

**Quantum corrections to vortex masses and energies**Yago Ferreirós<sup>1,\*</sup> and Antonio González-Arroyo<sup>2,3,†</sup><sup>1</sup>*Instituto de Ciencia de Materiales de Madrid, CSIC, Sor Juana Inés de la Cruz 3, Cantoblanco, 28049 Madrid, Spain*<sup>2</sup>*Instituto de Física Teórica UAM/CSIC, Nicolás Cabrera 13-15, UAM, Cantoblanco, 28049 Madrid, Spain*<sup>3</sup>*Departamento de Física Teórica, C-15, Universidad Autónoma de Madrid, E-28049 Madrid, Spain*

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We study the  $2 + 1$  dimensional Abelian Higgs model defined on a spatial torus at critical self-coupling. We propose a method to compute the quantum contribution to the mass of the Abrikosov-Nielsen-Olesen vortex and to multivortex energies. The one-loop quantum correction to multivortex energies is computed analytically at the critical value of the torus area (Bradlow limit). For other values of the area one can set up an expansion around this critical area (Bradlow parameter expansion). The method is explained and the next-to-leading term explicitly evaluated. To this order, the resulting energies depend on the torus periods, but not on the vortex positions.

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**I. INTRODUCTION**

Abrikosov-Nielsen-Olesen (ANO) vortices are stringlike objects which appear as classical solutions in spontaneously broken Abelian gauge theories. They are important structures occurring in ordinary superconductors [1] and corresponding solutions of the relativistic Abelian Higgs model [2]. Their stability has a topological origin and this has triggered many theoretical and mathematical works [3–5]. It is simpler to view these vortices as solutions in  $2 + 1$  dimensions. The total magnetic flux through space is quantized and can be referred to as *vortex number*. The dimensionality of the space of solutions is twice the vortex number and can be interpreted as given by the two-dimensional positions of a set of minimum-flux vortices. A particularly attractive situation occurs at a critical value of the Higgs self-coupling, at which the gauge and Higgs field masses coincide. The minimum energy equations reduce then to the first order Bogomol’nyi equations [6]. At the classical level the vortex mass is given simply by the minimum energy of the system with one unit of flux. It turns out that the minimum energy grows linearly with the vortex number. This can be described by saying that the interaction energy of vortices vanishes irrespective of their relative positions. This is quite remarkable given the nonlinear character of the field equations.

Although, many properties of the vortex solutions are known exactly, there is no analytic expression for these solutions. The single vortex case is easy to describe numerically since the solutions are rotational invariant [7]. Thus, it can be expressed in terms of functions of a

single variable (the distance to its center). Much more difficult is to obtain multivortex solutions numerically with *a priori* given vortex centers [8,9]. In a previous paper [10] one of the present authors and Alberto Ramos derived a method to obtain analytic control of the solutions. This follows by considering the Abelian Higgs model on a 2-torus and expanding around a particular value of the area, for which the solution is known analytically. Considering enough terms in the expansion one can obtain good approximations to the solution for large torus sizes and even extrapolate to infinite size, where the solution tends to that of the plane. What is more interesting is that the expansion can be developed for multivortex solutions as well, and for any location of the vortex centers. Having analytic control allows many possible applications involving vortices. One such case is to study vortex scattering, within the geodesic approximation [11], which was done in Ref. [12]. This leads to computations which, even for the plane, are at least as precise as those obtained by other numerical techniques.

In this paper we exemplify this idea even further by studying quantum corrections to vortex masses and interaction energies. Given the extension of the present work we will just explain the methodology and compute the leading and next-to-leading terms in the expansion (known as Bradlow parameter expansion). This fails too short for a reasonable extrapolation to vortices on the plane. Extension to higher orders can be done along the same guidelines, with a straightforward but technically demanding effort.

The quantum energies of topological objects to one-loop order receive two types of contributions. First, one has the Casimir energies, which follow by computing the difference of ground state energies between the topological nontrivial and trivial sector. This subtraction should get

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rid of the most ultraviolet divergent contributions, since topology is a global constraint. In addition, there are corrections to the classical energy due to the renormalization of the Lagrangian parameters. Both contributions are of order  $\hbar$ .

Although the final result should be finite, at intermediate steps one will be manipulating divergent quantities. In this work, we have made use of the zeta-function regularization technique. This method is commonly used in the literature of quantum corrections to topological defects [13–18]. In particular, it is interesting to mention the study of supersymmetric vortices in Refs. [14] and [19]. The situation here is much better than for ANO vortices because of the analytical control on the solutions (as for Bogomol'nyi-Prasad-Sommerfield monopoles [20]), but also because supersymmetry ensures cancellation of the contributions of the bosons and fermions to the vacuum energy. Nevertheless, one still has to deal with the contributions coming from finite renormalization.

For the purely bosonic case, the calculation of the quantum mass of self-dual vortices on the plane was addressed in Refs. [15–17], using a mixture of numerical and analytical techniques. The problem becomes easier to handle for circular invariant multivortices, including the single vortex case. Nevertheless, the situation for spatially separated vortices is important, as it answers the question of whether quantum effects produce an attraction or repulsion, absent at the classical level. From that respect our methodology is much more powerful, since one can fix the positions of the multiple vortices in any way and the analytical techniques apply equally well for all situations.

For the rest of this section, we will describe the layout of this paper. In Sec. II we particularize the Abelian Higgs model to the case of a spatial 2-torus with arbitrary constant metric tensor and any value of the vortex number  $q$ . The metric can be parametrized in terms of the total area of the torus  $\mathcal{A}$  and a complex parameter  $\tau$  with positive imaginary part. We perform several manipulations to simplify the study of the classical and quantum system. In particular, we recall that when the area attains a critical value  $\mathcal{A}_c$  the classical solutions become extremely simple. Furthermore, one can obtain analytical control on the classical solutions in an expansion on the parameter  $\epsilon = 1 - \mathcal{A}_c/\mathcal{A}$ . This is essentially the Bradlow parameter expansion proposed in Refs. [10,12]. Our presentation here is slightly different, and in our opinion more elegant, than the one used in those papers.

For the study of the quantum system we use quantization in the  $A_0 = 0$  gauge. This is the simplest and most appropriate for computing energies. In this gauge the physical Hilbert space is restricted to states that satisfy the Gauss constraint. Equivalently, physical states are those which are gauge invariant under the remaining time-independent gauge transformations. Gauge invariance implies that, when studying the spectrum of the quadratic fluctuations in the potential, gauge degrees of freedom are

zero modes. Thus, they have a vanishing contribution to the ground state quantum energy. Thus, in computing the vacuum energy in a given topology, it is not necessary to fix the spatial gauge and no ghosts have to be added.

The previous comments become clear in the derivation of the quantum energies for critical area, performed in Sec. III. The calculation is fairly simple, but we take advantage to present certain technical aspects necessary for the calculation at any order. In particular, the ideas explained earlier about the separation of gauge and nongauge degrees of freedom are easily checked. Finally, the Casimir energy calculation employing the zeta-function regularization shows the cancellation of the leading singularity, as expected. Indeed, this turns out to be the only singularity in the analytical continuation of the energies.

In the following section we explain the way in which the previous result can be extended to other values of the area using the Bradlow parameter expansion. As an example we perform all the steps to produce the next-to-leading order correction to the masses. Part of the result depends on the calculation of the spectrum of the quantum fluctuation operator, which uses standard perturbative methods of quantum mechanics. The calculation of the eigenvalues itself is presented in Appendix A. These results are then used in combination with the zeta-function technique to produce the Casimir energies to this order.

The contribution of the quantum correction induced by the renormalization of the parameters is performed in Appendix B. The results depend on the renormalization prescription. A prescription is adopted in which the renormalization of the parameters is based on the behavior of the theory in the trivial topological sector and for large areas. This makes sense, since typically one should not change the bare Lagrangian of the model when changing the area, the flat metric or the vortex number. Thus, we set up a renormalization prescription within our  $A_0 = 0$  context based on the behavior of the effective potential under space-time independent background fields. With the renormalization of parameters done in this way, we compute the counterterm contribution to the quantum energy which depends on the area and on the number of vortices, but not on the location of these vortices or the metric shape parameter  $\tau$ . With this result, all dependence of the quantum energies on these parameters should come from the Casimir energies themselves. This dependence is finite and emerges from our calculation. This and other aspects are analyzed in the concluding Sec. V.

## II. THE ABELIAN HIGGS MODEL ON THE TORUS

In this section we will present the basic details of the model that we are studying, and derive some of the formulas to be used later. We are considering the Abelian Higgs model living on a two-dimensional spatial torus with nonvanishing flux. The Lagrangian density of the model is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi)^*(D^\mu\phi) - \frac{e^2\lambda}{8}(|\phi|^2 - v^2)^2 \quad (1)$$

where  $\phi(x)$  is a complex scalar field, henceforth referred as Higgs field, and  $A_\mu(x)$  is the electromagnetic vector potential. In our notation the covariant derivative is given by  $D_\mu = \partial_\mu - ieA_\mu(x)$ .

Usually the model is considered in flat Minkowski space, but it is easily generalizable to compact spatial manifolds [21]. Here we will specialize to a two dimensional torus with flat metric. The torus is characterized by the two periods, given by two linearly independent vectors. It is possible to perform a linear change of coordinates and map the torus to a unit square with opposite sides identified. The points can then be parametrized by two real coordinates  $0 \leq x^i < 1$ . With this transformation the spatial Euclidean metric tensor is mapped onto a constant metric tensor  $g_{ij}$ . In these coordinates the area of the torus is  $\mathcal{A} = \det(g) \equiv |g|$ . As we will see, the properties of the model become simple at a particular value of this area. One can study other values of the area by a systematic expansion method introduced in Refs. [10,12], and referred as *Bradlow parameter expansion*. In the aforementioned references the expansion was carried to sufficiently high order so as to provide a good description of multivortex classical solutions on an infinite plane. Having analytical control enables many other possible calculations. Here we will show how the formalism allows us also to compute quantum corrections to the masses analytically. Our presentation differs slightly from the one used in Ref. [10]. The main difference is precisely the use of the square coordinates  $x^i$ . The Bradlow parameter expansion can then be viewed as an expansion around a particular value  $g^{(0)}$  of the constant Riemannian metric. This provides a more elegant formulation of the expansion.

Now let us introduce one complex vector  $w^i$  such that

$$(w^i)^*w^j = g^{ij} + i\epsilon_{ij}I \quad (2)$$

where we use the standard notation such that  $g^{ij}$  are the components of the inverse of  $g$ . The  $w^i$  vector is defined up to an overall phase. However, one can fix this freedom by imposing  $\Im(w^2) = 0$ ,  $\Re(w^2) > 0$  (the symbols  $\Im$  and  $\Re$  stand for the imaginary and real parts of a complex number). With this choice one easily finds that  $I = \pm \frac{1}{\sqrt{|g|}}$  where  $|g|$  is the determinant of the metric tensor. The optimal choice of sign is connected with the sign of the flux of the magnetic field through the torus. From now on, without loss of generality, we will take this flux to be positive and correspondingly  $I = \frac{1}{\sqrt{|g|}}$ .

The Higgs field is to be seen as a section of a U(1) associated bundle on the torus. The electromagnetic field is a connection on this bundle. As customarily done in the

physics literature, we will work with a trivialization of the bundle. The Higgs field  $\phi(x)$  can then be seen as an ordinary complex function of the coordinates satisfying peculiar boundary conditions:

$$\phi(x + e_{(i)}) = e^{i\theta_i(x)}\phi(x) \quad (3)$$

where  $e_{(i)}$  stands for the unit vector in the  $i$ th direction. The topology of the bundle is encoded in the transition functions  $e^{i\theta_i(x)}$ .

Now we construct the complex operator  $D$  as follows

$$D = w^1D_1 + w^2D_2 \quad (4)$$

where  $w^i$  are the components of the complex vector introduced before. Let us compute

$$\begin{aligned} D^\dagger D &= -(w^i)^*w^jD_iD_j \\ &= -g^{ij}D_iD_j - \epsilon_{ij}F_{ij}\frac{e}{2\sqrt{|g|}} \\ &= -D^iD_i - \frac{eB}{\sqrt{|g|}} \end{aligned} \quad (5)$$

where the magnetic field is  $F_{ij} = \epsilon_{ij}B$ . Similarly we arrive at

$$[D, D^\dagger] = \frac{2eB}{\sqrt{|g|}} \quad (6)$$

One can use the previous definitions and results to reexpress the potential energy of the model with this metric. It is given by

$$\begin{aligned} \prod_i \left( \int_0^1 dx^i \right) &\left[ \frac{\sqrt{|g|}}{2} |D\phi|^2 + \frac{eB}{2} |\phi|^2 + \frac{1}{2\sqrt{|g|}} B^2 \right. \\ &\left. + \frac{\sqrt{|g|}e^2\lambda}{8} (v^2 - |\phi|^2)^2 \right]. \end{aligned} \quad (7)$$

The integrand can be rewritten as

$$\begin{aligned} \frac{\sqrt{|g|}}{2} |D\phi|^2 &+ \frac{1}{2\sqrt{|g|}} \left( B + \frac{e\sqrt{|g|}}{2} (|\phi|^2 - v^2) \right)^2 \\ &+ \frac{eBv^2}{2} + \frac{\sqrt{|g|}e^2(\lambda - 1)}{8} (v^2 - |\phi|^2)^2. \end{aligned} \quad (8)$$

The integral of the third term is proportional to the flux of the magnetic field through the torus. The boundary conditions impose that this flux is quantized  $e \int B = 2\pi q$ , where  $q$  is an integer called first Chern number of the bundle (vortex number in the physics literature). Its value is determined by the choice of the transition functions  $e^{i\theta_i(x)}$ . Making use of a parity transformation we can always bring the flux and  $q$  to take positive values. A look at the

remaining terms of Eq. (8) shows that, at the critical value  $\lambda = 1$ , the potential energy attains its minimum for fields satisfying the Bogomol'nyi equations:

$$D\phi = 0 \quad (9)$$

$$B = \frac{e\sqrt{|g|}}{2}(v^2 - |\phi|^2). \quad (10)$$

From now on we will restrict ourselves to this critical case.

