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# Newtonian hydrodynamics in the comoving gauge

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Einstein's gravity has the exact Newtonian limit as we take the infinite-speed-of-light limit in the zero-shear gauge and the uniform-expansion gauge. Although lacking proper Newtonian gravitational potential, here we show that Newtonian hydrodynamics for density and velocity is also recovered in the comoving gauge using the weak gravity and negligible pressure limits but without using the slow-motion and subhorizon conditions. This curious correspondence, however, is available only for the irrotational fluid.

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## I. INTRODUCTION

Einstein's gravity is crucially demanded in many cosmological studies including the light propagation, early Universe with radiation era, inflation era, the dark energy, the evolution beyond horizon, etc. Newton's gravity is still very important in cosmology because of its relatively simple mathematical structure compared with Einstein's gravity. In practice, Newton's gravity is known to be quite successful in the weakly nonlinear/fully relativistic (near and beyond horizon) and the weakly realtivistic/fully nonlinear (small-scale clustering) situations in the matter dominated era with the cosmological constant as a dark energy [1].

Einstein's gravity has a proper Newtonian hydrodynamics as the infinite-speed-of-light limit [2]. Expansions to the next orders in the speed of light lead to the post-Newtonian approximation [2,3]. This is known even in the context of cosmological spacetime [4]. Although we need Einstein's gravity to handle the background world model [5], in the nonlinear perturbation level Newtonian equations indeed follow from Einstein's gravity as the infinite-speed-of-light limit (weak gravity, slow-motion, negligible pressure, and subhorizon scale) in the zero-shear gauge and the uniform-expansion gauge [6].

Meanwhile, despite its lacking proper Newtonian gravitational potential, the comoving gauge [7] is known to have striking Newtonian correspondence (in the density and velocity perturbations), even to higher order perturbations in all scales [1].

In this work, we study the Newtonian limit of Einstein's gravity in the comoving gauge condition. In this gauge we also have proper Newtonian hydrodynamics for density and velocity perturbations (modulo gravitational potential). Compared with the case in the zero-shear and the uniform-expansion gauges, the comoving gauge shows both strength (slow-motion and subhorizon conditions not demanded) and weakness (scalar-type perturbation only), which will be expounded in the following. Our proof is based on the fully nonlinear and exact perturbation theory in Einstein's gravity [8] summarized in Sec. II and the Appendix.

## II. NOTATIONS

We consider the scalar- and vector-type perturbations in a flat Friedmann background with the metric

$$ds^2 = -a^2(1+2\alpha)dx^0dx^0 - 2a\chi_i dx^0 dx^i + a^2(1+2\varphi)\delta_{ij}dx^i dx^j, \quad (1)$$

where $a(x^0)$ is the cosmic scale factor, and $\alpha$, $\varphi$, and $\chi_i$ are functions of spacetime with arbitrary amplitudes; the index of $\chi_i$ is raised and lowered by $\delta_{ij}$ as the metric; here, $x^0 = \eta$ with $ad\eta \equiv cdt$. The spatial part of the metric is simple because we have ignored the transverse-tracefree part and already have taken the spatial gauge condition without losing any generality to the fully nonlinear order [8,9].

We consider a fluid without anisotropic stress. The energy-momentum tensor is given as

$$\tilde{T}_{ab} = \tilde{\varrho}c^2\tilde{u}_a\tilde{u}_b + \tilde{p}(\tilde{g}_{ab} + \tilde{u}_a\tilde{u}_b), \quad (2)$$

where tildes indicate the covariant quantities; $\tilde{u}_a$ is the normalized fluid four-vector; and $\tilde{\varrho}$ and $\tilde{p}$ are the mass density and pressure, respectively. We decompose the fluid quantities into the background and perturbation as

$$\tilde{\varrho} = \varrho + \delta\varrho, \qquad \tilde{p} = p + \delta p, \qquad \tilde{u}_i \equiv a\frac{v_i}{c}, \quad (3)$$

where $\varrho$ and $p$ are functions of $x^0$ only, but the perturbed parts have arbitrary amplitudes; the index of $v_i$ is raised and lowered by $\delta_{ij}$ as the metric. In the explicit presence of the internal energy $\tilde{\varrho}\tilde{\Pi}$, $\tilde{\varrho}$ should be replaced by $\tilde{\varrho}(1+\tilde{\Pi}/c^2)$

