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Estimating detection rates of compact binary inspirals with networks of ground-based gravitational-wave detectors

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In a recent paper, Schutz proposed an analytical approximation for simplifying the treatment of the polarization angle and conveniently evaluating relative detection rates of compact binary inspirals for various networks of ground-based interferometers. We derive relative event rates by strictly handling the polarization angle, and we quantitatively examine the validity of Schutz's approximation. The associated error of the approximation is rigorously shown to be less than 1.02%, irrespective of the details of the detector networks.

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Currently, second-generation gravitational-wave (GW) interferometers are being installed, and constructed around the world. Their most promising targets are inspirals of compact binaries, and various scientific prospects have been actively discussed for the binaries.

One of the primary measures for such studies is the detection rate of the binaries. While the overall rate is highly uncertain, due to limitations of our astronomical knowledge, the relative detection rates depend mainly on the geometry of the source-network configuration (see, e.g., [1,2]) for a spatially homogeneous distribution of sources. The relative rates play critical roles in examining the performance of potential detector networks. The arguments related to the detection rates include the dependence on duty cycles of constituent detectors, the impact of an additional detector (e.g., LIGO-India), and designing appropriate strategies for counterpart searches with electromagnetic wave telescopes (see, e.g., [1,3,4]).

However, the signal-to-noise ratios (SNRs) of individual binaries depend not only on their sky positions, but also strongly on their orientations, which are specified by the inclination I and polarization angle ψ (explained below) [1,2]. In order to make solid estimations of the relative rates, we have traditionally applied cumbersome methods, such as Monte Carlo calculations, to incorporate binary orientations.

To conveniently evaluate the relative event rates, Schutz recently proposed an analytical approximation of taking a certain average for the polarization angle ψ [1] (see, e.g., [5] for its application); only a two-dimensional integral with respect to the sky position is then actually required for the relative event rates. However, in Schutz's paper, the accuracy of this approximation was left unexamined, with a comment that it can be tested by comparing it with Monte Carlo studies.

In this paper, we analytically evaluate the relative rates, with strict handling of the dependence on the polarization angle. After deriving our final expression given in Eq. (8), we show how Schutz's approximation can be understood in our formulation, and we rigorously clarify its accuracy.

We assume a coherent analysis of GWs with L-shaped interferometers labeled by $i = 1, \ldots, m$ (m is the total number of detectors). Because of the spin-two nature of GWs, we can generally express the responses of a detector i to the incoming two polarization modes $+$ and $\times$ as [1,2]

$$c_{i+}(\boldsymbol{n}, \psi) = a_i(\boldsymbol{n}) \cos 2\psi + b_i(\boldsymbol{n}) \sin 2\psi, \tag{1}$$

$$c_{i\times}(\boldsymbol{n}, \psi) = -a_i(\boldsymbol{n}) \sin 2\psi + b_i(\boldsymbol{n}) \cos 2\psi, \tag{2}$$

with the polarization angle ψ and the source direction $\boldsymbol{n}$.

For GW sources, we consider inspirals of circular binaries that are assumed to have random positions and orientations, and that emit two polarization modes proportional to

$$d_+(I) = \frac{I^2 + 1}{2}, \qquad d_\times(I) = I \tag{3}$$

with the inclination $I \equiv \cos i$ (i is the inclination angle). In Eqs. (1) and (2), the polarization angle ψ fixes the azimuthal direction of the orbital angular momentum of binaries around the sky direction $\boldsymbol{n}$.

Then, neglecting the precession of the orbital plane, the coherent SNR depends on the direction $\boldsymbol{n}$ and orientation (I, ψ) of a binary as

$$SNR^2 \propto \sum_{i=1}^{m} [(c_{i+} d_+)^2 + (c_{i\times} d_\times)^2] \equiv f(\boldsymbol{n}, I, \psi). \tag{4}$$

Here, applying trigonometric identities, the function f can be expressed as

$$f(\boldsymbol{n}, \psi, I) = \sigma(\boldsymbol{n})[(d_+^2 + d_\times^2) + \epsilon(\boldsymbol{n})(d_+^2 - d_\times^2) \cos 4\psi'] \tag{5}$$

with a shifted polarization angle $\psi' = \psi + \delta(\boldsymbol{n})$, and the two parameters $\sigma(\boldsymbol{n})$ and $\epsilon(\boldsymbol{n})$ that depend only on $\boldsymbol{n}$ for a given detector network as

$$\sigma(\boldsymbol{n}) \equiv \sum_{i=1}^{m} [a_i^2 + b_i^2], \tag{6}$$

$$\epsilon(\boldsymbol{n}) = \frac{\sqrt{[\sum_{i=1}^m (a_i^2 - b_i^2)]^2 + 4(\sum_{i=1}^m a_i b_i)^2}}{\sigma(\boldsymbol{n})}. \quad (7)$$

The latter represents the asymmetry of the network sensitivities to the two polarization modes. Using the Cauchy-Schwarz inequality, we can show $0 \le \epsilon(\boldsymbol{n}) \le 1$ with the identity $\epsilon(\boldsymbol{n}) = 1$ for a single detector network. Note that the expression (5) can be also found in [2].