One way to encode the flux condition is to write  $B = \frac{2\pi q}{e} + \delta B$ , where the integral of  $\delta B$  over space vanishes. We will parametrize the metric as  $g_{ij} = \kappa g_{ij}^{(0)}$  and fix the normalization of the reference metric  $g^{(0)}$  to satisfy  $\sqrt{|g^{(0)}|} = \frac{4\pi q}{v^2 e^2} \equiv \mathcal{A}_c$ . Notice that the complex vector  $w_0^i$  associated to  $g^{(0)}$  is related to the previous one by  $w^i = \frac{1}{\sqrt{\kappa}} w_0^i$ . In a similar fashion we can use the new vector  $w_0^i$  to define  $D^{(0)}$ , connected to the previous one by  $D = \frac{1}{\sqrt{\kappa}} D^{(0)}$ . The advantage of our construction is that of keeping the dependence on the conformal factor  $\kappa$  explicit. With these new definitions the potential energy (at the critical coupling  $\lambda = 1$ ) takes the form:

$$\begin{aligned} \pi q v^2 + \prod_i \left( \int_0^1 dx^i \right) & \left[ \frac{2\pi q}{e^2 v^2} |D^{(0)}\phi|^2 \right. \\ & \left. + \frac{v^2 e^2}{8\pi q \kappa} \left( \delta B + \frac{2\pi q(1-\kappa)}{e} + \frac{2\pi q \kappa}{v^2 e} |\phi|^2 \right)^2 \right] \quad (11) \end{aligned}$$

The choice of scale of the reference metric  $g^{(0)}$  is such that, for  $\kappa = 1$ , this potential energy takes its minimal value (equal to  $\pi q v^2$ ) for  $\delta B = \phi = 0$ . In other words, for the critical area ( $\mathcal{A} = \mathcal{A}_c = \frac{4\pi q}{v^2 e^2}$ ) the solution of the Bogomol'nyi equations is very simple: vanishing Higgs field and constant magnetic field.

We continue to fix our field redefinitions by rewriting the vector potential as

$$A_i = A_i^{(0)} + \delta A_i \quad (12)$$

where  $A_i^{(0)}$  is a specific vector potential leading to a constant magnetic field  $\frac{2\pi q}{e}$ . Rather than working with  $\delta A_i(x)$  directly, we will be working with the complex field  $\delta A(x) \equiv w_0^1 \delta A_1(x) + w_0^2 \delta A_2(x)$  and its complex conjugate  $\delta A^*(x)$ . With this choice, the complex covariant derivative  $D^{(0)}$  can be written as

$$D^{(0)} = w_0^i (\partial_i - ieA_i^{(0)} - ie\delta A_i) \equiv \tilde{D} - ie\delta A \quad (13)$$

where the operator  $\tilde{D}$  satisfies

$$[\tilde{D}, \tilde{D}^\dagger] = e^2 v^2 \quad (14)$$

which is, up to a scale, the commutation relation of creation and annihilation operators. Finally, one can express  $\delta B$  in terms of  $\delta A$  as

$$\delta B = -\frac{4\pi q}{v^2 e^2} \Im(\tilde{\partial}\delta A^*) \quad (15)$$

with  $\tilde{\partial} = w_0^1 \partial_1 + w_0^2 \partial_2$ .

Before reexpressing the potential energy in terms of the two complex fields  $\delta A(x)$  and  $\phi$ , let us write down the kinetic term of the Hamiltonian. In the  $A_0 = 0$  gauge we have:

$$\prod_i \left( \int_0^1 dx^i \right) \frac{2\pi q}{e^2 v^2} [\kappa |\dot{\phi}|^2 + |\delta \dot{A}|^2]. \quad (16)$$

It is convenient to eliminate the explicit dependence on the initial metric by rescaling the fields in an obvious way:

$$\phi \rightarrow \frac{\phi}{|g|^{1/4}} \quad (17)$$

$$\delta A \rightarrow \frac{\delta A}{|g^{(0)}|^{1/4}}. \quad (18)$$

The kinetic term then takes the canonical form

$$\prod_i \left( \int_0^1 dx^i \right) \frac{1}{2} [|\dot{\phi}|^2 + |\delta \dot{A}|^2]. \quad (19)$$

To simplify notation we have preserved the symbols  $\phi(x)$  and  $\delta A$  in referring to the rescaled fields.

The final expression of the potential energy after our field redefinitions and massaging is

$$\begin{aligned} \pi q v^2 + \prod_i \left( \int_0^1 dx^i \right) & \frac{1}{2\kappa} \left[ |D^{(0)}\phi|^2 + \left( -\Im(\tilde{\partial}\delta A^*) \right. \right. \\ & \left. \left. - v\sqrt{\pi q}(\kappa - 1) + \frac{ve^2}{4\sqrt{\pi q}} |\phi|^2 \right)^2 \right] \quad (20) \end{aligned}$$

with  $D^{(0)} = \tilde{D} - i\frac{e^2 v}{2\sqrt{\pi q}} \delta A$ . It is interesting to spend a few lines in explaining the dimensionality and dependencies of the previous expression. Obviously  $v^2$  has dimensions of energy and provides its natural unit. Neither  $q$ ,  $\kappa$  or the coordinates  $x^i$  have dimensions. By our choice of coordinates, the dimensions of length square are transferred to the metric tensor. Hence, both  $\tilde{D}$  and  $\tilde{\partial}$  have dimensions of inverse length, whose natural unit is  $ev$ . Thus, the fluctuation operator and its eigenvalues will be measured in  $e^2 v^2$  units. This implies that the quantum contribution to the energies will become proportional to  $ev$ . This can be easily understood if we realize that  $ev\hbar c$  has dimensions of energy. From now on, however, we will continue to work in

natural units  $\hbar = c = 1$ . Finally, the rescaled background fields  $\phi$  and  $\delta A$  will appear naturally proportional to  $1/e$ .

For later purposes it is convenient to write down an explicit parametrization of the critical area metric:

$$g^{(0)} = \mathcal{A}_c \bar{g}(\tau) \quad (21)$$

where  $\mathcal{A}_c = \frac{4\pi q}{v^2 e^2}$  is the critical area and  $\bar{g}(\tau)$  is a conformally equivalent metric of unit determinant. The metric  $\bar{g}(\tau)$  is dimensionless and can be parametrized in terms of a complex number  $\tau$  as follows:

$$\bar{g}(\tau) = \frac{1}{\Im(\tau)} \begin{pmatrix} 1 & \Re(\tau) \\ \Re(\tau) & |\tau|^2 \end{pmatrix}. \quad (22)$$

The two periods that define the torus can be changed into an equivalent set (generating the same lattice) by a change of coordinates belonging to  $SL(2, \mathbf{Z})$ . This generates a class of equivalent metric tensors  $\bar{g}$ , which amount to transforming  $\tau$  by an element of the modular group. Our results are then expected to be modular invariant. We recall that the modular group is generated by two transformations  $T: \tau \rightarrow \tau + 1$  and  $S: \tau \rightarrow -1/\tau$ . This allows us to restrict  $\tau$  to a fundamental domain, which can be chosen to be given by  $|\tau| \geq 1$  and  $|\Im(\tau)| \leq \frac{1}{2}$ .

Having presented the model and the basic fields, we will now briefly describe how the vortex masses and energies are defined up to one loop order. In each topological sector (characterized by a value of  $q$ ), we compute the minimum energy of the system. This has a classical contribution corresponding to the potential energy of the minimum energy solutions. In our case, the latter are the solutions of the Bogomol'nyi equations, and the corresponding energy is  $\pi q v^2$ . In addition, the ground-state energy receives quantum corrections. At one-loop this correction follows by expanding the Hamiltonian around classical solutions, and keeping only quadratic terms in the fluctuations. The quadratic form is given in terms of an operator  $\hat{V}$ , whose diagonalization defines the normal modes. Quantizing the system, we get a system of decoupled harmonic oscillators whose gap energies are given by the square root of the eigenvalues of the aforementioned operator. The ground-state energy is the corresponding for the system of oscillators, namely one half of the sum of the energies for each oscillator. As is well known, this sum is divergent and some regularization method is needed to handle the result.

The vortex Casimir mass  $\mathcal{E}_C$  is defined as the difference of minimum energies (classical + quantum) between the  $q = 1$  and  $q = 0$  sectors, for the same metric. Multivortex Casimir energies are equally defined by subtracting the energy of the zero-flux sector from that with  $q > 1$ . *A priori* this can depend not only on the metric parameter  $\tau$  but also on the location of the vortices. However, it is expected that the leading ultraviolet divergence is independent of these

positions and on the value of  $q$ . Hence, the subtraction of the regularized quantities might be convergent, or at least less divergent than the individual vacuum energies. This methodology is the standard one in computing finite Casimir energies.

To compute the vortex mass one must add an extra contribution, which is also of order  $\hbar$ . This comes from quantum corrections to the classical energies, which are due to the renormalization of the Lagrangian parameters. We will refer to this extra contribution as counterterm mass  $\mathcal{E}_R$ .

The difficulty in carrying out the procedure described above to compute the vortex mass and multivortex energies is that, in general, the solutions of the Bogomol'nyi equations are not known in closed analytic form. This forces the program to be performed numerically, as explained in the Introduction. On the contrary, our method allows an analytic treatment based on expanding the result around particular values of the metric associated to a critical value for the area of the torus. For that value, as mentioned earlier, the solutions of the Bogomol'nyi equations are known and very simple, allowing the whole program to be carried over to completion using analytical techniques. This will be done in the next section. For other values of the area of the torus, one can set up an expansion around the critical metric and compute all the terms in the expansion in a systematic way. The method is explained in Sec. IV, and the leading order correction evaluated explicitly as an example.

### III. VORTEX MASS FOR THE CRITICAL AREA CASE

To exemplify the whole procedure, let us consider here the case of critical area  $\kappa = 1$ . As mentioned previously, the classical solution for  $q \neq 0$  in this case is extremely simple:  $\phi = \delta A = 0$ . Thus, the quadratic piece in the expansion of the potential is given by

$$\prod_i \left( \int_0^1 dx_i \right) \frac{1}{2} [|\tilde{D}\phi|^2 + (\Im(\tilde{\partial}\delta A^*))^2], \quad (23)$$

where  $\Im$  stands for imaginary part. In the sector of Higgs fluctuations, the operator to diagonalize is  $\tilde{D}^\dagger \tilde{D}$ . Comparison with the creation-annihilation operator algebra shows that the eigenvalues of this operator are given by  $e^2 v^2 n$  for all non-negative integers  $n$ . In the gauge potential sector the operator to diagonalize is just obtainable in terms of  $\tilde{\partial}$ , which can be diagonalized with plane waves  $e^{ip_i x^i}$ . The eigenvalues are given by  $|w_0^i p_i|^2 = \|\vec{p}\|_0^2 = (g^{(0)})^{ij} p_i p_j$ .

Our previous discussion has been extremely naive. We have skipped several relevant technicalities: boundary conditions, zero modes and gauge invariance. In the following subsections we will consider them in turn. In so doing we will develop the necessary machinery to deal with the computation at any value of  $\kappa$ .

### A. Basis of field space

The question of boundary conditions is very relevant. The fields and their fluctuations satisfy homogeneous boundary conditions. For the Higgs field  $\phi(x)$ , they are given in Eq. (3). The space of fields satisfying these boundary conditions defines a pre-Hilbert space  $\mathcal{H}_q$ . We can follow a standard quantum mechanical formulation to study this space and the operators acting on it. This formalism was developed extensively in the appendix of Ref. [10], and here we will only review the necessary results. The reader is addressed to that reference for a detailed description.

One of the main results is that the space  $\mathcal{H}_q$  decomposes naturally into  $q$  orthogonal subspaces:

$$\mathcal{H}_q = \bigoplus_{s=0}^{q-1} \mathcal{H}_{q,s} \quad (24)$$

The decomposition is associated to a symmetry group. This group is a discrete subgroup of the translation group (combined with gauge transformations). The operators  $\tilde{D}^\dagger$  and  $\tilde{D}$  act on each of these subspaces without changing the value of  $s$ . As mentioned previously, these operators satisfy the same algebra as creation-annihilation operators up to a multiplicative factor. Hence, we can introduce a basis of  $\mathcal{H}_{q,s}$  using eigenstates of the number operator  $\tilde{D}^\dagger \tilde{D} / (e^2 v^2)$ . Incidentally, this basis is the same one that diagonalizes the fluctuation Lagrangian for the critical area Eq. (23). The spectrum is given by  $e^2 v^2 n$ , for any non-negative integer  $n$ . Thus, the result is the one anticipated previously, but now we know that each eigenvalue is  $q$ -fold degenerate, corresponding to the different values of  $s$ .

In what follows, we will not need the explicit form of the basis states  $\Psi_{n,s}(x)$ , which satisfy the standard orthogonality conditions

$$\prod_i \left( \int_0^1 dx^i \right) \Psi_{n,s}^*(x) \Psi_{n',s'}(x) = \delta_{n,n'} \delta_{s,s'}. \quad (25)$$

Furthermore, the action of the operator  $\tilde{D}$  on these states can be read out trivially from the harmonic oscillator formulas

$$\tilde{D} \Psi_{n,s}(x) = ev \sqrt{n} \Psi_{n-1,s}(x) \quad (26)$$

$$\tilde{D}^\dagger \Psi_{n,s}(x) = ev \sqrt{n+1} \Psi_{n+1,s}(x) \quad (27)$$

where  $n \in \mathbf{Z}^+ \cup \{0\}$  and  $s = 0, \dots, q-1$ .

