

Symmetric point quartic gluon vertex and momentum subtraction

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We compute the full one loop correction to the quartic vertex of QCD at the fully symmetric point. This allows us to define a new momentum subtraction (MOM) scheme in the class of schemes introduced by Celmaster and Gonsalves. Hence using properties of the renormalization group equation, the two loop renormalization group functions for this scheme are given.

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I. INTRODUCTION

In recent years there has been interest in studying the vertex structure of quantum chromodynamics (QCD) which is the quantum field theory corresponding to the strong nuclear force. The gluon and quark fields which are the constituent fields and which behave as free entities at high energy are not, however, observed in nature. This is due to confinement or infrared slavery. To understand why such approximately free particles are restricted to dwell inside the nucleons of the atomic nucleus requires studying their properties at low energy. Theoretically one uses the underlying quantum field theory, QCD, and endeavors to ascertain the nonperturbative behavior of the basic quanta interactions. The main computational tools for such investigations are lattice gauge theory and Schwinger-Dyson methods. Both techniques complement each other. For instance, there has been a large activity in extracting the zero momentum limit of the gluon and Faddeev-Popov ghost propagators in the Landau gauge. See, for example, various recent articles, [1–10]. While there is a good degree of agreement there is now interest in extending such analyses to the 3-point and 4-point vertex functions [11,12]. This is a level of difficulty beyond the 2-point function studies. While certain lattice data is available for several vertex functions [13–15], in say the Landau gauge, with general qualitative agreement between lattice and Schwinger-Dyson methods, the exploration of these vertex functions is not as mature a field as that for propagator analyses. Though, it should be emphasized that lattice gauge theory studies should improve with refined algorithms and supercomputers. While the infrared structure is related to confinement issues detailed knowledge is important when considering models of hadrons. These physical states of the theory can be examined by modeling the dynamics and interactions of the constituent fields via nonperturbative knowledge of the vertex functions.

From the point of view of perturbation theory, one can compute the vertex functions order by order in the loop expansion. Such a field theoretic program has been

ongoing over several decades. For instance, the one loop structure of the 3-point vertices has been studied in [16] at the fully symmetric point for the external momenta squared and for more general momentum configurations in [17]. Various extensions have been provided for a variety of cases. These include two loop corrections with one or more external legs on-shell [18–22]. Moreover, massive quarks have been included in some cases [23,24]. More recently the one loop analysis of [16] was extended to two loops in [25] with the momentum subtraction (MOM) scheme renormalization group functions determined for a linear covariant gauge. To assist model building and to allow Schwinger-Dyson practitioners to examine whether certain of their truncating approximations or vertex ansätze are reliable, the full off-shell massless vertex functions for each of the three QCD vertices were determined in [26]. Having completed the 3-point analysis the next stage is to examine the quartic vertex to complete the set. This is the purpose of this article where we consider the massless 4-point gluon vertex at the fully symmetric point, which is a nonexceptional point, for a general linear covariant gauge. While the extra leg on the Green's function sets it apart from the other vertices its Feynman rule does not contain a derivative and hence it should not be as problematic in that sense as the triple gluon vertex for, say, a lattice analysis. We note that, by contrast, in nonlinear covariant gauges one can have a quartic ghost vertex. Thus, in such gauges one would have to consider additional 4-point vertex computations. Previous work on the quartic gluon vertex has not been as intense as the 3-point ones. Though, we note the earlier relevant work of [18,27–30]. For instance, the quartic vertex with several on-shell gluons was studied in [27], while a Schwinger-Dyson analysis of the quartic vertex was considered in [28]. However, that analysis was not at the fully symmetric point but was for several nonexceptional momenta configurations. Moreover, an effective running coupling constant based on the vertex function was constructed [28]. An alternative recent approach to vertex functions has been provided in [29,30] using string

theory methods. Based on the triple gluon vertex experience of [29], off-shell quartic vertices have been studied in, for example, $\mathcal{N} = 4$ supersymmetric Yang-Mills theories in [30]. By contrast, in [18] the coupling constant renormalization constant was extracted for what was termed the Weinberg scheme. Though, unlike what we will present here no information on the *full* quartic vertex structure at the symmetric point was recorded as it was not required for the problem which was at hand. Here we will give the full structure. That this is possible is due to progress in recent years with computer algebra leading to automatic symbolic manipulation programs as well as with multiloop algorithms. One of the key ones has been the provision of the Laporta algorithm [31], which has been implemented in various computer packages including REDUZE [32]. While we report only on a one loop analysis which demonstrates the viability of studying the full quartic vertex and thus its potential extension to two loops, we will also provide a new renormalization scheme in the MOM class of schemes. This will be termed the MOMgggg scheme which will have the same ethos of definition as the three MOM schemes of [16] which was based on each of the 3-point QCD vertices. Although we do not envisage such a scheme overtaking the modified minimal subtraction, $\overline{\text{MS}}$, scheme which is the standard renormalization scheme, it in some sense completes the set begun in [16]. As a byproduct of the scheme definition we will determine its Λ parameter and evaluate it in relation to $\Lambda_{\overline{\text{MS}}}$. Moreover, once the one loop MOMgggg scheme renormalization of the QCD Lagrangian has been completed we will use the renormalization group equation to determine all the *two* loop renormalization group functions without having to complete an explicit two loop calculation. Such information will be the bedrock for any future computations at this order. Equally our values for the full quartic vertex will serve as an independent check if the fully off-shell vertex function is found at one loop.

The article is organized as follows. The background for the quartic vertex computation is discussed in Sec. II together with the relevant group theory and master integrals. Section III is devoted to explicit results for the 4-point function at the fully symmetric point while the MOMgggg scheme is defined in Sec. IV. There the two loop MOMgggg renormalization group functions are recorded. We provide conclusions in Sec. V while an Appendix records the full tensor basis into which we decompose our results.

II. BACKGROUND

In this section we discuss the technical details to our computations. First, the Green's function we will consider is

$$\begin{aligned} & \langle A_\mu^a(p) A_\nu^b(q) A_\sigma^c(r) A_\rho^d(-p-q-r) \rangle|_{\text{symm}} \\ & = \Sigma_{\mu\nu\sigma\rho}^{abcd}(p, q, r)|_{\text{symm}}, \end{aligned} \quad (2.1)$$

where p, q and r are the external momentum and we have substituted the momentum for the leg with color label d in terms of the others from energy-momentum conservation. The label *symm* indicates the fully symmetric point which is defined by

$$p^2 = q^2 = r^2 = -\mu^2, \quad pq = pr = qr = \frac{1}{3}\mu^2, \quad (2.2)$$

where μ is a mass scale. Following the same approach as [25,33] we use the same scale to ensure the coupling constant is dimensionless in d dimensions as we will regularize dimensionally throughout with $d = 4 - 2\epsilon$. This external momentum configuration is the same as [18] and we note that the Mandelstam variables are then

$$s = \frac{1}{2}(p+q)^2, \quad t = \frac{1}{2}(q+r)^2, \quad u = \frac{1}{2}(p+r)^2 \quad (2.3)$$

implying

$$s = t = u = -\frac{4}{3}\mu^2. \quad (2.4)$$

In (2.1) the right-hand side represents the Lorentz and color structure. For the 3-point vertices the color dependence, at least to low loop order, involves only one color tensor which can be readily factored off so that the focus is purely on the Lorentz component. For the quartic vertex there is more than one color structure as will be evident from later discussions. Therefore, for the moment we indicate how we extract the different Lorentz channels in (2.1). We follow the same method of [25] and use a projection method. Making no *a priori* assumptions about the final tensor structure of the vertex we formally write (2.1) in terms of a basis of 138 Lorentz tensors. This is the number of rank 4 Lorentz tensors that one can build from the three independent external momenta p, q and r as well as the metric $\eta_{\mu\nu}$. Given the size of this basis, the explicit forms given in terms of the labeling we use are recorded in the Appendix and denoted by $\mathcal{P}_{(k)\mu\nu\sigma\rho}(p, q, r)$ where k is our label with $1 \leq k \leq 138$. With these then we rewrite (2.1) as

$$\Sigma_{\mu\nu\sigma\rho}^{abcd}(p, q, r)|_{\text{symm}} = \sum_{k=1}^{138} \mathcal{P}_{(k)\mu\nu\sigma\rho}(p, q, r) \Sigma_{(k)}^{abcd}(p, q, r)|_{\text{symm}}, \quad (2.5)$$

where $\Sigma_{(k)}^{abcd}(p, q, r)$ are the Lorentz scalar but colorful amplitudes which will be determined. We could choose to write each Lorentz scalar in terms of color scalars.

However, as this emerges naturally from the color group algebra within the computation for the choice of $SU(N_c)$ we have not chosen to proceed that way. Moreover, as the computational method we will use requires Lorentz scalar Feynman integrals and the color structure factors off each integral the projection into Lorentz scalars is more crucial. For this we follow the earlier method of [25] and first construct the 138×138 matrix \mathcal{N}_{kl}^i defined by

$$\mathcal{N}_{kl}^i = \mathcal{P}_{(k)\mu\nu\sigma\rho}^i(p, q, r) \mathcal{P}_{(l)}^{i\mu\nu\sigma\rho}(p, q, r) \Big|_{\text{symm}}. \quad (2.6)$$

If we denote the inverse of \mathcal{N}_{kl}^i by \mathcal{M}_{kl}^i then the latter is the projection tensor we use. In other words,

$$\Sigma_{(k)}^{abcd}(p, q, r) \Big|_{\text{symm}} = \mathcal{M}_{kl}(\mathcal{P}_{(l)}^{\mu\nu\sigma\rho}(p, q, r) \langle A_\mu^a(p) A_\nu^b(q) A_\sigma^c(r) A_\rho^d(-p - q - r) \rangle) \Big|_{\text{symm}} \quad (2.7)$$

gives each individual amplitude and there is a sum over l .

Once this projection or decomposition (2.7) of the Green's function has been established we need to organize the actual computation. At one loop there are 24 Feynman graphs contributing to the quartic function which are generated using the QGRAF package [34]. Although this is a relatively small number given the number of tensors in the projection basis there are a large number of individual Feynman integrals within each graph to compute. The effect of the projection is to produce scalar integrals which have at most scalar products in the numerator which depend on the loop momentum and the three external momenta. To proceed these are rewritten in terms of the possible propagators that arise. The reason for this is that to perform the large number of integrals we chose to use the Laporta algorithm [31], for which this is the first step. The algorithm uses integration by parts to establish linear relations between integrals within a specific topology. These can be solved systematically to produce relations between all the integrals and a relatively small set of integrals which cannot be reduced to any other integrals. These are known as master integrals and they have to be evaluated by direct techniques not involving integration by parts. For the quartic vertex we have followed this Laporta approach and used its implementation in the REDUZE package [32]. It is a C++ based program which uses GiNAC [35]. This allows the user to build a database of requisite integrals from which one can extract the relations necessary for the particular Green's function of interest. In order to complete the automatic computation we use FORM and its threaded version TFORM [36,37] to handle the algebra. The REDUZE package allows for the integral relations to be extracted in FORM syntax. Indeed FORM is used to process the rearrangement of the scalar products in each Feynman graph into REDUZE input notation and REDUZE extracts the correct relations from the database but in FORM syntax. After applying the REDUZE algorithm [32], there are several basic classes of master integrals to insert from direct evaluation. If we regard the number of propagators as defining a class then the first set is those integrals with two propagators. This is the simple one loop

bubble. However, there are two specific integrals which are dependent on the value of the square of the momentum flowing through the graph and derive from how the basic one loop box of four propagators is shrunk. In one instance three of the original external legs of the box can be at one external point to the bubble and in the other case the external legs of the 2-propagator bubble are two pairs of the full box. For the 3-propagator master, given the symmetry of the squared momenta of the external legs there is one master. Here two of the legs are at $(-\mu^2)$ while the other is the corresponding Mandelstam variable and the value was given in [38–40]. Finally, the 4-propagator case is the basic one loop box. The general off-shell expression was provided in [41]. The explicit forms of the masters in the last two classes will be discussed in detail when the results are presented later. As a final part of the automatic computation setup description we note that to perform the renormalization we follow the method of [42]. This involves performing the calculation in terms of bare parameters throughout. Then the counterterms are automatically introduced by replacing the bare quantities with the renormalized counterparts and associated renormalization constant. Those renormalization constants, such as the wave function and gauge parameter, which are already known from the renormalization of 2-point functions are included in this redefinition. Hence, one fixes the unknown renormalization constant associated with the 4-point function with the divergences which remain after the rescaling.

