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Spectral dimension of bosonic string theory

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Given that the scale of quantum gravity is not experimentally accessible, one naturally resorts to mathematical consistency as a measure for a good candidate theory to replace general relativity at high energies. Reproducing the semiclassical results of black hole entropy has become a standard test for any prospective theory of quantum gravity. It is often argued that another such commonality, albeit less known, is the similar fractal behavior. It is shown that many, if not all, approaches to quantum gravity predict a spectral dimension of 2 in ultraviolet regime. In this paper, by computing the heat kernel, we show that the spectral dimension of closed bosonic string theory is 26. We discuss the implications of this disparity.

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I. INTRODUCTION

This year marks the thirtieth anniversary of the Green-Schwarz paper [1] on anomaly cancellation which convinced many theoretical physicists that string theory was a very promising candidate for unifying all the fundamental interactions in nature and prompted an era of intense research in the field. This is often called the "First Superstring Revolution." String theory provides an ultraviolet finite and order-by-order perturbative renormalizable theory of gravity. The problem of nonrenormalizability resulting from the direct quantization of general relativity is well known. Circumventing this disastrous difficulty without losing mathematical consistency is what makes string theory arguably the most interesting candidate for quantum gravity.

The dynamics of the string is described by a two-dimensional world sheet in the D-dimensional spacetime. The world sheet is central to all the physics of the string. When a closed string moves in a curved spacetime its coordinates feel the curvature. In order for there to be a consistent quantum theory, the target spacetime must be a solution to the Einstein field equations. Besides requiring general relativity to be a part of the theory, it adds corrections to it. General covariance of spacetime becomes an emergent concept in string theory.

The seemingly different approaches to quantum gravity have a few things in common, including the defining spin-2 graviton and Hawking-Bekenstein entropy of black holes. In addition, it has recently been found that causal dynamical triangulations [2], asymptotic safety [3], loop quantum gravity [4], Hořava-Lifshitz theory [5] and also Liouville quantum gravity [6] predict the same spectral dimension of 2 [7]. It has been suggested that such an agreement must contain some hints of a full theory of quantum gravity [8]. One exceptional theory is noncommutative geometry [9] which predicts spectral dimension of 3 in the ultraviolet regime.

In this context, the often-quoted result from string theory is that of Atick and Witten [10]. They studied the statistical mechanics of string theory: bosonic, type II and heterotic. Such results in the bosonic case were first obtained by Sathiapalan [11] and Kogan [12]. Here we only concentrate on the bosonic case. Their motivation for studying the thermal ensemble is to understand the underlying degrees of freedom in string theory. To this end, they compute the free energy $F = -T \ln Z$, where T is the temperature in natural units and Z is the partition function. In a free field theory in D-dimensions, for large T, the free energy per unit volume has the form,

$$\frac{FT}{V} \sim T^{D-1}. \tag{1.1}$$

The interactions will not make this any significantly lesser. In fact, below Hagaedorn temperature, the free energy grows much faster with temperature because of proliferation of string modes. However, above Hagaedorn temperature, the free energy grows linearly with temperature, i.e.,

$$\frac{FT}{V} \sim T. \tag{1.2}$$

This implies that the theory undergoes a phase transition only to act like a $(1+1)$-dimensional quantum field theory at each point of a lattice. That is to say, the high-temperature limit of the free energy in string theory is much less than in any known relativistic quantum field theory. But the effective string theory governing the high-temperature behavior is still 26 dimensional. This last point is what is not mentioned whenever this result is quoted in the context of spectral dimension.

In order to reconcile with other approaches, we compute the spectral dimension in the similar way as other approaches to quantum gravity. Besides throwing light on the

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spectral behavior of string theory, our calculations done using heat kernels precisely emphasize the last point of Atick and Witten.

II. HEAT KERNEL AND SPECTRAL DIMENSION

Besides the obvious applications in many branches of engineering, the study of heat kernel is of importance for algebraic topologists and differential geometers on the side of mathematics, and for quantum field theorists and general relativists on the side of physics. The study of the spectral theory of the Laplacian through the heat equation was, for many, popularized by Marc Kac [13].

A diffusion process on a D-dimensional smooth manifold $\mathcal{M}$ with boundary ∂M and metric $g_{\mu\nu}$ is described by a heat equation

$$\left(\frac{\partial}{\partial s}-\Delta\right)K(x,x';s)=0 \quad \text{with} \quad K(x,x',0)=\delta(x-x'), \tag{2.1}$$

where the Laplace-Beltrami operator is given as,

$$\Delta=\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu). \tag{2.2}$$

The function $K(x,x';s)$ is called the heat kernel, which satisfies certain boundary conditions and describes the probability for a random walker to go from point x to x' on the manifold in time s.

One of the basic defining properties of a manifold is its Hausdorff dimension, sometimes simply called the dimension. The spectral dimension is defined as the effective dimension of a diffusion process. In other words, it is the dimensions perceived by a random walker, a randomly moving particle. Mathematically it is defined as

$$d_s=-2\lim_{s\to 0}\frac{d\ln K(x,x';s)}{d\ln s}. \tag{2.3}$$

On a smooth manifold the spectral dimension is the same as Hausdorff dimension, but they are generally different on a fractal [14]. There are many ways of computing a heat kernel [15] and one of the most popular methods in quantum field theory is using the Feynman path integral [16].