Now we proceed to study the space of  $\delta A$  fields. From the previous considerations one concludes that they are periodic on the two-torus with period 1. To diagonalize the fluctuation Hamiltonian at critical area, one can indeed choose a basis of plane waves  $e^{i\vec{p}\cdot\vec{x}}$ . However, the boundary conditions impose that the momentum is given by  $\vec{p} = 2\pi\vec{k}$ , where  $\vec{k}$  is a vector of integers. Plugging this

plane-wave state into the fluctuation formula Eq. (23) we see that the corresponding eigenvalue is  $4\pi^2 \|\vec{k}\|_0^2$ , where  $\|\vec{k}\|_0 = \sqrt{g^{(0)ij} k_i k_j}$  as expected.

Thus, collecting the two results, we can write down the formula for the quantum contribution to the ground-state energy in the  $q \neq 0$  sector at critical area:

$$qev \sum_{n=0}^{\infty} \sqrt{n+\pi} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \|\vec{k}\|_0. \quad (28)$$

In getting to this formula we simply added one half of the square root of the previously found eigenvalues of the fluctuation operator. There are some subtleties, though, concerning the degeneracy of each eigenvalue. The eigenvectors have to be taken to define a real vector space. Thus, the  $q$ -fold degeneracy of the Higgs fluctuation potential turns into a  $2q$ -fold degeneracy for this vector space over the real numbers. For the gauge field part, each plane wave contributes a single eigenvalue as we will explain below.

### B. Zero modes and gauge invariance

Zero modes are eigenstates of eigenvalue zero of the fluctuation operator. Although they do not contribute to the quantum mass, it is interesting to take a look at them to understand their origin. By looking at Eq. (28) one sees, first of all, that there are  $2q$  zero modes associated to  $n = 0$ . In addition, there is another zero mode corresponding to constant vector potentials ( $\vec{k} = 0$ ). These zero modes reflect the dimensionality of the moduli space of classical solutions, which is  $2q$  (see the analysis later on).

In addition, there are an infinite number of zero modes associated to gauge invariance. At this level, this shows up in the fact that the potential depends on  $\Im(\tilde{\partial}\delta A^*)$ . Half of the degrees of freedom drop out when taking the imaginary part. To separate gauge-dependent and gauge-invariant degrees of freedom it is necessary to modify the Fourier decomposition as follows:

$$\delta A(x) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} e^{i2\pi\vec{k}\cdot\vec{x} + i\alpha(\vec{k})} \hat{G}(\vec{k}) \quad (29)$$

where  $\alpha(\vec{k})$  is defined by the expression

$$w_0^i k_i = e^{i\alpha(\vec{k})} \|\vec{k}\|_0. \quad (30)$$

Now let us write  $\hat{G}(\vec{k}) = \hat{G}_1(\vec{k}) + i\hat{G}_2(\vec{k})$  where  $\hat{G}_i(\vec{k}) = \hat{G}_i^*(-\vec{k})$ . If we now substitute in the expression for  $\Im(\tilde{\partial}\delta A^*)$ , one sees that only  $\hat{G}_1(\vec{k})$  appears in the result. Thus, the gauge degrees of freedom are associated to  $\hat{G}_2(\vec{k})$ . This can also be seen by Fourier analyzing a pure gauge term  $A_i(x) = \partial_i \varphi(x)$ , and noticing that it has only  $\hat{G}_2(\vec{k})$  coefficients.

Special treatment is required for the  $\vec{k} = 0$  modes of the vector potential which, as commented earlier, also give rise to zero modes. Strictly speaking, these two are gauge-invariant modes. On the other hand, one of the  $n = 0$  modes of the Higgs field is actually a gauge mode associated to global gauge transformations  $\phi \longrightarrow e^{ia}\phi$ . Thus, altogether we got  $2q + 1$  gauge invariant zero modes. This does not match with the  $2q$  parameters of the moduli. As we will see later, it turns out that one of the zero modes is only accidentally so for critical area. The reader is addressed to Ref. [10] for a more detailed explanation of the topology and dimensionality of the moduli space (See also Ref. [22]).

### C. Subtraction of the $q = 0$ energy

The calculation for vanishing flux is quite different. We show here the result for an arbitrary value of the metric  $g$ . The minimum energy solution is given by a constant Higgs field  $\phi(x) = v$  and a vanishing vector potential  $A_i(x) = 0$ , up to gauge transformations. The quadratic fluctuation terms around this vacuum are well known, being a simple example of the Higgs mechanism. In addition to gauge modes, the degrees of freedom correspond to 3 real massive fields. One is the real Higgs field and the other two are the components of the massive vector potential. At the critical value of the self-coupling ( $\lambda = 1$ ), the mass of the photon and of the Higgs field are both equal to  $ev$ . Hence, the vacuum energy becomes

$$\frac{3}{2} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sqrt{4\pi^2 \|\vec{k}\|^2 + e^2 v^2}. \quad (31)$$

Notice that the formula is valid for every value of the constant metric  $g_{ij}$ . The dependence appears through  $\|\vec{k}\|^2 = g^{ij}k_i k_j$ .

As mentioned previously, the vortex Casimir energy  $\mathcal{E}_C$  is obtained by subtracting the  $q = 0$  vacuum energy from the  $q = 1$  one. For larger values of the flux ( $q > 1$ ) the same procedure leads to the multivortex energy. Notice, however, that the same metric has to be used for the subtracted piece. Since the critical value of the area depends on  $q$ , so will be the case for the vacuum energy subtraction. In what follows we will try to work as much as possible keeping the flux  $q$  arbitrary, and write down the final formulas to make this dependence explicit.

In order to perform a subtraction of two divergent quantities we need to handle them by some regularization procedure. Here we will use the method of analytical continuation, also known as zeta-function technique. Let us explain the method in a generic way before applying it to our situation.

Let  $\lambda_i$  denote the eigenvalues of the fluctuation operator  $\hat{V}$  in increasing order. The quantum contribution to the ground-state energy at one loop is given by

$$\mathcal{E}_Q = \frac{ev}{2} \sum_{i=1}^{\infty} \sqrt{\lambda_i} \quad (32)$$

with  $0 < \lambda_i \leq \lambda_j$  for  $i < j$ . We have factored out from  $\hat{V}$  the quantity  $ev$  having dimensions of mass and providing, as explained earlier, the natural unit for quantum energies. The quantities  $\lambda_i$  are hence dimensionless.

Although, the previous expression for  $\mathcal{E}_Q$  is divergent, we can define a function of the complex variable  $s$  by

$$\mathcal{E}_Q(s) \equiv \frac{ev}{2} \sum_{i=1}^{\infty} (\lambda_i)^{\frac{1}{2}-s} \quad (33)$$

which will be convergent for  $\Re(s) > s_0 > 0$ . To make the expression well defined for  $s > 1/2$  one must, in addition, exclude zero modes from the sum. Formally, the quantum energy is the analytical continuation of this function to  $s = 0$ . Obviously, the divergence of the initial one-loop energy reflects itself in the appearance of singularities as we move from the region of analyticity to the point  $s = 0$ . It could happen, however, that if we subtract two divergent expressions, the corresponding  $s$ -dependent functions are such that the divergences cancel each other, and one gets a smooth continuation. This necessarily happens whenever the initial expression is finite. As we will see later, this is indeed the case for our vortex energies.

A good way to evaluate  $\mathcal{E}_Q(s)$  is to rewrite it as

$$\begin{aligned} \mathcal{E}_Q(s) &= \frac{ev}{2\Gamma(s-1/2)} \int_0^{\infty} dx x^{s-3/2} \sum_i e^{-x\lambda_i} \\ &= \frac{ev}{2\Gamma(s-1/2)} \int_0^{\infty} dx x^{s-3/2} \text{Tr}(e^{-x\hat{V}}) \end{aligned} \quad (34)$$

where the operator  $e^{-x\hat{V}}$  is called the heat-kernel of the operator  $\hat{V}$ . For  $x > 0$  its trace is well defined. The divergence at  $s = 0$  appears because the trace does not vanish strongly enough as  $x \longrightarrow 0$  to make the integration convergent at the lower limit.

Let us apply these relations to the case of the multivortex energy at critical metric (or area). The one-loop quantum energy at non-trivial topology  $\mathcal{E}_Q(s)$  is given by Eq. (28). It is the sum of two terms. The analytical continuation of the first one, coming from the Higgs field fluctuations, can be easily recognized as  $qev\zeta(s-1/2)$ , where  $\zeta(x)$  is Riemann zeta-function. This function is analytic for  $\Re(s) > 3/2$ , is well defined at  $s = 0$ , and has a simple pole with unit residue at  $s = 3/2$ . The second term, coming from the vector potential, can be defined using the corresponding heat kernel, whose trace is

$$\mathcal{F}(x/q) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} e^{-x\xi} \quad (35)$$

where the dimensionless quantity  $\xi$  is given by

$$\xi \equiv \frac{4\pi^2}{e^2 v^2} \|\vec{k}\|_0^2 = \frac{4\pi^2}{e^2 v^2} (g^{(0)})^{ij} k_i k_j = \frac{\pi}{q} \bar{g}^{ij}(\tau) k_i k_j \quad (36)$$

and  $\bar{g}(\tau)$  is the conformally equivalent metric of unit determinant introduced earlier. Notice that the function  $\mathcal{F}(x)$  is indeed equal to the 2-dimensional Riemann theta function  $\Theta(z, \Omega)$ , as given for example in Ref. [23], for  $z = 0$  and  $\Omega = -ix(\bar{g}(\tau))^{-1}$ . The properties of this function realize the invariance under transformations of the modular group in the complex parameter  $\tau$ .

Using the definition of  $\mathcal{F}(x)$  and combining it with the  $\zeta$ -function term, we get the analytical continuation of the one-loop quantum energy given by

$$\begin{aligned} \mathcal{E}_Q^{(0)}(s) &= evq\zeta(s-1/2) + \frac{evq^{s-1/2}}{2\Gamma(s-1/2)} \\ &\times \int_0^\infty dx x^{s-3/2} (\mathcal{F}(x) - 1) \end{aligned} \quad (37)$$

where the superscript (0) recalls that the result is valid for the critical area case. The function is well defined for  $\Re(s) > 3/2$ . The second term also develops a pole at  $s = 3/2$ . This can be deduced using the modular invariance of the theta function, which implies that the leading behavior of  $\mathcal{F}(x)$  for  $x \rightarrow 0$  is

$$\mathcal{F}(x) = (\det(\Omega))^{-1/2} + \dots = \frac{1}{x} + \dots$$

where the dots represent terms with powers of the exponential of  $-1/x$ . To display the singularity explicitly, we can add and subtract a term  $\frac{1}{x}e^{-x}$  to the integrand. One gets

$$\begin{aligned} \mathcal{E}_Q^{(0)}(s) &= evq\zeta(s-1/2) + \frac{evq^{s-1/2}}{2(s-3/2)} + \frac{evq^{s-1/2}}{2\Gamma(s-1/2)} \\ &\times \int_0^\infty dx x^{s-3/2} \left( \mathcal{F}(x) - 1 - \frac{e^{-x}}{x} \right) \end{aligned} \quad (38)$$

where the integral is well defined for  $\Re(s) > -\frac{1}{2}$ .

A similar treatment can be done for the one-loop vacuum energy in the trivial topology sector. This time, however, we will do the calculation for an arbitrary value of the metric  $g = \kappa g^{(0)}$ . Using Eq. (31) and the previous definitions we get

$$\begin{aligned} \mathcal{E}_\emptyset(s) &= \frac{3ev}{2} \sum_{\vec{k}} \left( \frac{\xi}{\kappa} + 1 \right)^{1/2-s} \\ &= \frac{3ev(\kappa q)^{s-1/2}}{2\Gamma(s-1/2)} \int_0^\infty dx x^{s-3/2} e^{-\kappa q x} \mathcal{F}(x). \end{aligned} \quad (39)$$

To explicitly display the singularity of the integral we might add and subtract  $1/x$  from  $\mathcal{F}(x)$  to get to

$$\begin{aligned} \mathcal{E}_\emptyset(s) &= \frac{3ev\kappa q}{2(s-3/2)} + \frac{3ev(\kappa q)^{s-1/2}}{2\Gamma(s-1/2)} \\ &\times \int_0^\infty dx x^{s-3/2} e^{-\kappa q x} \left( \mathcal{F}(x) - \frac{1}{x} \right). \end{aligned} \quad (40)$$

The integral part is now an entire function and the only singularity resides in the single pole at  $s = 3/2$ . The fact that the residue is proportional to  $\kappa$  shows that there is a divergent contribution to the energy which is extensive and, hence, proportional to the area. We expect a similar behavior for the most divergent contribution to the energy for nontrivial topology.

Coming back to the critical area case, we can set  $\kappa = 1$  in the previous formula to get  $\mathcal{E}_\emptyset^{(0)}(s)$ , which should be subtracted from  $\mathcal{E}_Q^{(0)}(s)$ . One sees that the pole at  $s = 3/2$  cancels out in the difference, and the whole expression becomes regular down to  $\Re(s) = -1/2$ . According to our previous considerations, we interpret this as evidence that the vortex and multivortex Casimir energies are indeed finite quantities, and their value can be obtained by setting  $s = 0$  in the difference. The result is given by

$$\begin{aligned} \mathcal{E}_C^{(0)} &= ev \left( q\zeta(-1/2) + q - \frac{1}{3\sqrt{q}} \right) + \frac{ev}{2\sqrt{q}\Gamma(-1/2)} \\ &\times \int_0^\infty dx x^{-3/2} \left( \mathcal{F}(x) (1 - 3e^{-qx}) \right. \\ &\left. - 1 + \frac{1}{x} (3e^{-qx} - e^{-x}) \right). \end{aligned} \quad (41)$$

As commented in the Introduction, the quantum vortex energy results from adding to this result the  $\hbar$  contribution to the classical energy  $\mathcal{E}_R^{(0)}$ . This follows from the renormalization of the parameters in the Lagrangian. This extra term, however, depends on the renormalization prescription that is adopted. In Appendix B we set up a prescription that comes out quite natural within our formulation. Combining Eq. (41) with this result [Eq. (B15)] we get

$$\mathcal{E}^{(0)} = \mathcal{E}_C^{(0)} - \frac{39}{32} qev. \quad (42)$$

The new term changes the numerical value but has no influence on the dependence of our result on  $q$  and  $\tau$  that we will now analyze.