As we will only be considering $SU(N_c)$ as the color rather than a general Lie group we recall key properties relevant to corrections to the quartic vertex which are based on [18,43]. As part of the QGRAF and FORM setup the color and Lorentz indices are appended to the QGRAF output before the projection and the application of the group theory FORM module. First, we recall that the product of two $SU(N_c)$ group generators, T^a , can be decomposed as

$$T^a T^b = \frac{1}{2N_c} \delta^{ab} + \frac{1}{2} d^{abc} T^c + \frac{i}{2} f^{abc} T^c, \quad (2.8)$$

where f^{abc} are the structure functions and d^{abc} is a totally symmetric tensor. The latter vanishes in $SU(2)$. For the other product of generators, we have

$$T_{IJ}^a T_{KL}^a = \frac{1}{2} \left[\delta_{IL} \delta_{KJ} - \frac{1}{N_c} \delta_{IJ} \delta_{KL} \right] \quad (2.9)$$

solely for $SU(N_c)$. To simplify notation we introduce related group tensors defined by

$$f_4^{abcd} \equiv f^{abefcde}, \quad d_4^{abcd} \equiv d^{abe} d^{cde}, \quad e_4^{abcd} \equiv d^{abefcde}. \quad (2.10)$$

So the Jacobi identities [43] are readily expressed as

$$\begin{aligned} e_4^{abcd} + e_4^{bcad} + e_4^{cabd} &= 0, \\ f_4^{abcd} + f_4^{acdb} + f_4^{adbc} &= 0. \end{aligned} \quad (2.11)$$

From [43] we note the relation between two products,

$$f_4^{abcd} = \frac{2}{N_c} [\delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc}] + d_4^{abcd} - d_4^{adbc}. \quad (2.12)$$

Also since there will be box diagrams with closed quark loops we need the decomposition of various traces over the group generators in the fundamental representation. From [18,43] we use

$$\begin{aligned} \text{Tr}(T^a T^b T^c T^d) &= \frac{1}{16} \left[\frac{4}{N_c} [\delta^{ab} \delta^{cd} - \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}] \right. \\ &\quad + 2[d_4^{abcd} - d_4^{acbd} + d_4^{adbc}] \\ &\quad \left. + 2i[e_4^{abcd} - e_4^{acdb} + e_4^{adbc}] \right]. \end{aligned} \quad (2.13)$$

In choosing to rewrite such tensors we are in effect making a choice of how to express the group structure of the graphs in the computation. In other words we eliminate all products of group generators in favor of products which involve at least one rank three symmetric tensor or the unit matrix. While this is a suitable choice of color tensor basis, from the point of view of defining a MOM-type renormalization scheme it is more appropriate to reexpress the final vertex function after computation in terms of the color tensor structures that are present in the original vertex in the Lagrangian which is f_4^{abcd} and one permutation of the indices. The Jacobi identity (2.11) can be used to recover the third tensor. However, due to the symmetry of the quartic vertex other tensor structures arise at one loop which are not unrelated to the unit matrices in (2.12) and (2.13). One could retain these as was the case in [18] but we have preferred to use the rank four fully symmetric tensors introduced in [44]

$$d_F^{abcd} = \frac{1}{6} \text{Tr}(T^a T^b T^c T^d), \quad d_A^{abcd} = \frac{1}{6} \text{Tr}(T_A^a T_A^b T_A^c T_A^d), \quad (2.14)$$

where the subscript A in the latter denotes the adjoint representation of the group generator. Hence, we have to express d_4^{abcd} in terms of these additional color tensors which is straightforward to do. We have

$$\begin{aligned} d_4^{abcd} &= -\frac{1}{3} \left[f_4^{abcd} - \frac{2}{N_c} [\delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc}] \right] \\ &\quad + \frac{2}{3} \left[f_4^{acbd} - \frac{2}{N_c} [\delta^{ab} \delta^{cd} - \delta^{ad} \delta^{bc}] \right] \\ &\quad + 8 \left[d_F^{abcd} - \frac{1}{12N_c} [\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}] \right], \\ d_4^{adbc} &= -\frac{1}{3} \left[f_4^{abcd} - \frac{2}{N_c} [\delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc}] \right] \\ &\quad - \frac{1}{3} \left[f_4^{acbd} - \frac{2}{N_c} [\delta^{ab} \delta^{cd} - \delta^{ad} \delta^{bc}] \right] \\ &\quad + 8 \left[d_F^{abcd} - \frac{1}{12N_c} [\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}] \right] \end{aligned} \quad (2.15)$$

which we use for diagrams involving quark loops and

$$\begin{aligned} d_4^{abcd} &= -\frac{1}{3} \left[f_4^{abcd} - \frac{2}{N_c} [\delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc}] \right] \\ &\quad + \frac{2}{3} \left[f_4^{acbd} - \frac{2}{N_c} [\delta^{ab} \delta^{cd} - \delta^{ad} \delta^{bc}] \right] \\ &\quad + \frac{4}{N_c} \left[d_A^{abcd} - \frac{2}{3} [\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}] \right], \\ d_4^{adbc} &= -\frac{1}{3} \left[f_4^{abcd} - \frac{2}{N_c} [\delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc}] \right] \\ &\quad - \frac{1}{3} \left[f_4^{acbd} - \frac{2}{N_c} [\delta^{ab} \delta^{cd} - \delta^{ad} \delta^{bc}] \right] \\ &\quad + \frac{4}{N_c} \left[d_A^{abcd} - \frac{2}{3} [\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}] \right] \end{aligned} \quad (2.16)$$

for one loop graphs which do not depend on N_f . Useful identities in constructing these mappings were [43]

$$\begin{aligned} d_4^{abcc} &= 0, \quad d_4^{acbc} = \frac{[N_c^2 - 4]}{N_c} \delta^{ab}, \\ d_4^{apbq} d_4^{cdpq} &= \frac{[N_c^2 - 12]}{2N_c} d_4^{abcd} \end{aligned} \quad (2.17)$$

in the present notation. This completes the summary of the computational setup for the quartic vertex function.

III. AMPLITUDES

We now turn to the task of recording our results and concentrate first on the amplitudes. Given the size of the final expression for all 138 channels for a nonzero gauge parameter α we focus on key parts and relegate the full results to the attached data file [45]. (The Landau gauge corresponds to $\alpha = 0$.) To give a flavor of the analytic structure which is typical of each channel we record the expressions for those which correspond to the Feynman rules for the quartic gluon vertex. For the $\overline{\text{MS}}$ scheme we find

$$\begin{aligned}
\Sigma_{(1)}^{abcd}(p, q, r) = & f_4^{abcd} + f_4^{adbc} \\
& + \left[\left[\frac{161}{80} + \frac{1}{16}\alpha - \frac{93}{80}\alpha^2 + \frac{7}{80}\alpha^3 + \frac{431}{75}\ln\left(\frac{4}{3}\right) - \frac{27}{80}\ln\left(\frac{4}{3}\right)\alpha \right. \right. \\
& - \frac{253}{200}\ln\left(\frac{4}{3}\right)\alpha^2 + \frac{9}{400}\ln\left(\frac{4}{3}\right)\alpha^3 - \frac{7219}{38400}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) - \frac{897}{2560}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right)\alpha + \frac{2709}{12800}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right)\alpha^2 \\
& \left. \left. - \frac{351}{12800}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right)\alpha^3 - \frac{85}{192}\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) - \frac{5}{32}\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right)\alpha + \frac{9}{64}\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right)\alpha^2 \right] f_4^{abcd} N_c \right. \\
& + \left[\frac{2}{3} - \frac{7}{6}\ln\left(\frac{4}{3}\right) + \frac{1}{24}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) - \frac{7}{24}\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \right] f_4^{abcd} N_f \\
& + \left[\frac{161}{80} + \frac{1}{16}\alpha - \frac{93}{80}\alpha^2 + \frac{7}{80}\alpha^3 + \frac{431}{75}\ln\left(\frac{4}{3}\right) - \frac{27}{80}\ln\left(\frac{4}{3}\right)\alpha \right. \\
& - \frac{253}{200}\ln\left(\frac{4}{3}\right)\alpha^2 + \frac{9}{400}\ln\left(\frac{4}{3}\right)\alpha^3 - \frac{7219}{38400}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) - \frac{897}{2560}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right)\alpha + \frac{2709}{12800}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right)\alpha^2 \\
& \left. \left. - \frac{351}{12800}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right)\alpha^3 - \frac{85}{192}\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) - \frac{5}{32}\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right)\alpha + \frac{9}{64}\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right)\alpha^2 \right] f_4^{adbc} N_c \right. \\
& + \left[\frac{2}{3} - \frac{7}{6}\ln\left(\frac{4}{3}\right) + \frac{1}{24}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) - \frac{7}{24}\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \right] f_4^{adbc} N_f \\
& + \left[\frac{3577}{1920} - \frac{123}{160}\alpha + \frac{21}{64}\alpha^2 - \frac{15}{32}\alpha^3 + \frac{243}{640}\alpha^4 + \frac{28113}{3200}\ln\left(\frac{4}{3}\right) \right. \\
& - \frac{903}{400}\ln\left(\frac{4}{3}\right)\alpha - \frac{867}{320}\ln\left(\frac{4}{3}\right)\alpha^2 - \frac{27}{20}\ln\left(\frac{4}{3}\right)\alpha^3 + \frac{1701}{3200}\ln\left(\frac{4}{3}\right)\alpha^4 + \frac{115493}{102400}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \\
& - \frac{5841}{25600}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right)\alpha + \frac{16443}{10240}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right)\alpha^2 - \frac{1701}{5120}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right)\alpha^3 - \frac{17739}{102400}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right)\alpha^4 \\
& \left. - \frac{145}{128}\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) + \frac{147}{128}\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right)\alpha - \frac{243}{128}\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right)\alpha^2 + \frac{81}{128}\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right)\alpha^3 \right] d_A^{abcd} \\
& + \left[-\frac{8}{3} - 6\ln\left(\frac{4}{3}\right) - \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) + \frac{5}{2}\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \right] d_F^{abcd} N_f \Big] a + O(a^2), \tag{3.1}
\end{aligned}$$

where we have used the Jacobi identity at one loop to recover the symmetry structure of the quartic vertex Feynman rule. For the other two channels given the similarity of the expressions we note that they satisfy

$$\begin{aligned}
\Sigma_{(2)}^{abcd}(p, q, r) &= \Sigma_{(1)}^{acbd}(p, q, r), \\
\Sigma_{(3)}^{abcd}(p, q, r) &= \Sigma_{(1)}^{adcb}(p, q, r) \tag{3.2}
\end{aligned}$$

which is a useful check on the computation. Other checks include the fact that the one loop Green's function is finite when the $\overline{\text{MS}}$ gluon wave function and coupling constant renormalization constants are included. The two specific values of the function $\Phi_1(x, y)$ appearing in (3.1) derive from the two master integrals referred to already. The general expression for $\Phi_1(x, y)$ includes the usual dilogarithm function $\text{Li}_2(z)$ via [38,39]

$$\Phi_1(x, y) = \frac{1}{\lambda} \left[2\text{Li}_2(-\rho x) + 2\text{Li}_2(-\rho y) + \ln\left(\frac{y}{x}\right) \ln\left(\frac{(1+\rho y)}{(1+\rho x)}\right) + \ln(\rho x) \ln(\rho y) + \frac{\pi^2}{3} \right], \quad (3.3)$$

where

$$\lambda(x, y) = \sqrt{\Delta_G}, \quad \rho(x, y) = \frac{2}{[1 - x - y + \lambda(x, y)]} \quad (3.4)$$

and

$$\Delta_G(x, y) = x^2 - 2xy + y^2 - 2x - 2y + 1 \quad (3.5)$$

is the Gram determinant. The appearance of $\Phi_1(x, y)$ at two different but symmetric arguments arises from two masters in the Laporta sense. One is for the 3-point function where the squares of the external momenta are p^2 , q^2 and s and is $\Phi_1(\frac{3}{4}, \frac{3}{4})$. The other is for the pure symmetric scalar box which was computed in [41] corresponding to $\Phi_1(\frac{9}{16}, \frac{9}{16})$. When one evaluates these functions from (3.3) the dilogarithms involve the Clausen function, $\text{Cl}_2(\theta)$, since the argument of the dilogarithm is complex. Though, the expression for each is ultimately real. In particular the argument of each dilogarithm function is $\frac{1}{3}(1 + 2\sqrt{2}i)$ for the triangle master and $\frac{1}{9}(1 + 4\sqrt{5}i)$ for the box master. Using the more symmetric definition of $\Phi_1(x, y)$ given in [46], our two basic master values can be written as

$$\begin{aligned} \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) &= \sqrt{2} \left[2\text{Cl}_2\left(2\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) + \text{Cl}_2\left(2\cos^{-1}\left(\frac{1}{3}\right)\right) \right], \\ \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) &= \frac{4}{\sqrt{5}} \left[2\text{Cl}_2\left(2\cos^{-1}\left(\frac{2}{3}\right)\right) + \text{Cl}_2\left(2\cos^{-1}\left(\frac{1}{9}\right)\right) \right]. \end{aligned} \quad (3.6)$$

Though, for compactness in presenting results we will use the shorthand notation $\Phi_1(\frac{3}{4}, \frac{3}{4})$ and $\Phi_1(\frac{9}{16}, \frac{9}{16})$ throughout.