 

[2]; in the latter expression $\tilde{\varrho}$ is the material density. We introduce $\hat{v}_i$ as

$$v_i \equiv \hat{\gamma}\hat{v}_i, \qquad \hat{\gamma} \equiv \frac{1}{\sqrt{1 - \frac{\hat{v}^k \hat{v}_k}{c^2(1+2\varphi)}}}. \tag{4}$$

Notice that the Lorentz factor is affected by gravity ($\varphi$) as well; in the Minkowski spacetime $\hat{v}_i$ can be identified as the velocity vector, but in the cosmological spacetime $\hat{v}_i$ is gauge dependent.

We can decompose $\chi_i$ and $\hat{v}_i$ into the scalar- and vector-type perturbations even to nonlinear order as

$$\chi_i = \chi_{,i} + \chi_i^{(v)}, \qquad \hat{v}_i \equiv -\hat{v}_{,i} + \hat{v}_i^{(v)}, \tag{5}$$

with $\chi^{(v)i}{}_{,i} \equiv 0 \equiv \hat{v}^{(v)i}{}_{,i}$.

The basic set of perturbation equations is made of seven fundamental equations presented in the Appendix. We emphasize that these equations are exact equations derived from Einstein equations based on our metric and energy-momentum conventions in Eqs. (1)–(4). Although the cosmic scale factor is introduced, we have not imposed any condition on all the perturbation amplitudes. In fact, the scale factors can be absorbed into the perturbation variables [e.g., $a^2(1+2\alpha) \equiv e^{2A}$, $a^2(1+2\varphi) \equiv e^{2B}$, $a\chi_i \equiv X_i$, and $a\hat{v}_i \equiv \hat{V}_i$ where $A$, $B$, $X_i$, and $\hat{V}_i$ are arbitrary spacetime variables], but we have intentionally introduced the scale factor in order to readily get the conventional background and perturbation equations in the Friedmann world model. Notice that in our basic equations we have not separated the background and perturbation. If we take a perturbative approach, we can separate the background and perturbation, and this can be continued to any higher order perturbation. In the present work we have not separated the background and perturbation; thus, the results are based on the fully nonlinear perturbation approach.

## III. NEWTONIAN CORRESPONDENCE IN THE COMOVING GAUGE

We take the comoving gauge, thus setting [8]

$$\hat{v} \equiv 0. \tag{6}$$

As the (temporal) comoving gauge together with our spatial gauge condition already taken in Eq. (1) completely fixes the gauge modes, all variables in this gauge condition have the unique gauge-invariant combinations to the fully nonlinear orders [8].

As the nonrelativistic limit we consider

$$\varphi \ll 1, \qquad \frac{\tilde{p}}{\tilde{\varrho}c^2} \ll 1, \tag{7}$$

which can be considered as the weak gravity and the negligible pressure conditions, respectively. But we do not assume both the slow-motion ($|\mathbf{v}|^2/c^2 \ll 1$) and small-scale [$\dot{a}/(kc) \ll 1$, the subhorizon] limits; $k$ is the wave number. The perturbed velocity $\mathbf{v}$ in the comoving gauge will be identified later: see Eqs. (12) and (14).

The covariant momentum conservation equation gives

$$\frac{1}{a}\left[\frac{\partial}{\partial t} + \frac{1}{a}\left(\mathcal{N}\hat{v}^{(v)k} + \frac{c}{a}\chi^k\right)\nabla_k\right](a\hat{v}_i^{(v)}) + \frac{c}{a^2}\hat{v}^{(v)k}\nabla_i\chi_k + \frac{c^2}{a}\nabla_i\mathcal{N} + \frac{\mathcal{N}}{a\tilde{\varrho}}\nabla_i\tilde{p} = 0. \tag{8}$$

To the linear order, we have $(a\hat{v}_i^{(v)})^{\cdot} = 0$ and $\mathcal{N} =$ constant in space; thus, $\mathcal{N} = 1$ and $\alpha = 0$; an overdot indicates the time derivative based on $t$. But, to the nonlinear order, the equation is nontrivial; the meaning of $v_i^{(v)}$ in the temporal comoving gauge condition is not clear; for example, the comoving gauge condition is not allowed in the Minkowski spacetime.