The quantum contribution to the vortex mass can be obtained by setting  $q = 1$  in Eq. (42). The terms which do not involve an integral add up to  $-0.75997$ . On the other hand, the integral concentrates all the dependence on the metric parameter  $\tau$ . It attains its minimum value  $0.169259$  for  $\tau = e^{i\pi/3}$ , which adds up to a quantum vortex mass of  $-0.590711$  in  $ev$  units. Going back to the Euclidean metric



coordinates, we can see that this value of  $\tau$  corresponds to the periods characteristic of a triangular lattice of vortices. The value of the energy corresponding to a square lattice ( $\tau = i$ ) is  $-0.589877$ , which gives a very small difference. It is remarkable that the quantum correction gives rise to a minimum energy configuration which coincides to the vortex lattice obtained in type II superconductors.

The situation changes considerably for other values of  $q$ . At large values, the dominant contribution comes from the first two terms in Eq. (41). This produces a linear dependence with  $q$  and slope  $-0.426636ev$ . Adding up the classical energy, which is also linear in  $q$ , we get  $\mathcal{M} = \pi v^2 - 0.426636ev$ . The latter value is, henceforth, the energy per vortex on a large vortex situation, and can be interpreted as an alternative estimate of the vortex mass. Clearly the quantum contribution becomes sizable when  $e/v$  is large enough.

Corrections to the linear behavior are proportional to  $1/\sqrt{q}$ , up to exponentially suppressed terms. The coefficient of the  $1/\sqrt{q}$  term is dominated by the photon energy contribution and depends on the metric parameter  $\tau$ . This time, however, it is maximal at  $\tau = e^{i\pi/3}$  and minimal along the line  $\tau = ir$ , becoming negative divergent at infinite  $r$ . Using properties of integrals of the Jacobi theta function one can calculate the coefficient of the  $1/\sqrt{q}$  term, up to terms exponentially suppressed in  $r$  to be

$$-\frac{r^{3/2} \zeta(3)}{\sqrt{q} 4\pi^{3/2}} - \frac{r^{-1/2} \sqrt{\pi}}{\sqrt{q} 12} \quad (43)$$

For  $r \sim 6q$  this term becomes comparable with the linear term in  $q$ .

#### IV. BRADLOW PARAMETER EXPANSION OF QUANTUM ENERGIES

In the previous section we evaluated the quantum correction to the vortex mass on a spatial torus of critical area. In this section we will show how it is possible to extend this result to other values of the area. This is done by setting up a power expansion in the conformal factor  $(\kappa - 1)$ . The methodology has been used previously in Ref. [10] to obtain an analytic expression for the multi-vortex field configurations at critical coupling: solutions of the Bogomol'nyi equations. With sufficiently high order calculations, one obtains a competitive description of the solutions on the plane. An advantage of these analytic expressions is that they facilitate other calculations involving vortices, such as their scattering behavior [12]. Here we will explain how one can set up a similar expansion for the quantum corrections to the vortex masses. The leading term is given by the result of the previous section and the next to leading term will be computed in the present one.

Before going into details, let us enumerate briefly the different steps involved in the procedure. As explained previously, the one-loop quantum contribution follows by

calculating the spectrum of the operator  $\hat{V}$  determining the quadratic fluctuations around classical solutions of the equations of motion. In our case, these are just the solutions of the Bogomol'nyi equations. It is precisely in this step where one makes use of the results of Ref. [10], by obtaining a series expansion of these solutions in powers of the square root of  $\epsilon = (\kappa - 1)/\kappa$ . Substituting these background fields onto the expression of the fluctuation operator, one arrives to an equivalent expansion for this operator

$$\hat{V} = \sum_{n=0}^{\infty} \epsilon^{n/2} \hat{V}^{(n)}. \quad (44)$$

Its eigenvalues also admit an expansion

$$\lambda_i = \lambda_i^{(0)} + \sum_{p=1}^{\infty} \epsilon^p \lambda_i^{(p)} \quad (45)$$

involving only integer powers of  $\epsilon$ . The coefficients  $\lambda_i^{(p)}$  can be obtained by applying the standard technique, analogous to that employed in quantum mechanics when using perturbation theory.

The final step is to plug this result into the analytically continued formulas for the ground-state quantum energy  $\mathcal{E}_Q(s)$  and expand the result in powers of  $\epsilon$  to give

$$\begin{aligned} \mathcal{E}_Q(s) &= \frac{ev}{2\Gamma(s-1/2)} \int_0^{\infty} dx x^{s-3/2} \\ &\times \sum_i^l e^{-x\lambda_i^{(0)}} \left( 1 + \sum_{p=1}^{\infty} \epsilon^p \sum_{l=1}^p \frac{(-x)^l}{l!} c_i(p, l) \right) \end{aligned} \quad (46)$$

where

$$c_i(p, l) = \sum_{k_1, k_2, \dots, k_l=1}^{\infty} \delta\left(\sum k_a = p\right) \prod_{a=1}^l \lambda_i^{(k_a)}. \quad (47)$$

Finally, as done before, the total quantum energy is obtained by subtracting the contribution of trivial topology, extrapolating to  $s = 0$  and adding the counterterm contribution.

In the following subsections we will apply the above procedure to the calculation of the quantum vortex energies to order  $\epsilon$ . In that case  $p = l = 1$  and the coefficient  $c_i(1, 1) = \lambda_i^{(1)}$ . This allows us to circumvent an important complication arising when there are degenerate levels at lowest order. If degeneracy is accidental it is broken by higher order corrections. This implies a diagonalization procedure within the subspace associated to the same lowest order eigenvalue  $\lambda_i^{(0)}$ . However, for the calculation of the mass to order  $\epsilon$ , all we need is the sum of the  $\lambda_i^{(1)}$  within that space, i.e. the trace of the operator in the degenerate space. This avoids the much more involved problem of computing the splittings and eigenstates.

### A. Solutions of the Bogomol'nyi equations

In terms of our main complex fields  $\phi$  and  $\delta A$ , the Bogomol'nyi equations can be read off from the form of the potential Eq. (20):

$$\tilde{D}\phi = i \frac{e^2 v}{2\sqrt{\pi q}} \delta A \phi \quad (48)$$

$$\Im(\tilde{\partial}\delta A^*) = -v\sqrt{\pi q}(\kappa - 1) + \frac{ve^2}{4\sqrt{\pi q}} |\phi|^2. \quad (49)$$

For  $\kappa = 1$  the solution is given by  $\phi = \delta A = 0$ . Hence, the idea is simple: express the solution as a power series in  $\sqrt{\kappa - 1}$  or  $\sqrt{\epsilon} = \sqrt{\kappa - 1}/\sqrt{\kappa}$ . The coefficients of this expansion can be obtained iteratively. The occurrence of the square root of  $\kappa - 1$  can be understood if we integrate over space the second Bogomol'nyi equation. One gets

$$\int d^2x |\phi|^2 = v_0^2(\kappa - 1) \quad (50)$$

where  $v_0 = \frac{\sqrt{4\pi q}}{e}$ . From this equation one concludes that the Higgs field  $\phi$  is proportional to  $v_0\sqrt{\kappa - 1}$ . Apart from this normalization factor, all the remaining corrections involve integer powers of  $\epsilon$ . For example, notice that the left-hand side of Eq. (49) is of order  $\epsilon$ . Hence,  $\delta A$  starts at order  $\epsilon$ . Plugging this in Eq. (48) one gets a correction of order  $\epsilon^{3/2}$  to  $\phi$ , and so on and so forth. In summary,  $\delta A$  can be expanded in integer powers of  $\epsilon$  and  $\phi$  in half-integer powers.

Obviously, in finding the solutions one must take proper care of the boundary conditions. The appropriate boundary conditions are valid for the solutions as well as for the fluctuations, and were explained in the previous section. The  $\phi$  field belongs to the space  $\mathcal{H}_q$ , and hence can be expanded in the basis  $\Psi_{n,s}$ . On the other hand,  $\delta A$  is periodic on the torus and can be expanded in our modified Fourier expansion [Eq. (29)] in terms of coefficients  $\hat{G}_1(\vec{k})$  and  $\hat{G}_2(\vec{k})$ .

One might wonder about how does the multiplicity of the Bogomol'nyi solutions arises. First of all, one can fix the gauge by requiring that the coefficients  $\hat{G}_2(\vec{k})$  vanish. Another freedom is associated with translation invariance and this can be fixed by setting  $\hat{G}_1(\vec{k} = 0) = 0$ . The remaining multiplicity is fixed, as we will see, by fixing the lowest order terms in the expansion. This is one of powers of this method, which allows obtaining multivortex solutions with arbitrary centers.

In any case, it is not our purpose to describe in detail the methodology to get results to higher order in  $\epsilon$ , since that was the subject of Ref. [10]. In that reference we gave two procedures to compute the corrections. One method, described in the appendix of that paper, uses the terminology of a quantum mechanical description, despite the fact that the problem is indeed classical. The pre-Hilbert space is given by the space of Higgs fields  $\mathcal{H}_q$ , a basis of which is provided by the functions  $\Psi_{n,s}(x)$ . One can define operators  $\hat{U}(\vec{k})$  acting on  $\mathcal{H}_q$ , and amounting to multiplication by  $e^{2\pi i \vec{k} \cdot \vec{x}}$ . The matrix elements of these operators on the basis states will be the main formula needed to perform all the calculations. The result, which we reproduce here, is

$$\begin{aligned} X_{mn}^{s's}(\vec{k}) &\equiv \langle m, s' | \hat{U}(\vec{k}) | n, s \rangle \\ &= \int d^2x \Psi_{m,s'}^*(x) e^{2\pi i \vec{k} \cdot \vec{x}} \Psi_{n,s}(x) \\ &= U_{s's}(\vec{k}) e^{i(m-n)\beta} (-1)^{(M+n)} e^{-\frac{\xi}{2}} \xi^{\frac{|m-n|}{2}} \times \sum_{j=0}^M (-1)^j \frac{\sqrt{n!m!} \xi^j}{j!(M-j)!(j+|m-n|)!} \\ &= U_{s's}(\vec{k}) e^{i(m-n)\beta} (-1)^{(M+n)} e^{-\frac{\xi}{2}} \xi^{\frac{|m-n|}{2}} \times \frac{\sqrt{M!}}{\sqrt{(M+|m-n|)!}} L_M^{(|m-n|)}(\xi) \end{aligned} \quad (51)$$

where  $M = \min(m, n)$ ,  $\beta = \alpha(\vec{k}) + \pi/2$  and  $\xi \propto \|\vec{k}\|_0^2$  is the quantity introduced in the previous section. The function  $L_M^{(|m-n|)}(x)$  denotes a generalized Laguerre polynomial. All the  $s, s'$  dependence sits in the unitary  $q \times q$  matrix  $U(\vec{k})$ , given by

$$U_{s's}(\vec{k}) = \delta_{s's+k_1} e^{-i\pi k_1 k_2/q} e^{-2\pi i s k_2/q} \quad (52)$$

where the  $\delta$  is to be taken modulo  $q$ . For the single vortex case ( $q=1$ ) this is just  $e^{-i\pi k_1 k_2}$ . The symbols  $X_{mn}^{s's}$  can be

regarded as the components of a matrix  $X$  which satisfies  $X(-\vec{k}) = X^\dagger(\vec{k})$ .

Let us conclude this subsection by performing the program up to leading order in  $\epsilon$ . The solution for  $\phi(x)$  to lowest order is given by the solution of Eq. (48) with vanishing right-hand side. This is proportional to the ground state of the corresponding harmonic oscillator:

$$\bar{\phi}(x) = \sqrt{\epsilon} v_0 \sum_{s=1}^q c_s \Psi_{0,s}(x) \quad (53)$$

where the constants  $c_s$  are the components of a  $q$ -dimensional complex vector of unit norm. These constants encode the multiplicity of the solutions and are related to the position of the vortices (see Ref. [10]). For the single vortex case ( $q = 1$ )  $c_1$  is just a phase.

To solve for  $\delta A$  to this order, we simply have to substitute the previous expression in the right-hand side of Eq. (49) and use Eq. (29) and Eq. (51) to obtain

$$\bar{G}_1(\vec{k}) = -\frac{\epsilon v_0}{2\sqrt{\xi}} e^{-\frac{\xi}{2}u(\vec{k})} \quad (54)$$

where  $u(\vec{k}) = c_i^* U_{ts}(-\vec{k}) c_s$  carries the dependence on the multivortex moduli parameters. Notice that the equation only constrains  $\bar{G}_1$ . Taking  $\bar{G}_2$  to zero amounts to a choice of gauge for the background field solution, which we will adopt. In addition, for simplicity we also fix to zero the component associated to  $\vec{k} = 0$ . Results do not depend on these choices.