The other main check on our one loop expression is to compare with the early work of [18]. In that article the quartic vertex was used to construct a renormalization scheme motivated by ideas of Weinberg [47]. More specifically a coupling constant renormalization, denoted there by Z_5 , for this Weinberg scheme was recorded for an arbitrary α and we have been able to virtually reproduce it. However, in order to do so we have had to convert to the same color tensor basis as [18] which involved d^{abc} and the unit matrix rather than our $\{f_4^{abcd}, d_F^{abcd}, d_A^{abcd}\}$ basis. This is straightforward to do. Also we have had to map our two main master functions to those present in [18] which are $R(\frac{2}{3})$ and $K(\frac{2}{3})$ in the notation of [18]. The relations between these and those which appear in our computation are

$$\begin{aligned} R\left(\frac{2}{3}\right) &= \frac{3}{4} \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right), \\ K\left(\frac{2}{3}\right) &= R\left(\frac{2}{3}\right) - \frac{15}{32} \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right). \end{aligned} \quad (3.7)$$

However, in [18] the values of the functions were only expressed numerically and not in analytic form involving $\text{Li}_2(z)$. Using (3.3) we have checked that the numerical values of (3.7) are in agreement with the values given in

[18]. For completeness and for a numerical evaluation we note that

$$\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) = 2.832045, \quad \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) = 3.403614. \quad (3.8)$$

In extracting the same renormalization constant Z_5 as [18] from our analysis we find agreement with all the terms given in Eq. (3.10) of [18], including that noted in the erratum associated with [18], except for two coefficients in the term linear in what corresponds to our α . Given that we get agreement with the other 18 coefficients which have various powers of α we assume that there is a minor discrepancy in [18]. For the interested reader we believe the correct coefficients are obtained if $\frac{315}{20}$ and $\frac{163}{100}$ given in [18] are replaced by $\frac{84}{5}$ and $\frac{22}{25}$ respectively. Although we are not in a position to comment on the effect these changes would have on the subsequent analysis performed in [18], we note that the numerical value of this term linear in α increases by about 10% primarily due to the large drop in the second value. Aside from these two terms and in light of the exact agreement with the other terms in Z_5 of [18], we believe we have the correct form for the one loop amplitudes.

In order to gauge the structure of the full one loop vertex function we have evaluated it numerically in the Landau gauge and find

$$\begin{aligned}
\Sigma^{abcd}(p, q, r)|_{\alpha=0} = & f_4^{abcd}[\mathcal{P}_{(2)} - \mathcal{P}_{(3)}] + f_4^{acbd}[\mathcal{P}_{(1)} - \mathcal{P}_{(3)}] + f_4^{adbc}[\mathcal{P}_{(1)} - \mathcal{P}_{(2)}] \\
& + [[5.021026\mathcal{P}_{(1)} + 5.021026\mathcal{P}_{(2)} + 5.021026\mathcal{P}_{(3)} + 4.028121\mathcal{P}_{(4)} + 6.174705\mathcal{P}_{(5)} - 1.359046\mathcal{P}_{(6)} \\
& + 6.174705\mathcal{P}_{(7)} + 4.028121\mathcal{P}_{(8)} - 1.359046\mathcal{P}_{(9)} + 3.872502\mathcal{P}_{(10)} + 3.872502\mathcal{P}_{(11)} + 2.513456\mathcal{P}_{(12)} \\
& + 4.028121\mathcal{P}_{(13)} - 1.359046\mathcal{P}_{(14)} + 6.174705\mathcal{P}_{(15)} + 3.872502\mathcal{P}_{(16)} + 2.513456\mathcal{P}_{(17)} \\
& + 3.872502\mathcal{P}_{(18)} + 6.174705\mathcal{P}_{(19)} - 1.359046\mathcal{P}_{(20)} + 4.028121\mathcal{P}_{(21)} - 4.293168\mathcal{P}_{(22)} \\
& - 2.146584\mathcal{P}_{(23)} - 2.146584\mathcal{P}_{(24)} - 2.146584\mathcal{P}_{(25)} + 4.028121\mathcal{P}_{(26)} + 5.387167\mathcal{P}_{(27)} \\
& - 2.146584\mathcal{P}_{(28)} + 0.155618\mathcal{P}_{(29)} + 4.028121\mathcal{P}_{(30)} + 2.513456\mathcal{P}_{(31)} + 3.872502\mathcal{P}_{(32)} \\
& + 3.872502\mathcal{P}_{(33)} - 1.359046\mathcal{P}_{(34)} + 4.028121\mathcal{P}_{(35)} + 6.174705\mathcal{P}_{(36)} - 1.359046\mathcal{P}_{(37)} \\
& + 6.174705\mathcal{P}_{(38)} + 4.028121\mathcal{P}_{(39)} + 4.028121\mathcal{P}_{(40)} - 2.146584\mathcal{P}_{(41)} + 0.155618\mathcal{P}_{(42)} \\
& - 2.146584\mathcal{P}_{(43)} - 4.293168\mathcal{P}_{(44)} - 2.146584\mathcal{P}_{(45)} + 5.387167\mathcal{P}_{(46)} - 2.146584\mathcal{P}_{(47)} \\
& + 4.028121\mathcal{P}_{(48)} + 4.028121\mathcal{P}_{(49)} + 0.155618\mathcal{P}_{(50)} - 2.146584\mathcal{P}_{(51)} + 5.387167\mathcal{P}_{(52)} \\
& + 4.028121\mathcal{P}_{(53)} - 2.146584\mathcal{P}_{(54)} - 2.146584\mathcal{P}_{(55)} - 2.146584\mathcal{P}_{(56)} - 4.293168\mathcal{P}_{(57)} \\
& + 0.397845\mathcal{P}_{(58)} + 0.397845\mathcal{P}_{(59)} + 0.397845\mathcal{P}_{(60)} + 2.493711\mathcal{P}_{(61)} + 0.198922\mathcal{P}_{(62)} \\
& + 0.198922\mathcal{P}_{(63)} - 2.095866\mathcal{P}_{(64)} + 2.493711\mathcal{P}_{(65)} + 0.198922\mathcal{P}_{(66)} + 0.198922\mathcal{P}_{(67)} \\
& - 2.095866\mathcal{P}_{(68)} + 2.493711\mathcal{P}_{(69)} + 0.198922\mathcal{P}_{(70)} - 2.095866\mathcal{P}_{(71)} + 0.198922\mathcal{P}_{(72)} \\
& + 2.493711\mathcal{P}_{(73)} - 2.095866\mathcal{P}_{(74)} + 0.198922\mathcal{P}_{(75)} + 0.198922\mathcal{P}_{(76)} + 2.493711\mathcal{P}_{(77)} \\
& - 2.095866\mathcal{P}_{(78)} + 0.198922\mathcal{P}_{(79)} + 0.198922\mathcal{P}_{(80)} + 2.493711\mathcal{P}_{(81)} + 0.198922\mathcal{P}_{(82)} \\
& - 2.095866\mathcal{P}_{(83)} + 0.198922\mathcal{P}_{(84)} + 0.198922\mathcal{P}_{(85)} + 2.493711\mathcal{P}_{(86)} - 2.095866\mathcal{P}_{(87)} \\
& + 2.493711\mathcal{P}_{(88)} - 2.095866\mathcal{P}_{(89)} + 0.198922\mathcal{P}_{(90)} + 2.493711\mathcal{P}_{(91)} + 0.198922\mathcal{P}_{(92)} \\
& - 2.095866\mathcal{P}_{(93)} + 2.493711\mathcal{P}_{(94)} + 0.198922\mathcal{P}_{(95)} - 2.095866\mathcal{P}_{(96)} + 2.493711\mathcal{P}_{(97)} \\
& - 2.095866\mathcal{P}_{(98)} + 0.198922\mathcal{P}_{(99)} + 0.198922\mathcal{P}_{(100)} + 2.493711\mathcal{P}_{(101)} - 2.095866\mathcal{P}_{(102)} \\
& + 3.917581\mathcal{P}_{(103)} + 2.663495\mathcal{P}_{(104)} - 0.956684\mathcal{P}_{(105)} + 0.681225\mathcal{P}_{(106)} + 0.368706\mathcal{P}_{(107)} \\
& + 1.622792\mathcal{P}_{(108)} + 2.663495\mathcal{P}_{(109)} + 3.917581\mathcal{P}_{(110)} - 0.956684\mathcal{P}_{(111)} - 2.210770\mathcal{P}_{(112)} \\
& + 1.622792\mathcal{P}_{(113)} + 0.368706\mathcal{P}_{(114)} + 2.663495\mathcal{P}_{(115)} - 0.956684\mathcal{P}_{(116)} + 3.917581\mathcal{P}_{(117)} \\
& + 1.622792\mathcal{P}_{(118)} - 2.210770\mathcal{P}_{(119)} + 1.622792\mathcal{P}_{(120)} + 3.917581\mathcal{P}_{(121)} - 0.956684\mathcal{P}_{(122)} \\
& + 2.663495\mathcal{P}_{(123)} + 0.368706\mathcal{P}_{(124)} + 0.681225\mathcal{P}_{(125)} + 0.368706\mathcal{P}_{(126)} - 0.956684\mathcal{P}_{(127)} \\
& + 3.917581\mathcal{P}_{(128)} + 2.663495\mathcal{P}_{(129)} + 1.622792\mathcal{P}_{(130)} + 0.368706\mathcal{P}_{(131)} + 0.681225\mathcal{P}_{(132)} \\
& - 0.956684\mathcal{P}_{(133)} + 2.663495\mathcal{P}_{(134)} + 3.917581\mathcal{P}_{(135)} + 0.368706\mathcal{P}_{(136)} + 1.622792\mathcal{P}_{(137)} \\
& - 2.210770\mathcal{P}_{(138)}]d_A^{abcd} \\
& + [-0.716260\mathcal{P}_{(1)} - 0.716260\mathcal{P}_{(2)} - 0.716260\mathcal{P}_{(3)} - 2.382957\mathcal{P}_{(4)} - 4.268597\mathcal{P}_{(5)} + 4.502776\mathcal{P}_{(6)} \\
& - 4.268597\mathcal{P}_{(7)} - 2.382957\mathcal{P}_{(8)} + 4.502776\mathcal{P}_{(9)} - 2.217185\mathcal{P}_{(10)} - 2.217185\mathcal{P}_{(11)} + 2.285591\mathcal{P}_{(12)} \\
& - 2.382957\mathcal{P}_{(13)} + 4.502776\mathcal{P}_{(14)} - 4.268597\mathcal{P}_{(15)} - 2.217185\mathcal{P}_{(16)} + 2.285591\mathcal{P}_{(17)} - 2.217185\mathcal{P}_{(18)} \\
& - 4.268597\mathcal{P}_{(19)} + 4.502776\mathcal{P}_{(20)} - 2.382957\mathcal{P}_{(21)} + 3.771280\mathcal{P}_{(22)} + 1.885640\mathcal{P}_{(23)} + 1.885640\mathcal{P}_{(24)} \\
& + 1.885640\mathcal{P}_{(25)} - 2.382957\mathcal{P}_{(26)} - 6.885733\mathcal{P}_{(27)} + 1.885640\mathcal{P}_{(28)} - 0.165772\mathcal{P}_{(29)} - 2.382957\mathcal{P}_{(30)} \\
& + 2.285591\mathcal{P}_{(31)} - 2.217185\mathcal{P}_{(32)} - 2.217185\mathcal{P}_{(33)} + 4.502776\mathcal{P}_{(34)} - 2.382957\mathcal{P}_{(35)} - 4.268597\mathcal{P}_{(36)}]
\end{aligned}$$

