Black holes in $\mathcal{N} = 8$ supergravity from SO(4,4) hidden symmetries

David D. K. Chow* and Geoffrey Compère†

Physique Théorique et Mathématique, Université Libre de Bruxelles and International Solvay Institutes, Campus Plaine C.P. 231, B-1050 Bruxelles, Belgium

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We detail the construction of the most general asymptotically flat, stationary, rotating, nonextremal, dyonic black hole of the four-dimensional $\mathcal{N} = 2$ supergravity coupled to 3 vector multiplets that describes the STU model. It generates through U dualities the most general asymptotically flat, stationary black hole of $\mathcal{N} = 8$ supergravity. We develop the solution generating technique based on $\mathrm{SO}(4,4)/\mathrm{SL}(2,\mathbb{R})^4$ coset model symmetries, with an emphasis on the 4-fold permutation symmetry of the gauge fields. We indicate how previously known nonextremal and extremal solutions of the STU model are recovered as limiting cases. Several properties of the general black hole solution are discussed, including its thermodynamics, the quadratic mass formula, the Bogomolny-Prasad-Sommerfield limit, the slow and fast extremal rotating limits, its properties in regards to the Kerr/conformal field theory correspondence, its Killing tensors and the separability of geodesic motion and probe scalars.

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I. INTRODUCTION

Black holes are some of the most important nonperturbative objects of quantum gravity. To understand their fundamental properties, such as their microscopic description, it is essential to have explicit black hole solutions and understand all their classical properties, such as their thermodynamics. In four-dimensional Einstein-Maxwell theory, the Kerr-Newman solution represents a general stationary, asymptotically flat black hole. More general theories, such as those arising from string theory, admit more general families of black hole solutions. One of the most studied string theory compactifications down to 4 dimensions is the reduction of M theory on T^7, which is described in the low-energy regime by maximal $\mathcal{N} = 8$ supergravity [1,2]. The bosonic sector, which is relevant for classical solutions, includes Einstein-Maxwell theory as a truncation, and also includes several other well-studied theories of gravity coupled to vectors and scalars. A number of black hole solutions of $\mathcal{N} = 8$ supergravity and its truncations have been discovered over the last 35 years, but the most general family had proved elusive. In this paper, we give a derivation of the most general stationary, asymptotically flat black hole of $\mathcal{N} = 8$ supergravity in a specific U-duality frame, as announced in [3].

$\mathcal{N} = 8$ supergravity admits a consistent truncation to an $\mathcal{N} = 2$ supergravity coupled to three vector multiplets, which is known as the STU model [4,5] (S, T and U are sometimes used to denote its three complex scalar fields). The STU supergravity is particularly useful because a suitable 5-charge solution of STU supergravity suffices to generate the general black hole of $\mathcal{N} = 8$ supergravity through U dualities [6,7]. Solutions of STU supergravity can also be used to generate solutions of pure $\mathcal{N} > 2$ supergravities and heterotic supergravity [7]. Such U dualities only act on the matter fields, while leaving the four-dimensional metric invariant.

While $\mathcal{N} = 8$ supergravity admits an $\mathrm{E}_{7(7)}(\mathbb{R})$ symmetry of its field equations, the STU supergravity action has an $\mathrm{SL}(2,\mathbb{R})^3$ symmetry, and also symmetry under permutations of the three $\mathrm{SL}(2,\mathbb{R})$ factors, which is commonly referred to as the "S-T-U" triality symmetry [5]. Upon dimensional reduction along time, the classical symmetry of the action enhances to SO(4,4), which contains an $\mathrm{SL}(2,\mathbb{R})^4$ subgroup. The extra $\mathrm{SL}(2,\mathbb{R})$ is associated with the Ehlers $\mathrm{SL}(2,\mathbb{R})$ that arises from reduction of Einstein gravity [8,9].

$\mathcal{N} = 8$ supergravity has been of considerable interest recently thanks to the identification of elegant ultraviolet cancellations in perturbation theory, see e.g. [10]. Using Kawai-Lewellen-Tye relations [11], amplitudes in $\mathcal{N} = 8$ supergravity are related to amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory. The latter theory is finite [12,13], prompting speculation that $\mathcal{N} = 8$ supergravity might be finite. However, pure $\mathcal{N} = 8$ supergravity cannot be decoupled from string theory [14], contrary to $\mathcal{N} = 4$ super-Yang-Mills theory [15].

The entropy of extremal black holes in $\mathcal{N} = 8$ supergravity is related to qubit entanglement measures in quantum information systems, as reviewed in [16,17]. There have been in particular studies of the STU supergravity, which corresponds to entanglement of three qubits [18–20] and four qubits [21–23]. More generally, extremal black hole entropy in $\mathcal{N} = 8$ supergravity corresponds to tripartite entanglement of seven qubits [24].

*david.chow@ulb.ac.be
†gcompere@ulb.ac.be

Ungauged $\mathcal{N} = 8$ supergravity can be generalized to gauged $\mathcal{N} = 8$ supergravities. Whereas the ungauged theory admits a Minkowski vacuum solution, the gauged theories admit anti–de Sitter (AdS) vacuum solutions, so are relevant for studying the AdS/CFT correspondence [15]. The gauged theories have attracted recent interest because the original $\mathcal{N} = 8$, SO(8) gauged supergravity [25,26], previously thought to be unique, has been generalized to a one-parameter family of $\mathcal{N} = 8$ gauged supergravities [27]. Black holes in the ungauged $\mathcal{N} = 8$ supergravity provide a starting point for finding black holes of the gauged $\mathcal{N} = 8$ supergravity. Systematic solution generating techniques, which work for the ungauged theory, fail for the gauged theories. Therefore, finding solutions of the gauged theory requires guesswork based on solutions of the ungauged theory. For some recent results in this direction, see [28–30] and references therein.

It is conceptually straightforward to find complicated charged black hole solutions of interest, such as the most general black hole of $\mathcal{N} = 8$ supergravity, given the existence of well-known algorithms and suitable uncharged black hole solutions, but it can be a difficult algebraic task. A common method of generating charged, stationary black holes is to dimensionally reduce the theory on the time coordinate to give Euclidean 3-dimensional gravity coupled to matter. After Hodge dualizing three-dimensional vectors to scalars, the resulting bosonic matter Lagrangian typically consists of a coset model of scalar fields, which admits symmetries forming a real Lie algebra. A solution can then be generated starting from an initial seed solution and acting on it with coset model symmetries. In this paper we will detail the coset model based on SO(4,4) symmetries and use it to obtain general black holes. The four-dimensional *STU* supergravity has four gauge fields on an equal footing. In this paper, we will present a formulation of the SO(4,4) coset model that keeps the permutation symmetry between the four gauge fields manifest.

The conceptual foundations of coset model symmetries have been known for years [31,32]. The main interest of such symmetries, when considering spacelike reductions down to 4 dimensions only, is their role as symmetries of string theory after quantization [33] (see e.g. [34–37] for reviews). Attempts have been made to similarly understand symmetries appearing in timelike reductions, which led to string theories in mixed time signatures [38,39], but it is not clear if such theories can be quantized. In the case of reductions down to 3 dimensions, it has been conjectured that the classical symmetry group is quantized in string theory [40,41] but only partial indications have been obtained in this direction [42,43]. In this paper we will only treat symmetries classically as a solution generating technique. A classification of the symmetries appearing in torus reductions of various maximal supergravities (on both space and time) has been performed [44–47]. Explicit algorithms for particular cosets have been developed extensively over the years, starting from the pioneering work on Einstein gravity [8,9], understood in terms of an $SL(2,\mathbb{R})$ coset [48], and on Einstein-Maxwell theory [49], understood in terms of an SU(2, 1) coset [50–53]. Other theories considered are Kaluza-Klein theory, understood in terms of an $SL(3,\mathbb{R})$ coset [54]; the particular Einstein-Maxwell-dilaton-axion theory used for generating solutions of $\mathcal{N} = 4$ supergravity written in terms of a $Sp(4,\mathbb{R})$ coset [55–62]; 5d minimal supergravity, which admits $G_{2(2)}$ symmetries in [47,63–70]; and *STU* supergravity in 4 and 5 dimensions, which admits SO(4,4) symmetries [71–75]. For the full $\mathcal{N} = 8$ supergravity, reduction to 3 Euclidean dimensions gives the maximal $\mathcal{N} = 16$ supergravity theory [31] with 128 scalars parametrizing the coset $E_{8(8)}/SO^*(16)$ [32,76,77].

The stationary asymptotically flat black hole which generates, under U dualities, all single-centered, stationary black holes of $\mathcal{N} = 8$ supergravity has been presented in [3]. The main purpose of this article is to present the details of the solution and its generation from SO(4,4) hidden symmetries. The solution generalizes previously known subcases [55,71,78–87]; see also [88–96] for extremal branches. It admits 8 independent electromagnetic charges (4 electric and 4 magnetic), in addition to mass and angular momentum [the generalization with Newman-Unti-Tamburino (NUT) charge is considered as well]. Since there are 4 gauge fields on an equal footing, it is simpler to present explicitly the more general solution with 8 independent charges rather than a 5-charge solution. Moreover, keeping the NUT charge on the same footing as the mass allows for a simplifying SO(2) symmetry that can be broken as a final step to specialize to asymptotically flat black holes.

Many physical properties of the general solution are as expected from its known subcases, such as the Kerr-Newman black hole. There are generically two horizons. The asymptotically flat solution obeys the first law of thermodynamics and the Smarr relation. The formal first law of thermodynamics and Smarr relation at the inner horizon also hold. The product of areas of the outer and inner horizons is quantized, i.e. independent of the mass. This product is proportional to the sum of the angular momentum squared and the Cayley hyperdeterminant, which is a quartic invariant of the electromagnetic charges. Rotating extremal limits exist, with both fast and slow rotation. The black hole entropy takes the expected chiral Cardy form in these extremal cases and the near-horizon limits have the expected $SL(2,\mathbb{R})$ enhanced symmetry. Supersymmetric black holes with finite horizon area are recovered in a specific nonrotating extremal limit.

We show that in a different conformal frame, the metric belongs to a class of spacetimes admitting a Killing-Stäckel tensor, similar to all other known charged generalizations of the Kerr black hole [97]. Consequently, the geodesics of the conformally related metric are completely integrable, and the null geodesics in Einstein frame are completely integrable. The massless Klein-Gordon equation is separable around the general stationary asymptotically flat black hole of $\mathcal{N}=8$ supergravity obtained from our solution by U dualities.

The entropy of extremal black holes in $\mathcal{N}=8$ supergravity is known to have a simple expression [98] in terms of the Cartan-Cremmer-Julia quartic $E_{7(7)}$ invariant, which is constructed solely from the electromagnetic charges. Here, we derive the formula for entropy of the nonextremal black hole, and show that it cannot be expressed as a function of the usual $E_{7(7)}$ invariants, namely the quartic invariant, the mass and angular momentum. Instead, the entropy of the generic nonextremal stationary asymptotically flat black hole of $\mathcal{N}=8$ supergravity depends upon an additional $E_{7(7)}$ invariant that remains to be understood. We identify this quantity for black holes in the U-duality frame of the STU model in terms of auxiliary parameters that are also used to parametrize the conserved charges of the black hole. In specific subcases including the dyonic Kerr-Newman black hole and the dyonic, rotating Kaluza-Klein black hole, we are able to provide the explicit expression for the invariant and therefore the entropy in terms of conserved charges.

The rest of the paper is organized as follows. We present the relevant supergravity theories in Sec. II. We outline the solution generating technique based on SO(4,4) symmetries in Sec. III, and then apply it to the particular case of a Kerr-Taub-NUT seed solution in Sec. IV. We summarize the general resulting solution in Sec. V, and present its physical properties in Sec. VI. Then we discuss particular limits of the general solution, recovering known nonextremal solutions in Sec. VII and finding some extremal limits in Sec. VIII. In Sec. IX, we consider a more general class of metrics, discuss Killing tensors and the separability of geodesic motion and the Klein-Gordon equation. We conclude in Sec. X.

II. *STU* SUPERGRAVITY

Four-dimensional maximal $\mathcal{N}=8$ supergravity can be obtained from T^7 reduction of 11-dimensional supergravity, via 10-dimensional type IIA supergravity. The bosonic fields of $\mathcal{N}=8$ supergravity are the metric, 28 U(1) gauge fields, and 70 scalar fields parametrizing $E_{7(7)}/SU(8)$. To obtain a generating solution for the most general black hole of $\mathcal{N}=8$ supergravity, global symmetries of the field equations (classical U dualities) imply that it suffices to truncate to a theory with only 4 gauge fields [7]. The relevant supergravity theory, sometimes called the STU model, is an $\mathcal{N}=2$ supergravity coupled to 3 vector multiplets. Each vector multiplet contains a gauge field, a dilaton, and an axion. The fourth gauge field belongs to the $\mathcal{N}=2$ supergravity multiplet. Together, the bosonic fields are the metric, four U(1) gauge fields A^I, three dilatons φ_i and three axions χ_i. We label the gauge fields by $I=1,2,3,4$, and label the dilatons and axions by $i=1,2,3$. It is convenient to denote[1]

$$x_i=\chi_i, \qquad y_i=\mathrm{e}^{-\varphi_i}, \tag{2.1}$$

which can be united as a complex scalar

$$z_i=x_i+\mathrm{i}y_i. \tag{2.2}$$

The scalars parametrize $(\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1))^3$. These complex scalars are sometimes denoted S,T,U, hence the name "STU supergravity."

Since we are in 4 dimensions, the gauge fields A^I may be dualized to dual gauge fields $\tilde{A}_I$. The field strengths are $F^I=\mathrm{d}A^I$ and the dual field strengths are $\tilde{F}_I=\mathrm{d}\tilde{A}_I$. We use the terminology "electric and magnetic according to the nature of the gauge fields A^I. Note that other literature often uses the terms electric and magnetic differently, depending on the choice of duality frame.

We choose a duality frame so that there is a 4-fold symmetry of the gauge fields A^I. One way to understand this is that the original gauged generalization of the theory, the original maximal $\mathcal{N}=8$, SO(8) gauged supergravity, arises from S^7 reduction of 11-dimensional supergravity [26]. An Abelian truncation then gives $\mathcal{N}=2$, $\mathrm{U}(1)^4$ gauged supergravity [100]. The four U(1) gauge fields originate from the $\mathrm{U}(1)^4$ Cartan subgroup of the full SO(8) gauge group, explaining why the four gauge fields A^I are on an equal footing. Taking the ungauged limit then gives the STU supergravity. Furthermore, setting all the gauge fields equal as $A^1=A^2=A^3=A^4$, with vanishing scalars, recovers Einstein-Maxwell theory.

The Lagrangian in terms of $(A^1,\tilde{A}_2,\tilde{A}_3,A^4)$ is relatively short,

$$\begin{aligned}\mathcal{L}_4 &= R\star 1-\frac{1}{2}\sum_{i=1}^{3}(\star\mathrm{d}\varphi_i\wedge\mathrm{d}\varphi_i+\mathrm{e}^{2\varphi_i}\star\mathrm{d}\chi_i\wedge\mathrm{d}\chi_i)\\ &\quad -\frac{1}{2}\mathrm{e}^{-\varphi_1}(\mathrm{e}^{\varphi_2+\varphi_3}\star\mathcal{F}^1\wedge\mathcal{F}^1+\mathrm{e}^{\varphi_2-\varphi_3}\star\tilde{\mathcal{F}}_2\wedge\tilde{\mathcal{F}}_2,\\ &\quad +\mathrm{e}^{-\varphi_2+\varphi_3}\star\tilde{\mathcal{F}}_3\wedge\tilde{\mathcal{F}}_3+\mathrm{e}^{-\varphi_2-\varphi_3}\star\mathcal{F}^4\wedge\mathcal{F}^4)\\ &\quad +\chi_1(F^1\wedge F^4+\tilde{F}_2\wedge\tilde{F}_3),\end{aligned} \tag{2.3}$$

where $\mathcal{F}^I$ and $\tilde{\mathcal{F}}_I$ are field strengths modified by "transgression" terms, given by

$$\begin{aligned}&\mathcal{F}^1=F^1+\chi_3\tilde{F}_2+\chi_2\tilde{F}_3-\chi_2\chi_3F^4, \qquad \mathcal{F}^4=F^4,\\ &\tilde{\mathcal{F}}_2=\tilde{F}_2-\chi_2F^4, \qquad \tilde{\mathcal{F}}_3=\tilde{F}_3-\chi_3F^4.\end{aligned} \tag{2.4}$$

Note that the parity-odd terms can also be written as $\chi_1(\mathcal{F}^1\wedge\mathcal{F}^4+\tilde{\mathcal{F}}_2\wedge\tilde{\mathcal{F}}_3)$. After relabeling and changing

[1] The literature has various conventions; our previous papers [3,28] stated $x_i=-\chi_i$, but used only χ_i. The convention in [99] is $x_i=-\chi_i$.

the signs of some axions, this matches the Lagrangian of [71,100].[2] A further advantage of this formulation is that it comes directly from T^2 reduction of a 6-dimensional supergravity, given in Sec. II E 2.

It is also useful to write the Lagrangian (2.3) in the general form

$$\mathcal{L}_4 = \mathrm{d}^4x\sqrt{-g}\bigg[R - \frac{1}{2}f_{AB}(\Phi)\partial_\mu\Phi^A\partial^\mu\Phi^B - \frac{1}{4}k_{IJ}(\Phi)\mathbf{F}^I_{\mu\nu}\mathbf{F}^{J\mu\nu} + \frac{1}{4}h_{IJ}(\Phi)\epsilon^{\mu\nu\rho\sigma}\mathbf{F}^I_{\mu\nu}\mathbf{F}^J_{\rho\sigma}\bigg], \quad (2.5)$$

where $\Phi^A = (\varphi_1,\varphi_2,\varphi_3,\chi_1,\chi_2,\chi_3)$ are the scalar fields, and $\mathbf{A}^I = (A^1,\tilde{A}_2,\tilde{A}_3,A^4)$ are the U(1) gauge fields, with field strengths $\mathbf{F}^I = \mathrm{d}\mathbf{A}^I$. The kinetic coefficients are

$$f_{AB} = \mathrm{diag}(1,1,1,\mathrm{e}^{2\varphi_1},\mathrm{e}^{2\varphi_2},\mathrm{e}^{2\varphi_3}),$$
$$h_{IJ} = -\frac{\chi_1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (2.6)$$

and k_{IJ} is a longer expression that can be easily deduced from the Lagrangian (2.3).

A. Symmetries: SL$(2,\mathbb{R})$ and triality

We define the three matrices of scalars $\mathcal{M}_i$ as (see e.g. [5])

$$\mathcal{M}_i = \frac{1}{y_i}\begin{pmatrix} 1 & x_i \\ x_i & x_i^2 + y_i^2 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\varphi_i} & \chi_i\mathrm{e}^{\varphi_i} \\ \chi_i\mathrm{e}^{\varphi_i} & \mathrm{e}^{-\varphi_i} + \chi_i^2\mathrm{e}^{\varphi_i} \end{pmatrix}. \quad (2.7)$$

The scalar matrix $\mathcal{M}_i$ transforms under the classical $\mathrm{SL}(2,\mathbb{R})_1 \times \mathrm{SL}(2,\mathbb{R})_2 \times \mathrm{SL}(2,\mathbb{R})_3$ U dualities in the trivial representation for two out of the three $\mathrm{SL}(2,\mathbb{R})$ groups. For the nontrivial corresponding $\mathrm{SL}(2,\mathbb{R})_i$ group, it transforms as

$$\mathcal{M}_i \to \omega_i^T\mathcal{M}_i\omega_i, \quad (2.8)$$

where $\omega_i \in \mathrm{SL}(2,\mathbb{R})_i$, given by

$$\omega_i = \begin{pmatrix} d & b \\ c & a \end{pmatrix}, \qquad ad - bc = 1. \quad (2.9)$$

In the quantum theory, a,b,c,d are integers. The scalar kinetic terms of the Lagrangian may be written as

$$\mathcal{L}_{\mathrm{scalar}} = -\frac{1}{2}\sum_{i=1}^{3}(\star\mathrm{d}\varphi_i \wedge \mathrm{d}\varphi_i + \mathrm{e}^{2\varphi_i}\star\mathrm{d}\chi_i \wedge \mathrm{d}\chi_i) = \frac{1}{4}\sum_{i=1}^{3}\mathrm{Tr}(\star\mathrm{d}\mathcal{M}_i^{-1} \wedge \mathrm{d}\mathcal{M}_i), \quad (2.10)$$

which is manifestly invariant under $\mathrm{SL}(2,\mathbb{R})^3$ and under permutation of the three pairs of scalars. Note that if the scalars (φ_i,χ_i), $i = 1, 2, 3$ vanish at infinity, then $\mathcal{M}_i = \mathbb{I} + O(1/r)$.

More generally, one can show that the equations of motion of the Lagrangian (2.3) can be written in a form manifestly invariant under $\mathrm{SL}(2,\mathbb{R})^3$ and under permutation of the three copies of $\mathrm{SL}(2,\mathbb{R})$. The symmetry is however not manifest in the action (2.3). However, there exist three actions that each make manifest a pair of $\mathrm{SL}(2,\mathbb{R})$ symmetries and that only differ by dualizations of gauge fields [5]. In this sense, the theory described by (2.3) admits a triality symmetry.

B. Dualization

There are several other formulations of STU supergravity that appear in the literature, corresponding to different duality frames. To obtain these, we need relations between gauge fields F^I and dual gauge fields $\tilde{F}_I$, for each I. We introduce the dual gauge potential as a Lagrange multiplier to enforce the Bianchi identity for the original gauge field strength, and then vary with respect to the original field strength. To dualize F^1 to $\tilde{F}_1$, we add to the Lagrangian (2.3) an extra term

$$-\tilde{A}_1 \wedge \mathrm{d}F^1 = -\tilde{F}_1 \wedge F^1 + \mathrm{d}(\tilde{A}_1 \wedge F^1). \quad (2.11)$$

Varying the modified Lagrangian with respect to F^1, we see that F^1 and $\tilde{F}^1$ are related by

$$\tilde{F}_1 - \chi_1F^4 = -\mathrm{e}^{-\varphi_1+\varphi_2+\varphi_3}\star\mathcal{F}^1. \quad (2.12)$$

Similarly, F^4 and $\tilde{F}_4$ are related by

$$\tilde{F}_4 - \chi_1F^1 = \mathrm{e}^{-\varphi_1}(-\mathrm{e}^{-\varphi_2-\varphi_3}\star\mathcal{F}^4 + \chi_2\chi_3\mathrm{e}^{\varphi_2+\varphi_3}\star\mathcal{F}^1 + \chi_2\mathrm{e}^{\varphi_2-\varphi_3}\star\tilde{\mathcal{F}}_2 + \chi_3\mathrm{e}^{-\varphi_2+\varphi_3}\star\tilde{\mathcal{F}}_3). \quad (2.13)$$

To dualize $\tilde{F}_2$ to F^2, we instead add to the Lagrangian (2.3) an extra term

$$A^2 \wedge \mathrm{d}\tilde{F}_2 = F^2 \wedge \tilde{F}_2 - \mathrm{d}(A^2 \wedge \tilde{F}_2), \quad (2.14)$$

and similarly for dualizing $\tilde{F}_3$ to F^3. We see that F^2 and F^3 are related to $\tilde{F}_2$ and $\tilde{F}_3$ by

$$F^2 + \chi_1\tilde{F}_3 = \mathrm{e}^{-\varphi_1+\varphi_2}(\mathrm{e}^{-\varphi_3}\star\tilde{\mathcal{F}}_2 + \chi_3\mathrm{e}^{\varphi_3}\star\mathcal{F}^1),$$
$$F^3 + \chi_1\tilde{F}_2 = \mathrm{e}^{-\varphi_1+\varphi_3}(\mathrm{e}^{-\varphi_2}\star\tilde{\mathcal{F}}_3 + \chi_2\mathrm{e}^{\varphi_2}\star\mathcal{F}^1). \quad (2.15)$$

[2] Our field strengths are related to the hatted field strengths of [71] by $\mathcal{F}^1 = \hat{F}_2$, $\tilde{\mathcal{F}}_2 = \hat{F}_1$, $\tilde{\mathcal{F}}_3 = \hat{\mathcal{F}}^1$, $\mathcal{F}^4 = \hat{\mathcal{F}}^2$ and the signs of χ_1 and χ_3 are flipped while the one of χ_2 is kept fixed.

To obtain a dual Lagrangian, we take the original Lagrangian modified by adding the extra term, and then substitute in the algebraic relation between a gauge field strength and its dual. Applying the procedure to replace F^1 in favor of $\tilde{F}_1$, we obtain the Lagrangian in terms of $(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, A^4)$,

$$\begin{aligned}\mathcal{L}_4 = {} & R\star 1 - \frac{1}{2}\sum_{i=1}^{3}(\star d\varphi_i \wedge d\varphi_i + e^{2\varphi_i}\star d\chi_i \wedge d\chi_i) - \frac{1}{2}e^{-\varphi_1-\varphi_2-\varphi_3}\star F^4 \wedge F^4 \\ & - \frac{1}{2}\sum_{i=1}^{3} e^{2\varphi_i-\varphi_1-\varphi_2-\varphi_3}\star(\tilde{F}_i - \chi_i F^4) \wedge (\tilde{F}_i - \chi_i F^4) + \chi_1\chi_2\chi_3 F^4 \wedge F^4 \\ & - (\chi_1\chi_2\tilde{F}_3 + \chi_2\chi_3\tilde{F}_1 + \chi_3\chi_1\tilde{F}_2) \wedge F^4 + \chi_1\tilde{F}_2 \wedge \tilde{F}_3 + \chi_2\tilde{F}_3 \wedge \tilde{F}_1 + \chi_3\tilde{F}_1 \wedge \tilde{F}_2. \end{aligned} \tag{2.16}$$

An advantage of this Lagrangian is that there is a manifest symmetry between 3 gauge fields, and it fits into a more general prepotential formalism for $\mathcal{N}=2$ supergravity coupled to vector multiplets, as discussed later in Sec. II C.

