Electron structure through a classical description of the *Zitterbewegung*

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The description of the *Zitterbewegung* at a classical level indicates possible predictions of the electron quantum properties even before quantization: this quivering motion is restricted to a plane, which leads us to only two possible orientations of the corresponding angular momentum, even in the absence of a quantized theory or external fields. Besides, the angular momentum associated with the *Zitterbewegung* turns out to be proportional to $\hbar$. Namely, using a standard constraint analysis we have proven this last result. Thus, assuming that this is an observable phenomenon, we recognize this oscillatory motion as a classical signature of the spin of the electron. We also propose here an interpretation of the *Zitterbewegung* based on geometrical grounds: it can be seen as the physical degrees of freedom of position variables constrained to a sphere, which enforces the hypothesis of assuming an electron internal structure.

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I. INTRODUCTION

Two years after Dirac published his seminal paper that established the basis of the electron quantum theory [1], Schrödinger analyzed the time evolution of the position operator which has appeared from the Dirac equation [2]. He found, besides the rectilinear movement with constant velocity, an oscillatory one, denoted by *Zitterbewegung* (ZB), which means trembling or quivering motion. It was long associated with the electron structure, and it is seen as being responsible for the electron spin. It was noticed for the first time by Schrödinger as a consequence of the noncommutative relation between the Hamiltonian and the position operator. It was faced as a problem, since we could expect no acceleration of the Dirac electron: the equation was supposed to describe a free particle.

Some years later, the ZB problem was related to a noncommutative Poisson bracket algebra like $\{X_1, X_2\} = \theta$ [3]. It was demonstrated in Ref. [4] that Mathisson's classical electron [3] shows interesting analogies with Schrödinger ZB. Solving Mathisson's equation of motion, we also obtain an oscillatory behavior, in the absence of an external force, showing a noncommutative algebra concerning the center of mass and the internal coordinates.

In technical terms, the ZB motion can be found by solving the Heisenberg equations of motion for $x(t)$ (the position operator) using Dirac's Hamiltonian given by

$$H = m\gamma + \vec{p}\cdot\vec{\alpha},$$

which is the generator for time translations. Consider that this coordinate operator has a term concerning the center of mass such as

$$\vec{X}(t) = H^{-1}\vec{p}t + \vec{a},$$

where $\vec{a}$ is a constant vector. This center of mass on momentum eigenstates moves with a uniform velocity and an oscillatory term that can be written as

$$\vec{\xi}(t) = \frac{i}{2}[\vec{\alpha}(0) - H^{-1}\vec{p}]H^{-1}e^{-2iHt},$$

which is known as *Zitterbewegung*. The position operator can be written as

$$\vec{x}(t) = \vec{X}(t) + \vec{\xi}(t),$$

and it can be seen as the center of charge for the electron. This last one oscillates quickly around the center of mass $X(t)$, and $\xi(t)$ can be called the relative or the internal position operator of the electron.

Since then, many efforts have been made to understand this oscillatory motion in different experimental areas, including, for example, spintronic systems [5], ultracold atoms [6], and graphene [7], and also in a theoretical manner [8]. Although recent advances claim that it should be an unobservable phenomenon [9–11], a new experimental setup advocates a direct observation of the ZB [12].

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Recently, the study of spinning particles was attacked in Refs. [13,14], where the question of identifying the appropriate position coordinates arises once again. Besides, simulations with trapped ions also indicate that the electron should experience ZB [15]. In order to reproduce this scenario, we present in this work an interpretation of the ZB predicted by a semiclassical[1] Hamiltonian model that produces both Dirac's equation and gamma matrices in the path to canonical quantization [16]. It uses both a variational principle and commuting variables for its description. Our main objective here is to discuss the classical counterpart of the model, which predicts quantum properties for the electron, even before quantization. In fact, the general solution of the classical equations of motion shows ZB behavior.

We will show that the coordinates $\tilde{x}^i$ and J^i are analogous to those of the center of mass and to the relative position of a two-body system subjected to a central field. The Dirac equation, which appears as a constraint of the model, dictates the perpendicularity of the J^i coordinates (which have trajectories that are ellipses) to the direction of the center-of-mass motion. The angular momentum associated with the ZB turns out to be proportional to $\hbar$. Besides, the polarization induced by Dirac's equation allows only two possible orientations concerning the evolution of J^i. The combination of both results reflect the well-known properties associated with the spin of the electron. A detailed analysis of the oscillatory motion also shows that it can be produced by a central potential of the form $V(J) \sim J^2$; $J = |J^i|$. The physical origin of this potential could be traced back to a free particle constrained to a sphere. Projection of its geodesic motion in the plane (where J^i evolves) can be described by $V = V(J)$. Hence, one is naturally led to interpret the electron as having an internal structure of a sphere with radius limited by its Compton wavelength, which is a new result.