$$\begin{aligned}
& + 4.502776\mathcal{P}_{(37)} - 4.268597\mathcal{P}_{(38)} - 2.382957\mathcal{P}_{(39)} - 2.382957\mathcal{P}_{(40)} + 1.885640\mathcal{P}_{(41)} - 0.165772\mathcal{P}_{(42)} \\
& + 1.885640\mathcal{P}_{(43)} + 3.771280\mathcal{P}_{(44)} + 1.885640\mathcal{P}_{(45)} - 6.885733\mathcal{P}_{(46)} + 1.885640\mathcal{P}_{(47)} - 2.382957\mathcal{P}_{(48)} \\
& - 2.382957\mathcal{P}_{(49)} - 0.165772\mathcal{P}_{(50)} + 1.885640\mathcal{P}_{(51)} - 6.885733\mathcal{P}_{(52)} - 2.382957\mathcal{P}_{(53)} + 1.885640\mathcal{P}_{(54)} \\
& + 1.885640\mathcal{P}_{(55)} + 1.885640\mathcal{P}_{(56)} + 3.771280\mathcal{P}_{(57)} - 1.307807\mathcal{P}_{(58)} - 1.307807\mathcal{P}_{(59)} - 1.307807\mathcal{P}_{(60)} \\
& - 0.838049\mathcal{P}_{(61)} - 0.653903\mathcal{P}_{(62)} - 0.653903\mathcal{P}_{(63)} - 0.469758\mathcal{P}_{(64)} - 0.838049\mathcal{P}_{(65)} - 0.653903\mathcal{P}_{(66)} \\
& - 0.653903\mathcal{P}_{(67)} - 0.469758\mathcal{P}_{(68)} - 0.838049\mathcal{P}_{(69)} - 0.653903\mathcal{P}_{(70)} - 0.469758\mathcal{P}_{(71)} - 0.653903\mathcal{P}_{(72)} \\
& - 0.838049\mathcal{P}_{(73)} - 0.469758\mathcal{P}_{(74)} - 0.653903\mathcal{P}_{(75)} - 0.653903\mathcal{P}_{(76)} - 0.838049\mathcal{P}_{(77)} - 0.469758\mathcal{P}_{(78)} \\
& - 0.653903\mathcal{P}_{(79)} - 0.653903\mathcal{P}_{(80)} - 0.838049\mathcal{P}_{(81)} - 0.653903\mathcal{P}_{(82)} - 0.469758\mathcal{P}_{(83)} - 0.653903\mathcal{P}_{(84)} \\
& - 0.653903\mathcal{P}_{(85)} - 0.838049\mathcal{P}_{(86)} - 0.469758\mathcal{P}_{(87)} - 0.838049\mathcal{P}_{(88)} - 0.469758\mathcal{P}_{(89)} - 0.653903\mathcal{P}_{(90)} \\
& - 0.838049\mathcal{P}_{(91)} - 0.653903\mathcal{P}_{(92)} - 0.469758\mathcal{P}_{(93)} - 0.838049\mathcal{P}_{(94)} - 0.653903\mathcal{P}_{(95)} - 0.469758\mathcal{P}_{(96)} \\
& - 0.838049\mathcal{P}_{(97)} - 0.469758\mathcal{P}_{(98)} - 0.653903\mathcal{P}_{(99)} - 0.653903\mathcal{P}_{(100)} - 0.838049\mathcal{P}_{(101)} - 0.469758\mathcal{P}_{(102)} \\
& + 5.472775\mathcal{P}_{(103)} - 0.497317\mathcal{P}_{(104)} - 2.746924\mathcal{P}_{(105)} - 0.497317\mathcal{P}_{(106)} - 0.313172\mathcal{P}_{(107)} + 5.656920\mathcal{P}_{(108)} \\
& - 0.497317\mathcal{P}_{(109)} + 5.472775\mathcal{P}_{(110)} - 2.746924\mathcal{P}_{(111)} - 8.717016\mathcal{P}_{(112)} + 5.656920\mathcal{P}_{(113)} - 0.313172\mathcal{P}_{(114)} \\
& - 0.497317\mathcal{P}_{(115)} - 2.746924\mathcal{P}_{(116)} + 5.472775\mathcal{P}_{(117)} + 5.656920\mathcal{P}_{(118)} - 8.717016\mathcal{P}_{(119)} + 5.656920\mathcal{P}_{(120)} \\
& + 5.472775\mathcal{P}_{(121)} - 2.746924\mathcal{P}_{(122)} - 0.497317\mathcal{P}_{(123)} - 0.313172\mathcal{P}_{(124)} - 0.497317\mathcal{P}_{(125)} - 0.313172\mathcal{P}_{(126)} \\
& - 2.746924\mathcal{P}_{(127)} + 5.472775\mathcal{P}_{(128)} - 0.497317\mathcal{P}_{(129)} + 5.656920\mathcal{P}_{(130)} - 0.313172\mathcal{P}_{(131)} - 0.497317\mathcal{P}_{(132)} \\
& - 2.746924\mathcal{P}_{(133)} - 0.497317\mathcal{P}_{(134)} + 5.472775\mathcal{P}_{(135)} - 0.313172\mathcal{P}_{(136)} + 5.656920\mathcal{P}_{(137)} \\
& - 8.717016\mathcal{P}_{(138)}]d_F^{abcd}N_f \\
& + [-0.353158\mathcal{P}_{(2)} + 0.353158\mathcal{P}_{(3)} + 0.396705\mathcal{P}_{(4)} - 0.086664\mathcal{P}_{(5)} + 0.374322\mathcal{P}_{(6)} + 0.861923\mathcal{P}_{(7)} \\
& - 0.009757\mathcal{P}_{(8)} - 0.032141\mathcal{P}_{(9)} + 0.545727\mathcal{P}_{(10)} + 0.139265\mathcal{P}_{(11)} + 0.513587\mathcal{P}_{(12)} - 0.386947\mathcal{P}_{(13)} \\
& - 0.342181\mathcal{P}_{(14)} - 0.775260\mathcal{P}_{(15)} - 0.684993\mathcal{P}_{(16)} - 1.027174\mathcal{P}_{(17)} - 0.684993\mathcal{P}_{(18)} - 0.775259\mathcal{P}_{(19)} \\
& - 0.342181\mathcal{P}_{(20)} - 0.386947\mathcal{P}_{(21)} - 0.388312\mathcal{P}_{(22)} + 0.076906\mathcal{P}_{(23)} + 0.483368\mathcal{P}_{(24)} - 0.871680\mathcal{P}_{(25)} \\
& - 0.009757\mathcal{P}_{(26)} + 0.022383\mathcal{P}_{(27)} - 0.465218\mathcal{P}_{(28)} - 0.149022\mathcal{P}_{(29)} + 0.396704\mathcal{P}_{(30)} + 0.513586\mathcal{P}_{(31)} \\
& + 0.139265\mathcal{P}_{(32)} + 0.545727\mathcal{P}_{(33)} - 0.032140\mathcal{P}_{(34)} - 0.009757\mathcal{P}_{(35)} + 0.861923\mathcal{P}_{(36)} + 0.374321\mathcal{P}_{(37)} \\
& - 0.086664\mathcal{P}_{(38)} + 0.396704\mathcal{P}_{(39)} - 0.386947\mathcal{P}_{(40)} + 0.388311\mathcal{P}_{(41)} + 0.298045\mathcal{P}_{(42)} + 0.388311\mathcal{P}_{(43)} \\
& + 0.776623\mathcal{P}_{(44)} + 0.388311\mathcal{P}_{(45)} - 0.044766\mathcal{P}_{(46)} + 0.388311\mathcal{P}_{(47)} - 0.386947\mathcal{P}_{(48)} + 0.396704\mathcal{P}_{(49)} \\
& - 0.149022\mathcal{P}_{(50)} - 0.465218\mathcal{P}_{(51)} + 0.022383\mathcal{P}_{(52)} - 0.009757\mathcal{P}_{(53)} - 0.871680\mathcal{P}_{(54)} + 0.483368\mathcal{P}_{(55)} \\
& + 0.076906\mathcal{P}_{(56)} - 0.388311\mathcal{P}_{(57)} + 0.553863\mathcal{P}_{(58)} - 1.107726\mathcal{P}_{(59)} + 0.553863\mathcal{P}_{(60)} + 0.596522\mathcal{P}_{(61)} \\
& + 0.844398\mathcal{P}_{(62)} + 0.168365\mathcal{P}_{(63)} - 0.048819\mathcal{P}_{(64)} + 0.602682\mathcal{P}_{(65)} + 0.385497\mathcal{P}_{(66)} - 0.290535\mathcal{P}_{(67)} \\
& - 0.042659\mathcal{P}_{(68)} - 1.199205\mathcal{P}_{(69)} - 0.553863\mathcal{P}_{(70)} + 0.091478\mathcal{P}_{(71)} - 0.553863\mathcal{P}_{(72)} + 0.596522\mathcal{P}_{(73)} \\
& - 0.048819\mathcal{P}_{(74)} + 0.168365\mathcal{P}_{(75)} + 0.844398\mathcal{P}_{(76)} + 0.602682\mathcal{P}_{(77)} - 0.042659\mathcal{P}_{(78)} - 0.290535\mathcal{P}_{(79)} \\
& + 0.385497\mathcal{P}_{(80)} - 1.199205\mathcal{P}_{(81)} - 0.553863\mathcal{P}_{(82)} + 0.091478\mathcal{P}_{(83)} - 0.553863\mathcal{P}_{(84)} + 0.953329\mathcal{P}_{(85)} \\
& - 0.157073\mathcal{P}_{(86)} + 0.060111\mathcal{P}_{(87)} + 0.307987\mathcal{P}_{(88)} - 0.126382\mathcal{P}_{(89)} - 0.585230\mathcal{P}_{(90)} - 0.150913\mathcal{P}_{(91)} \\
& - 0.368098\mathcal{P}_{(92)} + 0.066271\mathcal{P}_{(93)} - 0.150913\mathcal{P}_{(94)} - 0.368098\mathcal{P}_{(95)} + 0.066271\mathcal{P}_{(96)} + 0.307987\mathcal{P}_{(97)} \\
& - 0.126382\mathcal{P}_{(98)} - 0.585230\mathcal{P}_{(99)} + 0.953329\mathcal{P}_{(100)} - 0.157073\mathcal{P}_{(101)} + 0.060111\mathcal{P}_{(102)} + 1.184049\mathcal{P}_{(103)} \\
& + 0.149792\mathcal{P}_{(104)} - 0.270205\mathcal{P}_{(105)} + 0.149792\mathcal{P}_{(106)} - 0.067392\mathcal{P}_{(107)} + 0.290831\mathcal{P}_{(108)} + 0.454011\mathcal{P}_{(109)}
\end{aligned}$$