The Lagrangian (2.16) gives duality relations involving $(\tilde{F}_1, \tilde{F}_2, \tilde{F}_3, F^4)$, namely

$$\begin{aligned} e^{\varphi_1-\varphi_2-\varphi_3}\star(\tilde{F}_1 - \chi_1 F^4) &= F^1 + \chi_3\tilde{F}_2 + \chi_2\tilde{F}_3 - \chi_2\chi_3 F^4, \\ e^{\varphi_2-\varphi_3-\varphi_1}\star(\tilde{F}_2 - \chi_2 F^4) &= F^2 + \chi_1\tilde{F}_3 + \chi_3\tilde{F}_1 - \chi_3\chi_1 F^4, \\ e^{\varphi_3-\varphi_1-\varphi_2}\star(\tilde{F}_3 - \chi_3 F^4) &= F^3 + \chi_2\tilde{F}_1 + \chi_1\tilde{F}_2 - \chi_1\chi_2 F^4, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \tilde{F}_4 = {} & -e^{-\varphi_1-\varphi_2-\varphi_3}\star F^4 + \sum_{i=1}^{3} e^{2\varphi_i-\varphi_1-\varphi_2-\varphi_3}\chi_i\star(\tilde{F}_i - \chi_i F^4) + 2\chi_1\chi_2\chi_3 F^4 \\ & - (\chi_2\chi_3\tilde{F}_1 + \chi_3\chi_1\tilde{F}_2 + \chi_1\chi_2\tilde{F}_3). \end{aligned} \tag{2.18}$$

Alternatively, these can be obtained from the duality relations involving $(F^1, \tilde{F}_2, \tilde{F}_3, F^4)$ that arise from the first Lagrangian (2.3).

C. Prepotential formalism

Any $\mathcal{N}=2$ supergravity coupled to vector multiplets can be derived from a prepotential in a certain duality frame. We first define the gauge field and dual gauge field

$$A^0 \equiv -\tilde{A}_4, \qquad \tilde{A}_0 \equiv A^4. \tag{2.19}$$

In this formalism, STU supergravity has complex scalars X^Λ, $\Lambda = 0, 1, 2, 3$ and gauge fields $\tilde{F}_\Lambda = d\tilde{A}_\Lambda$ for $\Lambda = 0$, 1, 2, 3. The Lagrangian is

$$\mathcal{L}_4 = R\star 1 - 2g_{i\bar{j}}\star dX^i \wedge d\bar{X}^{\bar{j}} + \frac{1}{2}\tilde{F}_\Lambda \wedge \tilde{G}^\Lambda, \tag{2.20}$$

where $g_{i\bar{j}} = \partial_i\partial_{\bar{j}}K$ is a Kähler metric derived from a Kähler potential K, and $\tilde{G}^\Lambda$ depends on $\tilde{F}_\Lambda$ and its dual. The prepotential is

$$F(X) = -\frac{X^1X^2X^3}{X^0}. \tag{2.21}$$

One may define complex scalars $z_i = X^i/X^0$, fix the gauge $X^0 = 1$, and relate $z_i = x_i + iy_i = \chi_i + ie^{-\varphi_i}$. For more details, see e.g. [99].

D. Truncations

Some special cases of our general black hole solutions are already known in the literature. Most of these are solutions of theories that are consistent bosonic truncations of the STU model, and some of these are bosonic truncations of other supergravity theories. We therefore review these truncations (see also [101]). The relationships between these truncations are indicated in Fig. 1.

1. ST^2 supergravity

There is a consistent truncation of STU supergravity to an $\mathcal{N}=2$ supergravity coupled to two vector multiplets. We refer to it as ST^2 supergravity, since it involves setting the complex scalars $T = U$ in STU supergravity. There are 3 independent gauge fields, 2 dilatons and 2 axions. It is obtained by setting $A^2 = A^3$, $\varphi_2 = \varphi_3$ and $\chi_2 = \chi_3$, which implies that $\tilde{A}_2 = \tilde{A}_3$. The theory can be obtained by reduction of 5-dimensional supergravity coupled to a vector multiplet, as discussed in Sec. II E 1. This theory admits $SO(2,2) \sim SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetries which get enhanced to SO(4,3) upon dimensional reduction to 3 dimensions [101,102].

2. S^3 supergravity

There is a consistent truncation of STU supergravity to an $\mathcal{N}=2$ supergravity coupled to one vector multiplet. This is sometimes known as S^3 supergravity (or T^3

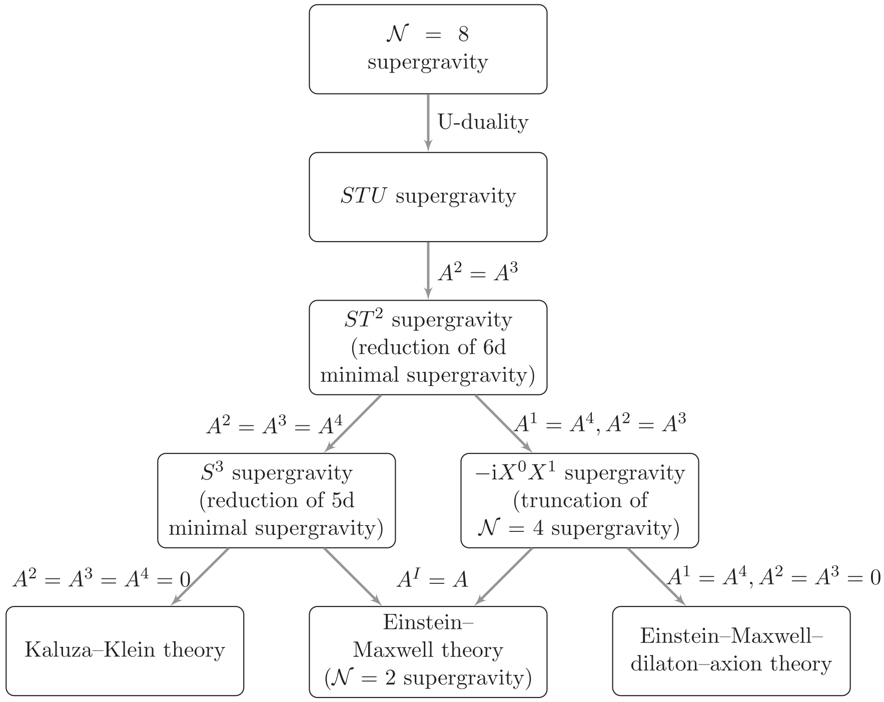

FIG. 1. Bosonic truncations of $\mathcal{N} = 8$ supergravity.

supergravity), since the truncation of the STU supergravity includes setting the three complex scalars equal, $S = T = U$. There are 2 independent gauge fields, 1 dilaton and 1 axion. It is obtained by setting equal the fields in each of the three vector multiplets of STU supergravity, namely $A/\sqrt{3} \equiv A^1 = A^2 = A^3$, $\varphi/\sqrt{3} \equiv \varphi_1 = \varphi_2 = \varphi_3$ and $\chi/\sqrt{3} \equiv \chi_1 = \chi_2 = \chi_3$. The $(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, A^4)$ Lagrangian (2.16) becomes

$$\mathcal{L}_4 = R\star 1 - \frac{1}{2}\star\mathrm{d}\varphi \wedge \mathrm{d}\varphi - \frac{1}{2}\mathrm{e}^{2\varphi/\sqrt{3}}\star\mathrm{d}\chi \wedge \mathrm{d}\chi - \frac{1}{2}\mathrm{e}^{-\varphi/\sqrt{3}}\star(\tilde{F} - \chi F^4) \wedge (\tilde{F} - \chi F^4) - \frac{1}{2}\mathrm{e}^{-\sqrt{3}\varphi}\star F^4 \wedge F^4 + \frac{\chi}{\sqrt{3}}\left(\tilde{F} \wedge \tilde{F} - \chi \tilde{F} \wedge F^4 + \frac{\chi^2}{3} F^4 \wedge F^4\right). \tag{2.22}$$

It can be obtained by reduction of 5-dimensional minimal supergravity, as discussed in Sec. II E 1. The 3-dimensional action obtained by dimensional reduction has $G_{2(2)}$ symmetries.

3. Kaluza-Klein theory

A further consistent bosonic truncation of S^3 supergravity is Kaluza-Klein theory, i.e. the reduction to 4 dimensions of 5-dimensional Einstein gravity. This comes from (2.22) by taking $\tilde{A} = 0$ and $\chi = 0$. Relabeling $A^4 \to A$, the Lagrangian is

$$\mathcal{L}_4 = R\star 1 - \frac{1}{2}\star\mathrm{d}\varphi \wedge \mathrm{d}\varphi - \frac{1}{2}\mathrm{e}^{-\sqrt{3}\varphi}\star F \wedge F. \tag{2.23}$$

The symmetry group obtained upon dimensional reduction to 3 dimensions is $\mathrm{SL}(3, \mathbb{R})$.

4. $-\mathrm{i}X^0X^1$ supergravity

A different set of consistent truncations from STU supergravity comes from setting the 4 gauge fields pairwise equal. From the Lagrangian (2.3), we set $A^1 = A^4$, $\tilde{A}_2 = \tilde{A}_3$, and $\varphi_2 = \varphi_3 = \chi_2 = \chi_3 = 0$, giving the Lagrangian

$$\begin{aligned}\mathcal{L}_4 = {} & R\star 1 - \frac{1}{2}\star\mathrm{d}\varphi \wedge \mathrm{d}\varphi - \frac{1}{2}\mathrm{e}^{2\varphi}\star\mathrm{d}\chi \wedge \mathrm{d}\chi \\ & - \mathrm{e}^{-\varphi}(\star F^1 \wedge F^1 + \star\tilde{F}_2 \wedge \tilde{F}_2) \\ & + \chi(F^1 \wedge F^1 + \tilde{F}_2 \wedge \tilde{F}_2),\end{aligned} \tag{2.24}$$

where $\varphi \equiv \varphi_1$ and $\chi \equiv \chi_1$. This is the bosonic truncation of an $\mathcal{N} = 2$ supergravity coupled to one vector mutiplet. This theory is also known in the literature as the $\mathrm{EM}_2\mathrm{DA}$ theory [59,61]. An important use is to generate solutions of $\mathcal{N} = 4$ supergravity, since it is a truncation of the SU(4) formulation of $\mathcal{N} = 4$ supergravity [103]. By dualizing $\tilde{F}_2$ to F^2, or equivalently making a symplectic transformation, the theory is equivalent to that obtained from a prepotential $F(X) = -\mathrm{i}X^0X^1$ [104]. A truncation of the SO(4) formulation of $\mathcal{N}=4$ supergravity [105,106] corresponds to the dual formulation [107]. Upon dimensional reduction to 3 dimensions, the theory admits $\mathrm{SU}(2,2) \sim \mathrm{SO}(4,2)$ symmetries.

5. Einstein-Maxwell-dilaton-axion theory

A further consistent bosonic truncation of the $-\mathrm{i}X^0X^1$ supergravity has just one gauge field. We take $\tilde{A}_2 = 0$ in (2.24), so the Lagrangian is

$$\mathcal{L}_4 = R\star 1 - \frac{1}{2}\star \mathrm{d}\varphi \wedge \mathrm{d}\varphi - \frac{1}{2}\mathrm{e}^{2\varphi}\star \mathrm{d}\chi \wedge \mathrm{d}\chi - \mathrm{e}^{-\varphi}\star F \wedge F + \chi F \wedge F, \tag{2.25}$$

where $F = F^1$. This is sometimes known as Einstein-Maxwell-dilaton-axion (EMDA) theory or dilaton-axion gravity. Again, the theory is used when generating solutions of $\mathcal{N}=4$ supergravity. Upon dimensional reduction to 3 dimensions, the theory admits $Sp(4,\mathbb{R}) \sim SO(3,2)$ symmetries.

6. Einstein-Maxwell theory

Einstein-Maxwell theory corresponds to setting the gauge fields equal, $A = A^1 = A^2 = A^3 = A^4$, and the scalars trivial, $\varphi_i = \chi_i = 0$. The Lagrangian is

$$\mathcal{L}_4 = R\star 1 - 2\star F \wedge F. \tag{2.26}$$

It is the bosonic sector of pure $\mathcal{N}=2$ supergravity. Upon dimensional reduction to 3 dimensions, the theory admits $SU(2,1)$ symmetries.

E. Oxidation to higher dimensions

Some special cases of our general black hole solutions have been discussed in the literature with a higher-dimensional interpretation. For example, a 4-dimensional black hole can be regarded as a 5-dimensional homogeneous black string. Also, the embedding in 10-dimensional or 11-dimensional supergravity allows for a microscopic interpretation of black holes in terms of string theory or M theory and its web of dual theories. We therefore quickly review several oxidations of 4-dimensional STU supergravity into higher-dimensional theories. A review of the lift to 5 and 6 dimensions, including truncations and a generalization to an SO(5, 4) coset model, is [101].

1. Uplift to 5 dimensions

The Lagrangian (2.16) has a direct uplift to a 5-dimensional $\mathcal{N}=2$ supergravity coupled to 2 vector multiplets, also known as the STU model or 5-dimensional $\mathrm{U}(1)^3$ supergravity [4,5]. This 5-dimensional theory has 3 gauge fields $\tilde{A}_i$, $i = 1, 2, 3$ on an equal footing. The Lagrangian is

$$\mathcal{L}_5 = R\star 1 - \frac{1}{2}\sum_{i=1}^{3} h_i^{-2}(\star \mathrm{d}h_i \wedge \mathrm{d}h_i + \star\tilde{F}_i \wedge \tilde{F}_i) + \tilde{F}_1 \wedge \tilde{F}_2 \wedge \tilde{A}_3, \tag{2.27}$$

subject to the constraint that $h_1h_2h_3 = 1$. A common parametrization of the scalars is

$$h_1 = \mathrm{e}^{-\varphi_1'/\sqrt{6}-\varphi_2'/\sqrt{2}}, \qquad h_2 = \mathrm{e}^{-\varphi_1'/\sqrt{6}+\varphi_2'/\sqrt{2}}, \qquad h_3 = \mathrm{e}^{2\varphi_1'/\sqrt{6}}. \tag{2.28}$$

Another parametrization of the scalars, which is useful for lifting to 6 dimensions, is

$$h_1 = \mathrm{e}^{2\phi_2/\sqrt{6}}, \qquad h_2 = \mathrm{e}^{\phi/\sqrt{2}-\phi_2/\sqrt{6}}, \qquad h_3 = \mathrm{e}^{-\phi/\sqrt{2}-\phi_2/\sqrt{6}}. \tag{2.29}$$

The scalar kinetic terms with these parametrizations are

$$\frac{1}{2}\sum_{i=1}^{3} h_i^{-2}\star \mathrm{d}h_i \wedge \mathrm{d}h_i = \frac{1}{2}\sum_{i=1}^{2}\star \mathrm{d}\varphi_i' \wedge \mathrm{d}\varphi_i' = \frac{1}{2}(\star \mathrm{d}\phi \wedge \mathrm{d}\phi + \star \mathrm{d}\phi_2 \wedge \mathrm{d}\phi_2). \tag{2.30}$$

We may dualize the third gauge field $\tilde{A}_3$ to a 2-form potential B. The usual dualization procedure gives $\tilde{F}_3 = \mathrm{d}\tilde{A}_3 = -h_1^{-2}h_2^{-2}\star\mathcal{H}$, where $\mathrm{d}\mathcal{H} = -\tilde{F}_1 \wedge \tilde{F}_2$, and the Lagrangian is

$$\mathcal{L}_5 = R\star 1 - \frac{1}{2}\sum_{i=1}^{3} h_i^{-2}\star \mathrm{d}h_i \wedge \mathrm{d}h_i - \frac{1}{2}\sum_{i=1}^{2} h_i^{-2}\star\tilde{F}_i \wedge \tilde{F}_i - \frac{1}{2}h_1^{-2}h_2^{-2}\star\mathcal{H} \wedge \mathcal{H}. \tag{2.31}$$

The Kaluza-Klein reduction ansatz is

$$\mathrm{d}s_5^2 = f^{-1}\mathrm{d}s^2 + f^2(\mathrm{d}z_5 - A^4)^2, \qquad \tilde{A}_{(5\mathrm{d})i} = \tilde{A}_i + \chi_i(\mathrm{d}z_5 - A^4). \tag{2.32}$$

Three of the four gauge fields $\tilde{A}_i$ are manifestly on an equal footing; the fourth gauge field A^4 is the graviphoton. Redefining $fh_i = \mathrm{e}^{-\varphi_i}$, the Lagrangian (2.16) is recovered.

There are some notable consistent truncations. Setting $\tilde{A}_2 = \tilde{A}_3$ and $h_2 = h_3$ gives an $\mathcal{N}=2$ supergravity coupled to 1 vector multiplet. If we set $h_1 = \mathrm{e}^{2\varphi/\sqrt{6}}$, then the Lagrangian is

$$\mathcal{L}_5 = R\star 1 - \frac{1}{2}\star \mathrm{d}\varphi \wedge \mathrm{d}\varphi - \frac{1}{2}\mathrm{e}^{-4\varphi/\sqrt{6}}\star\tilde{F}_1 \wedge \tilde{F}_1 - \mathrm{e}^{2\varphi/\sqrt{6}}\star\tilde{F}_2 \wedge \tilde{F}_2 + \tilde{F}_2 \wedge \tilde{F}_2 \wedge \tilde{A}_1. \tag{2.33}$$

Reduction to 4 dimensions gives the ST^2 supergravity. A further consistent truncation is to set all gauge fields equal, $\tilde{A}_i = \tilde{A}$, and trivial scalars $h_i = 1$. This gives the minimal pure $\mathcal{N} = 2$ supergravity, whose bosonic Lagrangian is

$$\mathcal{L}_5 = R\star 1 - \frac{3}{2}\star\tilde{F}\wedge\tilde{F} + \tilde{F}\wedge\tilde{F}\wedge\tilde{A}. \quad (2.34)$$

Reduction to 4 dimensions gives the S^3 supergravity (2.22).

2. Uplift to 6 dimensions

The 4-dimensional theory (2.3) has a higher-dimensional origin in minimal 6-dimensional $\mathcal{N} = (2, 0)$ supergravity coupled to a tensor multiplet. The Lagrangian is

$$\mathcal{L}_6 = R\star 1 - \frac{1}{2}\star d\phi \wedge d\phi - \frac{1}{2}e^{-\sqrt{2}\phi}\star H\wedge H, \quad (2.35)$$

where $H = dB$ is a 3-form field strength.

Directly reducing $\mathcal{L}_6$ on T^2, and then dualizing the 4-dimensional 2-form potential B to an axion χ_1 leads to the $(A^1, \tilde{A}_2, \tilde{A}_3, A^4)$ Lagrangian (2.3). If instead the 2-form B is dualized to a vector in 5 dimensions, and then reduced to 4 dimensions, then we obtain the $(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, A^4)$ Lagrangian (2.16). Either way, there is the same intermediate 5-dimensional STU supergravity theory in some duality frame.

Kaluza-Klein reduction of the 6-dimensional theory (2.35) directly gives the Lagrangian in terms of $(\tilde{A}_1, \tilde{A}_2, H)$. We make the reduction ansatz (see e.g. [108])

$$\begin{aligned} ds^2_{(6d)} &= e^{\phi_2/\sqrt{6}}ds^2 + e^{-3\phi_2/\sqrt{6}}(dz_6 + \tilde{A}_1)^2, \\ B_{(6d)} &= B + \tilde{A}_2 \wedge (dz_6 + \tilde{A}_1), \end{aligned} \quad (2.36)$$

decomposing the field strengths as

$$\begin{aligned} H_{(6d)} &= \mathcal{H} + \tilde{F}_2 \wedge (dz_6 + \tilde{A}_1), \\ \mathcal{H} &= dB - \tilde{A}_2 \wedge \tilde{F}_1, \qquad \tilde{F}_i = d\tilde{A}_i. \end{aligned} \quad (2.37)$$

This gives 5-dimensional STU supergravity in the form (2.31). The 5-dimensional fields $\tilde{F}_1$ and ϕ_2 come from reduction of the Einstein-Hilbert term; H and $\tilde{F}_2$ come from reduction of the 6-dimensional H.

There is a consistent truncation of the 6-dimensional theory (2.35) to the minimal pure $\mathcal{N} = (2, 0)$ supergravity by setting $\phi = 0$ and imposing the constraint that H is self-dual,

$$H = \star H. \quad (2.38)$$

The theory is obtained from the Lagrangian

$$\mathcal{L}_6 = R\star 1 - \frac{1}{2}\star H\wedge H, \quad (2.39)$$

with the self-duality condition imposed on the resulting field equations. Upon dimensional reduction to 5 dimensions, the latter condition is equivalent to $\tilde{A}_2 = \tilde{A}_3$ and $h_2 = h_3$. The resulting 5-dimensional theory is therefore given by (2.33).

3. Uplift to 10 dimensions

The 6-dimensional supergravity action (2.35) naturally uplifts to a consistent truncation of type IIB supergravity on T^4. The nontrivial 10-dimensional fields are the metric $g_{\mu\nu}$, the Ramond-Ramond two-form C and the dilaton Φ. The reduction ansatz is

$$\begin{aligned} ds^2_{10} &= ds^2_6 + e^{\phi/\sqrt{2}}(dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2), \\ \Phi &= \frac{\phi}{\sqrt{2}}, \qquad C \equiv B. \end{aligned} \quad (2.40)$$

4. Uplift to 11 dimensions

The 5-dimensional STU supergravity can be embedded in 11-dimensional supergravity as follows. The action of 11-dimensional supergravity is

$$\mathcal{L}_{11} = R\star 1 - \frac{1}{2}\star\mathcal{F}\wedge\mathcal{F} - \frac{1}{6}\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{A}, \quad (2.41)$$

where $\mathcal{A}$ is the 3-form and $\mathcal{F} = d\mathcal{A}$ its 4-form field strength. We Kaluza-Klein reduce on T^6 as (see e.g. [109])

$$\begin{aligned} ds^2_{11} &= ds^2_5 + h_1(dX_1^2 + dX_2^2) + h_2(dX_3^2 + dX_4^2) \\ &\quad + h_3(dX_5^2 + dX_6^2), \\ A &= \tilde{A}_1 \wedge dX_1 \wedge dX_2 + \tilde{A}_2 \wedge dX_3 \wedge dX_4 \\ &\quad + \tilde{A}_3 \wedge dX_5 \wedge dX_6, \end{aligned} \quad (2.42)$$

with the constraint that $h_1 h_2 h_3 = 1$ in order that T^6 has constant volume. The 11-dimensional action (2.41) then reduces to the 5-dimensional action (2.27).

III. GENERATING TECHNIQUE

Ungauged supergravity theories have global symmetries that can be used for solution generating techniques. When considering solutions with Killing vectors, one may dimensionally reduce the theories, leading to enhanced symmetries. If a 4-dimensional solution has a timelike Killing vector field, then we may perform a timelike dimensional reduction to a 3-dimensional theory. It has been generally shown that, if the 4-dimensional theory is gravity coupled to scalars parametrizing a symmetric space $\bar{G}/\bar{K}$ (a feature of all supergravity theories with enough supersymmetry) and vectors transforming in a representation of $\bar{G}$, then the 3-dimensional theory is a theory of gravity coupled to scalars that parametrize a larger symmetric space G/K [32]. In particular, the 3-dimensional symmetric space is $SO(4,4)/SL(2,\mathbb{R})^4$ for the STU model. These coset model techniques are described in for example [32,110,111], and in the particular case of SO(4,4) in [71,72,74]. The reduction down to three dimensions was already worked out explicitly for the STU model in terms of the so called c^* map, as done for example in [112].

There are other solution generating techniques available, but the reduction to 3 dimensions is particularly efficient. For example, an alternative method is to lift to higher dimensions, perform boosts to add charges, reduce back to 4 dimensions, apply permutations of gauge fields and electromagnetic duality, and repeat, but this requires multiple steps. Reduction to 3 dimensions is advantageous because the solution generating technique is essentially a one-step process once an appropriate group element has been identified.

A. Reduction to 3 dimensions

After Hodge dualizing 3-dimensional vectors to scalars, the 3-dimensional theory corresponding to the STU model is a theory of Euclidean-signature gravity coupled to 16 scalars: a scalar U corresponding to g_{tt}; a scalar σ dual to the Kaluza-Klein vector; 8 electromagnetic scalars ζ^I and $\tilde{\zeta}_I$; 3 dilatons $y_i = e^{-\varphi_i}$; and 3 axions $x_i = \chi_i$. The 8 scalars $\{U, \sigma, x_i, y_i\}$ arising from the 4-dimensional metric and scalars have the usual positive sign kinetic terms, whereas the 8 scalars $\{\zeta^I, \tilde{\zeta}_I\}$ arising from the 4-dimensional vectors have negative sign kinetic terms. The scalars parametrize a symmetric space $G/K = SO(4,4)/SL(2,\mathbb{R})^4$.