For binaries with precessing orbital planes, the orientation angles (I, ψ) change over time. Thus, in Eq. (5), they should be regarded as appropriately averaged angles. This complicates the problem mathematically. However, our simple treatment above would be a reasonable approximation, at least for double neutron stars [2].

Next, let us discuss the effective volume detectable with the detector network by the coherent signal analysis. With respect to a fixed detection threshold for the coherent SNR, the maximum detectable distance $r_{\max}$ scales as $r_{\max} \propto f(\boldsymbol{n}, \psi, I)^{1/2}$ for given angular parameters $(\boldsymbol{n}, \psi, I)$. Thus, the effective volume associated with a parameter space $d\boldsymbol{n} d\psi dI$ is simply proportional to $f(\boldsymbol{n}, \psi, I)^{3/2} d\boldsymbol{n} d\psi dI$.

By integrating out the source orientation angles (ψ, I), the effective volume (equivalently, the relative detection rate) for a given solid angle $d\boldsymbol{n}$ is proportional to

$$\sigma(\boldsymbol{n})^{3/2} g(\epsilon(\boldsymbol{n})) d\boldsymbol{n}, \quad (8)$$

where the new function $g(\epsilon)$ is defined by

$$g(\epsilon) \equiv \frac{1}{2^{5/2}\pi} \int_0^\pi d\psi \int_{-1}^1 dI[(d_+^2 + d_\times^2) + \epsilon(d_+^2 - d_\times^2)\cos 4\psi]^{3/2} \quad (9)$$

with the normalization factor $2^{5/2}\pi$ given for the double integrals with $d_+(1) = d_\times(1) = 1$ (corresponding to face-on binaries).

The function $g(\epsilon)$ monotonically increases in the relevant range $0 \le \epsilon \le 1$ with

$$g(0) = 0.290451, \quad g(1) = 0.293401 = 1.010125 \times g(0). \quad (10)$$

The numerical value $g(0)$ is identical to that given in [1]. By perturbatively expanding Eq. (9), we also have

$$g_{\exp}(\epsilon) = 0.290451(1 + 0.00978\epsilon^2 + 0.00026\epsilon^4 + O(\epsilon^6)) \quad (11)$$

with an accuracy of $|g_{\exp}(\epsilon)/g(\epsilon) - 1| < 10^{-4}$ [dropping the $o(e^4)$ terms] in the range $0 \le \epsilon \le 1$. We can anticipate the observed weak dependence on ϵ, considering that (i) the integral (9) becomes constant at the power index 1 close the original index of $3/2$, and (ii) we have $g'(0) = 0$ due to the symmetry of the integrand.

Now we discuss Schutz's approximation. In our formulation, this corresponds to taking an average of ψ at the Eq. (5) stage, before the nonlinear operation $[\cdots]^{3/2}$ in Eq. (9). This is equivalent to putting $\epsilon(\boldsymbol{n}) = 0$ in Eq. (9); the resultant expression is identical to

$$\sigma(\boldsymbol{n})^{3/2} g(0) d\boldsymbol{n}, \quad (12)$$

in contrast to Eq. (8) obtained in our strict derivation.

However, our results (10) and (11) show that, for evaluating the relative detection rates, disregarding the ϵ dependence [thus only using the leading term in Eq. (11)] is an excellent approximation, with an error less than 1.02%. Because the integrands in Eqs. (8) and (12) are non-negative, the quoted accuracy is also valid for the final results after the sky average. If necessary, we can readily include the ϵ dependence (11) for $g(\epsilon)$.

Within the guaranteed accuracy of 1.02%, we can now justify evaluating the relative detection rate in the solid angle $d\boldsymbol{n}$ simply by $\sigma(\boldsymbol{n})^{3/2} d\boldsymbol{n}$, or the total rate by

$$\int_{4\pi} \sigma(\boldsymbol{n})^{3/2} d\boldsymbol{n}, \quad (13)$$

without resorting to cumbersome Monte Carlo calculations to handle the orientations of the binaries.

If the detectors $i = 1, \ldots, m$ have different sensitivities (or, equivalently, horizon distances), we can straightforwardly apply our results by introducing appropriate weights to the response functions (a_i, b_i). Furthermore, the form (5) can be derived even in the presence of certain correlated noises between detectors with the corresponding functions $\sigma(\boldsymbol{n})$ and $\epsilon(\boldsymbol{n})$ [2]; our results are unchanged in such cases as well.

[1] B. F. Schutz, Classical Quantum Gravity **28**, 125023 (2011).
[2] C. Cutler and E. E. Flanagan, Phys. Rev. D **49**, 2658 (1994).
[3] P. Raffai, L. Gondán, I. S. Heng, N. Kelecsényi, J. Logue, Z. Márka, and S. Márka, Classical Quantum Gravity **30**, 155004 (2013).
[4] A. C. Searle, S. M. Scott, D. E. McClelland, and L. S. Finn, Phys. Rev. D **73**, 124014 (2006).
[5] H.-Y. Chen and D. E. Holz, Phys. Rev. Lett. **111**, 181101 (2013).