$$\begin{aligned}
& -0.447757\mathcal{P}_{(110)} + 0.127260\mathcal{P}_{(111)} - 0.137716\mathcal{P}_{(112)} - 0.199881\mathcal{P}_{(113)} + 0.025854\mathcal{P}_{(114)} - 0.603803\mathcal{P}_{(115)} \\
& + 0.142944\mathcal{P}_{(116)} - 0.736292\mathcal{P}_{(117)} - 0.090950\mathcal{P}_{(118)} + 0.275433\mathcal{P}_{(119)} - 0.090950\mathcal{P}_{(120)} - 0.736292\mathcal{P}_{(121)} \\
& + 0.142944\mathcal{P}_{(122)} - 0.603803\mathcal{P}_{(123)} + 0.041538\mathcal{P}_{(124)} - 0.299584\mathcal{P}_{(125)} + 0.041538\mathcal{P}_{(126)} + 0.127260\mathcal{P}_{(127)} \\
& - 0.447757\mathcal{P}_{(128)} + 0.454011\mathcal{P}_{(129)} + 0.290831\mathcal{P}_{(130)} - 0.067392\mathcal{P}_{(131)} + 0.149792\mathcal{P}_{(132)} - 0.270205\mathcal{P}_{(133)} \\
& + 0.149792\mathcal{P}_{(134)} + 1.184049\mathcal{P}_{(135)} + 0.025854\mathcal{P}_{(136)} - 0.199881\mathcal{P}_{(137)} - 0.137716\mathcal{P}_{(138)}]f_4^{abcd}N_f \\
& + [5.316244\mathcal{P}_{(2)} - 5.316244\mathcal{P}_{(3)} - 1.778964\mathcal{P}_{(4)} - 3.517872\mathcal{P}_{(5)} - 1.937055\mathcal{P}_{(6)} - 1.712767\mathcal{P}_{(7)} \\
& - 0.894191\mathcal{P}_{(8)} - 1.052282\mathcal{P}_{(9)} - 2.064702\mathcal{P}_{(10)} - 1.179930\mathcal{P}_{(11)} - 3.116984\mathcal{P}_{(12)} + 2.673156\mathcal{P}_{(13)} \\
& + 2.989336\mathcal{P}_{(14)} + 5.230640\mathcal{P}_{(15)} + 3.244632\mathcal{P}_{(16)} + 6.233969\mathcal{P}_{(17)} + 3.244632\mathcal{P}_{(18)} + 5.230640\mathcal{P}_{(19)} \\
& + 2.989336\mathcal{P}_{(20)} + 2.673156\mathcal{P}_{(21)} + 2.557484\mathcal{P}_{(22)} + 2.623681\mathcal{P}_{(23)} + 1.738908\mathcal{P}_{(24)} + 0.818576\mathcal{P}_{(25)} \\
& - 0.894191\mathcal{P}_{(26)} + 0.158090\mathcal{P}_{(27)} - 0.066197\mathcal{P}_{(28)} + 0.285738\mathcal{P}_{(29)} - 1.778964\mathcal{P}_{(30)} - 3.116984\mathcal{P}_{(31)} \\
& - 1.179930\mathcal{P}_{(32)} - 2.064702\mathcal{P}_{(33)} - 1.052282\mathcal{P}_{(34)} - 0.894191\mathcal{P}_{(35)} - 1.712767\mathcal{P}_{(36)} - 1.937055\mathcal{P}_{(37)} \\
& - 3.517872\mathcal{P}_{(38)} - 1.778964\mathcal{P}_{(39)} + 2.673151\mathcal{P}_{(40)} - 2.557486\mathcal{P}_{(41)} - 0.571477\mathcal{P}_{(42)} - 2.557486\mathcal{P}_{(43)} \\
& - 5.114962\mathcal{P}_{(44)} - 2.557486\mathcal{P}_{(45)} - 0.316180\mathcal{P}_{(46)} - 2.557486\mathcal{P}_{(47)} + 2.673151\mathcal{P}_{(48)} - 1.778964\mathcal{P}_{(49)} \\
& + 0.285732\mathcal{P}_{(50)} - 0.066193\mathcal{P}_{(51)} + 0.158090\mathcal{P}_{(52)} - 0.894194\mathcal{P}_{(53)} + 0.818572\mathcal{P}_{(54)} + 1.738904\mathcal{P}_{(55)} \\
& + 2.623680\mathcal{P}_{(56)} + 2.557486\mathcal{P}_{(57)} - 1.805822\mathcal{P}_{(58)} + 3.611644\mathcal{P}_{(59)} - 1.805822\mathcal{P}_{(60)} - 3.549941\mathcal{P}_{(61)} \\
& - 3.526038\mathcal{P}_{(62)} + 0.227060\mathcal{P}_{(63)} - 0.573359\mathcal{P}_{(64)} - 1.232465\mathcal{P}_{(65)} - 2.032884\mathcal{P}_{(66)} + 1.720206\mathcal{P}_{(67)} \\
& + 1.744128\mathcal{P}_{(68)} + 4.782416\mathcal{P}_{(69)} + 1.805822\mathcal{P}_{(70)} - 1.170762\mathcal{P}_{(71)} + 1.805822\mathcal{P}_{(72)} - 3.549941\mathcal{P}_{(73)} \\
& - 0.573359\mathcal{P}_{(74)} + 0.227060\mathcal{P}_{(75)} - 3.526038\mathcal{P}_{(76)} - 1.232465\mathcal{P}_{(77)} + 1.744128\mathcal{P}_{(78)} + 1.720208\mathcal{P}_{(79)} \\
& - 2.032883\mathcal{P}_{(80)} + 4.782416\mathcal{P}_{(81)} + 1.805823\mathcal{P}_{(82)} - 1.170769\mathcal{P}_{(83)} + 1.805823\mathcal{P}_{(84)} - 3.819839\mathcal{P}_{(85)} \\
& + 0.193345\mathcal{P}_{(86)} - 0.867162\mathcal{P}_{(87)} - 0.843246\mathcal{P}_{(88)} - 0.583153\mathcal{P}_{(89)} + 2.109430\mathcal{P}_{(90)} + 0.649901\mathcal{P}_{(91)} \\
& + 1.710409\mathcal{P}_{(92)} + 1.450315\mathcal{P}_{(93)} + 0.649901\mathcal{P}_{(94)} + 1.710409\mathcal{P}_{(95)} + 1.450315\mathcal{P}_{(96)} - 0.843246\mathcal{P}_{(97)} \\
& - 0.583153\mathcal{P}_{(98)} + 2.109430\mathcal{P}_{(99)} - 3.819839\mathcal{P}_{(100)} + 0.193345\mathcal{P}_{(101)} - 0.867162\mathcal{P}_{(102)} - 5.621879\mathcal{P}_{(103)} \\
& - 0.081106\mathcal{P}_{(104)} + 2.826341\mathcal{P}_{(105)} - 0.701413\mathcal{P}_{(106)} - 0.881520\mathcal{P}_{(107)} - 2.669203\mathcal{P}_{(108)} - 1.720159\mathcal{P}_{(109)} \\
& + 1.457590\mathcal{P}_{(110)} - 1.492175\mathcal{P}_{(111)} + 0.463316\mathcal{P}_{(112)} + 1.481505\mathcal{P}_{(113)} + 2.056848\mathcal{P}_{(114)} + 1.801265\mathcal{P}_{(115)} \\
& - 1.334166\mathcal{P}_{(116)} + 4.164290\mathcal{P}_{(117)} + 1.187697\mathcal{P}_{(118)} - 0.926631\mathcal{P}_{(119)} + 1.187697\mathcal{P}_{(120)} + 4.164290\mathcal{P}_{(121)} \\
& - 1.334166\mathcal{P}_{(122)} + 1.801265\mathcal{P}_{(123)} - 1.175328\mathcal{P}_{(124)} + 1.402826\mathcal{P}_{(125)} - 1.175328\mathcal{P}_{(126)} - 1.492175\mathcal{P}_{(127)} \\
& + 1.457590\mathcal{P}_{(128)} - 1.720159\mathcal{P}_{(129)} - 2.669203\mathcal{P}_{(130)} - 0.881520\mathcal{P}_{(131)} - 0.701413\mathcal{P}_{(132)} + 2.826341\mathcal{P}_{(133)} \\
& - 0.081106\mathcal{P}_{(134)} - 5.621879\mathcal{P}_{(135)} + 2.056848\mathcal{P}_{(136)} + 1.481505\mathcal{P}_{(137)} + 0.463316\mathcal{P}_{(138)}]f_4^{abcd} \\
& + [-0.353158\mathcal{P}_{(1)} + 0.353158\mathcal{P}_{(3)} - 0.386947\mathcal{P}_{(4)} - 0.775259\mathcal{P}_{(5)} - 0.342181\mathcal{P}_{(6)} - 0.775259\mathcal{P}_{(7)} \\
& - 0.386947\mathcal{P}_{(8)} - 0.342181\mathcal{P}_{(9)} - 0.684993\mathcal{P}_{(10)} - 0.684993\mathcal{P}_{(11)} - 1.027174\mathcal{P}_{(12)} + 0.396705\mathcal{P}_{(13)} \\
& + 0.374322\mathcal{P}_{(14)} - 0.086664\mathcal{P}_{(15)} + 0.545727\mathcal{P}_{(16)} + 0.513587\mathcal{P}_{(17)} + 0.139265\mathcal{P}_{(18)} + 0.861923\mathcal{P}_{(19)} \\
& - 0.032141\mathcal{P}_{(20)} - 0.009757\mathcal{P}_{(21)} - 0.388312\mathcal{P}_{(22)} - 0.465219\mathcal{P}_{(23)} - 0.871681\mathcal{P}_{(24)} + 0.483369\mathcal{P}_{(25)} \\
& + 0.396705\mathcal{P}_{(26)} + 0.022383\mathcal{P}_{(27)} + 0.076907\mathcal{P}_{(28)} - 0.149023\mathcal{P}_{(29)} - 0.009757\mathcal{P}_{(30)} + 0.513587\mathcal{P}_{(31)} \\
& + 0.545727\mathcal{P}_{(32)} + 0.139265\mathcal{P}_{(33)} + 0.374321\mathcal{P}_{(34)} + 0.396705\mathcal{P}_{(35)} - 0.086664\mathcal{P}_{(36)} - 0.032141\mathcal{P}_{(37)} \\
& + 0.861923\mathcal{P}_{(38)} - 0.009757\mathcal{P}_{(39)} + 0.396705\mathcal{P}_{(40)} - 0.465219\mathcal{P}_{(41)} - 0.149023\mathcal{P}_{(42)} + 0.483369\mathcal{P}_{(43)} \\
& - 0.388312\mathcal{P}_{(44)} + 0.076907\mathcal{P}_{(45)} + 0.022383\mathcal{P}_{(46)} - 0.871681\mathcal{P}_{(47)} - 0.009757\mathcal{P}_{(48)} - 0.386947\mathcal{P}_{(49)}
\end{aligned}$$

$$\begin{aligned}
& + 0.298046\mathcal{P}_{(50)} + 0.388312\mathcal{P}_{(51)} - 0.044766\mathcal{P}_{(52)} - 0.386947\mathcal{P}_{(53)} + 0.388312\mathcal{P}_{(54)} + 0.388312\mathcal{P}_{(55)} \\
& + 0.388312\mathcal{P}_{(56)} + 0.776624\mathcal{P}_{(57)} + 0.553863\mathcal{P}_{(58)} + 0.553863\mathcal{P}_{(59)} - 1.107727\mathcal{P}_{(60)} + 0.602682\mathcal{P}_{(61)} \\
& - 0.290535\mathcal{P}_{(62)} + 0.385497\mathcal{P}_{(63)} - 0.042659\mathcal{P}_{(64)} + 0.596523\mathcal{P}_{(65)} + 0.168366\mathcal{P}_{(66)} + 0.844399\mathcal{P}_{(67)} \\
& - 0.048819\mathcal{P}_{(68)} + 0.596523\mathcal{P}_{(69)} + 0.168366\mathcal{P}_{(70)} - 0.048819\mathcal{P}_{(71)} + 0.844399\mathcal{P}_{(72)} - 1.199205\mathcal{P}_{(73)} \\
& + 0.091478\mathcal{P}_{(74)} - 0.553863\mathcal{P}_{(75)} - 0.553863\mathcal{P}_{(76)} - 1.199205\mathcal{P}_{(77)} + 0.091478\mathcal{P}_{(78)} - 0.553863\mathcal{P}_{(79)} \\
& - 0.553863\mathcal{P}_{(80)} + 0.602682\mathcal{P}_{(81)} - 0.290535\mathcal{P}_{(82)} - 0.042659\mathcal{P}_{(83)} + 0.385497\mathcal{P}_{(84)} - 0.368099\mathcal{P}_{(85)} \\
& - 0.150914\mathcal{P}_{(86)} + 0.066271\mathcal{P}_{(87)} - 0.150914\mathcal{P}_{(88)} + 0.066271\mathcal{P}_{(89)} - 0.368099\mathcal{P}_{(90)} - 0.157073\mathcal{P}_{(91)} \\
& + 0.953329\mathcal{P}_{(92)} + 0.060111\mathcal{P}_{(93)} + 0.307987\mathcal{P}_{(94)} - 0.585230\mathcal{P}_{(95)} - 0.126382\mathcal{P}_{(96)} - 0.157073\mathcal{P}_{(97)} \\
& + 0.060111\mathcal{P}_{(98)} + 0.953329\mathcal{P}_{(99)} - 0.585230\mathcal{P}_{(100)} + 0.307987\mathcal{P}_{(101)} - 0.126382\mathcal{P}_{(102)} - 0.447757\mathcal{P}_{(103)} \\
& + 0.454011\mathcal{P}_{(104)} + 0.127261\mathcal{P}_{(105)} + 0.149792\mathcal{P}_{(106)} + 0.025855\mathcal{P}_{(107)} - 0.199881\mathcal{P}_{(108)} + 0.149792\mathcal{P}_{(109)} \\
& + 1.184049\mathcal{P}_{(110)} - 0.270205\mathcal{P}_{(111)} - 0.137716\mathcal{P}_{(112)} + 0.290832\mathcal{P}_{(113)} - 0.067392\mathcal{P}_{(114)} + 0.149792\mathcal{P}_{(115)} \\
& - 0.270205\mathcal{P}_{(116)} + 1.184049\mathcal{P}_{(117)} + 0.290832\mathcal{P}_{(118)} - 0.137716\mathcal{P}_{(119)} - 0.199881\mathcal{P}_{(120)} - 0.447757\mathcal{P}_{(121)} \\
& + 0.127261\mathcal{P}_{(122)} + 0.454011\mathcal{P}_{(123)} + 0.025855\mathcal{P}_{(124)} + 0.149792\mathcal{P}_{(125)} - 0.067392\mathcal{P}_{(126)} + 0.142944\mathcal{P}_{(127)} \\
& - 0.736292\mathcal{P}_{(128)} - 0.603804\mathcal{P}_{(129)} - 0.090951\mathcal{P}_{(130)} + 0.041538\mathcal{P}_{(131)} - 0.299585\mathcal{P}_{(132)} + 0.142944\mathcal{P}_{(133)} \\
& - 0.603804\mathcal{P}_{(134)} - 0.736292\mathcal{P}_{(135)} + 0.041538\mathcal{P}_{(136)} - 0.090951\mathcal{P}_{(137)} + 0.275433\mathcal{P}_{(138)}]f_4^{abcd}N_f \\
& + [5.316244\mathcal{P}_{(1)} - 5.316244\mathcal{P}_{(3)} + 2.673156\mathcal{P}_{(4)} + 5.230640\mathcal{P}_{(5)} + 2.989336\mathcal{P}_{(6)} + 5.230640\mathcal{P}_{(7)} \\
& + 2.673156\mathcal{P}_{(8)} + 2.989336\mathcal{P}_{(9)} + 3.244632\mathcal{P}_{(10)} + 3.244632\mathcal{P}_{(11)} + 6.233969\mathcal{P}_{(12)} - 1.778964\mathcal{P}_{(13)} \\
& - 1.937055\mathcal{P}_{(14)} - 3.517872\mathcal{P}_{(15)} - 2.064702\mathcal{P}_{(16)} - 3.116984\mathcal{P}_{(17)} - 1.179930\mathcal{P}_{(18)} - 1.712767\mathcal{P}_{(19)} \\
& - 1.052282\mathcal{P}_{(20)} - 0.894191\mathcal{P}_{(21)} + 2.557484\mathcal{P}_{(22)} - 0.066197\mathcal{P}_{(23)} + 0.818576\mathcal{P}_{(24)} + 1.738908\mathcal{P}_{(25)} \\
& - 1.778964\mathcal{P}_{(26)} + 0.158090\mathcal{P}_{(27)} + 2.623681\mathcal{P}_{(28)} + 0.285738\mathcal{P}_{(29)} - 0.894191\mathcal{P}_{(30)} - 3.116984\mathcal{P}_{(31)} \\
& - 2.064702\mathcal{P}_{(32)} - 1.179930\mathcal{P}_{(33)} - 1.937055\mathcal{P}_{(34)} - 1.778964\mathcal{P}_{(35)} - 3.517872\mathcal{P}_{(36)} - 1.052282\mathcal{P}_{(37)} \\
& - 1.712767\mathcal{P}_{(38)} - 0.894191\mathcal{P}_{(39)} - 1.778964\mathcal{P}_{(40)} - 0.066197\mathcal{P}_{(41)} + 0.285738\mathcal{P}_{(42)} + 1.738908\mathcal{P}_{(43)} \\
& + 2.557484\mathcal{P}_{(44)} + 2.623681\mathcal{P}_{(45)} + 0.158090\mathcal{P}_{(46)} + 0.818576\mathcal{P}_{(47)} - 0.894191\mathcal{P}_{(48)} + 2.673156\mathcal{P}_{(49)} \\
& - 0.571476\mathcal{P}_{(50)} - 2.557484\mathcal{P}_{(51)} - 0.316181\mathcal{P}_{(52)} + 2.673156\mathcal{P}_{(53)} - 2.557484\mathcal{P}_{(54)} - 2.557484\mathcal{P}_{(55)} \\
& - 2.557484\mathcal{P}_{(56)} - 5.114968\mathcal{P}_{(57)} - 1.805823\mathcal{P}_{(58)} - 1.805823\mathcal{P}_{(59)} + 3.611646\mathcal{P}_{(60)} - 1.232469\mathcal{P}_{(61)} \\
& + 1.720208\mathcal{P}_{(62)} - 2.032883\mathcal{P}_{(63)} + 1.744124\mathcal{P}_{(64)} - 3.549947\mathcal{P}_{(65)} + 0.227060\mathcal{P}_{(66)} - 3.526031\mathcal{P}_{(67)} \\
& - 0.573354\mathcal{P}_{(68)} - 3.549947\mathcal{P}_{(69)} + 0.227060\mathcal{P}_{(70)} - 0.573354\mathcal{P}_{(71)} - 3.526030\mathcal{P}_{(72)} + 4.782416\mathcal{P}_{(73)} \\
& - 1.170770\mathcal{P}_{(74)} + 1.805823\mathcal{P}_{(75)} + 1.805823\mathcal{P}_{(76)} + 4.782416\mathcal{P}_{(77)} - 1.170770\mathcal{P}_{(78)} + 1.805823\mathcal{P}_{(79)} \\
& + 1.805823\mathcal{P}_{(80)} - 1.232470\mathcal{P}_{(81)} + 1.720208\mathcal{P}_{(82)} + 1.744124\mathcal{P}_{(83)} - 2.032883\mathcal{P}_{(84)} + 1.710409\mathcal{P}_{(85)} \\
& + 0.649901\mathcal{P}_{(86)} + 1.450315\mathcal{P}_{(87)} + 0.649901\mathcal{P}_{(88)} + 1.450315\mathcal{P}_{(89)} + 1.710409\mathcal{P}_{(90)} + 0.193345\mathcal{P}_{(91)} \\
& - 3.819839\mathcal{P}_{(92)} - 0.867162\mathcal{P}_{(93)} - 0.843246\mathcal{P}_{(94)} + 2.109430\mathcal{P}_{(95)} - 0.583153\mathcal{P}_{(96)} + 0.193345\mathcal{P}_{(97)} \\
& - 0.867162\mathcal{P}_{(98)} - 3.819839\mathcal{P}_{(99)} + 2.109430\mathcal{P}_{(100)} - 0.843246\mathcal{P}_{(101)} - 0.583153\mathcal{P}_{(102)} + 1.457590\mathcal{P}_{(103)} \\
& - 1.720159\mathcal{P}_{(104)} - 1.492175\mathcal{P}_{(105)} - 0.701412\mathcal{P}_{(106)} + 2.056848\mathcal{P}_{(107)} + 1.481505\mathcal{P}_{(108)} - 0.081106\mathcal{P}_{(109)} \\
& - 5.621879\mathcal{P}_{(110)} + 2.826341\mathcal{P}_{(111)} + 0.463316\mathcal{P}_{(112)} - 2.669203\mathcal{P}_{(113)} - 0.881520\mathcal{P}_{(114)} - 0.081106\mathcal{P}_{(115)} \\
& + 2.826341\mathcal{P}_{(116)} - 5.621879\mathcal{P}_{(117)} - 2.669203\mathcal{P}_{(118)} + 0.463316\mathcal{P}_{(119)} + 1.481505\mathcal{P}_{(120)} + 1.457590\mathcal{P}_{(121)} \\
& - 1.492175\mathcal{P}_{(122)} - 1.720159\mathcal{P}_{(123)} + 2.056848\mathcal{P}_{(124)} - 0.701413\mathcal{P}_{(125)} - 0.881520\mathcal{P}_{(126)} - 1.334166\mathcal{P}_{(127)}
\end{aligned}$$