Let us first present the 3-dimensional Lagrangian in terms of the 16 scalars, before explaining the relationship to 4-dimensional fields. The set of 3-dimensional (pseudo) scalar fields is $\varphi^a = \{U, \sigma, x_i, y_i, \zeta^I, \tilde{\zeta}_I\}$. They parametrize the target space of the coset model whose Lagrangian is

$$\mathcal{L}_3 = R\star_3 1 - \frac{1}{2} G_{ab}\partial_\mu\varphi^a\partial^\mu\varphi^b\star_3 1. \tag{3.1}$$

The 3-dimensional moduli space metric G_{ab} is of the form

$$\begin{aligned} ds^2_{G/K} = {} & \sum_i \frac{dx_i^2 + dy_i^2}{y_i^2} + 4dU^2 + \frac{e^{-4U}}{4}\left(d\sigma + \sum_I (\tilde{\zeta}_I d\zeta^I - \zeta^I d\tilde{\zeta}_I)\right)^2 \\ & - e^{-2U}\sum_{I,J}\left(\frac{YY_IY_J}{X_{IJ}} d\zeta^I d\zeta^J + \frac{Y}{X_{IJ}Y_IY_J} d\tilde{\zeta}_I d\tilde{\zeta}_J + \frac{XX_{IJ}Y_I}{Y_J} 2d\zeta^I d\tilde{\zeta}_J\right). \end{aligned} \tag{3.2}$$

X_{IJ} is symmetric, $X_{IJ} = X_{(IJ)}$, and obeys the "self-duality" conditions $X_{12} = X_{34}$, $X_{13} = X_{24}$, $X_{23} = X_{14}$, with

$$\begin{aligned} X_{12} &= \frac{\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}}{x_1 x_2}, \\ X_{13} &= \frac{\sqrt{(x_1^2 + y_1^2)(x_3^2 + y_3^2)}}{x_1 x_3}, \\ X_{23} &= \frac{\sqrt{(x_2^2 + y_2^2)(x_3^2 + y_3^2)}}{x_2 x_3}, \\ X_{11} &= X_{22} = X_{33} = X_{44} = 1. \end{aligned} \tag{3.3}$$

The remaining functions are

$$\begin{aligned} Y_i &= \frac{\sqrt{x_i^2 + y_i^2}}{[(x_1^2 + y_1^2)(x_2^2 + y_2^2)(x_3^2 + y_3^2)]^{1/4}}, \\ Y_4 &= -[(x_1^2 + y_1^2)(x_2^2 + y_2^2)(x_3^2 + y_3^2)]^{1/4}, \\ X &= \frac{x_1 x_2 x_3}{y_1 y_2 y_3}, \\ Y &= \frac{\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)(x_3^2 + y_3^2)}}{y_1 y_2 y_3}. \end{aligned} \tag{3.4}$$

They obey the constraints

$$\begin{aligned} & Y_1Y_2Y_3Y_4 = -1, \qquad X^2X_{12}X_{13}X_{14} = Y^2, \\ & \frac{1}{X^2} = \left(\frac{Y^2}{X^2X_{12}^2} - 1\right)\left(\frac{Y^2}{X^2X_{13}^2} - 1\right)\left(\frac{Y^2}{X^2X_{14}^2} - 1\right). \end{aligned} \tag{3.5}$$

From varying with respect to σ, we have the field equation

$$d\left[e^{-4U}\star_3\left(d\sigma + \sum_I (\tilde{\zeta}_I d\zeta^I - \zeta^I d\tilde{\zeta}_I)\right)\right] = 0. \tag{3.6}$$

We may therefore dualize the scalar σ in favor of a 1-form potential ω_3 through the relation

$$d\omega_3 = -\frac{e^{-4U}}{2}\star_3\left(d\sigma + \sum_I (\tilde{\zeta}_I d\zeta^I - \zeta^I d\tilde{\zeta}_I)\right). \tag{3.7}$$

Similarly, we may dualize the electromagnetic scalars ζ^I and $\tilde{\zeta}_I$ to 1-form potentials $A^I_{(3d)}$ and $\tilde{A}_{I(3d)}$ through

$$\mathrm{d}A^I_{(3\mathrm{d})} = -\zeta^I \mathrm{d}\omega_3 + \mathrm{e}^{-2U} \star_3 \sum_J \left(\frac{Y}{X_{IJ} Y_I Y_J} \mathrm{d}\tilde{\zeta}_J + \frac{X X_{IJ} Y_J}{Y_I} \mathrm{d}\zeta^J \right),$$
$$\mathrm{d}\tilde{A}_{I(3\mathrm{d})} = -\tilde{\zeta}_I \mathrm{d}\omega_3 - \mathrm{e}^{-2U} \star_3 \sum_J \left(\frac{Y Y_I Y_J}{X_{IJ}} \mathrm{d}\zeta^J + \frac{X X_{IJ} Y_I}{Y_J} \mathrm{d}\tilde{\zeta}_J \right). \tag{3.8}$$

The 4-dimensional fields of STU supergravity are reconstructed as follows. The metric is

$$\mathrm{d}s^2 = -\mathrm{e}^{2U} (\mathrm{d}t + \omega_3)^2 + \mathrm{e}^{-2U} \mathrm{d}s_3^2, \tag{3.9}$$

and the gauge fields and dual gauge fields are

$$A^I = \zeta^I (\mathrm{d}t + \omega_3) + A^I_{(3\mathrm{d})}, \qquad \tilde{A}_I = \tilde{\zeta}_I (\mathrm{d}t + \omega_3) + \tilde{A}_{I(3\mathrm{d})}. \tag{3.10}$$

The dilatons φ_i and axions χ_i are the same in both 3 and 4 dimensions.

We have presented the 3-dimensional theory in a manner that emphasizes the 4-fold symmetry of the gauge fields. Other treatments in the literature dualize various 4-dimensional gauge fields, so use different notations in 3 dimensions. Consistently with (2.19), we define

$$\zeta^0 \equiv -\tilde{\zeta}_4, \qquad \tilde{\zeta}_0 \equiv \zeta^4, \tag{3.11}$$

and $\zeta^\Lambda = (\zeta^0, \zeta^1, \zeta^2, \zeta^3)$, $\tilde{\zeta}_\Lambda = (\tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2, \tilde{\zeta}_3)$. Then the scalar metric G_{ab} takes the form (see e.g. [74,99])

$$\mathrm{d}s^2_{G/K} = \sum_i \frac{\mathrm{d}x_i^2 + \mathrm{d}y_i^2}{y_i^2} + 4\mathrm{d}U^2 + \frac{\mathrm{e}^{-4U}}{4} (\mathrm{d}\sigma - \zeta^\Lambda \mathrm{d}\tilde{\zeta}_\Lambda + \tilde{\zeta}_\Lambda \mathrm{d}\zeta^\Lambda)^2 + \mathrm{e}^{-2U} [(\mathrm{Im}\mathcal{N})^{\Lambda\Sigma} \mathrm{d}\tilde{\zeta}_\Lambda \mathrm{d}\tilde{\zeta}_\Sigma$$
$$+ ((\mathrm{Im}\mathcal{N})^{-1})_{\Lambda\Sigma} (\mathrm{d}\zeta^\Lambda - (\mathrm{Re}\mathcal{N})^{\Lambda\Gamma} \mathrm{d}\tilde{\zeta}_\Gamma)(\mathrm{d}\zeta^\Sigma - (\mathrm{Re}\mathcal{N})^{\Sigma\Delta} \mathrm{d}\tilde{\zeta}_\Delta)]. \tag{3.12}$$

The period matrix $\mathcal{N}_{\Lambda\Sigma}$ is symmetric and given by (see e.g. [113])[3]

$$\mathcal{N} = \begin{pmatrix} -2x_1x_2x_3 - \mathrm{i}y_1y_2y_3\left(1 + \sum_{i=1}^3 \frac{x_i^2}{y_i^2}\right) & x_2x_3 + \mathrm{i}\frac{x_1y_2y_3}{y_1} & x_1x_3 + \mathrm{i}\frac{x_2y_1y_3}{y_2} & x_1x_2 + \mathrm{i}\frac{x_3y_1y_2}{y_3} \\ x_2x_3 + \mathrm{i}\frac{x_1y_2y_3}{y_1} & -\mathrm{i}\frac{y_2y_3}{y_1} & -x_3 & -x_2 \\ x_1x_3 + \mathrm{i}\frac{x_2y_1y_3}{y_2} & -x_3 & -\mathrm{i}\frac{y_1y_3}{y_2} & -x_1 \\ x_1x_2 + \mathrm{i}\frac{x_3y_1y_2}{y_3} & -x_2 & -x_1 & -\mathrm{i}\frac{y_1y_2}{y_3} \end{pmatrix}. \tag{3.13}$$

Note that if the scalars vanish at infinity, then $\mathcal{N} = -\mathrm{i}\mathbb{I} + O(1/r)$. As shown in [112], the pseudoscalar σ dual to ω_3 is given by

$$\mathrm{d}\omega_3 = -\frac{1}{2} \mathrm{e}^{-4U} \star_3 (\mathrm{d}\sigma + \tilde{\zeta}_\Lambda \mathrm{d}\zeta^\Lambda - \zeta^\Lambda \mathrm{d}\tilde{\zeta}_\Lambda). \tag{3.14}$$

The dualization relations for the 3-dimensional gauge fields and dual gauge fields are

$$\mathrm{d}A^\Lambda_{(3\mathrm{d})} = -\zeta^\Lambda \mathrm{d}\omega_3 - \mathrm{e}^{-2U} \star_3 [(\mathrm{Im}\mathcal{N})^{\Lambda\Sigma} \mathrm{d}\tilde{\zeta}_\Sigma + (\mathrm{Re}\mathcal{N})^{\Lambda\Gamma} ((\mathrm{Im}\mathcal{N})^{-1})_{\Gamma\Sigma} (\mathrm{d}\zeta^\Sigma - (\mathrm{Re}\mathcal{N})^{\Sigma\Delta} \mathrm{d}\tilde{\zeta}_\Delta)],$$
$$\mathrm{d}\tilde{A}_{\Sigma(3\mathrm{d})} = -\tilde{\zeta}_\Sigma \mathrm{d}\omega_3 + \mathrm{e}^{-2U} \star_3 ((\mathrm{Im}\mathcal{N})^{-1})_{\Sigma\Lambda} (\mathrm{d}\zeta^\Lambda - (\mathrm{Re}\mathcal{N})^{\Lambda\Sigma} \mathrm{d}\tilde{\zeta}_\Sigma). \tag{3.15}$$

These dualities are equivalent to the dualities (3.7) and (3.8).

To match the notation of [71], which essentially dualizes two of the gauge fields, apply the previous changes of x_i and y_i, and let

[3]Our conventions relates to the ones of [74,99] as $(\zeta^\Lambda_{\mathrm{ours}}, \tilde{\zeta}^{\mathrm{ours}}_\Lambda) = (-\tilde{\zeta}^{\mathrm{theirs}}_\Lambda, \zeta^\Lambda_{\mathrm{theirs}})$. Also, with respect to our conventions y_i and F are defined in [113] with an opposite sign.

$$(\zeta^1,\tilde{\zeta}_1)=(\sigma_2,-\psi_2),\qquad(\zeta^2,\tilde{\zeta}_2)=(\psi_1,\sigma_1),$$
$$(\zeta^3,\tilde{\zeta}_3)=(\psi_3,\sigma_3),\qquad(\zeta^4,\tilde{\zeta}_4)=(\sigma_4,-\psi_4),\tag{3.16}$$

and

$$\sigma=-2\chi_4-\zeta^1\tilde{\zeta}_1+\zeta^2\tilde{\zeta}_2+\zeta^3\tilde{\zeta}_3-\zeta^4\tilde{\zeta}_4.\tag{3.17}$$

B. Parametrizing $\mathfrak{so}(4,4)$

We choose an explicit parametrization of the Lie algebra $\mathfrak{so}(4,4)$ as given in [74]. However, to make the 4-fold permutation symmetry of the gauge fields manifest, we make some notational changes. We have the 4 Cartan generators

$$\begin{aligned}H_0&=E_{33}+E_{44}-E_{77}-E_{88},\\H_1&=E_{33}-E_{44}-E_{77}+E_{88},\\H_2&=E_{11}+E_{22}-E_{55}-E_{66},\\H_3&=E_{11}-E_{22}-E_{55}+E_{66},\end{aligned}\tag{3.18}$$

12 positive-root generators

$$\begin{aligned}E_0&=E_{47}-E_{38},&E_1&=E_{87}-E_{34},\\E_2&=E_{25}-E_{16},&E_3&=E_{65}-E_{12},\\E^{Q_1}&=E_{45}-E_{18},&E^{Q_2}&=E_{32}-E_{67},\\E^{Q_3}&=E_{36}-E_{27},&E^{Q_4}&=E_{41}-E_{58},\\E^{P^1}&=E_{57}-E_{31},&E^{P^2}&=E_{46}-E_{28},\\E^{P^3}&=E_{42}-E_{68},&E^{P^4}&=E_{17}-E_{35},\end{aligned}\tag{3.19}$$

and 12 negative-root generators

$$\begin{aligned}F_0&=E_{74}-E_{83},&F_1&=E_{78}-E_{43},\\F_2&=E_{52}-E_{61},&F_3&=E_{56}-E_{21},\\F^{Q_1}&=E_{54}-E_{81},&F^{Q_2}&=E_{23}-E_{76},\\F^{Q_3}&=E_{63}-E_{72},&F^{Q_4}&=E_{14}-E_{85},\\F^{P^1}&=E_{75}-E_{13},&F^{P^2}&=E_{64}-E_{82},\\F^{P^3}&=E_{24}-E_{86},&F^{P^4}&=E_{71}-E_{53},\end{aligned}\tag{3.20}$$

where E_{ij} is the 8×8 matrix with 1 in the (i,j) component, and zeros elsewhere. Our generators $(E^{Q_I},E^{P^I},F^{Q_I},F^{P^I})$ are related to the generators $(E_{q_\Lambda},E_{p^\Lambda},F_{q_\Lambda},F_{p^\Lambda})$ of [74] by

$$(E_{q_i},E_{p^i})=(E^{P^i},-E^{Q_i}),\qquad(E_{q_0},E_{p^0})=(E^{Q_4},E^{P^4}),$$
$$(F_{q_i},F_{p^i})=(F^{P^i},-F^{Q_i}),\qquad(F_{q_0},F_{p^0})=(F^{Q_4},F^{P^4}),\tag{3.21}$$

whilst we use the same notation for the generators H_Λ, E_Λ and F_Λ.

The generalized transpose $\sharp$ is defined to act on the generators as

$$H_\Lambda^\sharp=H_\Lambda,\qquad E_\Lambda^\sharp=F_\Lambda,\qquad F_\Lambda^\sharp=E_\Lambda,\tag{3.22}$$

and

$$(E^{Q_I})^\sharp=-F^{Q_I},\qquad(E^{P^I})^\sharp=-F^{P^I},$$
$$(F^{Q_I})^\sharp=-E^{Q_I},\qquad(F^{P^I})^\sharp=-E^{P^I}.\tag{3.23}$$

The following are elements of the eigenspace of the involution $\tau(x)=-x^\sharp$ with eigenvalue $+1$:

$$k_\Lambda=E_\Lambda-F_\Lambda,\qquad k^{Q_I}=E^{Q_I}+F^{Q_I},\qquad k^{P^I}=E^{P^I}+F^{P^I};\tag{3.24}$$

and the following have eigenvalue -1:

$$p_\Lambda=E_\Lambda+F_\Lambda,\quad p^{Q_I}=E^{Q_I}-F^{Q_I},\quad p^{P^I}=E^{P^I}-F^{P^I}.\tag{3.25}$$

k_Λ, p^{Q_I} and p^{P^I} are compact, and p_Λ, k^{Q_I} and k^{P^I} are noncompact. Equivalently, the generalized transpose $\sharp$ adapted to the coset is

$$A^\sharp=\eta A^T\eta^{-1},\tag{3.26}$$

where the 8×8 matrix

$$\eta=\mathrm{diag}(-1,-1,1,1,-1,-1,1,1)\tag{3.27}$$

is the quadratic form preserved by $\mathfrak{sl}(2,\mathbb{R})^4=\mathfrak{so}(2,2)^2$. The explicit generators of the four commuting $\mathfrak{sl}(2,\mathbb{R})$ subalgebras were detailed in [114,115].

The symmetric space G/K can then be parametrized by the group element

$$\begin{aligned}\mathcal{V}&=\exp(-UH_0)\exp\left(\frac{1}{2}\sum_i\varphi_iH_i\right)\exp\left(-\sum_i\chi_iE_i\right)\exp\left[-\sum_I(\zeta^IE^{Q_I}+\tilde{\zeta}_IE^{P^I})\right]\exp\left(-\frac{1}{2}\sigma E_0\right)\\&=\exp(-UH_0)\exp\left[-\frac{1}{2}\sum_i(\log y_i)H_i\right]\exp\left(-\sum_ix_iE_i\right)\exp\left[\sum_\Lambda(-\tilde{\zeta}_\Lambda E_{q_\Lambda}+\zeta^\Lambda E_{p^\Lambda})\right]\exp\left(-\frac{1}{2}\sigma E_0\right).\end{aligned}\tag{3.28}$$

The metric on G/K is then the right-invariant metric obtained from the Maurer-Cartan 1-form $\theta = \mathrm{d}\mathcal{V}\mathcal{V}^{-1}$,

$$\mathrm{d}s^2_{G/K} = \mathrm{Tr}(P_* P_*), \qquad P_* = \frac{1}{2}(\theta + \theta^\sharp). \tag{3.29}$$

Equivalently, one can define the matrix

$$\mathcal{M} = \mathcal{V}^\sharp \mathcal{V} \tag{3.30}$$

and the coset Lagrangian is then given by

$$-\frac{1}{2} G_{ab} \partial_\mu \varphi^a \partial^\mu \varphi^b \star_3 1 = -\frac{1}{8} \mathrm{Tr}[\star_3 (\mathcal{M}^{-1} \mathrm{d}\mathcal{M}) \wedge (\mathcal{M}^{-1} \mathrm{d}\mathcal{M})]. \tag{3.31}$$

Either way, we recover the 3-dimensional moduli space of (3.2).

A group element g acts as

$$\mathcal{V} \to k \mathcal{V} g \tag{3.32}$$

where $k \in \mathrm{SL}(2, \mathbb{R})^4$ is a local compensator, depending on the fields, defined to ensure that the coset element remains in Borel gauge, i.e. of the form (3.28). Since $k^\sharp k = \mathbb{I}$, $\mathcal{M}$ transforms as

$$\mathcal{M} \to g^\sharp \mathcal{M} g, \tag{3.33}$$

which is simpler than working with $\mathcal{V}$, because the compensator is not required.

C. Extracting 3-dimensional fields

1. Scalars

The 3-dimensional scalars are determined from the matrix $\mathcal{M}$ (3.28). For our choice of $\mathfrak{so}(4,4)$ parametrization, they can be extracted from $\mathcal{M}$ by inspection, using the following formulas. The scalar U, which corresponds to the g_{tt} component of the metric, is given by

$$\mathrm{e}^{-4U} = \mathcal{M}_{33} \mathcal{M}_{44} - \mathcal{M}_{34}^2. \tag{3.34}$$

The $i = 1$ dilaton and axion can be extracted from

$$x_1 = \frac{\mathcal{M}_{34}}{\mathcal{M}_{33}}, \qquad y_1^{-1} = \mathrm{e}^{2U} \mathcal{M}_{33}. \tag{3.35}$$

The remaining dilatons and axions are obtained from

$$\begin{aligned}
\frac{1}{y_2 y_3} &= \mathcal{M}_{11} + \mathrm{e}^{4U} (\mathcal{M}_{33} \mathcal{M}_{41}^2 + \mathcal{M}_{44} \mathcal{M}_{31}^2 - 2 \mathcal{M}_{31} \mathcal{M}_{34} \mathcal{M}_{41}), \\
\frac{x_2}{y_2 y_3} &= \mathcal{M}_{16} + \mathrm{e}^{4U} (\mathcal{M}_{34} \mathcal{M}_{41} \mathcal{M}_{63} + \mathcal{M}_{31} \mathcal{M}_{34} \mathcal{M}_{64} - \mathcal{M}_{31} \mathcal{M}_{44} \mathcal{M}_{63} - \mathcal{M}_{33} \mathcal{M}_{41} \mathcal{M}_{64}), \\
\frac{x_3}{y_2 y_3} &= \mathcal{M}_{12} + \mathrm{e}^{4U} (\mathcal{M}_{31} \mathcal{M}_{32} \mathcal{M}_{44} + \mathcal{M}_{33} \mathcal{M}_{41} \mathcal{M}_{42} - \mathcal{M}_{31} \mathcal{M}_{34} \mathcal{M}_{42} - \mathcal{M}_{32} \mathcal{M}_{34} \mathcal{M}_{41}), \\
\frac{x_3^2 + y_3^2}{y_2 y_3} &= \mathcal{M}_{22} + \frac{\mathcal{M}_{32}^2}{\mathcal{M}_{33}} + \mathrm{e}^{4U} \frac{(\mathcal{M}_{32} \mathcal{M}_{34} - \mathcal{M}_{33} \mathcal{M}_{42})^2}{\mathcal{M}_{33}}.
\end{aligned} \tag{3.36}$$

The electromagnetic scalars ζ^I and $\tilde{\zeta}_I$ are obtained from

$$\begin{aligned}
\mathrm{e}^{-4U} \zeta^1 &= \mathcal{M}_{35} \mathcal{M}_{34} - \mathcal{M}_{45} \mathcal{M}_{33}, & \mathrm{e}^{-4U} \tilde{\zeta}_1 &= \mathcal{M}_{31} \mathcal{M}_{44} - \mathcal{M}_{41} \mathcal{M}_{34}, \\
\mathrm{e}^{-4U} \zeta^2 &= \mathcal{M}_{42} \mathcal{M}_{34} - \mathcal{M}_{32} \mathcal{M}_{44}, & \mathrm{e}^{-4U} \tilde{\zeta}_2 &= \mathcal{M}_{64} \mathcal{M}_{33} - \mathcal{M}_{63} \mathcal{M}_{34}, \\
\mathrm{e}^{-4U} \zeta^3 &= \mathcal{M}_{63} \mathcal{M}_{44} - \mathcal{M}_{64} \mathcal{M}_{34}, & \mathrm{e}^{-4U} \tilde{\zeta}_3 &= \mathcal{M}_{32} \mathcal{M}_{34} - \mathcal{M}_{42} \mathcal{M}_{33}, \\
\mathrm{e}^{-4U} \zeta^4 &= \mathcal{M}_{31} \mathcal{M}_{34} - \mathcal{M}_{41} \mathcal{M}_{33}, & \mathrm{e}^{-4U} \tilde{\zeta}_4 &= \mathcal{M}_{35} \mathcal{M}_{44} - \mathcal{M}_{45} \mathcal{M}_{34}.
\end{aligned} \tag{3.37}$$

The scalar σ, dual to the Kaluza-Klein vector, is

$$\begin{aligned}
\sigma &= \frac{2\mathcal{M}_{38}}{\mathcal{M}_{33}} + \frac{\mathrm{e}^{4U}}{\mathcal{M}_{33}} (\mathcal{M}_{33} \mathcal{M}_{35} \mathcal{M}_{41} + \mathcal{M}_{31} \mathcal{M}_{33} \mathcal{M}_{45} + 2 \mathcal{M}_{32} \mathcal{M}_{34} \mathcal{M}_{63} - \mathcal{M}_{33} \mathcal{M}_{42} \mathcal{M}_{63} - \mathcal{M}_{32} \mathcal{M}_{33} \mathcal{M}_{64} - 2 \mathcal{M}_{31} \mathcal{M}_{34} \mathcal{M}_{35}) \\
&= \frac{2\mathcal{M}_{38}}{\mathcal{M}_{33}} - \zeta^4 \tilde{\zeta}_4 - \zeta^1 \tilde{\zeta}_1 + \zeta^2 \tilde{\zeta}_2 + \zeta^3 \tilde{\zeta}_3 + 2 x_1 \tilde{\zeta}_2 \tilde{\zeta}_3 - 2 x_1 \zeta^4 \zeta^1.
\end{aligned} \tag{3.38}$$

With the exception of U, the scalars do not depend on the overall factor in $\mathcal{M}$ but only on ratios of entries of $\mathcal{M}$, and in calculations it can be more practical to rescale $\mathcal{M}$ by a convenient factor.

2. Gauge fields

Three-dimensional gauge fields can be reconstructed from the 3-dimensional scalars using the dualizations (3.7) and (3.8). It is easier, however, to perform these dualizations initially in terms of the seed solution, and act with the solution generating technique on the gauge fields directly. This prevents the dualization of complicated expressions. For STU supergravity, this approach was noted in [73].