### B. Spectrum of quantum fluctuations

Having found the minima of the potential in the last subsection, we now expand the fields around these solutions as follows

$$\phi \longrightarrow \bar{\phi} + \varphi \quad (55)$$

$$\delta A \longrightarrow \bar{\delta A} + \delta a \quad (56)$$

and plug these into the expression of the Lagrangian keeping only terms quadratic in the quantum fluctuation fields  $\varphi$  and  $\delta a$ . In our case the result is

$$\begin{aligned} \delta V = \prod_i \left( \int_0^1 dx^i \right) \frac{1}{2\kappa} \left[ |(\bar{D} - iC\bar{\delta A})\varphi - iC\bar{\phi}\delta a|^2 \right. \\ \left. + (-\mathfrak{S}(\bar{\partial}\delta a^*) + \frac{C}{2}(\bar{\phi}\varphi^* + \bar{\phi}^*\varphi))^2 \right] \quad (57) \end{aligned}$$

where we introduced the constant  $C = \epsilon v/v_0$ . Indeed, the whole potential is proportional to  $\frac{\epsilon^2 v^2}{\kappa}$  so we can write (in a rather symbolic notation)

$$\delta V = \frac{\epsilon^2 v^2}{2\kappa} (\langle \varphi |, \langle \delta a |) \hat{V} \begin{pmatrix} |\varphi \rangle \\ |\delta a \rangle \end{pmatrix} \quad (58)$$

in terms of a Hermitian operator  $\hat{V}$ , acting on the space of fluctuation fields  $\varphi$  and  $\delta a$ , associated respectively to the Higgs field and the vector potential. Our goal is to obtain the eigenvalues  $\lambda_i$  of this operator. To do so, we substitute the background fields  $[\bar{\phi}(x)$  and  $\bar{\delta A}(x)]$  by their expansion in powers of  $\sqrt{\epsilon}$  derived in the previous section. We then obtain

$$\hat{V} = \hat{V}^{(0)} + \epsilon^{1/2}\hat{V}^{(1/2)} + \epsilon\hat{V}^{(1)} + \mathcal{O}(\epsilon^{3/2}). \quad (59)$$

We might use indices to specify on which of the fluctuation fields is the operator acting:

$$\hat{V} = \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} \\ \hat{V}_{21} & \hat{V}_{22} \end{pmatrix}. \quad (60)$$

Since  $\bar{\phi}$  and  $\bar{\delta A}$  are expandable in odd and even powers of  $\sqrt{\epsilon}$ , respectively, one concludes that the off-diagonal terms ( $\hat{V}_{12} = \hat{V}_{21}$ ) and the diagonal ones ( $\hat{V}_{ii}$ ) have the same property, respectively. From this, one easily concludes that the eigenvalues are expandable in integer powers of  $\epsilon$ :

$$\lambda = \sum_{m=0}^{\infty} \epsilon^m \lambda^{(m)}. \quad (61)$$

The coefficients  $\lambda^{(m)}$  can be determined by standard quantum mechanical techniques to be described below.

The first step is to diagonalize  $\hat{V}^{(0)}$ , the operator corresponding to critical area. This was done in the previous section. The eigenstates of  $\hat{V}_{11}^{(0)}$  will be labeled  $|n, s, \sigma\rangle$  and correspond to the functions  $\Psi_{n,s}(x)$  for  $\sigma = +$  and  $i\Psi_{n,s}(x)$  for  $\sigma = -$ . As mentioned earlier, we have to consider two states ( $\sigma = \pm 1$ ) because we want to work with real vector spaces. The corresponding eigenvalue is  $n$ , and does not depend on  $s$  or  $\sigma$ . Thus, to lowest order its degeneracy is  $2q$ , but this might be broken by higher order corrections. Nevertheless, for the purpose of computing the next correction all we need is the trace of  $\hat{V}$  in the space characterized by eigenvalue  $n$ :  $\delta\lambda_n$ . Applying standard perturbative techniques the linear correction in  $\epsilon$  is given by

$$\begin{aligned} \delta\lambda_n = \epsilon \sum_{s=0}^{q-1} \sum_{\sigma=\pm 1} \langle n, s, \sigma | (\hat{V}_{11}^{(1)} \\ + \hat{V}_{12}^{(1/2)}(n - \hat{V}_{22}^{(0)})^{-1} \hat{V}_{21}^{(1/2)}) | n, s, \sigma \rangle. \quad (62) \end{aligned}$$

To facilitate the reading of the paper we will collect the calculation in Appendix A, and give here only the final result:

$$\delta\lambda_{n \neq 0} = \epsilon \left( 2qn - \sum_{j=0}^{n-2} \rho(j) \right) \quad (63)$$

where

$$\rho(n) = \sum_k e^{-\xi} \frac{\xi^n}{n!}. \quad (64)$$

It is easy to see that  $\rho(j)$  oscillates around a constant value of  $q$ , with oscillations that are damped with increasing  $q$ . Hence,  $\delta\lambda_n$  should oscillate around  $2q$ . For  $q = 1$  the first few values are displayed in Fig. 1, showing the oscillatory pattern.

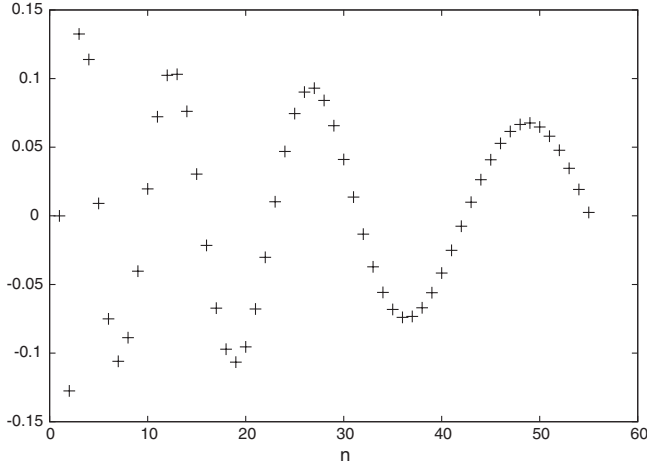


FIG. 1. We display  $(\delta\lambda_n - 2)/(2\sqrt{n})$  as a function of  $n$  for  $q = 1$ .

A similar procedure applies for the eigenvalues associated to the vector potential. The eigenfunctions of  $\hat{V}_{22}^{(0)}$  will be labeled  $|\vec{k}, L, \sigma\rangle$  and  $|\vec{k}, T, \sigma\rangle$ , for positive  $\vec{k}$ . The label  $\vec{k}$  corresponds to the modified Fourier modes Eq. (29), with  $T$  and  $L$  referring to the gauge invariant and gauge dependent parts, associated to the coefficients  $G_1(\vec{k})$  and  $G_2(\vec{k})$ , respectively. To work with real coefficients we take only half of the Fourier modes and split the coefficients  $G_i(\vec{k})$  into its real and imaginary parts labeled by the symbols  $\sigma = \pm 1$ , respectively. Thus, the vectors  $\vec{k}$  are restricted to positive values, meaning  $k_1 > 0$  or  $\{k_1 = 0 \text{ and } k_2 > 0\}$ . Summing up, the corresponding eigenfunctions are given by

$$\chi_{\vec{k}, \sigma}^\eta(x) \equiv P_\eta \frac{1}{\sqrt{2}} e^{i\pi(1-\sigma)/4} (e^{i2\pi\vec{k}\vec{x} + i\alpha(\vec{k})} + \sigma e^{-i2\pi\vec{k}\vec{x} + i\alpha(-\vec{k})}) \quad (65)$$

where  $\eta = T, L$  for transverse or longitudinal photons, respectively, while  $P_T = 1$  and  $P_L = i$ . To lowest order, the transverse photons  $|\vec{k}, T, \sigma\rangle$  have eigenvalue  $\xi$ , while the longitudinal photons have eigenvalue 0.

Calculating the correction to order  $\epsilon$  follows the same steps as for the Higgs. Once more we benefit from having to compute only the trace within each degenerate sector. Thus the goal is

$$\delta\lambda_{\vec{k}} = \epsilon \sum_{\sigma=\pm 1} \langle \vec{k}, T, \sigma | (\hat{V}_{22}^{(1)} + \hat{V}_{21}^{(1/2)} (\xi - \hat{V}_{11}^{(0)})^{-1} \hat{V}_{12}^{(1/2)}) | \vec{k}, T, \sigma \rangle. \quad (66)$$

The details of the calculation are collected in Appendix A. The result is very simple

$$\delta\lambda_{\vec{k} \neq 0} = 2\epsilon. \quad (67)$$

One can also compute the correction to the zero eigenvalue. Not surprisingly the correction vanishes, since it is associated to a gauge symmetry which is valid at all orders in  $\epsilon$ . Anyhow, it serves as a check of our manipulations.

We have left out from the previous spectrum corrections to the values at  $n = 0$  and  $k = 0$ . Together with the gauge modes of the longitudinal photons  $[G_2(\vec{k})]$ , these were zero modes at leading order. We emphasized then that  $2q$  of these zero modes are associated to the moduli of solutions of the Bogomol'nyi equation. Since this holds for any value of the area, they should remain zero modes at any order in our  $\epsilon$  expansion. Thus, as mentioned earlier, 2 of the  $2q + 2$  zero modes cannot be gauge-invariant zero modes. Indeed, we already mentioned that one of them is associated with global gauge transformations. The remaining zero mode was just accidentally so at critical area and is broken at order  $\epsilon$ . Indeed, in the Appendix we found  $\delta\lambda_{n=0} = 1$  and we explained that it is associated to a multiplicative rescaling of the solution,  $c_s \rightarrow tc_s$  with  $t$  real. Since this is a single state, we will leave it out from the analytical continuation and add its contribution to the final result.

### C. Vortex energies to order $\epsilon$

In this subsection we will use the results of the previous subsection to compute the quantum correction to the masses up to order  $\epsilon$ . The methodology was explained before. The mass receives contributions from quantum fluctuations of the Higgs field and of the photon. The contribution for trivial topology has to be subtracted out from the previous sum and added to the counterterm contribution to the energy. In order to manipulate these individually divergent quantities we make use of the zeta-function regularization method. Leaving out, for the time being, the contribution of  $\delta\lambda_{n=0}$ , the procedure can be summarized by the formula:

$$\mathcal{E}_C = \lim_{s \rightarrow 0} [\mathcal{E}_\phi^{(0)}(s) + \mathcal{E}_A^{(0)}(s) - \mathcal{E}_\theta^{(0)}(s) + \epsilon(\mathcal{E}_\phi^{(1)}(s) + \mathcal{E}_A^{(1)}(s) - \mathcal{E}_\theta^{(1)}(s))] + \mathcal{O}(\epsilon^2). \quad (68)$$

If the right-hand side is analytic for  $\Re(s) \geq 0$ , then we can say that the procedure is unambiguous and the quantum mass finite. This indeed turned out to be the case for the critical value of the area ( $\epsilon = 0$ ) as found in Sec. III. The corresponding limit is  $\mathcal{E}_C^{(0)}$  given in Eq. (41). Our goal now is to see if this continues to be the case up to order  $\epsilon$ .

The main formulas for obtaining the expansion of the analytically continued energies  $\mathcal{E}_Q(s)$  were explained earlier. In particular, given the expansion of the eigenvalues of the fluctuation operator  $\lambda_i = \lambda_i^{(0)} + \epsilon\lambda_i^{(1)} + \dots$  one obtains the order epsilon correction to  $\mathcal{E}_Q(s)$  as follows:

$$\mathcal{E}_Q^{(1)}(s) = -\frac{1}{2\Gamma(s-1/2)} \int_0^\infty dx x^{s-1/2} \sum_i e^{-x\lambda_i^{(0)}} \lambda_i^{(1)}. \quad (69)$$

However, in calculating the eigenvalues in Appendix A we factored out from the operator a coefficient  $\frac{ev}{\kappa}$ . As explained earlier,  $ev$  are just the natural units of quantum energies, and it seems more reasonable to analytically continue a dimensionless expression and put the units back at the end. Eliminating the factor  $1/\kappa$  was dictated only by simplicity, since it is trivial to correct for it in the final result:

$$\mathcal{E}_Q(s) \longrightarrow \kappa^{s-1/2} \mathcal{E}_Q(s) = (1 + (s-1/2)\epsilon + \dots) \mathcal{E}_Q(s). \quad (70)$$

The modification does not alter the cancellations between different terms, but induces an order  $\epsilon$  correction proportional to the leading order result:

$$\mathcal{E}_C = \left(1 - \frac{\epsilon}{2}\right) \mathcal{E}_C^{(0)} + \epsilon \bar{\mathcal{E}}^{(1)} + \dots \quad (71)$$

The new term  $\bar{\mathcal{E}}^{(1)}/(ev)$  can be constructed using Eq. (69) and the eigenvalues computed in Appendix A and given earlier in this section. Its evaluation is the goal of the rest of this section.