$$\begin{aligned}
 &+ 4.164290\mathcal{P}_{(128)} + 1.801265\mathcal{P}_{(129)} + 1.187697\mathcal{P}_{(130)} - 1.175328\mathcal{P}_{(131)} + 1.402826\mathcal{P}_{(132)} - 1.334166\mathcal{P}_{(133)} \\
 &+ 1.801265\mathcal{P}_{(134)} + 4.164290\mathcal{P}_{(135)} - 1.175328\mathcal{P}_{(136)} + 1.187697\mathcal{P}_{(137)} - 0.926631\mathcal{P}_{(138)}]f_4^{abcd} \\
 &+ [-0.353158\mathcal{P}_{(1)} + 0.353158\mathcal{P}_{(2)}]f_4^{abc}N_f + [5.316244\mathcal{P}_{(1)} - 5.316244\mathcal{P}_{(2)}]f_4^{abc}a + O(a^2), \tag{3.9}
 \end{aligned}$$

where we have dropped the common argument of $\mathcal{P}_{(k)}(p, q, r)$ and have only used the Jacobi identity in the channels with a nonzero tree term. Indeed for these three channels, 1, 2 and 3, the symmetry associated with color tensors d_F^{abcd} and d_A^{abcd} is evident.

IV. MOMgggg SCHEME

Having completely determined the 4-point function at one loop in the $\overline{\text{MS}}$ scheme at the symmetric point we can now consider the renormalization in other schemes. In [16] the symmetric point renormalization of the 3-point vertices led naturally into the definition of the momentum subtraction schemes. These are mass dependent schemes and are constructed in such a way that after renormalization at the subtraction point the respective vertices have no $O(a)$ corrections. This is in addition to the wave function renormalization constants being defined in the same way via the 2-point functions. As there are three 3-point vertices in QCD this leads to three separate MOM schemes which are denoted by MOMggg, MOMh and MOMq based on the respective triple gluon, ghost-gluon and quark-gluon vertices [16]. In light of this and the fact that we have considered the quartic gluon vertex at the fully symmetric point we can define an analogous momentum subtraction scheme which will be denoted by MOMgggg. More

specifically the scheme is defined as follows. We will retain the wave function MOM renormalization scheme approach. By this we mean that the 2-point functions are rendered finite by ensuring that at the subtraction point there are no $O(a)$ corrections. This does not mean that the wave function renormalization constants are the same as the three MOM schemes of [16]. This is because as one proceeds beyond one loop the renormalization constant of the coupling constant is required and this is different in different MOM schemes. Therefore, that observation will equally apply to MOMgggg. However, for the MOMgggg coupling constant renormalization its definition requires some care. This is because unlike the 3-point vertices there is more than one color tensor structure for the 4-point function. Therefore, we require the quartic vertex function to be written in terms of the color tensors of the original quartic gluon Feynman rule for $SU(N_c)$ in contrast to [18]. This was one of the reasons for already presenting our results in this format in the previous section. Thus the MOMgggg scheme coupling constant renormalization is defined so that after renormalization there are no $O(a)$ corrections to that part of the vertex function corresponding to the original quartic vertex Feynman rule. With this definition we find the coupling constant renormalization constant is

$$\begin{aligned}
 Z_g^{\text{MOMgggg}} &= 1 + \left[\frac{1}{3}N_f - \frac{11}{6}N_c \right] \frac{a}{\epsilon} \\
 &+ \left[-\frac{5329}{1440} - \frac{17}{32}\alpha + \frac{53}{160}\alpha^2 - \frac{7}{160}\alpha^3 - \frac{431}{150}\ln\left(\frac{4}{3}\right) + \frac{27}{160}\ln\left(\frac{4}{3}\right)\alpha \right. \\
 &+ \frac{253}{400}\ln\left(\frac{4}{3}\right)\alpha^2 - \frac{9}{800}\ln\left(\frac{4}{3}\right)\alpha^3 + \frac{7219}{76800}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \\
 &+ \frac{897}{5120}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right)\alpha - \frac{2709}{25600}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right)\alpha^2 + \frac{351}{25600}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right)\alpha^3 \\
 &+ \left. \frac{85}{384}\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) + \frac{5}{64}\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right)\alpha - \frac{9}{128}\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right)\alpha^2 \right] N_c \\
 &+ \left[\frac{7}{9} + \frac{7}{12}\ln\left(\frac{4}{3}\right) - \frac{1}{48}\Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) + \frac{7}{48}\Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \right] N_f a + O(a^2). \tag{4.1}
 \end{aligned}$$

Equipped with this renormalization constant we can construct the relation between the Λ parameters in MOMgggg and $\overline{\text{MS}}$. From standard formalism this ratio is defined as

$$\frac{\Lambda^{\text{MOMgggg}}}{\Lambda^{\overline{\text{MS}}}} = \exp\left[\frac{\lambda^{\text{MOMgggg}}(\alpha, N_f)}{b_0} \right], \tag{4.2}$$

where we take

$$b_0 = \frac{22}{3}N_c - \frac{4}{3}N_f \quad (4.3)$$

and $\lambda^{\text{MOMgggg}}(\alpha, N_f)$ is related to the finite part of the one loop coupling constant renormalization constant. From Z_g^{MOMgggg} we have

$$\begin{aligned} \lambda^{\text{MOMgggg}}(\alpha, N_f) = & \frac{1}{115200} \left[2592 \ln\left(\frac{4}{3}\right) \alpha^3 N_c - 145728 \ln\left(\frac{4}{3}\right) \alpha^2 N_c - 38880 \ln\left(\frac{4}{3}\right) \alpha N_c \right. \\ & + 662016 \ln\left(\frac{4}{3}\right) N_c - 134400 \ln\left(\frac{4}{3}\right) N_f \\ & - 3159 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^3 N_c + 24381 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^2 N_c - 40365 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha N_c \\ & - 21657 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) N_c + 4800 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) N_f \\ & + 16200 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha^2 N_c - 18000 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha N_c - 51000 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) N_c - 33600 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) N_f \\ & \left. + 10080 \alpha^3 N_c - 76320 \alpha^2 N_c + 122400 \alpha N_c + 852640 N_c - 179200 N_f \right]. \quad (4.4) \end{aligned}$$

To appreciate the behavior of $\lambda^{\text{MOMgggg}}(\alpha, N_f)$ we have computed it for various values of α and N_f and presented the results in Table I. There we reproduce the corresponding values for the three MOM schemes defined in [16] in relation to $\overline{\text{MS}}$ rather than the original minimal subtraction (MS) scheme values which were actually given there. For instance, the Landau gauge values are lower than those for MOMggg.

While we have introduced the new scheme MOMgggg we have not presented the associated one loop renormalization group functions for this scheme. This is because at this order these functions are scheme independent.

TABLE I. Values of $\Lambda^{\text{MOMgggg}}/\Lambda^{\overline{\text{MS}}}$ for $SU(3)$ in comparison with other MOM schemes of [16].