From (3.31), the coset matrix $\mathcal{M}$ obeys the equation of motion $\mathrm{d}(\mathcal{M}^{-1}\star_3\mathrm{d}\mathcal{M})=0$. Therefore, we can define the matrix of one-forms $\mathcal{N}$ as

$$\mathrm{d}\mathcal{N}=\mathcal{M}^{-1}\star_3\mathrm{d}\mathcal{M}. \tag{3.39}$$

The coset transformations act on $\mathcal{N}$ as

$$\mathcal{N}\to g^{-1}\mathcal{N}g. \tag{3.40}$$

The matrix $\mathcal{M}^{-1}\mathrm{d}\mathcal{M}$ is a combination of all 28 $\mathfrak{so}(4,4)$ generators with coefficients that depend on the 3-dimensional scalars. Some of these coefficients are directly related to 1-form potentials. In particular, we have

$$\begin{aligned}\mathrm{d}\mathcal{N}&=\mathcal{M}^{-1}\star_3\mathrm{d}\mathcal{M}\\&=\mathrm{d}\omega_3F_0+\sum_I(\mathrm{d}A^I_{(3\mathrm{d})}F^{P^I}-\mathrm{d}\tilde{A}_{I(3\mathrm{d})}F^{Q_I})+\ldots\\&=\mathrm{d}\omega_3F_0+\sum_\Lambda(\mathrm{d}\tilde{A}_{\Lambda(3\mathrm{d})}F_{p^\Lambda}+\mathrm{d}A^\Lambda_{(3\mathrm{d})}F_{q_\Lambda})+\ldots,\end{aligned} \tag{3.41}$$

where the Kaluza-Klein 1-form, gauge fields and dual gauge fields are related to 3-dimensional scalars through (3.7) and (3.8). The dots stands for the terms involving the remaining generators, whose coefficients involve more complicated dependence on the 3-dimensional scalars. From (3.20), one can extract the Kaluza-Klein 1-form

$$\omega_3=\mathcal{N}_{74}, \tag{3.42}$$

and the 3-dimensional electromagnetic 1-forms

$$\begin{aligned}&A^1_{(3\mathrm{d})}=\mathcal{N}_{75},\qquad A^2_{(3\mathrm{d})}=\mathcal{N}_{64},\qquad A^3_{(3\mathrm{d})}=\mathcal{N}_{24},\\&A^4_{(3\mathrm{d})}=\mathcal{N}_{71},\qquad \tilde{A}_{1(3\mathrm{d})}=\mathcal{N}_{81},\qquad \tilde{A}_{2(3\mathrm{d})}=\mathcal{N}_{76},\\&\tilde{A}_{3(3\mathrm{d})}=\mathcal{N}_{72},\qquad \tilde{A}_{4(3\mathrm{d})}=\mathcal{N}_{85}.\end{aligned} \tag{3.43}$$

D. Conserved charges

Consider solutions that are asymptotically flat, or more generally asymptotically Taub-NUT, with vanishing scalars at infinity. Taub-NUT spacetime is asymptotically flat at spatial infinity, in the sense that its metric has the appropriate falloff, so charges may be defined at spatial infinity. For the metric ansatz, we assume that $\mathrm{d}s_3^2$ is asymptotically Euclidean, and take r to be the usual radial coordinate. More precisely, we assume that

$$\begin{aligned}\mathrm{d}s_3^2=&\,\mathrm{d}r^2+(r^2-2mr)(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)+O(r^{-2})\mathrm{d}r^2\\&+O(r^0)\mathrm{d}\theta^2+O(r^0)\mathrm{d}\phi^2,\end{aligned} \tag{3.44}$$

where m is a constant. The asymptotic behavior of a solution gives 10 independent conserved charges at first order in the asymptotic radial expansion around Minkowski: mass M, NUT charge N, 4 electric charges Q_I, and 4 magnetic charges P^I. There is also the angular momentum J defined at second order in the radial expansion. We define Q_I and P^I to be associated with A^I. There are also 6 scalar charges, dilaton charges Σ_i and axion charges Ξ_i, but they are not independent for the solutions that we consider. These 16 charges are encoded in the first-order asymptotic behavior of the 16 3-dimensional scalars $\{U,\sigma,\zeta^I,\tilde{\zeta}_I,x_i,y_i\}$, using the reduction *Ansätze* and dualizations of Sec. III A. The angular momentum J appears in the second-order asymptotic behavior of σ.

More precisely, we assume that we have the expansions at infinity

$$\begin{aligned}&e^{2U}=1-\frac{2M}{r}+O(r^{-2}),\qquad \zeta^I=\frac{Q_I}{r}+O(r^{-2}),\\&\varphi_i=\frac{\Sigma_i}{r}+O(r^{-2}),\\&\omega_3=\left(2N\cos\theta+2J\frac{\sin^2\theta}{r}+O(r^{-2})\right)\mathrm{d}\phi,\\&\tilde{\zeta}_I=\frac{P^I}{r}+O(r^{-2}),\qquad \chi_i=\frac{\Xi_i}{r}+O(r^{-2}).\end{aligned} \tag{3.45}$$

Then M is the canonical Arnowitt-Deser-Misner mass and J is the canonical angular momentum obtained by the standard Komar integral. We have fixed the gauge so that ζ^I and $\tilde{\zeta}_I$ vanish at infinity.

Our convention for the 3-dimensional and 4-dimensional volume forms are $\epsilon_{r\theta\phi}>0$ and $\epsilon_{tr\theta\phi}>0$, so that as $r\to\infty$,[4]

$$\begin{aligned}\star_3 1&\sim r^2\sin\theta\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi,\\ \star 1&\sim r^2\sin\theta\mathrm{d}t\wedge\mathrm{d}r\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi.\end{aligned} \tag{3.46}$$

[4]The 4-dimensional orientation is the same as in [71], but the opposite of [99].

Dualizing ω_3, we have

$$\star_3 \mathrm{d}\omega_3 = -\frac{2N}{r^2}\mathrm{d}r - \frac{1}{2}\mathrm{d}\left(\frac{4J\cos\theta + c}{r^2}\right) + O(r^{-3}), \tag{3.47}$$

where c is a constant. The duality relation (3.14) then implies that

$$\sigma = -\frac{4N}{r} + \frac{4J\cos\theta + c}{r^2} + O(r^{-3}). \tag{3.48}$$

Therefore, the charges are

$$M = -\lim_{r\to\infty}(rU), \qquad Q_I = \lim_{r\to\infty}(r\zeta^I), \qquad \Sigma_i = \lim_{r\to\infty}(r\varphi_i),$$
$$J = \lim_{r\to\infty}\left(\frac{r(\omega_{3\phi} - 2N\cos\theta)}{2\sin^2\theta}\right), \qquad N = -\frac{1}{4}\lim_{r\to\infty}(r\sigma),$$
$$P^I = \lim_{r\to\infty}(r\tilde{\zeta}_I), \qquad \Xi_i = \lim_{r\to\infty}(r\chi_i). \tag{3.49}$$

For comparison with other duality frames, it is useful to define electromagnetic charges $\tilde{Q}^I$ and $\tilde{P}_I$ corresponding to $\tilde{F}_I$, analogous to the electromagnetic charges Q_I and P^I corresponding to F^I. These electromagnetic charges are related by

$$(Q_I, P^I) = (-\tilde{P}_I, \tilde{Q}^I). \tag{3.50}$$

Charges for A^0 are related to charges for A^4 by

$$(Q_0, P^0) = (-\tilde{Q}^4, -\tilde{P}_4), \qquad (\tilde{Q}^0, \tilde{P}_0) = (Q_4, P^4). \tag{3.51}$$

E. Charge matrices

The charge matrix $\mathcal{Q}$ is defined by a $1/r$ expansion of the matrix $\mathcal{M}$ as

$$\mathcal{M} = \mathbb{I} + \frac{\mathcal{Q}}{r} + \frac{\mathcal{Q}^{(2)}}{r^2} + O(r^{-3}). \tag{3.52}$$

Using the definition of $\mathcal{M}$ in terms of the 3-dimensional scalars and the expansions (3.45) and (3.48), the charge matrix is expressed in terms of physical charges as

$$\begin{aligned}\mathcal{Q} &= 2MH_0 + 2Np_0 - \sum_{I=1}^{4}(Q_I p^{Q_I} + P^I p^{P_I}) \\ &\quad + \sum_{i=1}^{3}(\Sigma_i H_i - \Xi_i p_i) \\ &= 2MH_0 + 2Np_0 + \sum_{\Lambda=0}^{3}(-Q_\Lambda p_{p^\Lambda} + P^\Lambda p_{q_\Lambda}) \\ &\quad + \sum_{i=1}^{3}(\Sigma_i H_i - \Xi_i p_i).\end{aligned} \tag{3.53}$$

From $\mathcal{Q}$ alone, one may therefore read off the charges without knowing full details of the solution. Since a group element g acts as $\mathcal{M} \to g^\sharp \mathcal{M} g$, to preserve asymptotic flatness at spatial infinity we should have $g^\sharp g = \mathbb{I}$. For S^3 supergravity, the charge matrix has been studied before in [116].

Using the generators of (3.18), (3.19) and (3.20), we have

$$\begin{aligned}\frac{1}{4}\,\mathrm{Tr}(\mathcal{Q}^2) &= 4(M^2 + N^2) - \sum_{I=1}^{4}[(Q_I)^2 + (P^I)^2] \\ &\quad + \sum_{i=1}^{3}(\Sigma_i^2 + \Xi_i^2).\end{aligned} \tag{3.54}$$

This quantity is invariant under transformations that preserve asymptotic flatness at spatial infinity.

The angular momentum does not appear in the charge matrix $\mathcal{Q}$, since it enters the $\mathcal{M}$ expansion (3.52) in $\mathcal{Q}^{(2)}$, at subleading order $1/r^2$. Using the expansions (3.45)–(3.48), one can show that

$$\mathcal{Q}^{(2)} = (-2J\cos\theta + a_0)p_0 + \ldots, \tag{3.55}$$

where a_0 is a constant and the dots are the other terms proportional to the Cartan generators H_Λ and the Lie algebra generators $p_i, p_{q_\Lambda}, p_{p^\Lambda}$ which all have eigenvalue -1 under the τ involution.

In [117] (see also [118,119]), it was proposed to define the charge matrix integral $\mathcal{Q}_{\partial_\phi}$ as

$$\mathcal{Q}_{\partial_\phi} \equiv -\frac{3}{8\pi}\int_{S^2_\infty}(\partial_\phi)_\mu \mathcal{M}^{-1}\partial_\nu \mathcal{M}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{3.56}$$

This may be written as

$$\mathcal{Q}_{\partial_\phi} = -\frac{3}{4}\int_0^\pi \mathrm{d}\theta \sin^2\theta\,\partial_\theta \mathcal{Q}^{(2)} = -2Jp_0 + \ldots, \tag{3.57}$$

where we used the 3-dimensional line element (3.44) at the first step and (3.55) at the second step. The angular momentum can therefore be extracted from $\mathcal{Q}_{\partial_\phi}$. The quantity

$$\frac{1}{16}\mathrm{Tr}(\mathcal{Q}_{\partial_\phi}^2) = J^2 + \ldots \tag{3.58}$$

contains the angular momentum square and is invariant under the action of transformations that preserve asymptotic flatness at spatial infinity.

IV. CHARGING UP THE BLACK HOLES

We apply the solution generating technique to the specific example of the Ricci-flat Kerr-Taub-NUT spacetime [78] to obtain dyonic rotating black holes. The resulting solutions of supergravity will in general carry 11 independent parameters, consisting of mass, NUT charge, angular momentum, 4 electric charges and 4 magnetic charges. It is convenient to keep the NUT charge on the same footing as the mass, which allows for an SO(2) symmetry that simplifies the solution. When discussing asymptotically flat black holes, we are free to restrict the solution to a 10 parameter family by solving the final zero NUT charge constraint. This constraint is a linear equation in terms of the NUT charge of the initial seed Kerr-Taub-NUT black hole and is therefore straightforwardly solved.

A. Seed solutions

We present here the initial seed solutions used in the solution generating technique.

1. Taub-NUT seed solution

Static solutions are obtained by starting with the Taub-NUT spacetime, whose metric is

$$\mathrm{d}s^2 = -\frac{r^2-2mr-n^2}{r^2+n^2}(\mathrm{d}t + 2n\cos\theta\mathrm{d}\phi)^2 + \frac{r^2+n^2}{r^2-2mr-n^2}\mathrm{d}r^2 + (r^2+n^2)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{4.1}$$

where m is the mass and n is the NUT charge. By Kaluza-Klein reduction on the t coordinate, it may be expressed in terms of 3-dimensional fields as

$$e^{-2U} = \frac{r^2+n^2}{r^2-2mr-n^2}, \qquad \omega_3 = 2n\cos\theta\mathrm{d}\phi,$$
$$\mathrm{d}s_3^2 = \mathrm{d}r^2 + (r^2-2mr-n^2)(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2). \tag{4.2}$$

By Hodge dualizing ω_3, using the orientation (3.46), we obtain the 3-dimensional scalar

$$\sigma = -\frac{4n(r-m)}{r^2+n^2}. \tag{4.3}$$

Since this is a Ricci-flat metric, all other 3-dimensional scalars are trivial. It is convenient to define the rescaled matrix $\bar{\mathcal{M}} = (r^2-2mr-n^2)\mathcal{M}$, which has polynomial entries.

2. Kerr-Taub-NUT seed solution

Our seed for rotating black holes is the Kerr-Taub-NUT solution [78], which can be written as

$$\mathrm{d}s^2 = -\frac{R}{r^2+u^2}\left(\mathrm{d}\bar{t} - \frac{\bar{a}^2-u^2}{\bar{a}}\mathrm{d}\bar{\phi}\right)^2 + \frac{U}{r^2+u^2}\left(\mathrm{d}\bar{t} - \frac{r^2+\bar{a}^2}{\bar{a}}\mathrm{d}\bar{\phi}\right)^2 + (r^2+u^2)\left(\frac{\mathrm{d}r^2}{R} + \frac{\mathrm{d}u^2}{U}\right), \tag{4.4}$$

where[5]

$$R = r^2 + \bar{a}^2 - 2mr, \qquad U = \bar{a}^2 - u^2 + 2nu. \tag{4.5}$$

Standard Boyer-Lindquist-like coordinates and parameters come from defining the coordinates (t, θ, ϕ) by

$$\frac{\phi}{a} = \frac{\bar{\phi}}{\bar{a}}, \qquad t = \bar{t} + \frac{2n^2}{\bar{a}}\bar{\phi}, \qquad u = n + a\cos\theta, \tag{4.6}$$

where the new angular parameter a and Kaluza-Klein 1-form ω_3 are defined by

$$\bar{a}^2 = a^2 - n^2, \qquad \mathrm{d}t + \omega_3 = \mathrm{d}\bar{t} + \bar{\omega}_3. \tag{4.7}$$

To recover the Taub-NUT solution (4.1), then take $a \to 0$. Note that if $a = 0$, then $\bar{a}^2 = -n^2$ leads to an imaginary rotation parameter $\bar{a}$, but this is not a physical feature since it can be removed by the reparametrization (4.7). In Kaluza-Klein form (3.9), the Kerr-Taub-NUT solution can be written as

$$\mathrm{d}s_3^2 = \frac{RU}{\bar{a}^2}\mathrm{d}\bar{\phi}^2 + (R-U)\left(\frac{\mathrm{d}r^2}{R} + \frac{\mathrm{d}u^2}{U}\right),$$
$$e^{-2U} = \frac{r^2+u^2}{R-U},$$
$$\bar{\omega}_3 = \frac{(r^2+\bar{a}^2)U - (\bar{a}^2-u^2)R}{\bar{a}(R-U)}\mathrm{d}\bar{\phi} = \frac{2(mrU+nuR)}{\bar{a}(R-U)}\mathrm{d}\bar{\phi}. \tag{4.8}$$

By Hodge dualizing ω_3, using the orientation (3.46), we obtain the 3-dimensional scalar

[5]The function U defined here should not be confused with U defined in (3.9). It should be clear to the reader which definition is valid depending on the context.

$$\sigma = \frac{4(mu - nr)}{r^2 + u^2}. \tag{4.9}$$

The nontrivial 3-dimensional scalars are e^{-2U} and σ. We also have $\zeta^I = \tilde{\zeta}_I = 0$, for $I = 1, 2, 3, 4$, and $x_i = 0$, $y_i = 1$, for $i = 1, 2, 3$. It is convenient to define, for the Kerr-Taub-NUT solution, the rescaled matrix

$$\bar{\mathcal{M}} \equiv (R - U)\mathcal{M}, \tag{4.10}$$

since its entries are polynomials rather than rational functions. Specifically, its entries are quadratic in r and u,

$$\bar{\mathcal{M}} = \begin{pmatrix} R-U & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & R-U & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r^2+u^2 & 0 & 0 & 0 & 0 & 2(mu-nr) \\ 0 & 0 & 0 & r^2+u^2 & 0 & 0 & -2(mu-nr) & 0 \\ 0 & 0 & 0 & 0 & R-U & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R-U & 0 & 0 \\ 0 & 0 & 0 & -2(mu-nr) & 0 & 0 & (r-2m)^2+(u-2n)^2 & 0 \\ 0 & 0 & 2(mu-nr) & 0 & 0 & 0 & 0 & (r-2m)^2+(u-2n)^2 \end{pmatrix}. \tag{4.11}$$

The static limit is obtained in the same way as discussed earlier.

The matrix of one-forms $\mathcal{N}$ takes the form $\mathcal{N} = \mathcal{N}_\phi \mathrm{d}\phi$. By definition, the components $\mathcal{N}_\phi$ obey

$$\partial_u \mathcal{N}_\phi = -\frac{R}{a}\mathcal{M}^{-1}\partial_r \mathcal{M}, \qquad \partial_r \mathcal{N}_\phi = \frac{U}{a}\mathcal{M}^{-1}\partial_u \mathcal{M}. \tag{4.12}$$

These are solved by (up to a gauge choice)

$$\mathcal{N}_\phi = \omega_{3\phi}(F_0 + E_0) - \frac{4(m^2 U + n^2 R)}{a(R-U)}E_0 + \frac{2(muR - nrU)}{a(R-U)}H_0. \tag{4.13}$$

B. Addition of charges

We act on the Kerr-Taub-NUT matrix $\mathcal{M}_{\mathrm{KTN}}$ with the group element

$$g = \exp\left(-\sum_I \gamma_I k^{P^I}\right)\exp\left(-\sum_I \delta_I k^{Q_I}\right). \tag{4.14}$$

The generators k^{Q_I} and k^{P^I} are given in (3.24). δ_I are electric charge parameters, and γ_I are magnetic charge parameters. The generator k is explicitly

$$k=\begin{pmatrix} c_{\gamma1}c_{\gamma4} & 0 & s_{\gamma1}c_{\gamma4} & 0 & s_{\gamma1}s_{\gamma4} & 0 & -c_{\gamma1}s_{\gamma4} & 0 \\ 0 & c_{\gamma2}c_{\gamma3} & 0 & -c_{\gamma2}s_{\gamma3} & 0 & s_{\gamma2}s_{\gamma3} & 0 & s_{\gamma2}c_{\gamma3} \\ s_{\gamma1}c_{\gamma4} & 0 & c_{\gamma1}c_{\gamma4} & 0 & c_{\gamma1}s_{\gamma4} & 0 & -s_{\gamma1}s_{\gamma4} & 0 \\ 0 & -c_{\gamma2}s_{\gamma3} & 0 & c_{\gamma2}c_{\gamma3} & 0 & -s_{\gamma2}c_{\gamma3} & 0 & -s_{\gamma2}s_{\gamma3} \\ s_{\gamma1}s_{\gamma4} & 0 & c_{\gamma1}s_{\gamma4} & 0 & c_{\gamma1}c_{\gamma4} & 0 & -s_{\gamma1}c_{\gamma4} & 0 \\ 0 & s_{\gamma2}s_{\gamma3} & 0 & -s_{\gamma2}c_{\gamma3} & 0 & c_{\gamma2}c_{\gamma3} & 0 & c_{\gamma2}s_{\gamma3} \\ -c_{\gamma1}s_{\gamma4} & 0 & -s_{\gamma1}s_{\gamma4} & 0 & -s_{\gamma1}c_{\gamma4} & 0 & c_{\gamma1}c_{\gamma4} & 0 \\ 0 & s_{\gamma2}c_{\gamma3} & 0 & -s_{\gamma2}s_{\gamma3} & 0 & c_{\gamma2}s_{\gamma3} & 0 & c_{\gamma2}c_{\gamma3} \end{pmatrix}$$
$$\times\begin{pmatrix} c_{\delta1}c_{\delta4} & 0 & 0 & -c_{\delta1}s_{\delta4} & s_{\delta1}s_{\delta4} & 0 & 0 & s_{\delta1}c_{\delta4} \\ 0 & c_{\delta2}c_{\delta3} & -s_{\delta2}c_{\delta3} & 0 & 0 & s_{\delta2}s_{\delta3} & c_{\delta2}s_{\delta3} & 0 \\ 0 & -s_{\delta2}c_{\delta3} & c_{\delta2}c_{\delta3} & 0 & 0 & -c_{\delta2}s_{\delta3} & -s_{\delta2}s_{\delta3} & 0 \\ -c_{\delta1}s_{\delta4} & 0 & 0 & c_{\delta1}c_{\delta4} & -s_{\delta1}c_{\delta4} & 0 & 0 & -s_{\delta1}s_{\delta4} \\ s_{\delta1}s_{\delta4} & 0 & 0 & -s_{\delta1}c_{\delta4} & c_{\delta1}c_{\delta4} & 0 & 0 & c_{\delta1}s_{\delta4} \\ 0 & s_{\delta2}s_{\delta3} & -c_{\delta2}s_{\delta3} & 0 & 0 & c_{\delta2}c_{\delta3} & s_{\delta2}c_{\delta3} & 0 \\ 0 & c_{\delta2}s_{\delta3} & -s_{\delta2}s_{\delta3} & 0 & 0 & s_{\delta2}c_{\delta3} & c_{\delta2}c_{\delta3} & 0 \\ s_{\delta1}c_{\delta4} & 0 & 0 & -s_{\delta1}s_{\delta4} & c_{\delta1}s_{\delta4} & 0 & 0 & c_{\delta1}c_{\delta4} \end{pmatrix}. \tag{4.15}$$

We use the notation $s_{\delta I}=\sinh\delta_I$, $c_{\delta I}=\cosh\delta_I$, $s_{\delta I\ldots J}=s_{\delta I}\ldots s_{\delta J}$, $c_{\delta I\ldots J}=c_{\delta I}\ldots c_{\delta J}$, and similarly for γ instead of δ.

This choice of group element is motivated by the 4-fold symmetry of the gauge fields F^I, and by examining the resulting charge matrix when acting on a simple uncharged solution such as the Schwarzschild solution. Asymptotic flatness at spatial infinity, which means that the scalars become trivial at infinity, implies that $k^\sharp k=\mathbb{I}$. The generators k_i do not alter the charge matrix of Schwarzschild, and furthermore leave the Schwarzschild solution invariant, up to a gauge transformation. The generator k_0 rotates the mass into a NUT charge; the group element $k=e^{\beta k_0}$ gives the Taub-NUT solution with mass $M=m\cos(2\beta)$ and NUT charge $N=m\sin(2\beta)$. This leaves the generators k^{Q_I} and k^{P^I} that we use. The new matrix

$$\mathcal{M}=k^\sharp\mathcal{M}_{\mathrm{KTN}}k, \tag{4.16}$$

with the generalized transpose $\sharp$ defined in (3.26), encodes the 16 3-dimensional scalars, which can be extracted using the formulas of Sec. III C 1.

In particular, the $O(r^{-1})$ part of $\mathcal{M}$ determines a new charge matrix $\mathcal{Q}$, from which we can read off the asymptotic charges. Since Taub-NUT and Kerr-Taub-NUT differ in $\mathcal{M}$ at order $O(r^{-2})$, the rotating and nonrotating cases share the same charge matrix.

We obtain the mass and NUT charge,

$$M=m\mu_1+n\mu_2,\qquad N=m\nu_1+n\nu_2, \tag{4.17}$$

where

$$\mu_1=1+\sum_I\left(\frac{s_{\delta I}^2+s_{\gamma I}^2}{2}-s_{\delta I}^2s_{\gamma I}^2\right)+\frac{1}{2}\sum_{I,J}s_{\delta I}^2s_{\gamma J}^2,\qquad \mu_2=\sum_I s_{\delta I}c_{\delta I}\left(\frac{s_{\gamma I}}{c_{\gamma I}}c_{\gamma1234}-\frac{c_{\gamma I}}{s_{\gamma I}}s_{\gamma1234}\right), \tag{4.18}$$

and

$$\nu_1 = \sum_I s_{\gamma I} c_{\gamma I} \left(\frac{c_{\delta I}}{s_{\delta I}} s_{\delta 1234} - \frac{s_{\delta I}}{c_{\delta I}} c_{\delta 1234} \right), \quad \nu_2 = \iota - D \quad (4.19)$$

where

$$\iota = c_{\delta 1234} c_{\gamma 1234} + s_{\delta 1234} s_{\gamma 1234} + \sum_{I<J} c_{\delta 1234} \frac{s_{\delta IJ}}{c_{\delta IJ}} \frac{c_{\gamma IJ}}{s_{\gamma IJ}} s_{\gamma 1234},$$
$$D = c_{\delta 1234} s_{\gamma 1234} + s_{\delta 1234} c_{\gamma 1234} + \sum_{I<J} c_{\delta 1234} \frac{s_{\delta IJ}}{c_{\delta IJ}} \frac{s_{\gamma IJ}}{c_{\gamma IJ}} c_{\gamma 1234}. \quad (4.20)$$

For asymptotically flat solutions, we cancel the NUT charge (4.17) by setting $n = n_0$ where

$$n_0 \equiv -m \frac{\nu_1}{\nu_2}. \quad (4.21)$$

The electric and magnetic charges admit elegant expressions in terms of derivatives of the mass and NUT charge with respect to δ_I,

$$Q_I = 2 \frac{\partial M}{\partial \delta_I}, \qquad P^I = -2 \frac{\partial N}{\partial \delta_I}. \quad (4.22)$$

Equivalently,

$$Q_I = m \rho_I^1 + n \rho_I^2, \qquad P^I = m \pi_1^I + n \pi_2^I, \quad (4.23)$$

where

$$\rho_I^1 = 2 \frac{\partial \mu_1}{\partial \delta_I}, \qquad \rho_I^2 = 2 \frac{\partial \mu_2}{\partial \delta_I},$$
$$\pi_1^I = -2 \frac{\partial \nu_1}{\partial \delta_I}, \qquad \pi_2^I = -2 \frac{\partial \nu_2}{\partial \delta_I}. \quad (4.24)$$

These explicit coefficients are

$$\rho_I^1 = 2 s_{\delta I} c_{\delta I} \left(1 - s_{\gamma I}^2 + \sum_{J \neq I} s_{\gamma J}^2 \right),$$
$$\rho_I^2 = 2(1 + 2 s_{\delta I}^2) \left(\frac{s_{\gamma I}}{c_{\gamma I}} c_{\gamma 1234} - \frac{c_{\gamma I}}{s_{\gamma I}} s_{\gamma 1234} \right), \quad (4.25)$$

and

$$\pi_1^I = 2 \left[s_{\gamma I} c_{\gamma I} (c_{\delta 1234} - s_{\delta 1234}) + \sum_{J \neq I} s_{\gamma J} c_{\gamma J} \left(c_{\delta 1234} \frac{s_{\delta IJ}}{c_{\delta IJ}} - s_{\delta 1234} \frac{c_{\delta IJ}}{s_{\delta IJ}} \right) \right],$$
$$\pi_2^I = -2 \left\{ (c_{\gamma 1234} - s_{\gamma 1234}) \left(c_{\delta 1234} \frac{s_{\delta I}}{c_{\delta I}} - s_{\delta 1234} \frac{c_{\delta I}}{s_{\delta I}} \right) + \sum_{J \neq I} \left[c_{\gamma 1234} \frac{s_{\gamma IJ}}{c_{\gamma IJ}} \left(\frac{c_{\delta J}}{s_{\delta J}} s_{\delta 1234} - \frac{s_{\delta J}}{c_{\delta J}} c_{\delta 1234} \right) + s_{\gamma 1234} \frac{c_{\gamma IJ}}{s_{\gamma IJ}} \left(\frac{s_{\delta J}}{c_{\delta J}} c_{\delta 1234} - \frac{c_{\delta J}}{s_{\delta J}} s_{\delta 1234} \right) \right] \right\}. \quad (4.26)$$

The angular momentum can be read from (3.55) and is

$$J = (\nu_2 m - \nu_1 n) a, \quad (4.27)$$

where ν_1, ν_2 are defined in (4.19).