In the following we ignore the vector-type perturbation, thus setting $\hat{v}_i^{(v)} \equiv 0$ and $\chi_i = \chi_{,i}$. Thus, $\hat{v}_i = 0$, $\hat{\gamma} = 1$ and Eq. (8) gives

$$\frac{\mathcal{N}_{,i}}{\mathcal{N}} = -\frac{\tilde{p}_{,i}}{\tilde{\varrho}c^2}. \tag{9}$$

For $\tilde{p}/(\tilde{\varrho}c^2) \ll 1$ we have $\mathcal{N} = 1$. However, in the case in which we have $c^2$ multiplied as in the $c^2\Delta\mathcal{N}$ term in the trace and tracefree parts of the Arnowitt-Deser-Misner (ADM) propagation equation, we should keep the pressure terms properly as

$$c^2\mathcal{N}_{,i} = -\frac{\tilde{p}_{,i}}{\tilde{\varrho}}. \tag{10}$$

The seven fundamental equations give the following. Covariant momentum conservation:

$$\mathcal{N} = 1, \quad \text{or} \quad c^2\nabla\mathcal{N} = -\frac{\nabla\tilde{p}}{\tilde{\varrho}}. \tag{11}$$

Here, we identify the perturbed velocity $\mathbf{v} \equiv \nabla v$ as

$$\chi \equiv \frac{a}{c}v. \tag{12}$$

ADM momentum constraint or the definition of $\kappa$:

$$\kappa = -c\frac{\Delta}{a^2}\chi. \tag{13}$$

Thus, we have

$$\kappa = -\frac{1}{a}\nabla\cdot\mathbf{v} = -\frac{\Delta}{a}v. \tag{14}$$

Covariant energy conservation:

$$\dot{\tilde{\varrho}} + 3\frac{\dot{a}}{a}\tilde{\varrho} + \frac{1}{a}\nabla\cdot(\tilde{\varrho}\mathbf{v}) = 0. \tag{15}$$

Trace of ADM propagation:

$$\frac{1}{a}\nabla\cdot\left(\dot{\mathbf{v}}+\frac{\dot{a}}{a}\mathbf{v}\right)+\frac{1}{a^2}\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{v})+4\pi G\delta\varrho + \frac{1}{a^2}\nabla\cdot\left(\frac{\nabla\tilde{p}}{\tilde{\varrho}}\right)=0. \quad (16)$$

ADM energy constraint or tracefree ADM propagation [using Eq. (16)]:

$$c^2\frac{\Delta}{a^2}\varphi=-4\pi G\delta\varrho+\frac{\dot{a}}{a}\frac{1}{a}\nabla\cdot\mathbf{v} + \frac{1}{4a^2}[(\nabla\cdot\mathbf{v})^2-v^{,ij}v_{,ij}]. \quad (17)$$

As we have used all the fundamental equations, the results are consistent in the context of Einstein's gravity.

Notice that Eqs. (15) and (16) provide a closed set of equations for the fluid variables $\tilde{\varrho}$ and $\mathbf{v}$; $\tilde{p}$ is provided by the equation of state. Equation (17) can be regarded as a relation determining $\varphi$ from the fluid variables. By identifying

$$\Delta U\equiv c^2\Delta\varphi-\dot{a}\nabla\cdot\mathbf{v}-\frac{1}{4}[(\nabla\cdot\mathbf{v})^2-v^{,ij}v_{,ij}], \quad (18)$$

we recover the Poisson's equation

$$\frac{\Delta}{a^2}U=-4\pi G\delta\varrho, \quad (19)$$

where $U$ can be identified as the Newtonian gravitational potential; notice that $U$ is an *ad hoc* combination of fundamental variables in the comoving gauge. Here, we have no intention to suggest that the Newtonian gravitational potential is properly recovered in the comoving gauge even in the Newtonian limit. Equation (16) can be written as

$$\frac{1}{a}\nabla\cdot\left(\dot{\mathbf{v}}+\frac{\dot{a}}{a}\mathbf{v}+\frac{1}{a}\mathbf{v}\cdot\nabla\mathbf{v}-\frac{1}{a}\nabla U+\frac{1}{a}\frac{\nabla\tilde{p}}{\tilde{\varrho}}\right)=0, \quad (20)$$

which is the divergence of the momentum conservation equation in Newtonian context.