α	N_f	MOMggggg	MOMggg	MOMh	MOMq
0	0	2.6551	3.3341	2.3236	2.1379
0	1	2.4965	3.0543	2.3250	2.1277
0	2	2.3274	2.7644	2.3267	2.1163
0	3	2.1474	2.4654	2.3286	2.1032
0	4	1.9560	2.1587	2.3308	2.0881
0	5	1.7529	1.8471	2.3336	2.0706
1	0	2.4543	2.8957	2.6166	1.9075
1	3	1.9506	2.0751	2.6924	1.8296
1	4	1.7631	1.7921	2.7265	1.7964
1	5	1.5658	1.5088	2.7670	1.7581
3	3	1.7693	1.8392	4.1918	1.3110
3	4	1.5868	1.5732	4.3978	1.2533
-2	4	2.6576	2.5437	2.0081	2.6597

Beyond one loop the coefficients of each term in the coupling constant expansion depend on the scheme. This applies to the β function too. Though, in mass independent renormalization schemes such as $\overline{\text{MS}}$ the two loop term of the β function is also scheme independent in theories with only one coupling constant [48]. However, while we have performed a one loop computation it is possible to construct the two loop renormalization group functions in the MOMgggg scheme using properties of the renormalization group. See, for example, [49] for background to this. To achieve this we have to compute the various conversion functions for each renormalization group function which in essence are the ratios of the respective renormalization constants in each scheme. In particular,

$$\begin{aligned} C_g^{\text{MOMgggg}}(a, \alpha) &= \frac{Z_g^{\text{MOMgggg}}}{Z_g^{\overline{\text{MS}}}}, \\ C_\phi^{\text{MOMgggg}}(a, \alpha) &= \frac{Z_\phi^{\text{MOMgggg}}}{Z_\phi^{\overline{\text{MS}}}}, \\ C_\alpha^{\text{MOMgggg}}(a, \alpha) &= \frac{Z_\alpha^{\text{MOMgggg}} Z_A^{\overline{\text{MS}}}}{Z_\alpha^{\overline{\text{MS}}} Z_A^{\text{MOMgggg}}}, \quad (4.5) \end{aligned}$$

where $\phi \in \{A, c, \psi\}$ denote the gluon, ghost and quark fields respectively. While in our conventions Z_α will be unity in a linear covariant gauge we have included it here so as to be formally correct. There are (nonlinear

covariant) gauges where the corresponding gauge parameter renormalization constant is not unity. It is important to realize that in these formal definitions the perturbative expansion is in powers of the coupling constant defined in one scheme. (Our convention is that that scheme is $\overline{\text{MS}}$.) Otherwise the conversion functions would have poles in ϵ . Therefore, we have to relate the MOMgggg

definition of the coupling constant to the $\overline{\text{MS}}$ one order by order. This is achieved by

$$a_{\text{MOMgggg}} = \frac{a_{\overline{\text{MS}}}}{(C_g^{\text{MOMgggg}}(a, \alpha))^2}. \quad (4.6)$$

For the MOMgggg scheme the explicit form of (4.1) gives

$$\begin{aligned} a_{\text{MOMgggg}} = a + & \left[2592 \ln\left(\frac{4}{3}\right) \alpha^3 N_c - 145728 \ln\left(\frac{4}{3}\right) \alpha^2 N_c - 38880 \ln\left(\frac{4}{3}\right) \alpha N_c + 662016 \ln\left(\frac{4}{3}\right) N_c - 134400 \ln\left(\frac{4}{3}\right) N_f \right. \\ & - 3159 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^3 N_c + 24381 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^2 N_c - 40365 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha N_c \\ & - 21657 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) N_c + 4800 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) N_f \\ & + 16200 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha^2 N_c - 18000 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha N_c - 51000 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) N_c - 33600 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) N_f \\ & \left. + 10080 \alpha^3 N_c - 76320 \alpha^2 N_c + 122400 \alpha N_c + 852640 N_c - 179200 N_f \right] \frac{a^2}{115200} + O(a^3), \end{aligned} \quad (4.7)$$

where the one loop correction coefficient is in effect $\lambda^{\text{MOMgggg}}(\alpha, N_f)$. In addition, from the one loop wave function renormalization constants [16] we have

$$\begin{aligned} C_A^{\text{MOMgggg}}(a, \alpha) &= 1 + [9\alpha^2 N_c + 18\alpha N_c + 97N_c - 40N_f] \frac{a}{36} + O(a^2), \\ C_c^{\text{MOMgggg}}(a, \alpha) &= 1 + N_c a + O(a^2), \\ C_\psi^{\text{MOMgggg}}(a, \alpha) &= 1 + \alpha[-N_c^2 + 1] \frac{a}{2N_c} + O(a^2). \end{aligned} \quad (4.8)$$

Once these are known explicitly the two loop MOMgggg corrections to the respective renormalization group functions can be computed from the formal relations

$$\beta^{\text{MOMgggg}}(a_{\text{MOMgggg}}, \alpha_{\text{MOMgggg}}) = \left[\beta^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}) \frac{\partial a_{\text{MOMgggg}}}{\partial a_{\overline{\text{MS}}}} + \alpha_{\overline{\text{MS}}} \gamma_\alpha^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \frac{\partial a_{\text{MOMgggg}}}{\partial \alpha_{\overline{\text{MS}}}} \right]_{\overline{\text{MS}} \rightarrow \text{MOMgggg}} \quad (4.9)$$

and

$$\begin{aligned} \gamma_\phi^{\text{MOMgggg}}(a_{\text{MOMgggg}}, \alpha_{\text{MOMgggg}}) &= \left[\beta^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}) \frac{\partial}{\partial a_{\overline{\text{MS}}}} \ln C_\phi^{\text{MOMgggg}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \right. \\ & \left. + \alpha_{\overline{\text{MS}}} \gamma_\alpha^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \frac{\partial}{\partial \alpha_{\overline{\text{MS}}}} \ln C_\phi^{\text{MOMgggg}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) + \gamma_\phi^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}) \right]_{\overline{\text{MS}} \rightarrow \text{MOMgggg}} \end{aligned} \quad (4.10)$$

which are constructed from the renormalization group equation itself. In (4.9) and (4.10) we have labeled the variables with the scheme they are defined in. In addition to having a coupling constant defined in a scheme the gauge parameter is also defined with respect to a scheme. Though, from the construction the Landau gauge is preserved between schemes. Also the renormalization

group functions are labeled with a scheme. However, the mapping, $\overline{\text{MS}} \rightarrow \text{MOMgggg}$, in (4.9) and (4.10) indicates that after the left-hand side has been determined in $\overline{\text{MS}}$ variables then they are mapped to their MOMgggg counterparts as indicated by the arguments on the right-hand side. From these expressions one can see that to construct the two loop

corrections only the one loop conversion functions are required since each term involving such a function is multiplied by a function which is $O(a)$. Further, as the $\overline{\text{MS}}$ two loop renormalization group functions are available

[50–55], then we have all the required information to extract the *two* loop renormalization group functions.

Therefore, using this formalism and the explicit values for the conversion functions we have

$$\begin{aligned}
\gamma_A^{\text{MOMgggg}}(a, \alpha) = & [3\alpha N_c - 13N_c + 4N_f] \frac{a}{6} \\
& + \left[-7776 \ln\left(\frac{4}{3}\right) \alpha^4 N_c^3 + 470880 \ln\left(\frac{4}{3}\right) \alpha^3 N_c^2 - 10368 \ln\left(\frac{4}{3}\right) \alpha^3 N_c^2 N_f - 1777824 \ln\left(\frac{4}{3}\right) \alpha^2 N_c^3 \right. \\
& + 582912 \ln\left(\frac{4}{3}\right) \alpha^2 N_c^2 N_f - 2491488 \ln\left(\frac{4}{3}\right) \alpha N_c^3 + 558720 \ln\left(\frac{4}{3}\right) \alpha N_c^2 N_f + 8606208 \ln\left(\frac{4}{3}\right) N_c^3 \\
& - 4395264 \ln\left(\frac{4}{3}\right) N_c^2 N_f + 537600 \ln\left(\frac{4}{3}\right) N_c N_f^2 + 9477 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^4 N_c^3 - 114210 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^3 N_c^3 \\
& + 12636 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^3 N_c^2 N_f + 438048 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^2 N_c^3 - 97524 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^2 N_c^2 N_f \\
& - 459774 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha N_c^3 + 147060 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha N_c^2 N_f - 281541 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) N_c^3 \\
& + 149028 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) N_c^2 N_f - 19200 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) N_c N_f^2 - 48600 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha^3 N_c^3 \\
& + 264600 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha^2 N_c^3 - 64800 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha^2 N_c^2 N_f - 81000 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha N_c^3 \\
& + 172800 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha N_c^2 N_f - 663000 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) N_c^3 - 232800 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) N_c^2 N_f + 134400 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) N_c N_f^2 \\
& - 30240 \alpha^4 N_c^3 + 273600 \alpha^3 N_c^3 - 40320 \alpha^3 N_c^2 N_f - 1071360 \alpha^2 N_c^3 + 190080 \alpha^2 N_c^2 N_f + 396480 \alpha N_c^3 \\
& \left. - 336000 \alpha N_c^2 N_f - 842080 N_c^3 + 736640 N_c^2 N_f + 204800 N_c N_f^2 - 691200 N_f \right] \frac{a^2}{691200 N_c} + O(a^3),
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
\gamma_c^{\text{MOMgggg}}(a, \alpha) = & [\alpha - 3] \frac{N_c a}{4} \\
& + \left[-2592 \ln\left(\frac{4}{3}\right) \alpha^4 N_c + 153504 \ln\left(\frac{4}{3}\right) \alpha^3 N_c - 398304 \ln\left(\frac{4}{3}\right) \alpha^2 N_c \right. \\
& - 778656 \ln\left(\frac{4}{3}\right) \alpha N_c + 134400 \ln\left(\frac{4}{3}\right) \alpha N_f + 1986048 \ln\left(\frac{4}{3}\right) N_c - 403200 \ln\left(\frac{4}{3}\right) N_f \\
& + 3159 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^4 N_c - 33858 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^3 N_c + 113508 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^2 N_c \\
& - 99438 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha N_c - 4800 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha N_f - 64971 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) N_c + 14400 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) N_f \\
& - 16200 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha^3 N_c + 66600 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha^2 N_c - 3000 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha N_c + 33600 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha N_f \\
& - 153000 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) N_c - 100800 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) N_f - 10080 \alpha^4 N_c + 135360 \alpha^3 N_c - 293760 \alpha^2 N_c \\
& \left. - 203840 \alpha N_c + 51200 \alpha N_f - 43680 N_c - 38400 N_f \right] \frac{N_c a^2}{460800} + O(a^3),
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
\gamma_{\psi}^{\text{MOMgggg}}(a, \alpha) = & [N_c^2 - 1] \frac{\alpha a}{2N_c} \\
& + \left[-2592 \ln\left(\frac{4}{3}\right) \alpha^4 N_c^4 + 2592 \ln\left(\frac{4}{3}\right) \alpha^4 N_c^2 + 145728 \ln\left(\frac{4}{3}\right) \alpha^3 N_c^4 \right. \\
& - 145728 \ln\left(\frac{4}{3}\right) \alpha^3 N_c^2 + 38880 \ln\left(\frac{4}{3}\right) \alpha^2 N_c^4 - 38880 \ln\left(\frac{4}{3}\right) \alpha^2 N_c^2 - 662016 \ln\left(\frac{4}{3}\right) \alpha N_c^4 \\
& + 134400 \ln\left(\frac{4}{3}\right) \alpha N_c^3 N_f + 662016 \ln\left(\frac{4}{3}\right) \alpha N_c^2 - 134400 \ln\left(\frac{4}{3}\right) \alpha N_c N_f + 3159 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^4 N_c^4 \\
& - 3159 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^4 N_c^2 - 24381 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^3 N_c^4 + 24381 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^3 N_c^2 \\
& + 40365 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^2 N_c^4 - 40365 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^2 N_c^2 + 21657 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha N_c^4 \\
& - 4800 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha N_c^3 N_f - 21657 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha N_c^2 + 4800 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha N_c N_f \\
& - 16200 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha^3 N_c^4 + 16200 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha^3 N_c^2 + 18000 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha^2 N_c^4 - 18000 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha^2 N_c^2 \\
& + 51000 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha N_c^4 + 33600 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha N_c^3 N_f - 51000 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha N_c^2 - 33600 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha N_c N_f \\
& - 10080 \alpha^4 N_c^4 + 10080 \alpha^4 N_c^2 + 105120 \alpha^3 N_c^4 - 105120 \alpha^3 N_c^2 + 21600 \alpha^2 N_c^4 - 21600 \alpha^2 N_c^2 \\
& - 139040 \alpha N_c^4 + 51200 \alpha N_c^3 N_f + 139040 \alpha N_c^2 - 51200 \alpha N_c N_f + 633600 N_c^4 - 115200 N_c^3 N_f \\
& \left. - 547200 N_c^2 + 115200 N_c N_f - 86400 \right] \frac{a^2}{230400 N_c^2} + O(a^3), \tag{4.13}
\end{aligned}$$

and

$$\begin{aligned}
\beta^{\text{MOMgggg}}(a, \alpha) = & [-11N_c + 2N_f] \frac{a^2}{3} \\
& + \left[-2592 \ln\left(\frac{4}{3}\right) \alpha^4 N_c^3 + 108384 \ln\left(\frac{4}{3}\right) \alpha^3 N_c^3 - 3456 \ln\left(\frac{4}{3}\right) \alpha^3 N_c^2 N_f \right. \\
& - 408032 \ln\left(\frac{4}{3}\right) \alpha^2 N_c^3 + 129536 \ln\left(\frac{4}{3}\right) \alpha^2 N_c^2 N_f - 56160 \ln\left(\frac{4}{3}\right) \alpha N_c^3 + 17280 \ln\left(\frac{4}{3}\right) \alpha N_c^2 N_f \\
& + 3159 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^4 N_c^3 - 29943 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^3 N_c^3 + 4212 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^3 N_c^2 N_f \\
& + 83889 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^2 N_c^3 - 21672 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha^2 N_c^2 N_f - 58305 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha N_c^3 \\
& + 17940 \Phi_1\left(\frac{9}{16}, \frac{9}{16}\right) \alpha N_c^2 N_f - 10800 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha^3 N_c^3 + 52800 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha^2 N_c^3 \\
& - 14400 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha^2 N_c^2 N_f - 26000 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha N_c^3 + 8000 \Phi_1\left(\frac{3}{4}, \frac{3}{4}\right) \alpha N_c^2 N_f \\
& - 10080 \alpha^4 N_c^3 + 94560 \alpha^3 N_c^3 - 13440 \alpha^3 N_c^2 N_f - 261280 \alpha^2 N_c^3 + 67840 \alpha^2 N_c^2 N_f + 176800 \alpha N_c^3 \\
& \left. - 54400 \alpha N_c^2 N_f - 870400 N_c^3 + 332800 N_c^2 N_f - 76800 N_f \right] \frac{a^3}{76800 N_c} + O(a^4). \tag{4.14}
\end{aligned}$$