C. Reconstruction of the 4d solution

We can determine the full 4-dimensional solution by extracting the 3-dimensional scalars and gauge fields, using the formulas of Secs. III C 1 and III C 2. The solution can then be simplified after lengthy algebraic manipulations and using the insights of previously known subcases. The procedure of identifying patterns and relationships among the various functions appearing in the solution is the most nontrivial part of the solution generating process. Here, we describe how to obtain the 4-dimensional fields, and then in Sec. V we summarize the solutions in the simplest presentation that we found.

1. Nonrotating, no NUT

For the static case with no NUT charge, the solution generating technique gives a 4-dimensional spherically symmetric metric of the form

$$ds^2 = -\frac{r^2 - 2mr - n_0^2}{W_0} dt^2 + W_0 \left(\frac{dr^2}{r^2 - 2mr - n_0^2} + d\theta^2 + \sin^2\theta d\phi^2 \right). \quad (4.28)$$

$W_0^2(r)$ is a quartic polynomial in r that can be written down concisely from the components of $\mathcal{M}$ using (3.34), namely

$$W_0^2(r) = \bar{\mathcal{M}}_{33} \bar{\mathcal{M}}_{44} - \bar{\mathcal{M}}_{34}^2. \quad (4.29)$$

The electromagnetic scalars ζ^I and $\tilde{\zeta}_I$ of (3.37) encode the gauge fields A^I. The scalars ζ^I are related by appropriate permutation of the indices $I = 1, 2, 3, 4$, and similarly for $\tilde{\zeta}_I$. We dualize the 3-dimensional scalars σ, ζ^I and $\tilde{\zeta}_I$ to 3-dimensional vectors, using (3.7) and (3.8), to obtain the $\mathrm{d}\phi$ coefficients of A^I, $\tilde{A}_I$ and ω_3. This is straightforward for spherically symmetric solutions with no NUT charge. In this case, $\omega_3 = 0$ and ζ^I, $\tilde{\zeta}_I$ only depend on r. Therefore, Eq. (3.15) implies that $\mathrm{d}\tilde{A}_{I(3\mathrm{d})} = \tilde{P}_I(r)\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\phi$ for some functions $\tilde{P}_I(r)$. Integrability implies that F_I are constants, which implies that $\tilde{A}_{I(3\mathrm{d})}$ are given in terms of the magnetic charges as $\tilde{A}_{I(3\mathrm{d})} = \tilde{P}_I\cos\theta\mathrm{d}\phi$. The gauge fields $A^I_{(3\mathrm{d})}$ are then most easily obtained by electromagnetic duality.

The 4-dimensional dilatons and axions are simply the 3-dimensional scalars derived from (3.36). The scalar fields x_i, y_i, are obtained from (3.35) and (3.36). The easiest way to obtain them is to read off x_1 and y_1 from (3.35), and then, from symmetry arguments, obtain x_2, x_3, y_2 and y_3 by permutation of indices.

2. General rotating

In the general rotating case, the solution generating technique will give a 4-dimensional metric of the form

$$\mathrm{d}s^2 = -\frac{R-U}{W}(\mathrm{d}t + \omega_{3\phi}\mathrm{d}\phi)^2 + W\left(\frac{\mathrm{d}r^2}{R} + \frac{\mathrm{d}u^2}{U} + \frac{RU}{a^2(R-U)}\mathrm{d}\phi^2\right), \tag{4.30}$$

where $R(r)$ and $U(u)$ are defined in (4.5) and $W^2(r,u)$ is a quartic polynomial in r and u that can be obtained from

$$W^2(r,u) = \bar{\mathcal{M}}_{33}\bar{\mathcal{M}}_{44} - \bar{\mathcal{M}}_{34}^2. \tag{4.31}$$

Here we define $a^2 = \bar{a}^2 + n^2$ as in (4.7). The Kaluza-Klein 1-form ω_3 can be obtained from (3.42). The scalars can be obtained from the same procedure as in the static case. Rather than dualizing electromagnetic scalars, the 4-dimensional gauge fields $\tilde{A}_1$, and A^4 are more conveniently obtained from the matrix $\mathcal{N}$ as (3.43) and (3.10). The other gauge fields $\tilde{A}_2$, $\tilde{A}_3$, $\tilde{A}_4$ and A^1, A^2, A^3 can then be obtained by appropriate permutation of indices.

V. SUMMARY OF GENERAL CHARGED BLACK HOLES

In this section, we summarize the explicit expressions for the general black hole solutions that we have constructed.

A. Static black hole

A general asymptotically flat, static generating solution for $\mathcal{N}=8$ supergravity was obtained in [80]. It is parametrized by a mass and 5 independent electromagnetic parameters, which are 6 electromagnetic charges with one constraint in order to cancel the NUT charge. Here, we present an 9-parameter asymptotically flat, static solution with 4 independent electric and 4 independent magnetic charges, including the explicit matter fields, which generalizes the seed solution of [80]. A NUT charge can also be included. Starting from this seed solution, one may then follow the procedure of [80] and generate, using U dualities, the static asymptotically flat solution of $\mathcal{N}=8$ supergravity with 56 electromagnetic parameters. Extreme, asymptotically flat, static black holes were studied in [88,89,91–93,95,96].

Including NUT charge, the solution is parametrized by 10 constants: mass parameter m, NUT parameter n, electric charge parameters δ_I and magnetic charge parameters γ_I, for $I = 1, 2, 3, 4$. The mass and NUT charges are defined in (4.17) and the NUT charge can be canceled by fixing $n = n_0$ defined in (4.21). The electric charges Q_I and magnetic charges P^I are given by (4.23). The orientation is given by (3.46).

1. Metric

The metric can be written as

$$\mathrm{d}s^2 = -\frac{R_0(r)}{W_0(r)}(\mathrm{d}t + 2N\cos\theta\mathrm{d}\phi)^2 + W_0(r)\left(\frac{dr^2}{R_0(r)} + \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \tag{5.1}$$

where

$$R_0(r) = r^2 - 2mr - n^2,$$
$$W_0^2(r) = R_0^2(r) + 2R_0(r)(2Mr + V) + (L(r) + 2Nn)^2. \tag{5.2}$$

Here M, N are the mass and NUT charge defined earlier in (4.17),

$$L(r) = \lambda_1 r + \lambda_0 \tag{5.3}$$

is a linear function in r, and the three remaining constants λ_0, λ_1, V are

$$\lambda_1 = 2(m\nu_2 - n\nu_1), \qquad \lambda_0 = 4(m^2 + n^2)D,$$
$$V = 2(-\mu_2 m + \mu_1 n)n + 2(m^2 + n^2)C, \tag{5.4}$$

where all quantities have been defined earlier in (4.18) and (4.19), except C, given by

$$C = 1 + \sum_I (s_{\delta I}^2 c_{\gamma I}^2 + s_{\gamma I}^2 c_{\delta I}^2) + \sum_{I<J} (s_{\delta IJ}^2 + s_{\gamma IJ}^2) + \sum_{I \neq J} s_{\delta I}^2 s_{\gamma J}^2 + \sum_I \sum_{J<K} (s_{\delta I}^2 s_{\gamma JK}^2 + s_{\gamma I}^2 s_{\delta JK}^2)$$
$$+ 2 \sum_{I<J} \left(s_{\delta 1234} c_{\delta 1234} \frac{s_{\gamma IJ}}{c_{\delta IJ}} \frac{c_{\gamma IJ}}{s_{\delta IJ}} + s_{\delta 1234}^2 \frac{s_{\gamma IJ}^2}{s_{\delta IJ}^2} + s_{\delta IJ} s_{\gamma IJ} c_{\delta IJ} c_{\gamma IJ} + s_{\delta IJ}^2 s_{\gamma IJ}^2 \right) - \nu_1^2 - \nu_2^2. \tag{5.5}$$

The metric is asymptotically flat when $n = n_0$ given in (4.21), which cancels the NUT charge $N = 0$. A global coordinate system is then achieved when the angular coordinates have the standard ranges $\theta \in [0, \pi]$, $\phi \sim \phi + 2\pi$.

2. Gauge fields

The gauge fields and dual gauge fields are

$$A^I = \zeta^I(r)(\mathrm{d}t + 2N\cos\theta \mathrm{d}\phi) + P^I \cos\theta \mathrm{d}\phi, \qquad \tilde{A}_I = \tilde{\zeta}_I(r)(\mathrm{d}t + 2N\cos\theta \mathrm{d}\phi) - Q_I \cos\theta \mathrm{d}\phi, \tag{5.6}$$

where it turns out that one can write the scalars $\zeta^I(r)$ in terms of the master function $W_0(r)$ as

$$\zeta^I = \frac{1}{2W_0^2} \frac{\partial W_0^2}{\partial \delta_I} = \frac{1}{W_0^2(r)} \left[R(r) \left(Q_I r + \frac{\partial V}{\partial \delta_I} \right) + (L(r) + 2Nn) \left(\frac{\partial L(r)}{\partial \delta_I} - P^I n \right) \right]. \tag{5.7}$$

In the case without NUT charge, one needs to take the derivative with generic n first, then set $n = n_0$ in the result. The dual scalars $\tilde{\zeta}_I(r)$ are

$$\tilde{\zeta}_I = \frac{R(r)(P^I r + \tilde{V}_I) + (L(r) + 2Nn)(\tilde{L}_I(r) + Q_I n)}{W_0^2(r)}, \tag{5.8}$$

where $\tilde{L}_I(r)$ is a linear function and $\tilde{V}_I$ a constant, given by

$$\tilde{L}_I(r) = (m\rho_I^2 - n\rho_I^1) r - 4(m^2 + n^2)\tilde{D}_I, \qquad \tilde{V}_I = (n\pi_1^I - m\pi_2^I) n + 2(m^2 + n^2)\tilde{C}_I, \tag{5.9}$$

with

$$\tilde{D}_I = \frac{s_{\gamma I}}{c_{\gamma I}} c_{\gamma 1234} s_{\delta I}^2 - \frac{c_{\gamma I}}{s_{\gamma I}} s_{\gamma 1234} c_{\delta I}^2,$$
$$\tilde{C}_I = (s_{\delta 1234} - c_{\delta 1234})\tilde{C}_{II} + 2 s_{\gamma I} c_{\gamma I} s_{\delta 1234} \left(2 + \sum_K s_{\gamma K}^2 \right) + \sum_{J \neq I} \left(c_{\delta 1234} \frac{s_{\delta IJ}}{c_{\delta IJ}} - s_{\delta 1234} \frac{c_{\delta IJ}}{s_{\delta IJ}} \right) \tilde{C}_{IJ}$$
$$+ 2 \sum_{J \neq I} s_{\gamma J} c_{\gamma J} \left(\frac{s_{\delta IJ}}{c_{\delta IJ}} c_{\delta 1234} (s_{\gamma I}^2 + s_{\gamma J}^2) - \frac{c_{\delta IJ}}{s_{\delta IJ}} s_{\delta 1234} \sum_{K \neq I,J} s_{\gamma K}^2 \right),$$
$$\tilde{C}_{IJ} = 2(1 + 2s_{\delta I}^2) s_{\gamma 1234} \left[\left(2 + \sum_{K \neq J} \frac{1}{s_{\gamma K}^2} \right) s_{\gamma 1234} \frac{c_{\gamma J}}{s_{\gamma J}} - (1 + 2s_{\gamma J}^2) \frac{c_{\gamma 1234}}{s_{\gamma J} c_{\gamma J}} \right] + 2 s_{\delta I}^2 s_{\gamma J} c_{\gamma J} \left(1 + \sum_K s_{\gamma K}^2 \right). \tag{5.10}$$

3. Scalar fields

The scalar fields are

$$\mathrm{e}^{\varphi_i} = \frac{r^2 + n^2 + g_i}{W}, \qquad \chi_i = \frac{f_i}{r^2 + n^2 + g_i}, \tag{5.11}$$

where

$$f_i = 2(mr + n^2)\xi_{i1} + 2n(m - n)\xi_{i2} + 4(m^2 + n^2)\xi_{i3},$$
$$g_i = 2(mr + n^2)\eta_{i1} + 2n(m - n)\eta_{i2} + 4(m^2 + n^2)\eta_{i3}. \tag{5.12}$$

The coefficients ξ_{i1}, ξ_{i2} and ξ_{i3} for $i=1$ are

$$\xi_{11}=[(s_{\delta123}c_{\delta4}-c_{\delta123}s_{\delta4})s_{\gamma_1}c_{\gamma_1}+(1\leftrightarrow4)]-((1,4)\leftrightarrow(2,3)),$$
$$\xi_{12}=\left[\frac{1}{2}(c_{\delta23}s_{\gamma14}+c_{\gamma14}s_{\delta23})(c_{\delta14}c_{\gamma23}+s_{\gamma23}s_{\delta14})+s_{\delta1}s_{\gamma4}c_{\delta4}c_{\gamma1}(s_{\delta2}s_{\gamma2}c_{\delta3}c_{\gamma3}+s_{\delta3}s_{\gamma3}c_{\delta2}c_{\gamma2})+(1\leftrightarrow4)\right]-((1,4)\leftrightarrow(2,3)),$$
$$\xi_{13}=[(s_{\delta134}c_{\delta2}c_{\gamma2}^2+c_{\delta134}s_{\delta2}s_{\gamma2}^2)s_{\gamma_3}c_{\gamma_3}+(2\leftrightarrow3)]-((1,4)\leftrightarrow(2,3)), \tag{5.13}$$

and the coefficients η_{i1}, η_{i2} and η_{i3} for $i=1$ are

$$\eta_{11}=s_{\delta2}^2+s_{\delta3}^2+s_{\gamma1}^2+s_{\gamma4}^2+(s_{\delta2}^2+s_{\delta3}^2)(s_{\gamma1}^2+s_{\gamma4}^2)+(s_{\delta2}^2-s_{\delta3}^2)(s_{\gamma3}^2-s_{\gamma2}^2),$$
$$\eta_{12}=2s_{\delta2}c_{\delta2}(c_{\gamma2}s_{\gamma134}-s_{\gamma2}c_{\gamma134})+(2\leftrightarrow3),$$
$$\eta_{13}=2s_{\delta23}c_{\delta23}(s_{\gamma23}c_{\gamma23}+s_{\gamma14}c_{\gamma14})+s_{\delta23}^2\left(1+\sum_I s_{\gamma I}^2\right)+(s_{\delta2}^2+s_{\delta3}^2+2s_{\delta23}^2)(s_{\gamma14}^2+s_{\gamma23}^2)$$
$$+s_{\delta2}^2s_{\gamma2}^2+s_{\delta3}^2s_{\gamma3}^2+s_{\gamma14}^2. \tag{5.14}$$

The results for $i=2$ and $i=3$ are obtained by respectively interchanging indices $1\leftrightarrow2$ and $1\leftrightarrow3$.

B. Rotating black hole

The general rotating solution depends on 11 independent parameters: the mass, NUT and rotation parameters (m, n, a); and electric (δ_I) and magnetic (γ_I) charge parameters. The mass and NUT charges are defined in (4.17) and the NUT charge can be canceled by fixing $n=n_0$ defined in (4.21). The electric charges Q_I and magnetic charges P^I are given by (4.23) and the angular momentum is given in (4.27). The orientation is given by (3.46).

1. Metric

The metric of the general solution is

$$\mathrm{d}s^2=-\frac{R-U}{W}(\mathrm{d}t+\omega_3)^2+W\left(\frac{\mathrm{d}r^2}{R}+\frac{\mathrm{d}u^2}{U}+\frac{RU}{a^2(R-U)}\mathrm{d}\phi^2\right), \tag{5.15}$$

where R and U are the quadratic functions

$$R(r)=r^2-2mr+a^2-n^2,\qquad U(u)=a^2-(u-n)^2. \tag{5.16}$$

The master function W and the Kaluza-Klein 1-form ω_3 can be expressed as

$$W^2=(R-U)^2+(2Nu+L)^2+2(R-U)(2Mr+V),$$
$$\omega_3=\frac{2N(u-n)R+U(L+2Nn)}{a(R-U)}\mathrm{d}\phi \tag{5.17}$$

in terms of $R(r)$, $U(u)$ and two linear functions $L(r)$ and $V(u)$ given by

$$L(r)=2(-n\nu_1+m\nu_2)r+4(m^2+n^2)D,$$
$$V(u)=2(n\mu_1-m\mu_2)u+2(m^2+n^2)C, \tag{5.18}$$

where ν_1, ν_2, μ_1, μ_2 and D have been defined in (4.18) and (4.19) and C has been defined in (5.5). The static limit is obtained by setting $u=n+a\cos\theta$ and taking $a\to0$. Then $\omega_3=2N\cos\theta\mathrm{d}\phi$, and the solution reduces to the static solution presented previously. The expression of W and ω_3 solely in terms of R, U and linear functions gives an elegant form of the metric.

2. Gauge fields

Astonishingly, the gauge fields can be expressed in the elegant form

$$A^I=-W\frac{\partial}{\partial\delta_I}\left(\frac{\mathrm{d}t+\omega_3}{W}\right), \tag{5.19}$$

which makes manifest that the gauge fields A^I can be built solely from functions already appearing in the metric. In terms of 3-dimensional fields, we have the equivalent relations

$$A^I=\zeta^I(\mathrm{d}t+\omega_3)+A^I_{(3\mathrm{d})}, \tag{5.20}$$

where

$$\zeta^I=\frac{1}{2W^2}\frac{\partial}{\partial\delta_I}(W^2)=\frac{1}{W^2}\left[(R-U)\left(Q_Ir+\frac{\partial V}{\partial\delta_I}\right)+(L+2Nu)\left(\frac{\partial L}{\partial\delta_I}-P^Iu\right)\right],$$
$$A^I_{(3\mathrm{d})}=-\frac{\partial}{\partial\delta_I}\omega_3=\left[P^I(u-n)+\frac{U}{R-U}\left(P^Iu-\frac{\partial L}{\partial\delta_I}\right)\right]\frac{\mathrm{d}\phi}{a}. \tag{5.21}$$

The dual gauge fields are

$$\tilde{A}_I = \tilde{\zeta}_I(\mathrm{d}t + \omega_3) + \tilde{A}_{I(3\mathrm{d})}, \tag{5.22}$$

where

$$\tilde{A}_{I(3\mathrm{d})} = -\left(Q_I(u-n) + \frac{U(Q_I u + \tilde{L}_I)}{R-U}\right)\frac{\mathrm{d}\phi}{a},$$
$$\tilde{\zeta}_I = \frac{1}{W^2}((R-U)(P^I r + \tilde{V}_I) + (L+2Nu)(\tilde{L}_I + Q_I u)), \tag{5.23}$$

where $\tilde{L}_I(r)$, $\tilde{V}_I(u)$ are the linear functions

$$\tilde{L}_I(r) = (m\rho_I^2 - n\rho_I^1)r - 4(m^2+n^2)\tilde{D}_I,$$
$$\tilde{V}_I(u) = (n\pi_1^I - m\pi_2^I)u + 2(m^2+n^2)\tilde{C}_I. \tag{5.24}$$

The coefficients ρ_I^1, ρ_I^2, π_1^I and π_2^I are defined in (4.25) and (4.26). The coefficients $\tilde{D}_I$, $\tilde{C}_I$ are defined in (5.10). We have not found an elegant expression for $\tilde{A}_I$ analogous to (5.19). The asymmetry between A^I and $\tilde{A}_I$ originates from the choice of SO(4,4) group element (4.14), which does not have symmetry under interchange of δ_I and γ_I.

3. Scalar fields

The scalar fields are

$$\mathrm{e}^{\varphi_i} = \frac{r^2+u^2+g_i}{W}, \qquad \chi_i = \frac{f_i}{r^2+u^2+g_i}, \tag{5.25}$$

where

$$f_i = 2(mr+nu)\xi_{i1} + 2(mu-nr)\xi_{i2} + 4(m^2+n^2)\xi_{i3},$$
$$g_i = 2(mr+nu)\eta_{i1} + 2(mu-nr)\eta_{i2} + 4(m^2+n^2)\eta_{i3}, \tag{5.26}$$

and the coefficients $\xi_{i1}, \xi_{i2}, \xi_{i3}, \eta_{i1}, \eta_{i2}, \eta_{i3}$ are the same as the static coefficients (5.13) and (5.14).

VI. PHYSICAL QUANTITIES

In this section, we restrict to asymptotically flat solutions, which have vanishing NUT charge, $N=0$, by setting $n=n_0$ given by (4.21), unless otherwise stated. Note that derivatives with respect to δ_I must be done before setting $n=n_0$.

A. Thermodynamics

In this subsection, we explicitly reinstate the 4-dimensional Newton constant G. Recall from Sec. IV B that the charge matrix provides the mass M in (4.17), and electric charges Q_I and magnetic charges P^I in (4.23). We normalize the electromagnetic charges as

$$\bar{Q}_I = \frac{1}{4G}Q_I, \qquad \bar{P}^I = \frac{1}{4G}P^I. \tag{6.1}$$

The angular momentum J is obtained from another charge matrix in (4.27). Canonical methods then associate the mass to ∂_t, the angular momentum to $-\partial_\phi$, the electric charges $\bar{Q}_I$ associated with A_I to the gauge parameter $\Lambda_I = -1$, and similarly magnetic charges $\bar{P}^I$ associated with $\tilde{A}_I$. To recapitulate, we have

$$M = \frac{m}{G}\left(\mu_1 - \frac{\nu_1\mu_2}{\nu_2}\right),$$
$$J = \frac{ma}{G}\frac{(\nu_1^2+\nu_2^2)}{\nu_2},$$
$$\bar{Q}_I = \frac{m}{2G}\left(\frac{\partial\mu_1}{\partial\delta_I} - \frac{\nu_1}{\nu_2}\frac{\partial\mu_2}{\partial\delta_I}\right),$$
$$\bar{P}^I = \frac{m}{2G}\left(\frac{\nu_1}{\nu_2}\frac{\partial\nu_2}{\partial\delta_I} - \frac{\partial\nu_1}{\partial\delta_I}\right), \tag{6.2}$$

where μ_1, μ_2, ν_1 and ν_2 are given in (4.18) and (4.19).