Let us start by computing the contribution of photon fluctuations  $\bar{\mathcal{E}}_A^{(1)}(s)$ . Plugging the correction to the eigenvalues in the general formula one gets:

$$\begin{aligned} \bar{\mathcal{E}}_A^{(1)}(s)/(ev) &= -\frac{1}{2\Gamma(s-1/2)} \int_0^\infty dx x^{s-1/2} (\mathcal{F}(x/q) - 1) \\ &= -\frac{q^{s+1/2}}{2\Gamma(s-1/2)} \int_0^\infty dx x^{s-1/2} (\mathcal{F}(x) - 1). \end{aligned} \quad (72)$$

The integral is divergent at  $s \leq 1/2$ . As done before, we may regulate it by adding and subtracting  $e^{-x}/x$  to  $\mathcal{F}$ . With this subtraction the integral converges down to  $s = 0$ . Curiously the singularity at  $s = 1/2$  of the subtracted term is canceled by the pole in the Gamma function, so that the whole expression is finite down to  $s = 0$ :

$$\begin{aligned} \bar{\mathcal{E}}_A^{(1)}(s)/(ev) &= -\frac{q^{s+1/2}}{2} - \frac{q^{s+1/2}}{2\Gamma(s-1/2)} \\ &\quad \times \int_0^\infty dx x^{s-1/2} \left( \mathcal{F}(x) - 1 - \frac{e^{-x}}{x} \right). \end{aligned} \quad (73)$$

Now let us compute the contribution from the Higgs sector  $\bar{\mathcal{E}}_\phi^{(1)}$ :

$$\begin{aligned} \bar{\mathcal{E}}_\phi^{(1)}(s)/(ev) &= -\frac{1}{\Gamma(s-1/2)} \\ &\quad \times \int_0^\infty dx x^{s-1/2} \sum_{n=1}^\infty e^{-xn} \left( qn - \sum_{j=0}^{n-2} \rho(j) \right). \end{aligned} \quad (74)$$

We can treat the first term in the integral using:

$$\sum_{n=1}^\infty n e^{-xn} = -\frac{d}{dx} \sum_{n=1}^\infty e^{-xn} = -\frac{d}{dx} \frac{1}{1-e^{-x}} = \frac{e^{-x}}{(1-e^{-x})^2}. \quad (75)$$

For the second term we use the explicit expression for  $\rho(j)$  and exchange the sums in  $j$  and  $n$  as follows

$$\begin{aligned} e^{-\xi} \sum_{j=0}^\infty \sum_{n=j+2}^\infty \frac{\xi^j}{j!} e^{-xn} &= e^{-\xi} \sum_{j=0}^\infty \frac{\xi^j}{j!} \frac{e^{-x(j+2)}}{1-e^{-x}} \\ &= \frac{e^{-2x}}{1-e^{-x}} e^{-\xi(1-e^{-x})}. \end{aligned} \quad (76)$$

Resumming over  $\vec{k}$  we get:

$$\begin{aligned} \bar{\mathcal{E}}_\phi^{(1)}(s)/(ev) &= \frac{1}{\Gamma(s-1/2)} \int_0^\infty dx x^{s-1/2} \frac{e^{-x}}{1-e^{-x}} \\ &\quad \times \left( e^{-x} \mathcal{F}((1-e^{-x})/q) - \frac{q}{1-e^{-x}} \right). \end{aligned} \quad (77)$$

Again the integral diverges. It is convenient as usual to rearrange the integrand subtracting the leading behavior of  $\mathcal{F}$  for small values of its argument. The remaining piece contains the divergent integral and takes the form:

$$\begin{aligned} &-\frac{q}{\Gamma(s-1/2)} \int_0^\infty dx x^{s-1/2} \frac{e^{-x}}{1-e^{-x}} \\ &= -q \frac{\Gamma(s+1/2) \zeta(s+1/2)}{\Gamma(s-1/2)} \\ &= -q(s-1/2) \zeta(s+1/2). \end{aligned} \quad (78)$$

Notice that, as before, this term is regular since the factor  $(s-1/2)$  cancels the pole of the Zeta function. We finally arrive to:

$$\begin{aligned} \bar{\mathcal{E}}_\phi^{(1)}(s)/(ev) &= -q(s-1/2) \zeta(s+1/2) \\ &\quad + \frac{1}{\Gamma(s-1/2)} \int_0^\infty dx x^{s-1/2} \frac{e^{-2x}}{1-e^{-x}} \\ &\quad \times \left( \mathcal{F}((1-e^{-x})/q) - \frac{q}{1-e^{-x}} \right) \end{aligned} \quad (79)$$

which is regular at all values of  $s$ .

The next step is to subtract from the previous terms the order  $\epsilon$  contribution to the vacuum energy for trivial topology. This is a simple matter since in Sec. III we computed this energy for arbitrary values of the area.

From it, we can separate out the part which is proportional to the critical area result and we are left with:

$$\begin{aligned} \bar{\mathcal{E}}_{\theta}^{(1)}(s)/(ev) &= -\frac{3q}{2} - \frac{3q^{s+1/2}}{2\Gamma(s-1/2)} \\ &\times \int dx x^{s-1/2} e^{-xq} \left( \mathcal{F}(x) - \frac{1}{x} \right). \end{aligned} \quad (80)$$

$$\begin{aligned} \bar{\mathcal{E}}^{(1)}/(ev) &= \left( \frac{3}{2} + \frac{\zeta(1/2)}{2} \right) q - \frac{\sqrt{q}}{2} + \frac{\sqrt{q}}{4\sqrt{\pi}} \int_0^\infty dx \frac{1}{\sqrt{x}} \left[ (1 - 3e^{-xq}) \mathcal{F}(x) - 1 - \frac{(e^{-x} - 3e^{-xq})}{x} \right] \\ &- \frac{1}{2\sqrt{\pi}} \int_0^\infty dx \frac{e^{-2x}}{\sqrt{x}(1-e^{-x})} \left( \mathcal{F}((1-e^{-x})/q) - \frac{q}{1-e^{-x}} \right). \end{aligned} \quad (81)$$

This quantity has a dependence on  $q$ , which is explicitly displayed, and a dependence on the metric (or the torus periods) hidden in the function  $\mathcal{F}$ .

Now we are ready to present the final result up to order  $\epsilon$  given by

$$\mathcal{E} = \mathcal{E}^{(0)} + \frac{\sqrt{\epsilon}}{2} + \epsilon \left( \bar{\mathcal{E}}^{(1)} - \frac{1}{2} \mathcal{E}_C^{(0)} + \mathcal{E}_R^{(1)} \right). \quad (82)$$

The second term comes from the correction to the  $n = 0$  eigenvalue, which was linear in  $\epsilon$ . Its contribution to the mass is finite, and can be added to the final result without any analytical continuation. The contribution  $\mathcal{E}_R^{(1)}$  depends on the scheme. In our scheme, presented in Appendix B, the result is  $-\frac{13}{32}q$ .

We may numerically evaluate the result to explore its dependence on  $q$  and  $\tau$ . For  $q = 1$  and  $\tau = e^{i\pi/3}$  (triangular lattice), at which the quantum energy had its minimum at critical area, we get

$$\mathcal{E}/(ev) = 0.628039 - \frac{39}{32} + 0.5\sqrt{\epsilon} - \epsilon \left( 0.1542505 + \frac{13}{32} \right) \quad (83)$$

where we have separated out the contributions of the Casimir energy and the renormalization counterterm. On the other hand, for  $\tau = i$  (square lattice) one gets

$$\mathcal{E}/(ev) = 0.628873 - \frac{39}{32} + 0.5\sqrt{\epsilon} - \epsilon \left( 0.155933 + \frac{13}{32} \right). \quad (84)$$

It is interesting to notice that the epsilon term goes in the direction of compensating the angle dependence. This is what is expected, since for large areas ( $\epsilon \rightarrow 1$ ) the dependence on the metric parameter  $\tau$  should disappear. With our numbers we see that for  $\epsilon \sim 1/2$  we get a common value of the mass of  $-0.5174$ . We can take this number as a

crude estimate of the vortex mass on the plane. Alternatively, we might take the value obtained at  $\epsilon = 1$  which is  $-0.652$ .

One can also explore the behavior for large  $q$ . The leading dependence goes linearly with  $q$ , as for the critical area result. Dividing out the linear term by  $q$ , we get an energy per vortex equal to

$$\begin{aligned} \mathcal{E}/(ev) &= \left( -\frac{7}{32} + \zeta(-1/2) \right) \\ &+ \epsilon \left( \frac{19}{32} + \frac{-\zeta(-1/2) + \zeta(1/2)}{2} \right) \\ &= -0.42663 - \epsilon 0.03249. \end{aligned} \quad (85)$$

Although, it is not possible to draw any rigorous conclusion from our 2 terms of the expansion, we see that all estimates give a value close to  $-0.5ev$  for the quantum mass of the vortex on the plane. For comparison with other numerical estimates, one should guarantee that the same renormalization prescription is adopted.

## V. CONCLUSIONS AND OUTLOOK

In this paper we have studied the one-loop quantum correction to the masses and energies of vortices in the  $2 + 1$  dimensional Abelian Higgs model formulated on the torus. For a critical value of the area  $\mathcal{A}_c$  the result can be computed analytically. Away from this value an expansion in powers of  $\epsilon = (\mathcal{A} - \mathcal{A}_c)/\mathcal{A}$  can be set up, of which we have computed the linear correction. In our formulation, the theory is defined on a unit square torus with constant metric tensor. A general metric of this type depends on a conformal factor and a complex number  $\tau$ . The conformal factor is directly proportional to the area, and measures the departure from the critical area case. On the other hand the modular parameter  $\tau$  can be mapped, after a change of variables to a Euclidean metric, onto the periods of the torus, i.e. their aspect ratio and relative angle. Hence, our

quantum masses are indeed functions of the complex parameter  $\tau$  and, as expected, are invariant under transformations of the modular group. Indeed, our analytic results are expressed in terms of integrals of the two-dimensional Riemann theta function having this property. For the critical area case we showed that the minimal energy is achieved for a torus which matches with a triangular lattice of vortices. It is remarkable that the quantum corrections induce a breaking of the classical degeneracy toward a configuration consistent with the standard vortex lattice in type II superconductors. However, this might be a simple coincidence with no particular significance.

Our methodology allows us also to study quantum energies for multivortex configurations. One of the advantages, compared to other numerical methods, is that we can fix from the start the position of the individual vortices, given by the zeroes of the Higgs field. At the classical level, the energy depends only on the number of vortices  $q$  and not on their positions. There is no known reason why this independence should be preserved at the quantum level. Thus, there could be attraction or repulsion of vortices induced by quantum corrections. Our result, however, shows that the degeneracy is preserved up to first order in  $\epsilon$ . There is no apparent symmetry underlying this degeneracy, so that it could still be broken at higher orders in  $\epsilon$ . This is an important conceptual issue which could be hard to settle in a purely numerical fashion. Our result here is analytic but only valid for the first two terms in the expansion. We hope this point could encourage other authors to extend our result to higher orders.

Another interesting piece of information is the dependence of the quantum energy on the number of vortices  $q$ . Notice, however, that the critical area scales with the number of vortices. Hence, one should actually talk about a critical value of the vortex density. If we scale the area and the relative distances among vortices at the same rate, one should expect that for large areas the quantum energy scales linearly with  $q$ . Dividing the multivortex energies by the number of vortices one gets another estimate of the vortex mass. For small values of the area, this linear dependence is modulated by corrections of order  $\sqrt{q}$ ,  $1/\sqrt{q}$  and subleading.

Since vortices are exponentially localized objects it is quite plausible that the expansion can be extrapolated to infinite area, obtaining the one-loop quantum energies for vortices on the plane. Indeed, once the area is a few times larger than the vortex size the effect of the periodic boundary conditions should be exponentially suppressed, which would suggest a fast approach. In Ref. [10] we investigated the shape of the Bogomol'nyi solutions themselves using a Bradlow parameter expansion up to order  $\epsilon^{51}$ . The result could be numerically extrapolated to infinite area (vortices on the plane) and compared successfully with other approaches. The expansion of the solution is only part of the program in computing quantum energies, and getting

to high orders in all the steps certainly demands more efficient and powerful automatization techniques. From our two terms in the expansion, we have played the game of extrapolating to infinite volume. The results are certainly not crazy but, due to the limited information involved, a serious comparison with other results lacks any rigour. A more interesting comparison could be to use the methodology of Ref. [15] for vortices on the torus, and compare the results with our exact results. These might give a new measure of the numerical errors involved in that method.

Finally, we should comment that the idea of a Bradlow parameter expansion is quite general and extends to other types of vortices, Abelian and non-Abelian [24] and in noncommutative space [25,26]. Indeed, the idea itself emerged from a related type of expansion that occurs for four-dimensional Yang-Mills theories [27]. It is also applicable in principle to nonself-dual vortices and to vortices in  $3 + 1$  dimension, for which the mass turns into the string tension [18], and to vortices coupled to fermions [28]. It is quite plausible that our technique can be extended to the computation of quantum corrections in all those cases.

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## APPENDIX A: CORRECTIONS TO THE EIGENVALUES OF THE FLUCTUATION OPERATOR

In this appendix we will collect the details of the calculation of the eigenvalues of the quadratic quantum fluctuation operator  $\hat{V}$  to order  $\epsilon$ . The methodology is explained in Sec. IV. The potential energy of fluctuations has two terms. The first term is given by

$$\frac{1}{2\kappa} \int dx |\tilde{D}\varphi - iC\tilde{\delta}A\varphi - iC\tilde{\phi}\delta a|^2. \quad (\text{A1})$$

The second term is

$$\frac{1}{2\kappa} \int dx \left( \Im(\tilde{\delta}^* \delta a) + \frac{C}{2} (\tilde{\phi}^* \varphi + \tilde{\phi} \varphi^*) \right)^2 \quad (\text{A2})$$

where  $C = ev/v_0$ . The operator for fluctuations  $\hat{V}$ , whose eigenvalues we have to calculate, can be read from the

potential divided by  $ev/(2\kappa)$ . Leaving the factor  $1/\kappa$  out is dictated by simplicity, since it is trivial to correct for it in the final result. We can see in the previous expressions, that the background fields  $\delta\bar{A}$  and  $\bar{\phi}$  appear divided by  $v_0$ , canceling out the multiplicative  $v_0$  appearing in their expression. Thus, since the dependence on all the constants is obvious, we can simplify the calculation of the spectrum by choosing units  $ev = v_0 = 1$  (implying  $C = 1$ ).

Furthermore, as explained in Sec. IV, we will benefit from the fact that we do not need to get involved into technicalities associated with degenerate perturbation theory, since all we need is to sum of the eigenvalues within each sector which is degenerate at leading order.

### 1. Calculation of $\delta\lambda_n$

In this subsection we will explain the calculation of  $\delta\lambda_n$  according to the formula Eq. (62). There are two terms: one coming from the diagonal Higgs-Higgs part  $\hat{V}_{11}^{(1)}$ , and the second coming from the mixed terms  $\hat{V}_{12}^{(1/2)} = \hat{V}_{21}^{(1/2)}$ .