With Eqs. (15), (20), and (19) we have recovered the well-known Newtonian hydrodynamic equations [10]: the mass conservation, (divergence of) momentum conservation, and Poisson's equations, respectively. Equation (17) can be regarded as a relation determining $\varphi$ (the spatial curvature perturbation in the comoving gauge) from hydrodynamic quantities ($\delta\varrho$ and $\mathbf{v}$). The curvature perturbation $\varphi$ in the comoving gauge is a well-known conserved quantity (in super-sound-horizon scale) in the linear perturbation theory [11]. From Eqs. (15)–(17) we can show $\dot{\varphi}=0$ to the linear order.

## IV. DISCUSSION

Considering our previous works on the relativistic/Newtonian correspondence in the comoving gauge [1], our similar conclusion in this work may not be an unexpected one. However, in this work we have probed different aspects of the correspondence: here, we probe the case in the Newtonian limit but in a fully nonlinear context, whereas previous ones were based on general relativistic nonlinear perturbation theory. One byproduct is that to get the correspondence to fully nonlinear order in the Newtonian limit we do not need to assume slow-motion and subhorizon limit. Compared with the proper Newtonian limit available in the zero-shear (and the uniform-expansion) gauge as the infinite-speed-of-light limit, the situation in the comoving gauge shows both a weak point and strong point as we explain below.

In order to show the relativistic/Newtonian correspondence in the comoving gauge we have considered the weak gravity ($\varphi\ll 1$) and negligible relativistic pressure [$\tilde{p}/(\tilde{\varrho}c^2)\ll 1$] conditions, and have ignored the vector-type (rotational) perturbation. However, it is remarkable that we have not assumed both the small-scale [$\dot{a}/(kc)\ll 1$, thus subhorizon] and the slow-motion ($|\mathbf{v}|^2/c^2\ll 1$) conditions. Therefore, in the comoving gauge no pure Einstein's gravity correction appears for general peculiar velocity in all scales as long as we take the weak gravity and negligible relativistic pressure and ignore the rotational perturbations. This is consistent with the striking relativistic/Newtonian correspondence in the nonlinear perturbation theory available in the comoving gauge [1], whereas our results in this work are fully nonlinear.

This situation in the comoving gauge can be compared with the case in the other gauges. In the zero-shear gauge and the uniform-expansion gauge we have Newtonian hydrodynamic equations together with the Poisson's equation exactly recovered in the infinite-speed-of-light (weak gravity, negligible relativistic pressure, subhorizon, and slow-motion) limit [6], whereas in the comoving gauge we have recovered the Newtonian hydrodynamic equations for irrotational fluid without the proper gravitational potential identified as a fundamental variable, but without assuming the subhorizon and the slow-motion limits.

In the zero-shear gauge we also have properly extended the equations to include the relativistic pressure [i.e., without assuming $\tilde{p}/(\tilde{\varrho}c^2)\ll 1$] [12], whereas at the moment such a luxury looks not feasible in the comoving gauge. The variables in all the gauge conditions mentioned in this work are free from the gauge mode and can be regarded as gauge-invariant ones to the fully nonlinear order [8].

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## APPENDIX: FULLY NONLINEAR PERTURBATION EQUATIONS

Here, we present the complete set of fully nonlinear perturbation equations without taking the temporal gauge [6,8].

Definition of $\kappa$:

$$\kappa \equiv 3\frac{\dot{a}}{a}\left(1-\frac{1}{\mathcal{N}}\right)-\frac{1}{\mathcal{N}(1+2\varphi)}\left[3\dot{\varphi}+\frac{c}{a^2}\left(\chi^k{}_{,k}+\frac{\chi^k\varphi_{,k}}{1+2\varphi}\right)\right]. \tag{A1}$$

ADM energy constraint:

$$-\frac{3}{2}\left(\frac{\dot{a}^2}{a^2}-\frac{8\pi G}{3}\tilde{\varrho}-\frac{\Lambda c^2}{3}\right)+\frac{\dot{a}}{a}\kappa+\frac{c^2\Delta\varphi}{a^2(1+2\varphi)^2} = \frac{1}{6}\kappa^2-4\pi G\left(\tilde{\varrho}+\frac{\tilde{p}}{c^2}\right)(\hat{\gamma}^2-1)+\frac{3}{2}\frac{c^2\varphi^{,i}\varphi_{,i}}{a^2(1+2\varphi)^3}-\frac{c^2}{4}\bar{K}^i_j\bar{K}^j_i. \tag{A2}$$

