive. Similarly, c_{Γ_c,α_s} also receives a positive contribution, which explains the small difference from c_{Γ_u,α_s}.

Now that we have understood the difference among c_{Γ_q,α_s} and are confident about their numerical values, we can calculate $c_{\Gamma_{\rm had},\alpha_s}$ by a weighted average, and the result is, by accident, very close to c_{Γ_b,α_s} (see Table VIII). As a result, $c_{R_b,\alpha_s} = c_{\Gamma_b,\alpha_s} - c_{\Gamma_{\rm had},\alpha_s}$ is much smaller than either of c_{Γ_b,α_s}, $c_{\Gamma_{\rm had},\alpha_s}$, and can thus have large uncertainty though the uncertainties in the latter are small.

Finally, a few comments are in order regarding future improvements of the Z decay calculation. Recent developments, including the complete $\mathcal{O}(\alpha_s^4)$ QCD corrections [27,28] and fermionic electroweak two-loop corrections [29] will be implemented in future versions of ZFITTER [30], which will certainly help improve the accuracy of our results. Meanwhile, we note two other aspects of the ZFITTER calculation that could be improved. First, the $\Delta_{\rm EW/QCD}$ term in Eq. (B1) is implemented as fixed numbers in ZFITTER, so the dependence on input observables is lost, which is especially relevant in the expansion formalism. Second, the $\mathcal{O}(\alpha_s^3)$ difference between $\Gamma_{\rm had}$ and $\sum_q \Gamma_q$ mentioned in a footnote in Sec. II, though calculated and stored in ZPAR(29) = QCDCOR(13), is not included in the calculation of $\Gamma_{\rm had}$ or the total width Γ_Z. The size of this term is only on the order of $10^{-5}\Gamma_{\rm had}$ [9], but the error might be magnified when the expansion coefficients are calculated.

[1] G. Aad *et al.* (ATLAS Collaboration), Phys. Lett. B **716**, 1 (2012).
[2] S. Chatrchyan *et al.* (CMS Collaboration), Phys. Lett. B **716**, 30 (2012).
[3] S. Chatrchyan *et al.* (CMS Collaboration), Phys. Rev. Lett. **110**, 081803 (2013).
[4] L. G. Almeida, S. J. Lee, S. Pokorski, and J. D. Wells, Phys. Rev. D **89**, 033006 (2014).
[5] D. Y. Bardin, P. Christova, M. Jack, L. Kalinovskaya, A. Olchevski, S. Riemann, and T. Riemann, Comput. Phys. Commun. **133**, 229 (2001).
[6] A. B. Arbuzov, M. Awramik, M. Czakon, A. Freitas, M. W. Grunewald, K. Monig, S. Riemann, and T. Riemann, Comput. Phys. Commun. **174**, 728 (2006).
[7] J. Beringer *et al.* (Particle Data Group Collaboration), Phys. Rev. D **86**, 010001 (2012) (and 2013 partial update for the 2014 edition).
[8] See http://physics.nist.gov/cuu/Constants/.
[9] K. G. Chetyrkin, J. H. Kuhn, and A. Kwiatkowski, Phys. Rep. **277**, 189 (1996).
[10] S. Schael *et al.* (ALEPH, DELPHI, L3, OPAL, SLD, LEP Electroweak Working Group, SLD Electroweak Group, and SLD Heavy Flavour Group Collaborations), Phys. Rep. **427**, 257 (2006).
[11] V. M. Abazov *et al.* (D0 Collaboration), Phys. Rev. D **84**, 012007 (2011).
[12] J. Han (CDF Collaboration), arXiv:1110.0153.
[13] Electroweak Working Group (CDF and D0 Collaborations), arXiv:1204.0042.
[14] J. Alcaraz *et al.* (ALEPH, DELPHI, L3, OPAL, and LEP Electroweak Working Group Collaborations), arXiv:hep-ex/0612034.
[15] M. Muether *et al.* (Tevatron Electroweak Working Group and CDF and D0 Collaborations), arXiv:1305.3929.
[16] W. Buchmuller and D. Wyler, Nucl. Phys. **B268**, 621 (1986).
[17] B. Grzadkowski, M. Iskrzynski, M. Misiak, and J. Rosiek, J. High Energy Phys. 10 (2010) 085.
[18] M. E. Peskin and J. D. Wells, Phys. Rev. D **64**, 093003 (2001).
[19] See, e.g., P. Bamert, C. P. Burgess, J. M. Cline, D. London, and E. Nardi, Phys. Rev. D **54**, 4275 (1996).
[20] An early comprehensive paper on this possibility is M. E. Peskin and T. Takeuchi, Phys. Rev. D **46**, 381 (1992).

[21] J. D. Wells, arXiv:hep-ph/0512342.

[22] S. Heinemeyer and G. Weiglein, arXiv:1007.5232.

[23] M. Bicer *et al.* (TLEP Design Study Working Group Collaboration), J. High Energy Phys. 01 (2014) 164.

[24] W. H. Press, S. A. Teukosky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C: The Art of Scientific Computing*, 2nd ed. (Cambridge University Press, Cambridge, England, 1992).

[25] B. A. Kniehl and J. H. Kuhn, Phys. Lett. B **224**, 229 (1989).

[26] B. A. Kniehl and J. H. Kuhn, Nucl. Phys. **B329**, 547 (1990).

[27] P. A. Baikov, K. G. Chetyrkin, J. H. Kuhn, and J. Rittinger, Phys. Rev. Lett. **108**, 222003 (2012).

[28] P. A. Baikov, K. G. Chetyrkin, J. H. Kuhn, and J. Rittinger, *Proc. Sci.*, RADCOR2011 (2011) 030 [arXiv:1210.3594].

[29] A. Freitas, J. High Energy Phys. 04 (2014) 070.

[30] A. Akhundov, A. Arbuzov, S. Riemann, and T. Riemann, Phys. Part. Nucl. **45**, 529 (2014); T. Riemann (private communication).