Let us start with the contribution of the diagonal part. This comes from two terms in the fluctuation potential:

$$\int dx [(\Re(\bar{\phi}^* \varphi))^2 + 2\Im((\tilde{D}\varphi)^* \delta\bar{A}\varphi)]. \quad (A3)$$

To calculate the correction to the leading order eigenvalue  $n$ , we have to replace  $\varphi = (a_{ns} + ib_{ns})\Psi_{n,s}$  for  $n$  and  $s$  fixed. Since we are interested in the trace we only need to the coefficients that multiply  $a_{ns}^2$  and  $b_{ns}^2$ , and we can drop all mixed terms. After a simple calculation we arrive at

$$\begin{aligned} \sum_s \int dx [|\bar{\phi}|^2 |\Psi_{n,s}|^2 + 4\Im((\tilde{D}\Psi_{n,s})^* \delta\bar{A}\Psi_{n,s})] \\ \equiv \epsilon(\tilde{K}_n + K_n). \end{aligned} \quad (A4)$$

In order to perform the integral of the first term we make use of the Fourier expansion of the product  $\Psi_{n,s}^* \bar{\phi}$ :

$$V_{n,s} \equiv \Psi_{n,s}^*(x) \bar{\phi}(x) = \sqrt{\epsilon} \sum_{\vec{k}} \sum_{s'} c_{s'} X_{n,0}^{s,s'}(\vec{k}) e^{-2\pi i \vec{k} \cdot \vec{x}}. \quad (A5)$$

Now we can plug the Fourier expansion of  $V_{n,s}$  and that of its complex conjugate and perform the integration over  $x$ . Hence, we get

$$\tilde{K}_n = \sum_{\vec{k}} \sum_s \left| \sum_{s'} c_{s'} X_{n,0}^{s,s'}(\vec{k}) \right|^2 = \sum_{\vec{k}} \frac{\xi^n}{n!} e^{-\xi} \equiv \rho(n). \quad (A6)$$

Notice that due to the unitarity of  $U(\vec{k})$  the dependence on the moduli parameters  $c_s$  has dropped completely. All that was left was the norm of  $c_s$  which is fixed by the Bogomol'nyi equation to be equal to 1.

Now we proceed to calculate the integral of the second term, by applying the  $\tilde{D}$  operator and substituting the Fourier expansion of  $\delta\bar{A}$ :

$$\begin{aligned} \epsilon K_n &= 4\sqrt{n} \sum_s \int dx \Im(\delta\bar{A} \Psi_{n-1,s}^* \Psi_{n,s}) \\ &= 4\sqrt{n} \sum_s \sum_{\vec{k}} \Im(e^{i\alpha(\vec{k})} X_{n-1,n}^{ss}(\vec{k}) \bar{G}_1(\vec{k})) \\ &= -2\epsilon \sum_{\vec{k}} e^{-\xi} L_{n-1}^{(1)}(\xi) u(\vec{k}) \text{Tr}(U(\vec{k})) \end{aligned} \quad (A7)$$

where we recall that  $L_{n-1}^{(1)}$  is a generalized Laguerre polynomial and  $u(\vec{k}) = c_i^* U_{i,s}(-\vec{k}) c_s$ . The trace of the unitary matrix  $U(\vec{k})$  imposes that the Fourier component  $k_i$  should be a multiple of  $q$ . This restriction eliminates the dependence on the moduli parameters contained in  $u(\vec{k})$ . Altogether, the result becomes

$$K_n = -2q \sum_{\vec{k}} e^{-q^2 \xi} L_{n-1}^{(1)}(q^2 \xi) \quad (A8)$$

where the primed sum runs over over all integer vectors excluding  $\vec{k} = 0$ .

There is an alternative evaluation of  $K_n$  which turns out to be more useful. This follows by going back to the definition of  $K_n$  as an integral involving the operator  $\tilde{D}$  and integrating by parts, passing the operator  $\tilde{D}^\dagger$  to act on the other factors. We leave the details to the reader and give here the relation that one gets:

$$K_n = K_{n+1} - \frac{4}{\epsilon} \sum_s \int dx \Im(\tilde{\delta}^* \delta\bar{A}) |\Psi_{n,s}(x)|^2. \quad (A9)$$

The last term can be evaluated by using the second Bogomol'nyi equation. The final relation is then

$$K_n = K_{n+1} + 2(\tilde{K}_n - q). \quad (A10)$$

Using our previous result, we conclude

$$K_n = -2 \sum_{m=0}^{n-1} (\rho(m) - q) \quad (A11)$$

valid for  $n \geq 1$ . The two expressions of  $K_n$  look very different but they can be verified to give the same numerical values. By the definition it is clear that  $K_{n=0} = 0$ .

Now we need to compute the contributions to  $\lambda_n$  coming from the mixed terms  $\hat{V}_{12} = \hat{V}_{21}$ . Our first step is then precisely to give the matrix elements of this operator in the basis of eigenstates of the lowest order potential. This can be read out from the corresponding terms in the potential



$$\Im \left( \int dx (\tilde{D}\varphi)^* \bar{\phi} \delta a \right) + \int dx \Re(\varphi^* \bar{\phi}) \Im(\tilde{\partial}^* \delta a) \quad (\text{A12})$$

where  $\varphi$  has to be treated as before. On the other hand the fluctuation of the photon field  $\delta a$  has two components, labeled  $\delta a_T$  and  $\delta a_L$ , corresponding to the gauge independent and gauge dependent part. Operationally the separation follows by computing  $(\tilde{\partial}^* \delta a)$ . For the transverse part it is purely imaginary, while for the longitudinal one it is real. Obviously, the second term in Eq. (A12) involves only the transverse fluctuations, while the first term involves both types. For the transverse modes it is convenient to perform an integration by parts similar to that done before for the diagonal term. After combining it with the second term we arrive at

$$\Im \left( \int dx (\tilde{D}\varphi)^* \bar{\phi} \delta a_L \right) + \Im \left( \int dx \varphi^* (\tilde{D}^\dagger \bar{\phi}) \delta a_T \right) \quad (\text{A13})$$

which separates neatly the longitudinal and transverse contributions.

Now we are ready to obtain the matrix elements of the mixed terms. We replace  $\varphi = (a_{n,s} + ib_{n,s})\Psi_{n,s}(x)$  and  $\delta a_\eta = r_{\vec{k},\sigma}^\eta \mathcal{X}_{\vec{k},\sigma}^\eta(x)$ , where  $\eta = L, T$  and the basis vectors

$\mathcal{X}_{\vec{k},\sigma}^\eta(x)$  are given in Eq. (65). We recall that  $\vec{k}$  is restricted to positive values. Later on we will consider the contribution of  $\vec{k} = 0$ . We will use the index  $\sigma = +$  for the terms proportional to  $a_{n,s}$  and  $\sigma = -$  for the terms proportional to  $b_{n,s}$ . With this notation we will write

$$\langle n, s, \sigma | V_{12}^{(1/2)} | \vec{k}, \eta, \sigma' \rangle \equiv A_\eta(n, s, \sigma; \vec{k}, \sigma') \quad (\text{A14})$$

where  $\eta$  should be replaced by  $T$  and  $L$  for the longitudinal and transverse modes respectively.

For the calculation of the transverse modes we will also need  $X_{n1}$  given by Eq. (51), which evaluates explicitly to

$$\begin{aligned} X_{n1}^{s'}(\vec{k}) &= U_{ss'}(\vec{k}) i^{(n-1)} e^{i(n-1)\alpha(\vec{k})} Y_{n1}(\xi) \\ &= U_{ss'}(\vec{k}) i^{(n-1)} e^{i(n-1)\alpha(\vec{k})} \frac{\xi^{(n-1)/2}}{\sqrt{n!}} e^{-\xi/2} (n - \xi) \end{aligned} \quad (\text{A15})$$

valid for any  $n$ , and which defines the real quantity  $Y_{n1}(\xi)$ . With the previous definitions and substitution into the second term of Eq. (A13) we get:

$$A_T(n, s, \sigma; \vec{k}, \sigma') = \frac{\sqrt{\epsilon} Y_{n1}(\xi)}{\sqrt{2}} \Im [i^{n-1} e^{-i\pi(1-\sigma)/4} e^{i\pi(1-\sigma')/4} (e^{i\alpha(\vec{k})} U_{ss'}(\vec{k}) + \sigma' e^{i\alpha(-\vec{k})} U_{ss'}(-\vec{k})) c_{s'}]. \quad (\text{A16})$$

A similar calculation can be done for the longitudinal terms, given by the first term of Eq. (A13). The result is

$$A_L(n, s, \sigma; \vec{k}, \sigma') = \frac{\sqrt{n\epsilon} Y_{n-10}(\xi)}{\sqrt{2}} \Re [i^{n-1} e^{-i\pi(1-\sigma)/4} e^{i\pi(1-\sigma')/4} (e^{i\alpha(\vec{k})} U_{ss'}(\vec{k}) + \sigma' e^{i\alpha(-\vec{k})} U_{ss'}(-\vec{k})) c_{s'}] \quad (\text{A17})$$

where  $Y_{n0}(\xi) = \xi^{n/2} e^{-\xi/2} / \sqrt{n!}$ . The previous results are complicated and depend on the moduli parameters  $c_s$ , related to the location of the vortices (zeroes of the background Higgs field). As we will see in a minute this dependence drops when computing the contributions to the vortex mass.

With the mixed matrix elements given before it is relatively straightforward to compute the contribution to the Higgs field eigenvalue of the form  $\hat{V}_{12}^{(1/2)}(n - \hat{V}_{22}^{(0)})^{-1} \hat{V}_{21}^{(1/2)}$ . The contribution coming from transverse photons becomes

$$\begin{aligned} \sum_{\vec{k}>0} \sum_{s\sigma\sigma'} \frac{(A_T(n, s, \sigma; \vec{k}, \sigma'))^2}{(n - \xi)} &= \epsilon \sum_{\vec{k}>0} \frac{2Y_{n1}^2(\xi)}{n - \xi} \\ &= \epsilon(\rho'(n-1) - \rho'(n)) \end{aligned} \quad (\text{A18})$$

where  $\rho'(n)$  is the same value as  $\rho(n)$  excluding the  $\vec{k} = 0$  contribution:

$$\rho'(n) = \rho(n) - \delta_{n0}. \quad (\text{A19})$$

The formula (A18) is valid for  $n = 0$  if we take  $\rho'(-1) = 0$ . As mentioned previously, the computation simplifies considerably due to summation over  $s$  and  $\sigma$ , which is all we need for the vortex mass calculation. In particular,  $\sigma$  appears as a factor  $e^{-i\pi(1-\sigma)/4}$  multiplying the remaining argument of the imaginary part (which we call  $W$  for simplicity). The sum then operates in Eq. (A18) as follows:

$$(\Im(W))^2 + (\Im(iW))^2 = |W|^2. \quad (\text{A20})$$

Hence, all phases drop from  $W$ . The sum over  $\sigma'$  then simplifies as follows

$$\sum_{\sigma'} |W(\vec{k}) + \sigma' W(-\vec{k})|^2 = 2(|W(\vec{k})|^2 + |W(-\vec{k})|^2) \quad (\text{A21})$$

canceling the phase proportional to  $\alpha(\vec{k})$ . Finally, the sum over  $s$ , the unitarity of  $U_{ss'}(\vec{k})$  and the normalization of  $c_s$  removes all dependence on the vortex locations and produces a fairly simple result.

Repeating the same steps we can obtain the contribution of the longitudinal modes, given by

$$\begin{aligned} \sum_{\vec{k}>0} \sum_{s\sigma\sigma'} \frac{(A_L(n, s, \sigma; \vec{k}, \sigma'))^2}{n} &= \epsilon \sum_{\vec{k}>0} \frac{2nY_{n-10}^2(\xi)}{n} \\ &= \epsilon \rho'(n-1) \end{aligned} \quad (\text{A22})$$

valid for all values of  $n$  with the prescription  $\rho'(-1) = 0$  adopted earlier.

To complete the calculation of the mixed terms we need to consider the contribution of the  $\vec{k} = 0$  states. It comes from an expression similar to that involving longitudinal photons and is proportional to  $X_{n-10}(\vec{k} = 0)$ , which vanishes for all values except  $n = 1$ . After a little bit of algebra one obtains that the contribution is  $2\epsilon\delta_{n1}$ .

Combining everything together we get the final result for the Higgs correction

$$\delta\lambda_n = \epsilon \left( 2qn - 2 \sum_{j=0}^{n-2} \rho(j) + \delta_{n0} \right). \quad (\text{A23})$$

The second term vanishes for  $n < 2$ . Notice that the  $q$  dependence is explicit in the first term, but is also present in the second term through the metric. Finally, we stress that there is, indeed, a correction to the eigenvalues at  $n = 0$ . One can look into the details of the calculation to see that there is only one of the  $2q$  eigenvalues, which acquires a correction. The remaining zero modes can be obtained by differentiating the Bogomol'nyi solution with respect to  $c_s$ . All variations orthogonal to  $c_s$  should correspond to a zero mode since the energy does not depend on  $c_s$ . Variations of the type  $\varphi = i\vec{\phi}$  decouple entirely from the potential. This is so because they correspond to global gauge transformations  $\phi \rightarrow e^{i\theta}\phi = \phi + i\theta\phi + \dots$ . We are only left with variations of the type  $\phi = a_0\vec{\phi}$  with  $a_0$  real. Since the vector  $c_s$  is normalized to 1, a rescaling does not correspond to a new solution. Hence, we do not have an associated zero mode, and its contribution is captured by the trace and is equal to  $\epsilon$  at this order.