The black hole has outer and inner horizons at $r=r_\pm$, the roots of the radial function $R(r)$. The angular velocity Ω_+ at the outer horizon is determined by the Killing vector

$$\xi^\mu\partial_\mu = \partial_t + \Omega_+\partial_\phi \tag{6.3}$$

that becomes null at the horizon, and is

$$\Omega_+ = \frac{a}{L(r_+)}, \tag{6.4}$$

where L is given in (5.3). The entropy and temperature are

$$S_+ = \frac{\pi}{G}L(r_+), \qquad T_+ = \frac{R'(r_+)}{4\pi L(r_+)} = \frac{r_+ - m}{2\pi L(r_+)}. \tag{6.5}$$

In the static case, the function $W(r_+, u)$ defined in (5.17) reduces to $L(r_+)$, and so these quantities can be expressed in terms of W. The electric potential $\Phi_+^I = \xi_+^\mu A_\mu^I$ and magnetic potential $\Psi_I^+ = \xi_+^\mu \tilde{A}_{I\mu}$ at the horizon are

$$\Phi_+^I = \Omega_+ A_{(3\mathrm{d})\phi}^I(r_+) = \frac{1}{L}\left(\frac{\partial L}{\partial\delta_I} - n_0 P^I\right)\bigg|_{r=r_+},$$
$$\Psi_I^+ = \Omega_+\tilde{A}_{(3\mathrm{d})\phi}^I(r_+) = \frac{\tilde{L}_I + n_0 Q_I}{L}\bigg|_{r=r_+}, \tag{6.6}$$

where $\tilde{L}_I$ is given in (5.9). These quantities obey the first law of thermodynamics

$$\delta M = T_+\delta S_+ + \Omega_+\delta J + \Phi_+^I\delta\bar{Q}_I + \Psi_I^+\delta\bar{P}^I, \tag{6.7}$$

and the Smarr relation

$$M = 2T_+S_+ + 2\Omega_+J + \Phi^I_+\bar{Q}_I + \Psi^+_I\bar{P}^I. \tag{6.8}$$

Technically, the Smarr relation follows from nontrivial identities obeyed by the parameters,

$$\sum_I(\rho_I^2\pi_1^I - \rho_I^1\pi_2^I) = 8(\mu_1\nu_2 - \mu_2\nu_1 - \imath - D),$$
$$\sum_I\left(Q_I\frac{\partial D}{\partial\delta_I} - P^I\tilde{D}_I\right) = 4D(\mu_1+1)m + (4D\mu_2 + 2\nu_1)n_0. \tag{6.9}$$

B. Cayley hyperdeterminant

For regular, static, extremal black holes of $\mathcal{N}=8$ supergravity, the entropy is expressed in terms of the electromagnetic charges as [98][6]

$$S_+ = 2\pi\sqrt{|\diamond|}, \tag{6.10}$$

where $\diamond$ is the Cartan-Cremmer-Julia $E_{7(7)}$ quartic invariant. See [120] for further details of the definitions of $\diamond$. Specializing to STU supergravity, the $E_{7(7)}$ quartic invariant $\diamond$ reduces to an $SL(2,\mathbb{R})^3$ invariant, the Cayley hyperdeterminant Δ. Consequently, the entropy reduces to (see e.g. [18])

$$S_+ = 2\pi\sqrt{|\Delta|}, \tag{6.11}$$

where the hyperdeterminant is

$$\Delta(Q_I,P^I) = \frac{1}{16}(4(Q_1Q_2Q_3Q_4 + P^1P^2P^3P^4) + 2\sum_{J<K}Q_JQ_KP^JP^K - \sum_J(Q_J)^2(P^J)^2). \tag{6.12}$$

Some special cases of Δ are: all gauge fields equal ($Q_I=Q$, $P^I=P$), with $\Delta=\frac{1}{4}(Q^2+P^2)^2$; only electric charges ($P^I=0$), with $\Delta=\frac{1}{4}Q_1Q_2Q_3Q_4$; only one nonvanishing gauge field ($Q_I=P^I=0$ for $I=2,3,4$), with $\Delta=-\frac{1}{16}(Q_1)^2(P^1)^2$; and pairwise equal gauge fields $[(Q_1,P^1)=(Q_4,P^4)$ and $(Q_2,P^2)=(Q_3,P^3)]$, with $\Delta=\frac{1}{4}(Q_1Q_2+P^1P^2)^2$.

The hyperdeterminant is invariant under permutations of the four gauge fields. It is also manifestly invariant under $SL(2,\mathbb{R})^3$ upon rewriting as

$$\Delta = \frac{1}{32}\epsilon^{ab}\epsilon^{cd}\epsilon^{a'b'}\epsilon^{c'd'}\epsilon^{a''c''}\epsilon^{b''d''}\gamma_{aa'a''}\gamma_{bb'b''}\gamma_{cc'c''}\gamma_{dd'd''}, \tag{6.13}$$

where $\epsilon^{ab}=\epsilon^{[ab]}$, $\epsilon^{01}=1$, and the components $\gamma_{aa'a''}$ are

$$(\gamma_{000},\gamma_{111}) = -(Q_1,P^1), \qquad (\gamma_{001},\gamma_{110}) = (P^2,Q_2),$$
$$(\gamma_{010},\gamma_{101}) = (P^3,Q_3), \qquad (\gamma_{011},\gamma_{100}) = (Q_4,P^4). \tag{6.14}$$

The sets of indices (a,b,c,d), (a',b',c',d') and (a'',b'',c'',d'') each correspond to different copies of $SL(2,\mathbb{R})$. Using Schouten identities such as $\epsilon^{a[b}\epsilon^{cd]}=0$, the hyperdeterminant may be rewritten as

$$\Delta = \frac{1}{32}\epsilon^{a'b'}\epsilon^{c'd'}\epsilon^{a''b''}\epsilon^{c''d''}\epsilon^{ac}\epsilon^{bd}\gamma_{aa'a''}\gamma_{bb'b''}\gamma_{cc'c''}\gamma_{dd'd''}$$
$$= \frac{1}{32}\epsilon^{a''b''}\epsilon^{c''d''}\epsilon^{ab}\epsilon^{cd}\epsilon^{a'c'}\epsilon^{b'd'}\gamma_{aa'a''}\gamma_{bb'b''}\gamma_{cc'c''}\gamma_{dd'd''}, \tag{6.15}$$

so the hyperdeterminant is invariant when the three copies of $SL(2,\mathbb{R})$ are cycled. Since each expression is also manifestly invariant under interchange of two copies of $SL(2,\mathbb{R})$, the hyperdeterminant is invariant under the triality symmetry of permuting the three copies of $SL(2,\mathbb{R})$.

For the general NUT-free, nonextremal black hole solution that we derived, the hyperdeterminant can be expressed in terms of the parameters m, δ_I and γ_I as

$$\Delta = \frac{m^4(\nu_1^2+\nu_2^2)^2(4\imath D - \nu_1^2)}{\nu_2^4}, \tag{6.16}$$

where ν_1, ν_2, $\imath$ and D are given in (4.19).

C. Inner horizon thermodynamics

Associating thermodynamic quantities to the inner horizon of a black hole is an old idea [121–123], but the physical interpretation of these quantities remains unclear. Two particularly interesting inner horizon quantities are the "temperature" T_- and "entropy" S_- which are defined from geometrical quantities at the horizon, through $S_-=A_-/4$ and $T=\kappa_-/2\pi$, where A_- is the area of the inner horizon (defined with a particular orientation) and κ_- is the surface gravity corresponding to the null generator $\xi^\mu_-\partial_\mu = \partial_t + \Omega_-\partial_\phi$ of the inner horizon. All inner horizon thermodynamic quantities T_-, S_-, Ω_-, Φ^I_- and Ψ^-_I are those defined at the outer horizon, but with r_+ replaced by r_-. It is then easy to see that

$$S_-T_- \le 0, \tag{6.17}$$

which makes the physical interpretation of T_- and S_- unclear. We emphasize that

[6]Our convention for the normalization of $\diamond$ differs by a factor of 4 from [98].

$$S_- = \frac{\pi}{G} L(r_-) \tag{6.18}$$

is not necessarily non-negative, and therefore whether the negative sign in $S_- T_-$ comes from T_- or S_- depends on the particular solution.

The inner horizon thermodynamic quantities also obey the first law of thermodynamics and the Smarr relation,

$$\delta M = T_- \delta S_- + \Omega_- \delta J + \Phi^I_- \delta \bar{Q}_I + \Psi^-_I \delta \bar{P}^I,$$
$$M = 2T_- S_- + 2\Omega_- J + \Phi^I_- \bar{Q}_I + \Psi^-_I \bar{P}^I. \tag{6.19}$$

There are relations between the outer and inner entropies, temperatures and angular velocities,

$$\frac{S_-}{S_+} = \frac{T_+}{-T_-} = \frac{\Omega_+}{\Omega_-} = \frac{L(r_-)}{L(r_+)}, \tag{6.20}$$

generalizing formulas known for the Kerr solution [123]. We also notice the relations

$$S_+ = \frac{\pi^2}{3} c_J \frac{-2T_-}{\Omega_- - \Omega_+} = \frac{\pi^2}{3} c_{Q_I} \frac{-2T_-}{\Phi^I_- - \Phi^I_+} = \frac{\pi^2}{3} c_{P^I} \frac{-2T_-}{\Psi^-_I - \Psi^+_I}, \tag{6.21}$$

for each $I = 1, 2, 3, 4$, where we define the "central charges"

$$c_J = 6 \frac{\partial \Delta_J}{\partial J}, \qquad c_{Q_I} = 6 \frac{\partial \Delta_J}{\partial \bar{Q}_I}, \qquad c_{P^I} = 6 \frac{\partial \Delta_J}{\partial \bar{P}^I}, \tag{6.22}$$

and

$$\Delta_J = \Delta + J^2. \tag{6.23}$$

The quantities (6.22) can be obtained as central charges of a Virasoro algebra for the class of extremal fast and slow rotating black holes, as discussed in Secs. VIII B and VIII C. In the case of general nonextremal black holes, there is no known derivation of these central charges from a Virasoro algebra. The first relation in (6.21) can also be written using (6.20) as

$$8\pi^2 J = \Omega_+ S_+ \left(\frac{1}{T_+} + \frac{1}{T_-} \right). \tag{6.24}$$

The thermodynamics of the inner horizon has been considered in higher-derivative theories in [124].

D. Product of horizon areas

The product of the two horizon areas is independent of the mass and quantized in terms of the angular momentum and electromagnetic charges as

$$\frac{A_+ A_-}{64\pi^2 G^2} = J^2 + \Delta(Q_I, P^I) = \Delta_J. \tag{6.25}$$

Some special cases have been considered previously: the 4-charge Cvetič-Youm black hole [125]; the dyonic Kerr-Newman black hole [126] and the dyonic black hole of Kaluza-Klein theory [127]. Since the metric is unaltered by U dualities, this result generalizes to black holes of $\mathcal{N} = 8$ supergravity with 28 electric and 28 magnetic charges by replacing the hyperdeterminant Δ with the quartic $E_{7(7)}$ invariant $\diamond$.

A natural interpretation of the product of areas formula is given in terms of auxiliary left and right "entropies"

$$S_L \equiv \frac{1}{2}(S_+ + S_-), \qquad S_R \equiv \frac{1}{2}(S_+ - S_-), \tag{6.26}$$

which are clearly non-negative. The cases where $S_- < 0$ are then rephrased as cases where $S_R > S_L$. The product formula becomes a level-matching condition,

$$S_L^2 - S_R^2 = 4\pi^2 (J^2 + \Delta). \tag{6.27}$$

Generalizing a result of Einstein gravity [128], in Einstein-Maxwell theory, it has been shown [129,130] (see [131] for a review) that universally

$$A_+ A_- = (8\pi J)^2 + (4\pi Q^2)^2, \tag{6.28}$$

for any electrically charged stationary axisymmetric black hole with surrounding matter. Furthermore, there are inequalities involving the area $\mathcal{A}$ of a smooth stable axisymmetric marginally outer trapped surface [132–134], for example

$$\mathcal{A}^2 \geq (8\pi J)^2 + (4\pi Q^2)^2. \tag{6.29}$$

These types of inequalities are reviewed in [135]. The inequalities can generalize to Einstein-Maxwell-dilaton theory, in particular to Kaluza-Klein theory [136]. We expect that these results further generalize using the appropriate quartic invariant in the charges, to the STU model as

$$A_+ A_- = (8\pi J)^2 + (8\pi)^2 \Delta, \qquad \mathcal{A}^2 \geq (8\pi J)^2 + (8\pi)^2 \Delta, \tag{6.30}$$

and to $\mathcal{N} = 8$ supergravity as

$$A_+ A_- = (8\pi J)^2 + (8\pi)^2 \diamond, \qquad \mathcal{A}^2 \geq (8\pi J)^2 + (8\pi)^2 \diamond. \tag{6.31}$$

E. Nonextremal entropy and F invariant

The nonextremal black hole entropy can be rewritten in the Cardy form [81,137,138]

$$S_+ = 2\pi(\sqrt{\Delta+F} + \sqrt{-J^2+F}), \qquad (6.32)$$

where

$$F(M,Q_I,P^I) = \frac{m^4(\nu_1^2+\nu_2^2)^3}{\nu_2^4}. \qquad (6.33)$$

Indeed, the equality $\Omega_+/T_+ = -\Omega_-/T_-$ (6.20) implies that $\partial S_L/\partial J = 0$ after using the first law at the outer and inner horizons and the definition of S_L in (6.26). Differentiating (6.27) with respect to J, one has $\partial(S_R^2)/\partial J = -8\pi^2 J$. Then integrating gives $S_R = 2\pi\sqrt{-J^2+F}$. The constant of integration F is fixed by the actual value of S_R to be (6.33). Using (6.27), we deduce that $S_L = 2\pi\sqrt{\Delta+F}$. The result for $S_+ = S_L + S_R$ follows.

Since the entropy, the quartic invariant and J are all $E_{7(7)}$ invariant, F admits an $E_{7(7)}$-invariant generalization, depending also on the moduli. We will therefore refer to F defined in (6.33) as the F invariant.

F. BPS bound

For the general rotating black hole, from (2.8) we have

$$\mathcal{M}_i(r,u) = \frac{1}{W}\begin{pmatrix} r^2+u^2+g_i & f_i \\ f_i & (W^2+f_i^2)/(r^2+u^2+g_i)\end{pmatrix}. \qquad (6.34)$$

At infinity, we find the identity since all scalar moduli are trivial,

$$\mathcal{M}_i = \mathbb{I} + O(r^{-1}). \qquad (6.35)$$

In generality, we define the moduli-dependent $\mathrm{SL}(2,\mathbb{R})^3$ invariant

$$\begin{aligned} M_2 = \frac{1}{16}\gamma_{aa'a''}[&(\mathcal{M}_1^{-1})^{ab}(\mathcal{M}_2^{-1})^{a'b'}(\mathcal{M}_3^{-1})^{a''b''} \\ &- (\mathcal{M}_1^{-1})^{ab}\epsilon^{a'b'}\epsilon^{a''b''} - \epsilon^{ab}(\mathcal{M}_2^{-1})^{a'b'}\epsilon^{a''b''} \\ &- \epsilon^{ab}\epsilon^{a'b'}(\mathcal{M}_3^{-1})^{a''b''}]\gamma_{bb'b''}, \end{aligned} \qquad (6.36)$$

which, for trivial moduli evaluated at infinity, is

$$\begin{aligned} M_2^\infty &= \frac{1}{16}\gamma_{aa'a''}(\delta^{ab}\delta^{a'b'}\delta^{a''b''} - \delta^{ab}\epsilon^{a'b'}\epsilon^{a''b''} \\ &\quad - \epsilon^{ab}\delta^{a'b'}\epsilon^{a''b''} - \epsilon^{ab}\epsilon^{a'b'}\delta^{a''b''})\gamma_{bb'b''} \\ &= \frac{1}{16}\sum_{I,J}(Q_IQ_J + P^IP^J). \end{aligned} \qquad (6.37)$$

The quantity $M_2^\infty = |Z(P,Q,z_\infty)|^2$ is also the modulus of the central charge of the $\mathcal{N}=2$ algebra [139]

$$Z(P,Q,z,\bar{z}) = \frac{1}{\sqrt{2}}e^{K(z,\bar{z})/2}(X^\Lambda(z)Q_\Lambda - F_\Lambda(z)P^\Lambda) \quad (6.38)$$

where $K = -\log(-8y_1y_2y_3)$ is the Kähler potential of the STU model and $F_\Lambda = \partial_\Lambda F$. We have the Bogomolny bound on the square mass,

$$M^2 \geq M_2^\infty. \qquad (6.39)$$

G. Quadratic mass formula

We define the moduli-dependent symplectic invariants [139]

$$\begin{aligned} I_2(r,u) &= -\frac{1}{4}(\tilde{P}^\Lambda,\tilde{Q}_\Lambda)\begin{pmatrix} \mathrm{Im}\mathcal{N} + \mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & -\mathrm{Re}\mathcal{N}(\mathrm{Im}\mathcal{N})^{-1} \\ -(\mathrm{Im}\mathcal{N})^{-1}\mathrm{Re}\mathcal{N} & (\mathrm{Im}\mathcal{N})^{-1}\end{pmatrix}\begin{pmatrix}\tilde{P}^\Lambda \\ \tilde{Q}_\Lambda\end{pmatrix}, \\ J_2(r,u) &= \frac{1}{4}(\tilde{P}^\Lambda,\tilde{Q}_\Lambda)\begin{pmatrix} \mathrm{Im}F + \mathrm{Re}F(\mathrm{Im}F)^{-1}\mathrm{Re}F & -\mathrm{Re}F(\mathrm{Im}F)^{-1} \\ -(\mathrm{Im}F)^{-1}\mathrm{Re}F & (\mathrm{Im}F)^{-1}\end{pmatrix}\begin{pmatrix}\tilde{P}^\Lambda \\ \tilde{Q}_\Lambda\end{pmatrix}, \end{aligned} \qquad (6.40)$$

where $F_{\Lambda\Sigma} = \partial_\Lambda\partial_\Sigma F$ and $F = -X^1X^2X^3/X^0$ is the prepotential of the STU model. Here, asymptotic flatness at spatial infinity fixes the scalar moduli at infinity as $x_i = 0$, $y_i = 1$, at $r = \infty$. The invariants read at infinity

$$I_2^\infty \equiv I_2(\infty,u) = \frac{1}{4}\sum_I[(Q_I)^2 + (P^I)^2], \qquad J_2^\infty \equiv J_2(\infty,u) = \frac{1}{4}\sum_I[(Q_I)^2+(P^I)^2] - \frac{1}{8}\sum_{I,J}(P_IP_J + Q_IQ_J), \qquad (6.41)$$

where we used (3.51).

For any $\mathcal{N}=2$ model,

$$|Z|^2 + |Z_i|^2 = I_2^\infty, \qquad -|Z|^2 + |Z_i|^2 = J_2^\infty, \qquad (6.42)$$

where Z is the central charge and $Z_i = D_iZ$ is the Kähler derivative of the central charge. Therefore, J_2^∞ can simply be expressed as $J_2^\infty = I_2^\infty - 2M_2^\infty$.

It is useful to define the invariant

$$S_2^\infty = \frac{1}{4}G_{AB}\partial_r\Phi^A\partial_r\Phi^B|_{r=\infty}. \qquad (6.43)$$

For the STU model, we have

$$S_2^\infty = \frac{1}{4}\sum_i (\Sigma_i^2 + \Xi_i^2). \tag{6.44}$$

It was observed by Gibbons [140] that for static configurations, the black hole mass obeys the condition

$$M^2 + N^2 + S_2^\infty = I_2^\infty + 4S_+^2 T_+^2. \tag{6.45}$$

This relation was interpreted in [141] as the statement that the total self-force on the black hole due to the attractive self-force of gravity and the scalar fields is not exceeded by the repulsive self-force due to the gauge fields, and vanishes only at extremality. For static black holes of Einstein-Maxwell theory and the Einstein-Maxwell-dilaton-axion theory (2.25), similar relations were derived using the 3-dimensional coset model in [142], and further generalized in [46].

We find that when rotation is present, the relation generalizes to

$$M^2 + N^2 + S_2^\infty = I_2^\infty + 4S_+^2\left(T_+^2 + \frac{\Omega_+^2}{4\pi^2}\right). \tag{6.46}$$

The angular velocity leads to an additional repulsive centrifugal force.

In fact, the last term on the right-hand side can also be written in terms of seed parameters m, n or quantities defined at the inner horizon as

$$4G^2 S_+^2\left(T_+^2 + \frac{\Omega_+^2}{4\pi^2}\right) = m^2 + n^2 = 4G^2 S_-^2\left(T_-^2 + \frac{\Omega_-^2}{4\pi^2}\right). \tag{6.47}$$

Using the latter relation, the identity (6.46) amounts to the statement that the quantity $Tr(\mathcal{Q}^2)$ defined in (3.54) is invariant under coset model transformations and therefore has the same value on the seed and final solutions. The identity therefore follows from a conservation law associated with the 3d coset model.

VII. NONEXTREMAL SPECIAL CASES

The general solution that we have constructed unifies many solutions in the literature. We now show how these are special cases of our general solution. We first describe nonextremal special solutions while some extremal limits will be discussed in Sec. VIII. In all cases, the black hole entropy is given by (6.32) in terms of the angular momentum, the quartic invariant Δ (6.12) and the F invariant (6.33).

A. Dyonic Kerr-Newman-Taub-NUT

If $\delta_I = \gamma_I = 0$, then all electromagnetic charges vanish. This gives the Ricci-flat Kerr-Taub-NUT solution, which we used as the starting point of the solution generating technique.

More generally, if $\delta_I = \delta$ and $\gamma_I = \gamma$ for all gauge fields, then all electric charges are equal and all magnetic charges are equal. This gives the dyonic Kerr-Newman-Taub-NUT solution [78] of Einstein-Maxwell theory (2.26). The conserved charges are

$$\begin{aligned} M &= m\cosh(2\delta)\cosh(2\gamma) + n\sinh(2\delta)\sinh(2\gamma), \\ N &= n\cosh(2\delta)\cosh(2\gamma) - m\sinh(2\delta)\sinh(2\gamma), \\ Q \equiv Q_I &= m\sinh(2\delta)\cosh(2\gamma) + n\sinh(2\gamma)\cosh(2\delta), \\ P \equiv P^I &= m\sinh(2\gamma)\cosh(2\delta) - n\sinh(2\delta)\cosh(2\gamma), \\ J &= aM, \end{aligned} \tag{7.1}$$

and the quartic invariant is

$$\Delta = \frac{1}{4}(Q^2 + P^2)^2. \tag{7.2}$$

Specializing to the dyonic Kerr-Newman solution, we set $n = n_0$, so that $N = 0$. Then the F invariant is

$$F = M^2(M^2 - Q^2 - P^2). \tag{7.3}$$

B. Kaluza-Klein black hole

If $\delta_I = \gamma_I = 0$ for $I = 2, 3, 4$ and $N = 0$, then we have the asymptotically flat, dyonic, rotating black hole [79,82,83] (see also [143]) of Kaluza-Klein theory (2.23). The conserved charges are

$$\begin{aligned} M &= \frac{1}{2}m(c_{\delta 1}^2 c_{\gamma 1}^2 - 1), \qquad Q_1 = \frac{2ms_{\delta 1}(c_{\delta 1}^2 + s_{\delta 1}^2 s_{\gamma 1}^2)}{c_{\delta 1}}, \\ J &= \frac{mac_{\gamma 1}(c_{\delta 1}^2 + s_{\delta 1}^2 s_{\gamma 1}^2)}{c_{\delta 1}}, \qquad N = 0, \\ P^1 &= \frac{2ms_{\gamma 1}c_{\gamma 1}}{c_{\delta 1}}. \end{aligned} \tag{7.4}$$

The quartic invariant and F invariant are

$$\Delta = -\frac{1}{16}(Q_1)^2(P^1)^2, \qquad F = m^4\frac{c_{\gamma_1}^2}{c_{\delta_1}^4}(c_{\delta_1}^2 + s_{\delta_1}^2 s_{\gamma_1}^2)^3, \tag{7.5}$$

but the F invariant is not easily expressed in terms of the conserved charges. For this purpose, we define the monotonic function

$$H(\psi) = 2\cos\psi\cos(\psi/3) + 6\sin\psi\sin(\psi/3) - 2, \tag{7.6}$$

where $0 \le \psi \le \pi/2$. We take

$$\sin^2\psi(M,Q_1,P^1)=\frac{54M^2[(Q_1)^2-(P^1)^2]^2}{[8M^2+(Q_1)^2+(P^1)^2]^3}, \tag{7.7}$$

which satisfies $0\le\psi\le\pi/2$ for regular black hole configurations obeying $4M\ge[(Q_1)^{2/3}+(P^1)^{2/3}]^{3/2}$. Then, after some lengthly algebra, we obtain

$$F=\left[M^2-\frac{1}{4}(Q_1)^2\right]\left[M^2-\frac{1}{4}(P^1)^2\right]+\frac{1}{3}\left\{M^2+\frac{1}{8}[(Q_1)^2+(P^1)^2]\right\}^2H(\psi(M,Q_1,P^1)). \tag{7.8}$$

For this class of solutions, the triality invariance reduces to the $\mathbb{Z}_2$ invariance $Q_1\to P^1$, $P^1\to -Q_1$, under which F is manifestly invariant. The function $H(\psi)$ was found by first expanding F in terms of the sum and difference of squares of electric and magnetic charges in a perturbation series. The Taylor coefficients of the function $H(\psi)$ were then recognized as belonging to a hypergeometric series using an algorithm for integer sequence recognition,[7] then simplified in terms of trigonometric functions. The final result was then tested numerically. Finally, note that when $P^1=0$ we have

$$F=\frac{1}{64}[32M^4-40M^2(Q_1)^2-(Q_1)^4+4M(4M^2+2(Q_1)^2)^{3/2}]. \tag{7.9}$$

We therefore obtained a novel expression for the entropy of the Kaluza-Klein black hole

$$S_+=2\pi\left(\sqrt{F-\frac{1}{16}(Q_1)^2(P^1)^2}+\sqrt{F-J^2}\right) \tag{7.10}$$

where F is given in (7.8), which could be used to study its thermodynamics further.

C. Four electric charges (Cvetič-Youm)

If $\gamma_I=0$ and $n=0$, then the NUT charge vanishes, $N=0$, and we have the asymptotically flat, 4-charge Cvetič-Youm solution [81]. The full explicit solution, including expressions for the gauge fields, was given in [71]. If we include nonvanishing n, then we recover the Kerr-Taub-NUT solution with 4 electric charges given in [71].[8]

[7]The algorithm can be found at http://www.oeis.org.

[8]We swap parameters δ_1 and δ_2, and correct a typographical error in the sign of χ_2 for the solution with NUT charge presented there.

In our parametrization, $\mu_2=\nu_1=0$. The conserved charges are

$$M=\frac{m}{4}\sum_I\cosh(2\delta_I),\qquad N=n(c_{\delta1234}-s_{\delta1234}),$$
$$Q_I=m\sinh(2\delta_I),\qquad P^I=2n(c_{\delta1}s_{\delta234}-s_{\delta1}c_{\delta234}). \tag{7.11}$$

The NUT charge can be set to zero by setting $n=0$, which we assume from now on. The angular momentum is then

$$J=ma(c_{\delta1234}-s_{\delta1234}). \tag{7.12}$$

The quartic invariant (6.12) and F invariant (6.33) are

$$\Delta=\frac{1}{4}Q_1Q_2Q_3Q_4,$$
$$F=\frac{1}{8}\left(m^4-4\Delta+\prod_I\sqrt{m^2+Q_I^2}+m^2\sum_{I<J}\sqrt{m^2+Q_I^2}\sqrt{m^2+Q_J^2}\right). \tag{7.13}$$

We have not found a closed form expression for the F invariant in terms of physical charges only.