For more practical purposes the numerical values are beneficial and we have

$$\begin{aligned}
\beta^{\text{MOMgggg}}(a, \alpha)|_{SU(3)} &= [0.666667N_f - 11.000000]a^2 \\
&\quad + [-0.008632\alpha^4 - 0.003836\alpha^3N_f - 0.792230\alpha^3 - 0.368726\alpha^2N_f + 6.608703\alpha^2 + 1.339388\alpha N_f \\
&\quad - 13.059036\alpha + 12.666667N_f - 102.000000]a^3 + O(a^4), \\
\gamma_A^{\text{MOMgggg}}(a, \alpha)|_{SU(3)} &= [1.500000\alpha + 0.666667N_f - 6.500000]a \\
&\quad + [-0.002877\alpha^4 - 0.001279\alpha^3N_f - 1.527349\alpha^3 - 0.684363\alpha^2N_f + 8.561163\alpha^2 + 3.535793\alpha N_f \\
&\quad - 27.533402\alpha + 0.976180N_f^2 - 3.284136N_f - 15.652749]a^2 + O(a^3), \\
\gamma_c^{\text{MOMgggg}}(a, \alpha)|_{SU(3)} &= [0.750000\alpha - 2.250000]a \\
&\quad + [-0.001439\alpha^4 + 0.359407\alpha^3 + 3.254037\alpha^2 + 1.098202\alpha N_f - 15.132618\alpha - 2.544606N_f \\
&\quad - 2.475952]a^2 + O(a^3), \\
\gamma_\psi^{\text{MOMgggg}}(a, \alpha)|_{SU(3)} &= 1.333333\alpha a \\
&\quad + [-0.002557\alpha^4 + 0.631274\alpha^3 + 7.678777\alpha^2 + 1.952359\alpha N_f - 3.866103\alpha - 1.333333N_f \\
&\quad + 22.333333]a^2 + O(a^3)
\end{aligned} \tag{4.15}$$

for the $SU(3)$ color group. One can see, for instance, that in the Landau gauge the usual two loop QCD β function of [52,53] emerges for the MOMgggg scheme.

V. DISCUSSION

We have completed the full symmetric point evaluation of the quartic vertex in QCD by providing the exact decomposition of the vertex into the full tensor basis. This extends the earlier work of [18] which was motivated by a different interest. Broadly we have agreement with [18] where there is overlap. One consequence of the determination of the full vertex structure is that we are able to define a new momentum subtraction scheme in the same class as those proposed in [16]. Unlike the 3-point vertices which the schemes of [16] were based on, one has first to be careful in organizing the color group tensors. In other words, one has to write the vertex function in terms of the structure of the original Feynman and a set of completely symmetric color tensors. The latter are natural for the fully symmetric momentum configuration. Once the preferred basis has been determined the definition of the MOMgggg scheme emerges naturally. Although we have carried out a one loop renormalization properties of the renormalization group equation have allowed us to construct all the *two* loop

renormalization group functions ahead of an explicit two loop computation. Such an extension would require a sizable calculation. Although there are a significantly large number of graphs, some [41] but not all the basic two loop box master integrals are known analytically at the fully symmetric point. This is not an insurmountable obstacle as a numerical evaluation can suffice in the interim much as (3.7) and (3.8) did for the one loop case. One reason why such a two loop computation would be of interest, albeit at one specific momentum point, is that it would give an estimate of the extent that the two loop corrections are significant for, say, Schwinger-Dyson comparisons. Though, a more general computation of the fully off-shell quartic vertex would support future Schwinger-Dyson analyses beyond that carried out, for example, in [28].

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APPENDIX: TENSOR BASIS

In this Appendix we give the complete set of basis tensors for the quartic vertex. We have

$$\begin{aligned}
\mathcal{P}_{(1)}^{\mu\nu\sigma\rho}(p, q, r) &= \eta^{\mu\nu}\eta^{\sigma\rho}, & \mathcal{P}_{(2)}^{\mu\nu\sigma\rho}(p, q, r) &= \eta^{\mu\sigma}\eta^{\nu\rho}, & \mathcal{P}_{(3)}^{\mu\nu\sigma\rho}(p, q, r) &= \eta^{\mu\rho}\eta^{\nu\sigma}, & \mathcal{P}_{(4)}^{\mu\nu\sigma\rho}(p, q, r) &= \eta^{\mu\nu}\frac{p^\sigma p^\rho}{\mu^2}, \\
\mathcal{P}_{(5)}^{\mu\nu\sigma\rho}(p, q, r) &= \eta^{\mu\nu}\frac{p^\sigma q^\rho}{\mu^2}, & \mathcal{P}_{(6)}^{\mu\nu\sigma\rho}(p, q, r) &= \eta^{\mu\nu}\frac{p^\sigma r^\rho}{\mu^2}, & \mathcal{P}_{(7)}^{\mu\nu\sigma\rho}(p, q, r) &= \eta^{\mu\nu}\frac{q^\sigma p^\rho}{\mu^2}, & \mathcal{P}_{(8)}^{\mu\nu\sigma\rho}(p, q, r) &= \eta^{\mu\nu}\frac{q^\sigma q^\rho}{\mu^2}, \\
\mathcal{P}_{(9)}^{\mu\nu\sigma\rho}(p, q, r) &= \eta^{\mu\nu}\frac{q^\sigma r^\rho}{\mu^2}, & \mathcal{P}_{(10)}^{\mu\nu\sigma\rho}(p, q, r) &= \eta^{\mu\nu}\frac{r^\sigma p^\rho}{\mu^2}, & \mathcal{P}_{(11)}^{\mu\nu\sigma\rho}(p, q, r) &= \eta^{\mu\nu}\frac{r^\sigma q^\rho}{\mu^2}, & \mathcal{P}_{(12)}^{\mu\nu\sigma\rho}(p, q, r) &= \eta^{\mu\nu}\frac{r^\sigma r^\rho}{\mu^2}, \\
\mathcal{P}_{(13)}^{\mu\nu\sigma\rho}(p, q, r) &= \eta^{\mu\sigma}\frac{p^\nu p^\rho}{\mu^2}, & \mathcal{P}_{(14)}^{\mu\nu\sigma\rho}(p, q, r) &= \eta^{\mu\sigma}\frac{p^\nu q^\rho}{\mu^2}, & \mathcal{P}_{(15)}^{\mu\nu\sigma\rho}(p, q, r) &= \eta^{\mu\sigma}\frac{p^\nu r^\rho}{\mu^2}, & \mathcal{P}_{(16)}^{\mu\nu\sigma\rho}(p, q, r) &= \eta^{\mu\sigma}\frac{q^\nu p^\rho}{\mu^2},
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{(101)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{r^\mu q^\nu r^\sigma q^\rho}{\mu^4}, & \mathcal{P}_{(102)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{r^\mu r^\nu q^\sigma q^\rho}{\mu^4}, & \mathcal{P}_{(103)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{p^\mu p^\nu q^\sigma r^\rho}{\mu^4}, & \mathcal{P}_{(104)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{p^\mu q^\nu p^\sigma r^\rho}{\mu^4}, \\
\mathcal{P}_{(105)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{q^\mu p^\nu p^\sigma r^\rho}{\mu^4}, & \mathcal{P}_{(106)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{p^\mu q^\nu r^\sigma p^\rho}{\mu^4}, & \mathcal{P}_{(107)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{q^\mu p^\nu r^\sigma p^\rho}{\mu^4}, & \mathcal{P}_{(108)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{q^\mu r^\nu p^\sigma p^\rho}{\mu^4}, \\
\mathcal{P}_{(109)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{p^\mu p^\nu r^\sigma q^\rho}{\mu^4}, & \mathcal{P}_{(110)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{p^\mu r^\nu p^\sigma q^\rho}{\mu^4}, & \mathcal{P}_{(111)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{r^\mu p^\nu p^\sigma q^\rho}{\mu^4}, & \mathcal{P}_{(112)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{p^\mu r^\nu q^\sigma p^\rho}{\mu^4}, \\
\mathcal{P}_{(113)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{r^\mu p^\nu q^\sigma p^\rho}{\mu^4}, & \mathcal{P}_{(114)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{r^\mu q^\nu p^\sigma p^\rho}{\mu^4}, & \mathcal{P}_{(115)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{q^\mu q^\nu r^\sigma p^\rho}{\mu^4}, & \mathcal{P}_{(116)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{q^\mu r^\nu q^\sigma p^\rho}{\mu^4}, \\
\mathcal{P}_{(117)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{r^\mu q^\nu q^\sigma p^\rho}{\mu^4}, & \mathcal{P}_{(118)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{q^\mu r^\nu p^\sigma q^\rho}{\mu^4}, & \mathcal{P}_{(119)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{r^\mu q^\nu p^\sigma q^\rho}{\mu^4}, & \mathcal{P}_{(120)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{r^\mu p^\nu q^\sigma q^\rho}{\mu^4}, \\
\mathcal{P}_{(121)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{q^\mu q^\nu p^\sigma r^\rho}{\mu^4}, & \mathcal{P}_{(122)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{q^\mu p^\nu q^\sigma r^\rho}{\mu^4}, & \mathcal{P}_{(123)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{p^\mu q^\nu q^\sigma r^\rho}{\mu^4}, & \mathcal{P}_{(124)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{q^\mu p^\nu r^\sigma q^\rho}{\mu^4}, \\
\mathcal{P}_{(125)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{p^\mu q^\nu r^\sigma q^\rho}{\mu^4}, & \mathcal{P}_{(126)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{p^\mu r^\nu q^\sigma q^\rho}{\mu^4}, & \mathcal{P}_{(127)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{r^\mu r^\nu p^\sigma q^\rho}{\mu^4}, & \mathcal{P}_{(128)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{r^\mu p^\nu r^\sigma q^\rho}{\mu^4}, \\
\mathcal{P}_{(129)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{p^\mu r^\nu q^\sigma q^\rho}{\mu^4}, & \mathcal{P}_{(130)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{r^\mu p^\nu q^\sigma r^\rho}{\mu^4}, & \mathcal{P}_{(131)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{p^\mu r^\nu q^\sigma r^\rho}{\mu^4}, & \mathcal{P}_{(132)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{p^\mu q^\nu r^\sigma r^\rho}{\mu^4}, \\
\mathcal{P}_{(133)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{r^\mu r^\nu q^\sigma p^\rho}{\mu^4}, & \mathcal{P}_{(134)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{r^\mu q^\nu r^\sigma p^\rho}{\mu^4}, & \mathcal{P}_{(135)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{q^\mu r^\nu r^\sigma p^\rho}{\mu^4}, & \mathcal{P}_{(136)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{r^\mu q^\nu p^\sigma r^\rho}{\mu^4}, \\
\mathcal{P}_{(137)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{q^\mu r^\nu p^\sigma r^\rho}{\mu^4}, & \mathcal{P}_{(138)}^{\mu\nu\sigma\rho}(p, q, r) &= \frac{q^\mu p^\nu r^\sigma r^\rho}{\mu^4}. & & & &
\end{aligned} \tag{A1}$$

In previous similar and related work we have at this point given the corresponding projection matrix, \mathcal{M}_{kl}^i , which allows one to determine each channel of the Green's function. However, due to the size of this matrix we have relegated the explicit matrix elements to the data file.

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