## 2. Calculation of $\delta\lambda_{\vec{k}}$

We are now ready to calculate the corrections to the photon eigenvalues. For transverse photons of momentum  $\vec{k}$ , the leading order eigenvalue was  $\xi$ . We will now evaluate the correction  $\delta\lambda_{\vec{k}}$ . As for the Higgs field there is a contribution coming from diagonal parts  $\hat{V}_{22}^{(1)}$ . This can be read out from the potential of fluctuations

$$\int dx |\vec{\phi}|^2 |\delta a|^2. \quad (\text{A24})$$

Now we should replace  $\delta a = \chi_{\vec{k}, \sigma'}^T$  for  $\vec{k} > 0$ . Summing over the two values of  $\sigma'$  and using the normalization of the background field  $\vec{\phi}$ , the result is just equal to  $2\epsilon$ .

The off-diagonal contribution can be easily evaluated using the matrix elements computed earlier. The result vanishes:

$$\sum_n \sum_{s\sigma\sigma'} \frac{(A_T(n, s, \sigma; \vec{k}, \sigma'))^2}{(\xi - n)} = \epsilon \sum_n \frac{2Y_{n1}^2(\xi)}{\xi - n} = 0 \quad (\text{A25})$$

so that the final result is

$$\delta\lambda_{\vec{k}} = 2\epsilon. \quad (\text{A26})$$

We might also investigate the correction to the eigenvalue corresponding to longitudinal photons. To leading order the eigenvalue vanishes since the potential is gauge invariant and the longitudinal photons are just gauge transformations. To order  $\epsilon$  the contribution coming from the diagonal term is equal to  $2\epsilon$ , as for transverse photons. The off-diagonal contribution can be easily evaluated, giving

$$\sum_n \sum_{s\sigma\sigma'} \frac{(A_L(n, s, \sigma; \vec{k}, \sigma'))^2}{(-n)} = \epsilon \sum_n \frac{2nY_{n-10}^2(\xi)}{-n} = -2\epsilon. \quad (\text{A27})$$

Hence, the sum of both contributions vanishes as it should, since the argument of gauge invariance is valid for all values of  $\epsilon$ .

A similar conclusion can be drawn for the corrections to the zero momentum photon field  $\vec{k} = 0$ . If we evaluate the correction to these zero modes, there are two contributions which cancel each other. Again this result follows from a symmetry that is valid for all values of  $\epsilon$ : translation invariance (which is part of the moduli of classical solutions).

## APPENDIX B: RENORMALIZATION OF THE LAGRANGIAN PARAMETERS

In this appendix we study the renormalization of the Lagrangian parameters and its contribution to the vortex mass. The main point is that, in order for the physical quantities to have finite values, we need to consider that the parameters in the Lagrangian are not equal to the physically defined quantities. Thus, we are led to consider the initial Lagrangian with the constants  $e, v, \lambda$  replaced by their bare values  $e_B, v_B, \lambda_B$ . In general, if one wants to construct finite Green functions one also needs a multiplicative renormalization of the fields:  $\phi \rightarrow \phi^B = Z_\phi \phi$  and  $A_\mu \rightarrow A_\mu^B = Z_A A_\mu$ . Within a given regularization method,

the physical observables expressed in terms of the bare parameters seem divergent as the regularization is removed. However, when replacing the bare parameters in terms of the renormalized ones the divergence cancels out and one gets a finite result. Thus, in order to implement the procedure one needs to express the bare quantities as a function of the renormalized ones. Typically, what one does is to select a set of physical quantities and impose that they take a prescribed form (usually the result of lowest order perturbation theory) as a function of the physical parameters. If the number of selected quantities is equal to the number of bare constants, one can solve for the latter in terms of the physical (renormalized) ones and the cutoff. These equations define the renormalization prescription or renormalization scheme. The final expression of any physical quantity as a function of the renormalized constants is different for each prescription, but the difference can be accounted for by regarding the numerical value of the constants as scheme dependent. Very often the adopted prescription is dictated by simplicity (minimal subtraction scheme, etc.) at the expense of making the relation with physical quantities more complex. This is the renormalization idea in a nutshell. Now let us apply these ideas to our particular case.

From the previous perspective, the manipulations done in Sec. II, in which we did not take renormalization into account, are valid by simply replacing the coupling constants by their bare counterparts. There are just two places in which we assumed particular values of the constants:  $\lambda = 1$  and  $\sqrt{|g^{(0)}|} = \frac{4\pi q}{e^2 v^2}$ . It makes no sense to replace these identities by the corresponding ones in terms of bare couplings. Thus, we have to maintain these relations as valid for the renormalized constants.

The theory is superrenormalizable in  $2 + 1$  dimensions so that the set of divergent diagrams is very small. Thus, it is possible to impose simple renormalization conditions on nondivergent quantities. In particular, it could be possible to assume no renormalization of the fields  $Z_\phi = Z_A = 1$ . Nevertheless, if we focus upon physical quantities as the energies, the results are not affected by the renormalization of the fields. Thus, we will consider here only the renormalization of the constants of the theory:  $\lambda = 1$ ,  $v$  and  $e$ . Hence, we should replace them by the corresponding bare quantities in the initial Lagrangian:  $\lambda_B = 1 + \delta\lambda$ ,  $v_B^2 = v^2 + \delta v^2$  and  $e_B = e + \delta e$ . Substituting these expressions into the bare potential density given in Eq. (8), we recover the original potential involving the renormalized constants, plus a correction linear in  $\hbar$ , which is called the counterterm potential:

$$\begin{aligned}
 & -\sqrt{|g|}\delta e\mathfrak{I}(A^*D\phi) + \delta e\frac{B|\phi|^2}{2} - \delta v^2\frac{e^2\sqrt{|g|}}{4}(|\phi|^2 - v^2) \\
 & + \delta(\lambda e^2)\frac{\sqrt{|g|}}{8}(|\phi|^2 - v^2)^2. \tag{B1}
 \end{aligned}$$

The expression is valid for any value of the flux and of the metric. Notice that we have combined the renormalization of  $\lambda$  and  $e$  into the combination  $(\lambda e^2)$  which appears more natural.

Starting from this point one can repeat the procedure to find the quantum Casimir energy of the vortices: find the fields that minimize the classical energy and expand around them up to quadratic fluctuations. Since the only modification is the addition of the counterterm, which is of order  $\hbar$ , there is no modification in the calculation of the quantum Casimir energies. However, there is an additional contribution to the energy coming from integral of the counterterm Eq. (B1) evaluated at the classical values of the fields. In our case, for  $q = 0$  these classical fields are  $B = 0$  and  $\phi = v$ , so that the correction vanishes. For  $q \neq 0$ , using the Bogomol'nyi equations, the result is extremely simple:

$$\frac{\delta(ev^2)}{2} \int dx B + \frac{\delta\lambda}{2A} \int dx B^2. \tag{B2}$$

To evaluate the integrals we use our parametrization  $B = \frac{2\pi q}{e} + \delta B$ , where  $\delta B$  is of order  $\epsilon$  and  $\int dx \delta B = 0$ . The resulting counterterm contribution up to order  $\epsilon$  becomes

$$\begin{aligned}
 \mathcal{E}_R &= \pi q \left( \frac{\delta(ev^2)}{e} + \frac{v^2 \delta\lambda}{2} (1 - \epsilon) \right) \\
 &= \pi q \left( \delta v^2 + \frac{v^2 \delta(\lambda e^2)}{2e^2} (1 - \epsilon) + \epsilon \frac{v^2 \delta e}{e} \right) \tag{B3}
 \end{aligned}$$

valid in all schemes. Notice that we have expressed the result in terms of the coefficients of the different terms in the counterterm potential. The choice of these coefficients depends on the renormalization scheme. In principle, only  $\delta v^2$  gets contributions from divergent diagrams, so one could take  $\delta e = \delta\lambda = 0$ . Furthermore, given that the renormalization of the parameters has to do with the ultraviolet properties of the theory, one could select a renormalization scheme in which the coefficients are independent of the boundary conditions (hence on  $q$ ) and of the metric tensor. With this choice, the counterterm contribution to the mass is linear in  $q$  and independent of the metric. In the following paragraphs we will present a particular renormalization scheme based on the effective potential, which employs much of the machinery used earlier to compute the Casimir energy.

Our prescription is based on the computation of the dependence of the energy on external background fields. If we choose space-time independent fields, what we are actually computing is the effective potential of the theory. On the basis of the previous considerations we will perform the calculation in the sector with trivial topology ( $q = 0$ ) and for large values of the area. Let us begin by taking a vanishing background vector potential and a real and constant Higgs field  $\phi(x) = \chi$ . Although this seems to

be gauge dependent, the final result is actually the same if we transform the background field by an arbitrary gauge transformation.

The effective potential, just as the energy, is the sum of three contributions. First of all, we have the classical potential. which for our background field becomes

$$V_{(0)}(\chi) = \frac{e^2 \mathcal{A}}{8} (\chi^2 - v^2)^2. \quad (\text{B4})$$

The remaining two contributions start at order  $\hbar$ . One is precisely the counterterm potential Eq. (B1) evaluated at the background field, whose form is

$$\delta V(\chi) = -\delta v^2 \frac{e^2 \mathcal{A}}{4} (\chi^2 - v^2) + \delta(\lambda e^2) \frac{\mathcal{A}}{8} (\chi^2 - v^2)^2. \quad (\text{B5})$$

The final contribution is obtained by integrating out quadratic quantum fluctuations around the background field. No gauge fixing is performed on the fluctuation fields, explaining why the result is gauge invariant. Analyzing the quadratic form, one sees that the photon acquires a mass equal to  $e\chi$ , while the Higgs field gets a mass square equal to  $\frac{3e^2}{2}\chi^2 - \frac{e^2}{2}v^2 \equiv e^2 v^2 \mu^2$ . With these considerations it is trivial to compute the resulting quantum potential contribution

$$V_Q(\chi) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sqrt{4\pi^2 \|\vec{k}\|^2 + e^2 \chi^2} + \frac{1}{2} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sqrt{4\pi^2 \|\vec{k}\|^2 + e^2 v^2 \mu^2}. \quad (\text{B6})$$

The quantity can be treated, as for the quantum energy, by analytical continuation in the complex variable  $s$ . We essentially repeat the steps that we employed in Eq. (39). For example, for the Higgs field fluctuation part we get

$$\frac{ev}{2\Gamma(s-1/2)} \int_0^\infty x^{s-3/2} e^{-x\mu^2} \mathcal{F}\left(\frac{4\pi x}{e^2 v^2 \mathcal{A}}\right). \quad (\text{B7})$$

Neglecting terms which are exponentially suppressed for large values of the area, the result becomes

$$\frac{e^3 v^3 \mathcal{A} (\mu^2)^{3/2-s}}{8\pi s-3/2}. \quad (\text{B8})$$

Adding the contribution of the photon fluctuations obtained along similar lines we obtain

$$V_Q(\chi, s) = \frac{e^3 v^3 \mathcal{A}}{8\pi(s-3/2)} (\mu^2)^{3/2-s} + 2 \left(\frac{\chi^2}{v^2}\right)^{3/2-s}. \quad (\text{B9})$$

If we set  $\chi = v$ , we recover the full calculation of the ground state quantum energy in the trivial topology sector.

On the other hand, we may set  $y = \frac{\chi^2}{v^2} - 1$  and recall that  $\mu^2 = 1 + 3y/2$  to obtain a series expansion of the previous formula in powers of  $y$ :

$$\frac{e^3 v^3 \mathcal{A}}{8\pi(s-3/2)} \left(3 + (s-3/2) \left(-\frac{7}{2}y + \frac{17}{8}(s-1/2)y^2 + \dots\right)\right). \quad (\text{B10})$$

Notice that all but the first term are analytic at any value of  $s$ . We may now set  $s = 0$  and add it to the classical potential and counterterm potential, which are both polynomials of degree 2 in  $y$ . Our renormalization conditions amount to imposing that the coefficients in  $y$  and  $y^2$  are given by the naive potential. Hence, the contribution of the counterterm must exactly cancel the effective potential coefficients. This gives two equations which allow us to fix two of the renormalization parameters.

$$\delta v^2 = -\frac{7}{4\pi} ev; \quad \delta(\lambda e^2) = \frac{17}{16\pi} \frac{e^3}{v}. \quad (\text{B11})$$

To fix the remaining renormalization of the charge, we consider a different background field configuration. This time we take  $\phi(x) = v$  and  $A_i(x) = \tilde{V}_i$ . Repeating the same procedure as before, we compute the effective potential to be

$$\frac{v^2 \mathcal{A}}{2} |\tilde{V}|^2 (e^2 + \delta(e^2)) + V_Q(\tilde{V}). \quad (\text{B12})$$

We skip the details of the calculation of the quantum energy  $V_Q(\tilde{V})$  which is elaborate but straightforward. The coefficient of  $|\tilde{V}|^2$  turns out to be

$$-\frac{e}{4v} \sum_{\vec{k}} (\|\vec{p}\|^2 + 1)^{-3/2} \longrightarrow -\frac{e^3 v \mathcal{A}}{8\pi} \quad (\text{B13})$$

with  $\vec{p} = 2\pi\vec{k}/(ev)$ . The sum is convergent, and in the large area limit tends to the expression on the right. Now imposing the renormalization condition that the coefficient of  $|\tilde{V}|^2$  equals the classical result, we find

$$\delta e = \frac{e^2}{8\pi v} \quad (\text{B14})$$

which completes our renormalization of parameters. The last step is to substitute these results onto the counterterm contribution to the quantum energy:

$$\begin{aligned} \mathcal{E}_R &= qev \left(-\frac{7}{4} + \frac{17}{32}(1-\epsilon) + \frac{\epsilon}{8}\right) \\ &= qev \left(-\frac{39}{32} - \epsilon \frac{13}{32}\right) \equiv \mathcal{E}_R^{(0)} + \epsilon \mathcal{E}_R^{(1)}. \end{aligned} \quad (\text{B15})$$

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