Let us also present the metric in our notation. The master function (5.17) takes the almost factorized form

$$W^2(r,u)=(r^2-2mr+u^2)(r^2+2(2M-m)r+u^2)+4\nu_2^2m^2r^2. \tag{7.14}$$

The metric is then given by

$$ds^2=-\frac{r^2-2mr+u^2}{W(r,u)}(dt+\omega_3)^2+W(r,u)\times\left(\frac{dr^2}{R(r)}+\frac{du^2}{a^2-u^2}+\frac{R(r)(a^2-u^2)}{a^2(r^2-2mr+u^2)}d\phi^2\right), \tag{7.15}$$

where $R(r)=r^2-2mr+a^2$ and the Kaluza-Klein 1-form is

$$\omega_3=\frac{2\nu_2m(a^2-u^2)r}{a(r^2-2mr+u^2)}d\phi. \tag{7.16}$$

D. $-iX^0X^1$ supergravity black hole

If we set the electric and magnetic charges pairwise equal, which is equivalent to $(\delta_1,\gamma_1)=(\delta_4,\gamma_4)$ and $(\delta_2,\gamma_2)=(\delta_3,\gamma_3)$, then we have the dyonic rotating black hole [84] of $-iX^0X^1$ supergravity (2.24). The dyonic Kerr-Newman-Taub-NUT is recovered upon setting $Q_2=Q_1$, $P^2=P^1$.

The solution is substantially simpler in this truncation.

To simplify the solution and physical quantities, it is convenient to define

$$\begin{aligned}\Delta r_1 &= m[\cosh(2\delta_1)\cosh(2\gamma_2) - 1] + n\sinh(2\delta_1)\sinh(2\gamma_1),\\ \Delta r_2 &= m[\cosh(2\delta_2)\cosh(2\gamma_1) - 1] + n\sinh(2\delta_2)\sinh(2\gamma_2),\\ \Delta u_1 &= n[\cosh(2\delta_1)\cosh(2\gamma_2) - 1] - m\sinh(2\delta_1)\sinh(2\gamma_1),\\ \Delta u_2 &= n[\cosh(2\delta_2)\cosh(2\gamma_1) - 1] - m\sinh(2\delta_2)\sinh(2\gamma_2),\end{aligned} \tag{7.17}$$

and

$$r_1 = r + \Delta r_1, \qquad r_2 = r + \Delta r_2, \qquad u_1 = u + \Delta u_1, \qquad u_2 = u + \Delta u_2. \tag{7.18}$$

Then $W = r_1 r_2 + u_1 u_2$ and the metric takes the simplified form

$$\begin{aligned}\mathrm{d}s^2 = &-\frac{R}{W}\left(\mathrm{d}t - \frac{a^2 - u_1 u_2 + (\Delta u_1 + n)(\Delta u_2 + n)}{a}\mathrm{d}\phi\right)^2 + \frac{W}{R}\mathrm{d}r^2\\ &+\frac{U}{W}\left(\mathrm{d}t - \frac{r_1 r_2 + a^2 + (\Delta u_1 + n)(\Delta u_2 + n)}{a}\mathrm{d}\phi\right)^2 + \frac{W}{U}\mathrm{d}u^2.\end{aligned} \tag{7.19}$$

The gauge fields and duals are

$$\begin{aligned}A^1 = &\frac{Q_1 r_2}{W}\left(\mathrm{d}t - \frac{a^2 - u_1 u_2 + (\Delta u_1 + n)(\Delta u_2 + n)}{a}\mathrm{d}\phi\right)\\ &-\frac{P^1 u_2}{W}\left(\mathrm{d}t - \frac{r_1 r_2 + a^2 + (\Delta u_1 + n)(\Delta u_2 + n)}{a}\mathrm{d}\phi\right) + \frac{(\Delta u_2 + n)}{2a}\frac{\partial(\Delta u_1)}{\partial\delta_1}\mathrm{d}\phi,\end{aligned} \tag{7.20}$$

and

$$\begin{aligned}\tilde{A}_1 = &\frac{P^1 r_1}{W}\left(\mathrm{d}t - \frac{a^2 - u_1 u_2 + (\Delta u_1 + n)(\Delta u_2 + n)}{a}\mathrm{d}\phi\right)\\ &+\frac{Q_1 u_1}{W}\left(\mathrm{d}t - \frac{r_1 r_2 + a^2 + (\Delta u_1 + n)(\Delta u_2 + n)}{a}\mathrm{d}\phi\right) + \frac{(\Delta u_1 + n)}{2a}\frac{\partial(\Delta r_1)}{\partial\delta_1}\mathrm{d}\phi,\end{aligned} \tag{7.21}$$

with A^2 and $\tilde{A}_2$ obtained by interchanging $1\leftrightarrow 2$. Here, the partial derivatives with respect to δ_I are performed after setting $(\delta_1, \gamma_1) = (\delta_4, \gamma_4)$ and $(\delta_2, \gamma_2) = (\delta_3, \gamma_3)$. The nontrivial scalar fields are

$$\mathrm{e}^{\varphi_1} = \frac{r_2^2 + u_2^2}{W}, \qquad \chi_1 = \frac{r_2 u_1 - r_1 u_2}{r_2^2 + u_2^2}. \tag{7.22}$$

Using a linear coordinate transformation of the coordinates t and ϕ, and a gauge transformation, the metric and gauge fields may be written in the simpler form

$$\begin{aligned}\mathrm{d}s^2 = &-\frac{R}{W}(\mathrm{d}\tau + u_1 u_2 \mathrm{d}\psi)^2 + \frac{U}{W}(\mathrm{d}\tau - r_1 r_2 \mathrm{d}\psi)^2\\ &+ W\left(\frac{\mathrm{d}r^2}{R} + \frac{\mathrm{d}u^2}{U}\right),\end{aligned} \tag{7.23}$$

and

$$\begin{aligned}A^1 &= \frac{Q_1 r_2}{W}(\mathrm{d}\tau + u_1 u_2 \mathrm{d}\psi) - \frac{P^1 u_2}{W}(\mathrm{d}\tau - r_1 r_2 \mathrm{d}\psi),\\ \tilde{A}_1 &= \frac{P^1 r_1}{W}(\mathrm{d}\tau + u_1 u_2 \mathrm{d}\psi) + \frac{Q_1 u_1}{W}(\mathrm{d}\tau - r_1 r_2 \mathrm{d}\psi).\end{aligned} \tag{7.24}$$

Guided by this simplified form of the solution, asymptotically AdS generalizations in gauged supergravity were obtained in [28].

The parameters for the mass and NUT charge are

$$\begin{aligned}\nu_1 = -\mu_2 &= -\frac{1}{2}[\sinh(2\delta_1)\sinh(2\gamma_1) + \sinh(2\delta_2)\sinh(2\gamma_2)],\\ \nu_2 = \mu_1 &= \frac{1}{2}[\cosh(2\delta_1)\cosh(2\gamma_2) + \cosh(2\delta_2)\cosh(2\gamma_1)].\end{aligned} \tag{7.25}$$

The conserved charges are therefore

$$\begin{aligned} &M = m + \frac{1}{2}(\Delta r_1 + \Delta r_2), \qquad N = n + \tfrac{1}{2}(\Delta u_1 + \Delta u_2),\\ &Q_1 = \frac{\partial M}{\partial \delta_1} = \frac{1}{2}\frac{\partial(\Delta r_1)}{\partial \delta_1}, \qquad P^1 = -\frac{\partial N}{\partial \delta_1} = -\frac{1}{2}\frac{\partial(\Delta u_1)}{\partial \delta_1},\\ &Q_2 = \frac{\partial M}{\partial \delta_2} = \frac{1}{2}\frac{\partial(\Delta r_2)}{\partial \delta_2}, \qquad P^2 = -\frac{\partial N}{\partial \delta_2} = -\frac{1}{2}\frac{\partial(\Delta u_2)}{\partial \delta_2},\\ &J = Ma. \end{aligned} \tag{7.26}$$

Since the gauge fields are set pairwise equal before taking δ_I derivatives, there is a factor of 2 difference for the electromagnetic charges compared with the general formulas (4.22).

Setting the NUT charge to zero, we obtain the quartic and F invariants

$$\begin{aligned} \Delta &= \left(\frac{1}{2}I_2^\infty - M_2^\infty\right)^2 = \frac{1}{4}(Q_1Q_2 + P^1P^2)^2,\\ F &= \left(M^2 - \frac{1}{2}I_2^\infty\right)^2 - \Delta = (M^2 - M_2^\infty)(M^2 + M_2^\infty - I_2^\infty) \end{aligned} \tag{7.27}$$

in terms of other invariants defined in (6.37) and (6.41), and which read here

$$\begin{aligned} I_2^\infty &= \frac{1}{2}[(Q_1)^2 + (P^1)^2 + (Q_2)^2 + (P^2)^2],\\ M_2^\infty &= \frac{1}{4}[(Q_1 + P^1)^2 + (Q_2 + P^2)^2]. \end{aligned} \tag{7.28}$$

If $(\delta_1, \gamma_1) = (\delta_4, \gamma_4)$ and $\delta_2 = \delta_3 = \gamma_2 = \gamma_3 = 0$, then we have the Einstein-Maxwell-dilaton-axion solution of [55], which is labeled by its conserved charges Q_1, P^1, M, J.

E. Reduction of the black string of minimal 5d supergravity

If $\delta_2 = \delta_3 = \delta_4$, $\gamma_2 = \gamma_3 = \gamma_4$ and $P^1 = N = 0$, then we have the Kaluza-Klein reduction of the most general asymptotically Kaluza-Klein homogeneous 5-dimensional black string of minimal $\mathcal{N}=1$ 5d supergravity [87]. The solution is labeled by its conserved charges M, J, Q_1, Q_2, P^2. The charge Q_1 is the momentum along the string in the Kaluza-Klein direction while Q_2 and P^2 are the 4-dimensional electromagnetic charges.

F. One dyonic gauge field and two magnetic gauge fields

If $P^4 = Q_2 = Q_3 = Q_4 = 0$ and $N = 0$, then we have an analytic continuation of the Kaluza-Klein black hole solution with two additional magnetic charges of [85].

VIII. EXTREMAL BLACK HOLES

An extremal black hole is characterized by the property that its Hawking temperature vanishes. All extremal black holes enjoy the attractor mechanism, which states that at the horizon all scalar moduli reach an extremum value, which is solely a function of the electromagnetic charges and angular momentum carried by the black hole. In terms of 3-dimensional coset models, extremal black holes lie on nilpotent orbits of the symmetry algebra of the coset model.

There are a number of different extremal solutions that may be obtained as limits of our general nonextremal solution.[9] We will not attempt a classification but simply present 3 extremal limits of general interest that lead to black holes with finite area: the 1/8-Bogomolny-Prasad-Sommerfield (BPS) static black hole, the extremal fast rotating black hole and the extremal slow rotating black hole which includes as a subcase the regular static extremal non-BPS black hole.

A. Static 1/8-BPS limit

Supersymmetric black holes of $\mathcal{N}=8$ supergravity which are 1/2-BPS or 1/4-BPS have zero area in the supergravity regime, see e.g. [144]. Instead, the 1/8-BPS black holes have finite area. Such black holes can be generated through U dualities from a 1/8-BPS black hole of the STU model, as constructed in [88,89,145,146]. In this section, we will show how the 1/8-BPS black hole can be obtained as a specific extremal limit of the nonextremal solution.

In the static case $a = 0$, we take the limit $\epsilon \to 0$ while scaling

$$m \sim \epsilon^2, \qquad \delta_I \sim \epsilon^0, \qquad e^{\gamma_I} \sim \epsilon^{-1}. \tag{8.1}$$

The solution admits 4 independent electric and 4 independent magnetic charges. The mass saturates the BPS bound

$$M^2 = M_2^\infty \tag{8.2}$$

where M_2^∞ is defined in (6.37), which indicates that the solution is supersymmetric. The F invariant is zero in the limit. The quartic invariant is non-negative, $\Delta \geq 0$, and the entropy (6.32) is

$$S_+ = 2\pi\sqrt{\Delta}, \tag{8.3}$$

which reproduces the known entropy formula [18]. Since the area is generically nonvanishing, the black hole is 1/8-BPS. The scalar fields also obey the particular property

[9] See [118,119] for developments on a limiting procedure for relating nonextremal to extremal coset orbits.

$$S_2^\infty = I_2^\infty - M_2^\infty \tag{8.4}$$

where these quantities have been defined in Sec. VI.

The metric (5.1) takes the isotropic form

$$\mathrm{d}s^2 = -r^2 W_0^{-1}(r)\mathrm{d}t^2 + W_0(r) r^{-2}(\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2), \tag{8.5}$$

and the scalar fields admit a nontrivial radial profile interpolating between the attractor values at the horizon and trivial values at infinity, as imposed by asymptotic flatness. dualities, the black hole is expected to reduce to the one discussed in [88,89,145,146].

B. Extremal fast rotating solution

The extremal, fast rotating solution is achieved for $a = \sqrt{m^2 + n_0^2}$. The solution admits 4 independent electric and 4 independent magnetic charges as well as angular momentum. There is a degenerate horizon at $r = r_+ = r_- = m$. Using (6.46) and (6.47), the mass obeys the remarkable formula

$$M^2 = I_2^\infty - S_2^\infty + a^2. \tag{8.6}$$

In our parametrization, we have $J/a = m(\nu_1^2 + \nu_2^2)/\nu_2$. Therefore, the F invariant can be evaluated as

$$F = J^2. \tag{8.7}$$

The entropy (6.32) then becomes

$$S_+ = 2\pi\sqrt{\Delta + J^2}. \tag{8.8}$$

The entropy of the generic extremal rotating black hole is therefore independent of scalar moduli in general, since it is only a functional of the quartic invariant and the angular momentum. This is a feature of the attractor mechanism.

Angular momentum breaks supersymmetry. In the BPS limit (8.1), $a \to 0$ and $J/a \sim \epsilon^0$, then $J \to 0$ and all conserved quantities coincide with those of Sec. VIII A. Therefore, one can also consider the BPS limit as a special limit of the extremal fast rotating solution.

The near-horizon limit is defined as

$$t \to r_0 \lambda^{-1} t, \qquad r \to r_+ + \lambda r_0 r, \qquad \phi \to \phi + \Omega_+^{\text{ext}} \lambda^{-1} r_0 t, \tag{8.9}$$

and

$$A^I \to A^I - \Phi^I_{+,\text{ext}} \lambda^{-1} r_0 \mathrm{d}t, \qquad \tilde{A}_I \to \tilde{A}_I - \Psi^+_{I,\text{ext}} \lambda^{-1} r_0 \mathrm{d}t, \tag{8.10}$$

where $\lambda \to 0$, Ω_+^{ext}, $\Phi^I_{+,\text{ext}}$, $\Psi^+_{I,\text{ext}}$ are the chemical potentials at extremality and r_0 is an overall constant that we choose to be $r_0^2 = L(r_+)$. The near-horizon metric is

$$\mathrm{d}s^2 = W_+\left(-r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + \frac{\mathrm{d}u^2}{U} + \Gamma^2(\mathrm{d}\phi + kr\mathrm{d}t)^2\right), \tag{8.11}$$

where $W_+(u) = W(r_+, u)$, and

$$\Gamma^2(u) = \frac{L(r_+)^2 U(u)}{a^2 W_+^2(u)}, \qquad k = 2(m\nu_2 - n_0\nu_1)\Omega_+ = \frac{2\pi J}{S_+}. \tag{8.12}$$

The near-horizon gauge fields are

$$\begin{aligned} A^I &= f^I(\mathrm{d}\phi + kr\mathrm{d}t) + \frac{e^I}{k}\mathrm{d}\phi, \\ \tilde{A}_I &= \tilde{f}_I(\mathrm{d}\phi + kr\mathrm{d}t) + \frac{\tilde{e}_I}{k}\mathrm{d}\phi, \end{aligned} \tag{8.13}$$

where

$$\begin{aligned} f^I(u) &= -\frac{L(r_+)}{a}\left(\zeta^I(r_+, u) + \frac{\nu_1\pi_1^I + \nu_2\pi_2^I}{2(\nu_1^2 + \nu_2^2)}\right), \\ e^I &= 2(m\nu_2 - n_0\nu_1)\Phi_+^I - n_0\pi_1^I + m\pi_2^I, \\ \tilde{f}_I(u) &= -\frac{L(r_+)}{a}\left(\tilde{\zeta}_I(r_+, u) - \frac{\nu_1\rho_I^1 + \nu_2\rho_I^2}{2(\nu_1^2 + \nu_2^2)}\right), \\ \tilde{e}_I &= 2(m\nu_2 - n_0\nu_1)\Psi_I^+ + n_0\rho_I^1 - m\rho_I^2. \end{aligned} \tag{8.14}$$

The geometry has the expected enhanced $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)$ symmetry [147] and the expected functional form [148]. In the BPS limit, $k = 0$ and the geometry reduces to $\mathrm{AdS}_2 \times S^2$.

Following the Kerr/conformal field theory (CFT) conjecture [149], the entropy is reproduced by Cardy's formula

$$S_+ = \frac{1}{3}\pi^2 c_J T_J \tag{8.15}$$

for a chiral sector of a CFT with central charge and temperature

$$c_J = 12J, \qquad T_J = \frac{1}{2\pi k}. \tag{8.16}$$

We expect that boundary conditions exist when a Virasoro algebra acts as asymptotic symmetry algebra, as in all known subcases (see [150] for references). A distinct description of the entropy is in terms of Cardy's formula

$$S_+ = \frac{1}{3}\pi^2 c_{Q_1} T_{Q_1} \tag{8.17}$$

for a chiral sector of a CFT with central charge and temperature

$$c_{Q_1} = 24\frac{\partial\Delta}{\partial Q_1}, \qquad T_{Q_1} = \frac{1}{2\pi e^1}, \tag{8.18}$$

which generalizes [151,152]. More explicitly,

$$c_{Q_1} = 6Q_2Q_3Q_4 + 3P^1(P^2Q_2 + P^3Q_3 + P^4Q_4 - P^1Q_1). \tag{8.19}$$

There are similar expressions corresponding to the other electromagnetic charges.

C. Extremal slow rotating and non-BPS static limit

An extremal limit with slow rotation is defined as

$$m \sim \epsilon^2 m, \qquad n \sim \epsilon n, \qquad a \sim \epsilon a, \qquad \mathrm{e}^{\gamma_1} \sim \epsilon^{-1}\mathrm{e}^{\gamma_1} \tag{8.20}$$

with $\epsilon \to 0$ and the remaining parameters $(\gamma_2, \gamma_3, \gamma_4, \delta_I, I = 1, 2, 3, 4)$ unscaled. The non-BPS static limit is defined analogously but with $a = 0$. There are four distinct limits depending on the choice of γ_I, $I = 1, 2, 3, 4$ that is blown up. By permutation symmetry, all limits lead to the same metric. We expect that the four different limits are related by field redefinitions of the charging parameters without changing the physics. Since $n_0 = O(\epsilon)$, one can set the NUT charge to zero by setting the final $n = n_0$. Besides angular momentum, the solution admits 4 independent electric and 4 independent magnetic charges.

In the limit (8.20) the temperature T_+ and angular velocity Ω_+ vanish. The horizon is located at $r = 0$. In the limiting procedure $r_+ = -r_- + O(\epsilon^2)$, which implies that $S_- = -S_+$. From (6.25), we deduce that $J^2 + \Delta \leq 0$, the F invariant is $F = -\Delta$ and the entropy (6.32) becomes

$$S_+ = 2\pi\sqrt{-\Delta - J^2}. \tag{8.21}$$

Since $\Delta \leq 0$, there are no BPS solutions with finite area in this class. One can explicitly check that the mass obeys

$$M^2 = I_2^\infty - S_\infty^2 \tag{8.22}$$

and, in particular, it does not depend upon the angular momentum J. Upon setting to zero all magnetic charges, the solution reduces to the extremal slow rotating four-charge extremal solution studied in Sec. 5 of [86].[10]

Regular extremal static non-BPS black holes with 8 independent electromagnetic charges are obtained by setting $J = 0$. One such class of black holes labeled by 2 independent parameters was obtained in [94]. In that case, we have the charge assignments $\tilde{P}_1 = 1$, $\tilde{P}_2 = \tilde{P}_3$, $\tilde{Q}_2 = \tilde{Q}_3$ and $P^4 = 0$.

[10] Note that contrary to the claim of [86], at least five independent electromagnetic charges are necessary to obtain a generating solution.

The general metric can be obtained by taking the limit (8.20). It turns out that the functions $L(r)$ and $V(u)$ defined in (5.18) blow up as $L = O(\epsilon^{-1})$, $V = O(\epsilon^{-2})$. Therefore, the form of the W^2 and ω_3 functions is not adapted to the description of the extremal slow rotating limit. However, these functions are finite in the limit, and we find

$$\begin{aligned} W^2 &= r^4 + 4Mr^3 + (M^2b_1 + b_2J\cos\theta)r^2 \\ &\quad + b_3M^3r - 4J^2\cos^2\theta - 4\Delta, \\ \omega_3 &= \frac{2J}{r}\sin^2\theta \mathrm{d}\phi, \end{aligned} \tag{8.23}$$

where b_1, b_2, b_3 only depend on the charging parameters (δ_I, γ_I), $I = 1, 2, 3, 4$. We have $b_1 \geq 0$, $b_3 \geq 0$. The form of ω_3 is exceptionally simple and only depends on the physical angular momentum. Since $J^2 \leq -\Delta$, W^2 is indeed positive near $r = 0$, which is the location of the extremal horizon. Finally, the metric is

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{r^2}{W(r,\theta)}\left(\mathrm{d}t + \frac{2J}{r}\sin^2\theta\mathrm{d}\phi\right)^2 \\ &+ \frac{W(r,\theta)}{r^2}[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)]. \end{aligned} \tag{8.24}$$

The matter fields can be obtained from the limit and we will not display them here.

In the near-horizon limit, we replace

$$t \to \lambda^{-1}r_0\sqrt{-\Delta - J^2}t, \qquad r \to \lambda r_0 r, \tag{8.25}$$

and

$$\begin{aligned} A^I &\to A^I - \mathrm{d}(\Phi_+^I\lambda^{-1}r_0\sqrt{-\Delta - J^2}t), \\ \tilde{A}_I &\to \tilde{A}_I - \mathrm{d}(\Psi_I^+\lambda^{-1}r_0\sqrt{-\Delta - J^2}t), \end{aligned} \tag{8.26}$$

and then take $\lambda \to 0$. For convenience, we fix $r_0 = \sqrt{2}$ for convenience, we obtain

$$\mathrm{d}s^2 = W_+\left(-r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2 + \Gamma^2(\mathrm{d}\phi - kr\mathrm{d}t)^2\right) \tag{8.27}$$

with

$$\begin{aligned} W_+ &= 2\sqrt{-\Delta - J^2\cos^2\theta}, \qquad k = \frac{J}{\sqrt{-\Delta + J^2}}, \\ \Gamma^2 &= \sin^2\theta\frac{-\Delta - J^2}{-\Delta - J^2\cos^2\theta}. \end{aligned} \tag{8.28}$$

The geometry only depends upon the quartic invariant and the angular momentum and it admits the expected enhanced $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)$ symmetry [147]. The gauge fields in the near-horizon limit can be most easily obtained by taking the extremal limit with slow rotation (8.20) followed by the near-horizon limit (8.25) of the expression (5.19). In order to evaluate the latter expression, we need to keep n general, and take $n = n_0$ only after taking the derivative with respect to δ_I. We first note that $C = -\nu_1^2 + O(\epsilon^{-2})$ and $\Phi_+^I = \partial_{\delta_I} \log \nu_1 + O(\epsilon)$. Then, we obtain after the limit $\epsilon \to 0$,

$$W^2 = 4\nu_1^2 n^2 (n^2 - a^2 \cos^2\theta) + O(\lambda),$$
$$\omega_3 = -\frac{2an\nu_1}{\lambda r_0 r} \sin^2\theta \mathrm{d}\phi + O(\lambda^0). \tag{8.29}$$

After using (4.22), we get $\xi^I = \partial_{\delta_I} \log \nu_1$ and $A^I = P^I \cos\theta \mathrm{d}\phi - \partial_{\delta_I} \log \nu_1 \omega_3$, which results finally in

$$A^I = f^I (\mathrm{d}\phi - kr\mathrm{d}t) - \frac{e^I}{k} \mathrm{d}\phi,$$
$$f^I = \frac{P^I(-\Delta - J^2)(\hat{\pi}^I + J\cos\theta)}{J(-\Delta - J^2 \cos^2\theta)}, \qquad e^I = \frac{P^I \hat{\pi}^I}{\sqrt{-\Delta - J^2}},$$
$$\tilde{A}^I = \tilde{f}_I (\mathrm{d}\phi - kr\mathrm{d}t) - \frac{\tilde{e}_I}{k} \mathrm{d}\phi,$$
$$\tilde{f}_I = \frac{Q_I(-\Delta - J^2)(\hat{\rho}_I - J\cos\theta)}{J(-\Delta - J^2 \cos^2\theta)}, \qquad \tilde{e}_I = \frac{Q_I \hat{\rho}_I}{\sqrt{-\Delta - J^2}}, \tag{8.30}$$

after performing the gauge transformation as indicated in (8.26). After analysis, we find

$$\hat{\pi}^I = -2\frac{1}{P^I} \frac{\partial\Delta}{\partial Q_I}, \qquad \hat{\rho}_I = -2\frac{1}{Q_I} \frac{\partial\Delta}{\partial P^I}. \tag{8.31}$$

Following the Kerr/CFT conjecture [149], the entropy is reproduced by Cardy's formula

$$S_+ = \frac{1}{3}\pi^2 c_J T_J \tag{8.32}$$

for a chiral sector of a CFT with central charge and temperature

$$c_J = 12J, \qquad T_J = \frac{1}{2\pi k}. \tag{8.33}$$

Eight other Cardy formulas hold,

$$S_+ = \frac{1}{3}\pi^2 c_{Q_I} T_{Q_I} = \frac{1}{3}\pi^2 c_{P^I} T_{P^I}, \tag{8.34}$$

one for each electric or magnetic charge, with central charges and temperatures

$$c_{Q_I} = -6\frac{\partial\Delta}{\partial\bar{Q}_I}, \qquad T_{Q_I} = \frac{1}{2\pi e^I},$$
$$c_{P^I} = -6\frac{\partial\Delta}{\partial\bar{P}^I}, \qquad T_{P^I} = \frac{1}{2\pi\tilde{e}_I}. \tag{8.35}$$

1. Kaluza-Klein black hole

Let us present the details of the extremal slow rotating solution in the case where the only nonzero electromagnetic charges are $Q \equiv Q_1$, $P \equiv P^1$, corresponding to the charging parameters $\delta \equiv \delta_1$ and $\gamma \equiv \gamma_1$. This is a 4-dimensional solution of Kaluza-Klein theory, about by reduction of the 5-dimensional Einstein gravity [79,82,83] (see also [153]).