II. SEMICLASSICAL SPIN MODELS

The true understanding of the electron spin was achieved by the Dirac equation

$$(\hat{p}_\mu \Gamma^\mu + mc)\Psi(x^\mu) = 0, \tag{1}$$

where $\hat{p}_\mu = -i\hbar\partial_\mu$. Spin degrees of freedom are described by gamma matrices Γ^μ and $\Gamma^{\mu\nu} \equiv \frac{i}{2}[\Gamma^\mu, \Gamma^\nu]$; see [17]. Recently, it has been proposed that different semiclassical Hamiltonian models produce both the Dirac equation and gamma matrices in the path to quantization [16,18]. One of its specific features is to assume a variational problem. So, we can analyze the time evolution of the configuration variables even before quantization. Let us aim our attention to the model introduced in Ref. [16]. Classically, it is parameterized by phase-space coordinates x^μ and p_μ $(\mu, \nu, \ldots = 0, 1, 2, 3)$, with standard interpretations, together with J^{AB} variables, where A, $B = (\mu, 5)$, which will be called spin variables since they produce gamma matrices during quantization. The Dirac equation is obtained by imposing the constraint

$$T = p_\mu J^{5\mu} + mc\hbar = 0 \tag{2}$$

over the state vector also in the process of quantization, according to the rules

$$p_\mu \to \hat{p}_\mu = -i\hbar\partial_\mu, \tag{3}$$

$$J^{5\mu} \to \hat{J}^{5\mu} = \hbar\Gamma^\mu. \tag{4}$$

Let us now discuss the general solution of the equations of motion [16]. They are given by

$$x^i(t) = X^i + c\frac{p^i}{p^0}t + \frac{1}{2|p|}(A^i \sin(\omega t) - B^i \cos(\omega t)), \tag{5}$$

$$J^{0i} = \frac{p^0}{|p|}(A^i \sin(\omega t) - B^i \cos(\omega t)), \tag{6}$$

$$J^{5i}(t) = \frac{mc\hbar}{|p|^2}p^i + A^i \cos(\omega t) + B^i \sin(\omega t), \tag{7}$$

$$J^{ij}(t) = \frac{|p|}{mc\hbar}B^{[i}A^{j]} + \frac{1}{|p|}p^{[i}A^{j]}\sin(\omega t) - \frac{1}{|p|}p^{[i}B^{j]}\cos(\omega t), \tag{8}$$

$$J^{50}(t) = \frac{mc\hbar p^0}{|p|^2}, \tag{9}$$

where X^i, A^i, and B^i are integration constants, p_μ is a constant four vector, $|p| \equiv \sqrt{-p_\mu p^\mu}$ and $\omega = \frac{2|p|^3}{m\hbar p^0}$, and m, c, and $\hbar$ have standard meanings. As usual, Latin letters (i, j) run the values 1,2,3. The expressions above make explicit that both position and spin variables show ZB behavior. In a theory of constrained systems, one can introduce the constraints into the equations of motion before solving them. Or, equivalently, it is also possible to find a general solution, and after that, one obtains which conditions over the integration constants can be imposed by the constraints. Using the solutions (7) and (9) of the equations of motion in the constraint (2), we will obtain two special features that will be crucial for later discussions. One possible solution is that

$$p_i A^i = p_i B^i = 0, \tag{10}$$

which means that the ZB oscillations take place in a plane perpendicular to the rectilinear movement with direction

[1]The term "semiclassical" in this context means that $\hbar$ is present in the model before canonical quantization.

p_i. Notice the first two terms in $x^i = x^i(t)$. This is exactly the classical Dirac equation (2) that leads to this "polarization" condition. Besides, the vectors A^i and B^i are bounded such as

$$|A^i| < \frac{mc\hbar}{|p|}, \qquad |B^i| < \frac{mc\hbar}{|p|}. \tag{11}$$

This condition is related to the causal motion of the spinning particle described by the model. The details can be found in Ref. [19].

III. SPACE-TIME INTERPRETATION OF CONFIGURATION AND SPIN VARIABLES

In order to fathom the evolution of x^i and J^{AB}, we recombine both as

$$\tilde{x}^i(t) = x^i - \frac{J^{0i}}{2p^0}; \qquad \tilde{p}^i = \frac{p^i}{p^0}; \tag{12}$$

$$J^i = \frac{J^{0i}}{2p^0}; \qquad V^i = \frac{J^{5i}}{J^{50}} - \frac{p^i}{p^0}. \tag{13}$$

The spatial coordinates obey the equations

$$\frac{d\tilde{x}^i}{dt} = c\tilde{p}^i, \qquad \frac{dJ^i}{dt} = cV^i, \tag{14}$$

and, by taking into account Eqs. (5)–(9), the general solutions of the equations in (14) are

$$\tilde{x}^i(t) = X^i + c\frac{p^i}{p^0}t, \tag{15}$$

$$J^i(t) = \frac{1}{2|p|}(A^i \sin\omega t - B^i \cos\omega t), \tag{16}$$

where (15) and (16) are analogous to the center-of-mass coordinates and the relative position of a two-body problem subjected to a central field. It is possible to show that $J^i = J^i(t)$ is described by the potential of an isotropic harmonic oscillator [20]:

$$V(J) = \frac{1}{2}m\omega^2 J^2, \tag{17}$$

where $J \equiv |J^i|$. On the other hand, the time evolution followed by $J^i = J^i(t)$, whose trajectory is an ellipse [19], can be described by the physical degrees of freedom of a particle constrained by a sphere. Actually, both descriptions turn out to be equivalent, and one can show that the force constant $k = m\omega^2$ is proportional to the scalar curvature of the surface, where the model can be found in Ref. [20] and the proportionality is a characteristic particle energy

$$k = m\omega^2 = \frac{2}{a^2}E, \tag{18}$$

where $2/a^2$ is the curvature of a sphere of radius a and E is the energy of the particle to be calculated. Then we are led to see the electron as a sphere. Let us compute its radius. We will consider the solutions where $p_\mu p^\mu = -m^2c^2$, since it is valid for all massive particles. For a slowly moving center of mass, $p^0 \approx mc$, and using the expression for ω, we have

$$\frac{2}{a^2}E = m\omega^2 \Rightarrow \frac{E}{a^2} = \frac{mc^2}{\left(\frac{\hbar}{\sqrt{2}mc}\right)^2}. \tag{19}$$

Hence, as we expect, E turns out to be the rest energy of the electron $E = mc^2$, while its radius is limited by the corresponding Compton wavelength $a = \frac{\hbar}{\sqrt{2}mc}$. Taking into account the bounds (11), we conclude that the Compton wavelength is just a superior bound for the radius a of the electron. We finally realize that this interpretation provides a natural origin for the ZB as physical configuration variables are constrained to a sphere. A different approach for the ZB that claims an internal structure for the electron can be found in Ref. [21]. The idea of a composed electron lead us to the work in Ref. [22], where Dirac has treated it as an extensible particle. We can also be driven to his seminal paper upon the unitary irreducible particle representations of the anti—de Sitter group [23].

IV. CONNECTION BETWEEN THE ANGULAR MOMENTUM AND THE ELECTRON SPIN USING ZB

Let us show now that the angular momentum associated with the ZB is connected to some specific characteristics of the electron spin, even though this last one is a quantum property of matter. Let us consider the angular momentum related to the coordinates $J^i = J^i(t)$:

$$L^i = m\varepsilon^{ijk}J^j\dot{J}^k = \frac{m\omega}{4|p|^2}\varepsilon^{ijk}A^jB^k. \tag{20}$$

First we can stress that L^i has only two possible orientations. In fact, $J^i = J^i(t)$ is restricted to a plane perpendicular to the direction of p^i. So, it evolves clock- or counterclockwise, and in this case L^i has the same or the opposite direction of p^i. To obtain an approximate value for $|L^i|$, we will use again the approximation for a slow particle $p^0 \approx mc$, and taking into account the superior bound for $|A^i|$ and $|B^i|$ in (11), one obtains the surprising result

$$|L^i| = \frac{2m|p|^3}{4|p|^2 m\hbar p^0}\left(\frac{mc\hbar}{|p|^2}\right)^2 = \frac{\hbar}{2}. \tag{21}$$

The two observations above lead us to see the angular momentum L^i as the classic analogue to the quantum spin of the electron. This is a direct demonstration that the ZB motion leads us to the electron spin. In his work [2], Schrödinger suggested the electron spin to be the result of a local circulatory motion, constituting the ZB motion. It would be also a result from the interference between negative and positive energy solutions of Dirac's equation, he has stressed.

This is a surprising demonstration, since we know that the Dirac particle orbital angular momentum is not a constant of motion given by $L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$, but we also know $\dot{L}_{\mu\nu} = v_\mu p_\nu - v_\nu v_\mu = -\dot{S}_{\mu\nu}$, and hence the sum $J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$ is a constant of motion, where $S_{\mu\nu}$ is the spin variables and $J_{\mu\nu}$ is the total angular momentum. So, $\dot{J}_{\mu\nu} = 0$ for a free particle with the internal variables being the coordinates oscillating with the ZB frequency $2m$ [24]. The result in (21) does not modify the $\dot{J}_{\mu\nu} = 0$ one but is a consequence of the solutions of (14) for the spatial coordinates.

Since then, of course, there have been other ways to associate ZB with spin, and the issue attracts interest until now [25]. However, what is new here is the constraint analysis that has begun with Eq. (2).

V. CONCLUDING REMARKS

We have analyzed in this paper the classical counterpart of a model whose quantization has led to the Dirac quantum theory of the electron. A general solution of the equations of motion presents the so-called *Zitterbewegung*. A geometrical interpretation of the time evolution of configuration variables leads us to consider the geometry of the electron as a sphere of radius bounded by its Compton wavelength. This picture has provided us with a simple origin of the ZB as physical position variables constrained to a sphere. The transversal evolution of the J^i oscillating variables caused by Dirac's equation provided only two possible orientations of the angular momentum L^i associated with the ZB, even in the absence of a quantized theory or external magnetic fields. Besides, $|L^i|$ is restricted by $\hbar/2$. Thus we interpret L^i as the classical analogue to the spin of the electron, a quantum feature. In this case, the *Zitterbewegung* can be seen as a classical signature of the spin.

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