ADM momentum constraint:

$$\frac{2}{3}\kappa_{,i}+\frac{c}{2a^2\mathcal{N}(1+2\varphi)}\left(\Delta\chi_i+\frac{1}{3}\chi^k{}_{,ik}\right)+8\pi G\left(\tilde{\varrho}+\frac{\tilde{p}}{c^2}\right)a\hat{\gamma}^2\frac{\hat{v}_i}{c^2} = \frac{c}{a^2\mathcal{N}(1+2\varphi)}\left\{\left(\frac{\mathcal{N}_{,j}}{\mathcal{N}}-\frac{\varphi_{,j}}{1+2\varphi}\right)\left[\frac{1}{2}(\chi^j{}_{,i}+\chi_i{}^{,j})-\frac{1}{3}\delta^j_i\chi^k{}_{,k}\right] - \frac{\varphi^{,j}}{(1+2\varphi)^2}\left(\chi_i\varphi_{,j}+\frac{1}{3}\chi_j\varphi_{,i}\right)+\frac{\mathcal{N}}{1+2\varphi}\nabla_j\left[\frac{1}{\mathcal{N}}\left(\chi^j\varphi_{,i}+\chi_i\varphi^{,j}-\frac{2}{3}\delta^j_i\chi^k\varphi_{,k}\right)\right]\right\}. \tag{A3}$$

Trace of ADM propagation:

$$-3\frac{1}{\mathcal{N}}\left(\frac{\dot{a}}{a}\right)^{\cdot}-3\frac{\dot{a}^2}{a^2}-4\pi G\left(\tilde{\varrho}+3\frac{\tilde{p}}{c^2}\right)+\Lambda c^2+\frac{1}{\mathcal{N}}\dot{\kappa}+2\frac{\dot{a}}{a}\kappa+\frac{c^2\Delta\mathcal{N}}{a^2\mathcal{N}(1+2\varphi)} = \frac{1}{3}\kappa^2+8\pi G\left(\tilde{\varrho}+\frac{\tilde{p}}{c^2}\right)(\hat{\gamma}^2-1)-\frac{c}{a^2\mathcal{N}(1+2\varphi)}\left(\chi^i\kappa_{,i}+c\frac{\varphi^{,i}\mathcal{N}_{,i}}{1+2\varphi}\right)+c^2\bar{K}^i_j\bar{K}^j_i. \tag{A4}$$

Tracefree ADM propagation:

$$\left(\frac{1}{\mathcal{N}}\frac{\partial}{\partial t}+3\frac{\dot{a}}{a}-\kappa+\frac{c\chi^k}{a^2\mathcal{N}(1+2\varphi)}\nabla_k\right)\left\{\frac{c}{a^2\mathcal{N}(1+2\varphi)} \times\left[\frac{1}{2}(\chi^i{}_{,j}+\chi_j{}^{,i})-\frac{1}{3}\delta^i_j\chi^k{}_{,k}-\frac{1}{1+2\varphi}\left(\chi^i\varphi_{,j}+\chi_j\varphi^{,i}-\frac{2}{3}\delta^i_j\chi^k\varphi_{,k}\right)\right]\right\} - \frac{c^2}{a^2(1+2\varphi)}\left[\frac{1}{1+2\varphi}\left(\nabla^i\nabla_j-\frac{1}{3}\delta^i_j\Delta\right)\varphi+\frac{1}{\mathcal{N}}\left(\nabla^i\nabla_j-\frac{1}{3}\delta^i_j\Delta\right)\mathcal{N}\right]$$
$$= 8\pi G\left(\tilde{\varrho}+\frac{\tilde{p}}{c^2}\right)\left[\frac{\hat{\gamma}^2\hat{v}^i\hat{v}_j}{c^2(1+2\varphi)}-\frac{1}{3}\delta^i_j(\hat{\gamma}^2-1)\right]+\frac{c^2}{a^4\mathcal{N}^2(1+2\varphi)^2}\left[\frac{1}{2}(\chi^{i,k}\chi_{j,k}-\chi_{k,j}\chi^{k,i}) + \frac{1}{1+2\varphi}(\chi^{k,i}\chi_k\varphi_{,j}-\chi^{i,k}\chi_j\varphi_{,k}+\chi_{k,j}\chi^k\varphi^{,i}-\chi_{j,k}\chi^i\varphi^{,k})+\frac{2}{(1+2\varphi)^2}(\chi^i\chi_j\varphi^{,k}\varphi_{,k}-\chi^k\chi_k\varphi^{,i}\varphi_{,j})\right] - \frac{c^2}{a^2(1+2\varphi)^2}\left[\frac{3}{1+2\varphi}\left(\varphi^{,i}\varphi_{,j}-\frac{1}{3}\delta^i_j\varphi^{,k}\varphi_{,k}\right)+\frac{1}{\mathcal{N}}\left(\varphi^{,i}\mathcal{N}_{,j}+\varphi_{,j}\mathcal{N}^{,i}-\frac{2}{3}\delta^i_j\varphi^{,k}\mathcal{N}_{,k}\right)\right]. \tag{A5}$$