Extremality fixes the mass in terms of the electromagnetic charges. In our parametrization, we find

$$M = \frac{me^{2\gamma}\cosh^2\delta}{8}, \quad \bar{Q} = \frac{me^{2\gamma}\sinh^3\delta}{8\cosh\delta}, \quad \bar{P} = \frac{me^{2\gamma}}{8\cosh\delta}, \tag{8.36}$$

which satisfy

$$M^{2/3} = \bar{Q}^{2/3} + \bar{P}^{2/3}. \tag{8.37}$$

Let us assume for simplicity and without loss of generality that $\bar{Q} \geq 0$, $\bar{P} \geq 0$. Then we have the factorization $W^2 = W_Q W_P$, where

$$W_Q = r^2 + 4\bar{Q}^{2/3}\sqrt{\bar{Q}^{2/3} + \bar{P}^{2/3}}\, r + 8\bar{Q}^{1/3}\bar{P}^{-1/3}(\bar{Q}\,\bar{P} - J\cos\theta),$$
$$W_P = r^2 + 4\bar{P}^{2/3}\sqrt{\bar{Q}^{2/3} + \bar{P}^{2/3}}\, r + 8\bar{P}^{1/3}\bar{Q}^{-1/3}(\bar{Q}\,\bar{P} + J\cos\theta). \tag{8.38}$$

Note that reversing the spacetime orientation would lead to a change $J \to -J$ in W_Q and W_P as a consequence of the equations of motion.

The coefficients in the gauge fields in the near-horizon limit are given by

$$\hat{\pi}^1 = \hat{\rho}_1 = \sqrt{-\Delta} = \bar{Q}\,\bar{P}. \tag{8.39}$$

At the horizon $r = 0$, the scalar moduli reduce to

$$x_i = 0, \qquad y_2 = y_3 = \frac{1}{y_1} = \frac{\bar{P}^{2/3}(\bar{Q}\,\bar{P} + J\cos\theta)}{\bar{Q}^{2/3}(\bar{Q}\,\bar{P} - J\cos\theta)}. \tag{8.40}$$

Introducing $\psi \sim \psi + 2\pi$, one can reconstruct a 5-dimensional Ricci-flat metric as

$$\mathrm{d}s_5^2 = f^2(\theta)\left[R_\psi \mathrm{d}\psi - \frac{\bar{P}r}{\bar{Q}\,\bar{P} - J\cos\theta}\left(\mathrm{d}t + \frac{J}{r}\sin^2\theta \mathrm{d}\phi\right) + \bar{P}\cos\theta \mathrm{d}\phi\right]^2 + \frac{G(\theta)}{2f(\theta)}\left[-\frac{r^2}{G^2(\theta)}(\mathrm{d}t + \frac{J}{r}\sin^2\theta \mathrm{d}\phi)^2 + \frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right], \tag{8.41}$$

where R_ψ is arbitrary and

$$f(\theta) = \left(\frac{\bar{Q}}{\bar{P}}\right)^{1/3}\left(\frac{\bar{Q}\,\bar{P} - J\cos\theta}{\bar{Q}\,\bar{P} + J\cos\theta}\right)^{1/2}, \qquad G(\theta) = \sqrt{\bar{Q}^2\bar{P}^2 - J^2\cos^2\theta}. \tag{8.42}$$

The near-horizon metric is obtained by replacing $r \to \lambda r$, $t \to t/\lambda$, and then taking the limit $\lambda \to 0$; it falls into the classification of [154]. The geometry of the horizon is globally S^3. The metric can be put in the form

$$\mathrm{d}s^2 = \Gamma(\theta)\left(-r^2\mathrm{d}\bar{t}^2 + \frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2 + \sum_{A,B=1}^{2}\gamma_{AB}(\theta)(\mathrm{d}\phi^A - k^A r\mathrm{d}\bar{t})(\mathrm{d}\phi^B - k^B r\mathrm{d}\bar{t})\right), \tag{8.43}$$

where $\bar{t} = t/(\bar{Q}^2\bar{P}^2 - J^2)^{1/2}$, $\phi^1 = \phi$, $\phi^2 = R_\psi\psi$,

$$\Gamma(\theta) = \frac{\bar{P}^{1/3}(\bar{Q}\,\bar{P} + J\cos\theta)}{2\bar{Q}^{1/3}}, \qquad k^1 = \frac{J}{(\bar{Q}^2\bar{P}^2 - J^2)^{1/2}}, \qquad k^2 = \frac{\bar{Q}\bar{P}^2}{(\bar{Q}^2\bar{P}^2 - J^2)^{1/2}}, \tag{8.44}$$

and

$$\gamma_{AB} = \frac{1}{(\bar{Q}\,\bar{P} + J\cos\theta)^2}\begin{pmatrix} \bar{Q}^2\bar{P}^2 - J^2\cos^2\theta + (\bar{Q}\,\bar{P}\cos\theta - J)^2 & 2\bar{Q}(\bar{Q}\,\bar{P}\cos\theta - J) \\ 2\bar{Q}(\bar{Q}\,\bar{P}\cos\theta - J) & 2\bar{Q}(\bar{Q}\,\bar{P} - J\cos\theta)/\bar{P} \end{pmatrix}. \tag{8.45}$$

It admits an $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)^2$ symmetry. The Killing vectors $\xi_{-1} = \partial_t$, $\xi_0 = r\partial_r - t\partial_t$ and

$$\xi_1 = \left(\frac{1}{2r^2} + \frac{\bar{t}^2}{2}\right)\partial_{\bar{t}} - \bar{t}r\partial_r + \frac{k^1}{r}\partial_\phi + \frac{k^2}{R_\psi r}\partial_\psi \tag{8.46}$$

satisfy the $\mathrm{SL}(2,\mathbb{R})$ commutators $[\xi_0, \xi_1] = -\xi_1$, $[\xi_0, \xi_{-1}] = \xi_{-1}$ and $[\xi_{-1}, \xi_1] = -\xi_0$.

Following the Kerr/CFT conjecture [149], the entropy is reproduced by either of Cardy's formulas

$$S_+ = \frac{1}{3}\pi^2 c_J T_J = \frac{1}{3}\pi^2 c_Q T_Q \tag{8.47}$$

for a chiral sector of CFTs with central charges and temperatures

$$c_J = 12J, \qquad T_J = \frac{1}{2\pi k^1}, \tag{8.48}$$

$$c_Q = -6\frac{\partial\Delta}{\partial\bar{Q}}, \qquad T_Q = \frac{1}{2\pi k^2}, \tag{8.49}$$

as obtained in [155] (see also [156]).

IX. KILLING TENSORS AND SEPARABILITY

It is well known that the Kerr solution possesses various types of Killing tensors. These tensors are related to the integrability of geodesic motion, and the separability of the Klein-Gordon equation and the Dirac equation. Black hole solutions of $\mathcal{N}=8$ supergravity also involve metrics that possess various types of Killing tensors as we will now demonstrate. Using (5.17), the metric (5.15) can be written in the form

$$\mathrm{d}s^2 = -\frac{R-U}{W}\mathrm{d}t^2 - \frac{(L_u R + L_r U)}{aW}2\mathrm{d}t\mathrm{d}\phi + \frac{(W_r^2 U - W_u^2 R)}{a^2 W}\mathrm{d}\phi^2 + W\left(\frac{\mathrm{d}r^2}{R} + \frac{\mathrm{d}u^2}{U}\right), \tag{9.1}$$

where

$$W^2 = (R-U)\left(\frac{W_r^2}{R} - \frac{W_u^2}{U}\right) + \frac{(L_u R + L_r U)^2}{RU}. \tag{9.2}$$

Its determinant is $\sqrt{-g} = W$. For the black hole solution, the functions $R(r)$ and $U(u)$ are given in (5.16), and

$$\begin{aligned} L_r(r) &= L + 2Nn, \\ W_r^2(r) &= R^2 + 4MrR + (L+2Nn)^2, \\ L_u(u) &= 2N(u-n), \\ W_u^2(u) &= U^2 - 2UV + 4N^2(u-n)^2, \end{aligned} \tag{9.3}$$

where $L(r)$ and $V(u)$ are given in (5.18).

Henceforth, in this section we consider a more general class of metrics of the form (9.1). We generalize so that: R, W_r and L_r are arbitrary functions of r; U, W_u and L_u are arbitrary functions of u; and W satisfies (9.2). There are two conformally related metrics of interest: the usual Einstein frame metric $\mathrm{d}s^2$, and the string frame metric

$$\mathrm{d}\tilde{s}^2 = \frac{r^2+u^2}{W}\mathrm{d}s^2, \tag{9.4}$$

whose inverse $(\partial/\partial\tilde{s})^2$ is given by

$$(r^2+u^2)\left(\frac{\partial}{\partial\tilde{s}}\right)^2 = R\partial_r^2 + U\partial_u^2 + \left(\frac{W_u^2}{U} - \frac{W_r^2}{R}\right)\partial_t^2 - a\left(\frac{L_r}{R} + \frac{L_u}{U}\right)2\partial_t\partial_\phi + a^2\left(\frac{1}{U} - \frac{1}{R}\right)\partial_\phi^2. \tag{9.5}$$

Let us recall some definitions of Killing tensors. A (rank-2) Killing-Stäckel tensor is a symmetric tensor $K_{\mu\nu} = K_{(\mu\nu)}$ that satisfies $\nabla_{(\mu}K_{\nu\rho)} = 0$. A (rank-2) conformal Killing-Stäckel tensor is a symmetric tensor $Q_{\mu\nu} = Q_{(\mu\nu)}$ that satisfies $\nabla_{(\mu}Q_{\nu\rho)} = q_{(\mu}g_{\mu\nu)}$ for some q_μ, given in 4 dimensions by $q_\mu = \frac{1}{6}(\partial_\mu Q^\nu{}_\nu + 2\nabla_\nu Q^\nu{}_\mu)$.

For black hole solutions of supergravity, usually only the string frame metric admits Killing tensors, whereas the Einstein frame metric usually only admits conformal Killing tensors [97]. Here we note that in general, the string frame metric has a Killing-Stäckel tensor given by

$$\tilde{K}^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{r^2+u^2}\left[\left(\frac{u^2W_r^2}{R} + \frac{r^2W_u^2}{U}\right)\partial_t^2 - a\left(\frac{u^2L_r}{R} + \frac{r^2L_u}{U}\right)2\partial_t\partial_\phi + a^2\left(\frac{r^2}{U} - \frac{u^2}{R}\right)\partial_\phi^2 - u^2R\partial_r^2 + r^2U\partial_u^2\right]. \tag{9.6}$$

It is generically irreducible, i.e. not a linear combination of the metric and products of Killing vectors. In general, if a metric $\mathrm{d}\tilde{s}^2$ possesses a Killing-Stäckel tensor $\tilde{K}_{\mu\nu}$, then for any conformally related metric $\mathrm{d}s^2$ there is an induced conformal Killing-Stäckel tensor with components given by $Q^{\mu\nu} = \tilde{K}^{\mu\nu}$; see e.g. [157]. In particular, the string frame Killing-Stäckel tensor induces a conformal Killing-Stäckel tensor for the Einstein frame metric. Note that the existence of a conformal frame admitting a Killing-Stäckel tensor is a more restrictive condition than the existence of a conformal Killing-Stäckel tensor in Einstein frame. This conformal Killing-Stäckel tensor was identified for the subcases with 4 electric charges in [97], and for the nonextremal rotating Kaluza-Klein black hole in [158].

If we specialize to $L_r = W_r$ and $L_u = W_u$, then we can write without loss of generality $W = W_r + W_u$. Then the Einstein frame metric is of the form

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{R}{W_r+W_u}\left(\mathrm{d}t + \frac{W_u}{a}\mathrm{d}\phi\right)^2 \\ &+ \frac{U}{W_r+W_u}\left(\mathrm{d}t - \frac{W_r}{a}\mathrm{d}\phi\right)^2 \\ &+ (W_r+W_u)\left(\frac{\mathrm{d}r^2}{R} + \frac{\mathrm{d}u^2}{U}\right). \end{aligned} \tag{9.7}$$

This class of metrics has been studied in detail [28], and has the property that both the string frame and Einstein frame metrics possess Killing-Yano tensors with torsion. It implies that both the Einstein and string frame metrics admit Killing-Stäckel tensors. The class of metrics includes the general black hole metric truncated to $-\mathrm{i}X^0X^1$ supergravity, by setting the gauge fields pairwise equal, say $(\delta_1,\gamma_1) = (\delta_4,\gamma_4)$ and $(\delta_2,\gamma_2) = (\delta_3,\gamma_3)$.

A. Geodesics

The Killing tensor in string frame guarantees the complete integrability of geodesic motion in this frame, which we now demonstrate explicitly. In string frame, the Hamilton-Jacobi equation for geodesic motion is

$$\frac{\partial S}{\partial\lambda} + \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu S\partial_\nu S = 0, \tag{9.8}$$

where S is Hamilton's principal function, $\partial_\mu S = p_\mu = \mathrm{d}x_\mu/\mathrm{d}\lambda$, p_λ are momenta conjugate to x^μ, and λ is an affine parameter. Consider the ansatz

$$S = \frac{1}{2}\mu^2\lambda - Et + L\phi + S_r(r) + S_u(u). \tag{9.9}$$

The constants $p_t = -E$ and $p_\phi = L$ are momenta conjugate to the ignorable coordinates t and ϕ, related to energy and angular momentum. The particle mass is μ, so that $p^\mu p_\mu = -\mu^2$. The components $(r^2+u^2)\tilde{g}^{\mu\nu}$ are additively separable into functions of r and of u, and so the Hamilton-Jacobi equation is additively separable. Explicitly, we have

$$\left(\frac{W_u^2}{U}-\frac{W_r^2}{R}\right)E^2+2a\left(\frac{L_r}{R}+\frac{L_u}{U}\right)EL+a^2\left(\frac{1}{U}-\frac{1}{R}\right)L^2+R\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2+U\left(\frac{\mathrm{d}S_u}{\mathrm{d}u}\right)^2+\mu^2(r^2+u^2)=0, \tag{9.10}$$

and so

$$\frac{\mathrm{d}S_r}{\mathrm{d}r}=\frac{1}{R}\sqrt{W_r^2E^2-2aL_rEL+a^2L^2-(C+\mu^2r^2)R},$$
$$\frac{\mathrm{d}S_u}{\mathrm{d}u}=\frac{1}{U}\sqrt{-W_u^2E^2-2aL_uEL-a^2L^2+(C-\mu^2u^2)U}, \tag{9.11}$$

where C is a separation constant. We then determine $r(\lambda)$ and $u(\lambda)$ by integrating

$$\frac{\mathrm{d}r}{\mathrm{d}\lambda}=\tilde{g}^{rr}p_r=\frac{R}{r^2+u^2}\frac{\mathrm{d}S_r}{\mathrm{d}r},\qquad \frac{\mathrm{d}u}{\mathrm{d}\lambda}=\tilde{g}^{uu}p_u=\frac{U}{r^2+u^2}\frac{\mathrm{d}S_u}{\mathrm{d}u}. \tag{9.12}$$

Finally, we determine $t(\lambda)$ and $\phi(\lambda)$ by integrating

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda}=\tilde{g}^{tt}p_t+\tilde{g}^{t\phi}p_\phi=\frac{E}{r^2+u^2}\left(\frac{W_r^2}{R}-\frac{W_u^2}{U}\right)-\frac{aL}{r^2+u^2}\left(\frac{L_r}{R}+\frac{L_u}{U}\right),$$
$$\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}=\tilde{g}^{t\phi}p_t+\tilde{g}^{\phi\phi}p_\phi=\frac{aE}{r^2+u^2}\left(\frac{L_r}{R}+\frac{L_u}{U}\right)+\frac{a^2L}{r^2+u^2}\left(\frac{1}{U}-\frac{1}{R}\right). \tag{9.13}$$

In Einstein frame, generically only the $\mu=0$ massless Hamilton-Jacobi equation separates.

B. Klein-Gordon equation

Separability of the massless Klein-Gordon equation makes the analysis of [159] applicable to the general black hole of $\mathcal{N}=8$ supergravity, which will therefore admit hidden conformal symmetries in the near-horizon region.

The massive Klein-Gordon equation for the Einstein frame metric is

$$\Box\Phi=\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi)=\mu^2\Phi. \tag{9.14}$$

Consider the ansatz

$$\Phi=\Phi_r(r)\Phi_u(u)\mathrm{e}^{\mathrm{i}(k\phi-\omega t)}. \tag{9.15}$$

Then the Klein-Gordon equation gives

$$\mu^2W=\frac{\omega^2W_r^2-2a\omega kL_r+a^2k^2}{R}-\frac{\omega^2W_u^2+2a\omega kL_u+a^2k^2}{U}+\frac{1}{\Phi_r}\frac{\mathrm{d}}{\mathrm{d}r}\left(R\frac{\mathrm{d}\Phi_r}{\mathrm{d}r}\right)+\frac{1}{\Phi_u}\frac{\mathrm{d}}{\mathrm{d}u}\left(U\frac{\mathrm{d}\Phi_u}{\mathrm{d}u}\right). \tag{9.16}$$

In the particular case where the Einstein metric takes the form (9.7), such as for generic black holes of $-\mathrm{i}X_0X_1$ supergravity, the $\mu\neq0$ massive Klein-Gordon equation in Einstein frame separates. Generically, there is separation only in the massless case $\mu=0$, leading to

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(R\frac{\mathrm{d}\Phi_r}{\mathrm{d}r}\right)+\left(\frac{\omega^2W_r^2-2a\omega kL_r+a^2k^2}{R}+C\right)\Phi_r=0,$$
$$\frac{\mathrm{d}}{\mathrm{d}u}\left(U\frac{\mathrm{d}\Phi_u}{\mathrm{d}u}\right)-\left(\frac{\omega^2W_u^2+2a\omega kL_u+a^2k^2}{U}+C\right)\Phi_u=0, \tag{9.17}$$

where C is an integration constant. Specializing to the black hole solutions we constructed, the radial equation has regular singular points at the locations of the horizons, $r=r_\pm$, and an irregular singular point at infinity, similar to what happens for the Kerr solution. The solutions are Heun functions. The angular equation involving u can be analyzed similarly.

X. CONCLUSION AND FURTHER DIRECTIONS

We have constructed a generating solution for the most general stationary, asymptotically flat black hole of $\mathcal{N}=8$ supergravity. We checked that this black hole reduces in specific subcases to all previously known solutions of the STU model with 4 independent (combinations of) electromagnetic charges [71,81,84,85,87]. Unlike many other treatments of STU supergravity, we have emphasized the 4-fold permutation symmetry of the gauge fields in the 3-dimensional coset model, not just the triality symmetry. We discussed several extremal limits of interest, but a comprehensive examination of all extremal limits of our solution remains to be done. The generic black hole that we constructed admits a conformal Killing-Stäckel tensor, and the massless Klein-Gordon equation separates, so we can deduce the presence of hidden conformal symmetries in the near-horizon region.

The solution generating technique that we detailed is general and could be used for a wider class of stationary seed solutions, beyond the Kerr-Taub-NUT solution that we used. Different choices of group element can be used, allowing for more general asymptotic behavior. One example is the application to "subtracted geometries" [138,160,161], obtained by solution generating techniques in [99,162]. Another example is obtaining charged black holes in a magnetic Melvin universe [163]. The techniques can also be applied to the various theories in 5 and 6 dimensions that we discussed.

The issue of black hole uniqueness has not been fully addressed (see [164] for a recent review). It was shown in [32] that, with certain assumptions, all charged black holes in coset models lie in the orbit of the Kerr black hole. These assumptions were clarified in [165] in the static case, where it was shown that all scalar fields should be regular on the horizon in order to apply the theorem of [32]. Clarifying the theorem of [32] in the stationary case seems a natural step to prove uniqueness.

Under general assumptions, stationary 4-dimensional black holes are axisymmetric [166], so can be Kaluza-Klein reduced to 2 spatial dimensions. For Einstein gravity and Einstein-Maxwell theory, inverse scattering techniques can then be used to generate solutions, as reviewed in [167]. These techniques can be generalized to certain theories of gravity coupled to matter, in particular supergravities. They have been developed for the S^3 supergravity in [168], and more generally for the STU supergravity in [114]. One may gain more insights into the algebraic structure of the general black hole solution by deriving it using these techniques.

Inverting the relation between conserved charges and auxiliary parameters that parametrize the 4-dimensional fields would allow for expressing the entropy, or equivalently the F invariant that we defined, in terms of physical charges. Our formula for the entropy of a general non-extremal black hole is not manifestly invariant under $\mathrm{SL}(2,\mathbb{R})^3$ or triality. We therefore are unable to provide here a manifestly $\mathrm{E}_{7(7)}$-invariant entropy formula for nonextremal black holes in $\mathcal{N}=8$ supergravity. Even in the simpler case of Kaluza-Klein theory, i.e. reduction of 5-dimensional Einstein gravity, the F invariant for the dyonic black hole takes an intricate form that we were able to present. We leave this difficult algebraic problem for future investigations.

We checked that the first law of thermodynamics closes both at the outer and inner horizon and that the Smarr formula holds at the outer and inner horizons. We derived a generalization of the quadratic mass formula in the presence of rotation and NUT charge. We also presented some relationships between physical charges defined at the outer and inner horizon that generalize previously known subcases. A microscopic understanding of these relationships remains to be uncovered.

Extremal black holes have been of interest recently with regards to the Kerr/CFT conjecture and its generalizations. The general black hole admits two distinct extremal rotating limits, the fast and slow rotating cases. In each case, similar to each subset of solutions that has been previously studied under that viewpoint, we reproduced all expected properties of extremal black holes, such as the existence of an $\mathrm{SL}(2,\mathbb{R})\times\mathrm{U}(1)$ symmetric near-horizon region and the Cardy form of the entropy. We noted the property that the generic near-horizon metric of extremal slow rotating black holes only depends upon the angular momentum and the quartic invariant. These results, if combined with a general asymptotic symmetry group analysis, would allow a microscopic counting of these extremal black holes.

Several avenues for microscopically accounting for the entropy of specific nonextremal black holes in STU supergravity have been proposed [159,169–171]. It would be very interesting to try to unify these approaches and propose a microscopic model for the general black hole.

We have given a generating solution for the most general black hole of maximal supergravity in four dimensions. Without the complication of magnetic charges and with fewer gauge fields, the same had been done a long time ago for black holes in maximal supergravity in five dimensions [172] and higher dimensions [173]. Black rings are a further class of exact solutions in five dimensions, and are known in Einstein gravity with two independent rotations [174]. Several charged generalizations are known; see [175] for references. Using U dualities, a generating black ring solution for maximal supergravity is expected to involve 21 parameters, including mass, 2 angular momenta, 3 electric monopole charges, and 15 dipole charges [176]. Its construction would be a formidable task, and even its truncation to the 5-dimensional STU supergravity is not known.

Partial generalizations to asymptotically AdS black holes in the $\mathrm{U}(1)^4$ truncation of maximal $\mathcal{N}=8$, SO(8) gauged supergravities (including the recently discovered one-parameter family of ω-deformed theories [27,30]) have been found; see [28,29,177]. The asymptotically flat solution presented here has been generalized in [28] to two classes of asymptotically AdS solutions: static solutions with 4 independent electric charges and 4 independent magnetic charges; and rotating solutions with pairwise equal gauge fields, generalizing the solution of $-\mathrm{i}X^0X^1$ supergravity, which has 2 independent electric charges and 2 independent magnetic charges. However, they are difficult to find, since the solution generating techniques of ungauged supergravity rely on hidden symmetries. These symmetries of the bosonic theory are mostly broken in gauged supergravity by a scalar potential, in STU

supergravity from $SL(2,\mathbb{R})^3$ to $SO(2)^3$ [100] (see also [177]). The most general AdS generalizations of our ungauged solutions remain to be found. One guide to finding these solutions is that they are expected to involve metrics that allow separability. The class of metrics that we defined that admit a Killing-Stäckel tensor in string frame might therefore be useful in this context.

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