Covariant energy conservation:

$$\left[\frac{\partial}{\partial t}+\frac{1}{a(1+2\varphi)}\left(\mathcal{N}\hat{v}^k+\frac{c}{a}\chi^k\right)\nabla_k\right]\tilde{\varrho}+\left(\tilde{\varrho}+\frac{\tilde{p}}{c^2}\right)\left\{\mathcal{N}\left(3\frac{\dot{a}}{a}-\kappa\right)+\frac{(\mathcal{N}\hat{v}^k)_{,k}}{a(1+2\varphi)}+\frac{\mathcal{N}\hat{v}^k\varphi_{,k}}{a(1+2\varphi)^2}+\frac{1}{\hat{\gamma}}\left[\frac{\partial}{\partial t}+\frac{1}{a(1+2\varphi)}\left(\mathcal{N}\hat{v}^k+\frac{c}{a}\chi^k\right)\nabla_k\right]\hat{\gamma}\right\}=0. \tag{A6}$$

Covariant momentum conservation:

$$\frac{1}{a\hat{\gamma}}\left[\frac{\partial}{\partial t}+\frac{1}{a(1+2\varphi)}\left(\mathcal{N}\hat{v}^k+\frac{c}{a}\chi^k\right)\nabla_k\right](a\hat{\gamma}\hat{v}_i)+\hat{v}^k\nabla_i\left(\frac{c\chi_k}{a^2(1+2\varphi)}\right)-\left(1-\frac{1}{\hat{\gamma}^2}\right)\frac{c^2\mathcal{N}\varphi_{,i}}{a(1+2\varphi)}+\frac{c^2}{a}\mathcal{N}_{,i}+\frac{1}{\tilde{\varrho}+\frac{\tilde{p}}{c^2}}\left\{\frac{\mathcal{N}}{a\hat{\gamma}^2}\tilde{p}_{,i}+\frac{\hat{v}_i}{c^2}\left[\frac{\partial}{\partial t}+\frac{1}{a(1+2\varphi)}\left(\mathcal{N}\hat{v}^k+\frac{c}{a}\chi^k\right)\nabla_k\right]\tilde{p}\right\}=0. \tag{A7}$$

We have

$$\mathcal{N}\equiv\sqrt{1+2\alpha+\frac{\chi^k\chi_k}{a^2(1+2\varphi)}},\qquad \bar{K}^i_j\bar{K}^j_i=\frac{1}{a^4\mathcal{N}^2(1+2\varphi)^2}\left\{\frac{1}{2}\chi^{i,j}(\chi_{i,j}+\chi_{j,i})-\frac{1}{3}\chi^i{}_{,i}\chi^j{}_{,j}-\frac{4}{1+2\varphi}\left[\frac{1}{2}\chi^i\varphi^{,j}(\chi_{i,j}+\chi_{j,i})-\frac{1}{3}\chi^i{}_{,i}\chi^j\varphi_{,j}\right]+\frac{2}{(1+2\varphi)^2}\left(\chi^i\chi_i\varphi^{,j}\varphi_{,j}+\frac{1}{3}\chi^i\chi^j\varphi_{,i}\varphi_{,j}\right)\right\}. \tag{A8}$$


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