As an example, we derive the spectral dimension of a D-dimensional Euclidean space. On a flat manifold $\mathcal{M}=\mathbb{R}^D$, the heat kernel for a scalar field of mass m [17] is given by,

$$K(x,x';s)=(4\pi s)^{-D/2}\exp\left(-\frac{(x-x')^2}{4s}-m^2 s\right) \tag{2.4}$$

where $(x-x')^2$ is essentially the square of the geodesic distance, and the associated Laplacian is

$$\Delta=-g_{\mu\nu}\nabla^\mu\nabla^\nu+m^2. \tag{2.5}$$

In order to calculate d_s, we first take the logarithm of the kernel Eq. (2.4), i.e.,

$$d_s=-2\lim_{s\to 0}\frac{d}{d\ln s}\left[-\frac{D}{2}\ln 4\pi-\frac{D}{2}\ln s-\frac{(x-x')^2}{4s}-m^2 s\right].$$

Now differentiating with respect to $\ln s$ and taking the limit as $s\to 0$ and also requiring the geodesic distance $(x-x')\to 0$, we get

$$d_s=D$$

which is same as its Hausdorff dimension as expected.

III. SPECTRAL DIMENSION OF CLOSED BOSONIC STRING THEORY

We consider the simplest and most basic class of string theories: closed bosonic string theory in Minkowski background. The graviton appears as a particular state of the closed string. It is described by the Polyakov action

$$S[g,X]=\frac{1}{2}\int d^2\sigma\sqrt{g}g^{\alpha\beta}\partial_\alpha X^\mu(\sigma)\partial_\beta X^\nu(\sigma)\eta_{\mu\nu}(X), \tag{3.1}$$

where $g_{\alpha\beta}$ is the world sheet metric with determinate g, X^μ are local coordinates on target spacetime, σ^α are the world sheet coordinates, and $\eta_{\mu\nu}$ is the D-dimensional Minkowski metric. It is invariant under world sheet diffeomorphism, Weyl rescalings of the metric and Poincaré transformations of the target space.

Invariance under Weyl transformations implies that the two-dimensional classical field theory described by the action in Eq. (3.1) is a conformal field theory. However, demanding conformal invariance after quantization leads to severe constraints on the theory and makes the theory only consistent in $D=26$ dimensions. The world sheet action can be thought of as a field theory in two dimensions with 26 scalar fields, while the target spacetime metric behaves like "coupling constants" [18].

A. Heat equation method

In order to find the spectral dimension, we need the heat kernel. One can follow the usual method and obtain the kernel by solving the heat equation constructed with a suitable Laplace-Beltrami operator. Following this method, we derive a heat equation given by,

$$\frac{\partial}{\partial\lambda}K[X,\lambda]=\frac{1}{2}\left(\eta^{\mu\nu}\frac{\delta^2}{\delta X^\mu\delta X^\nu}-\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma}\frac{\partial X^\nu}{\partial\sigma}\right)K[X,\lambda]. \tag{3.2}$$

This agrees for the ghost-free case with the results, obtained in two different ways, in Refs. [19] and [20]. The solution of this obtained using Fourier transforms is given by,

$$K(X,X';\lambda) = (4\pi\lambda)^{-13}\exp\left(-\frac{(X-X')^2}{2\lambda} - \frac{m^2\lambda}{2}\right) \quad (3.3)$$

with $m^2 \equiv \eta_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma}\frac{\partial X^\nu}{\partial\sigma}$. Substituting this kernel into Eq. (2.3), we get

$$d_s = 26. \quad (3.4)$$

B. Path integral method

One can employ the method developed by Feynman [16] to compute the kernel which satisfies the "imaginary-time" Schördinger equation. It is straightforward, although it involves tedious book keeping. This way, one generally computes the heat kernel and from which the propagator or amplitude is obtained. We do the reverse here. The ghost-free amplitude for closed bosonic string was first computed by Cohen *et al.* [21]. See [22] for closed-string propagator with ghosts. The heat kernel extracted from such an amplitude [23] is given by,

$$\begin{aligned} K[X_i,X_f;\lambda] = {} & \frac{e^{4\pi\lambda}}{\lambda^{13}}\prod_{n=1}^{\infty}[1-e^{-4\pi n\lambda}]^{-24} \\ & \times \exp\left\{\frac{-1}{4\pi\alpha'}\sum_{m=-\infty}^{\infty}\frac{2\pi m}{\sinh(2\pi m\lambda)}\right. \\ & \times [(|X_m^i|^2 + |X_m^f|^2)\cosh(2\pi m\lambda) \\ & \left. - 2\Re(X_m^i\cdot X_m^{*f})]\right\} \end{aligned} \quad (3.5)$$

where λ is the moduli parameter which plays the role of imaginary time.

We introduce the well-known Dedekind η-function defined in the upper half complex plane $\mathcal{H}$. For $s\in\mathcal{H}$ it is given by

$$\eta(s) = e^{\pi i s/12}\prod_{n=1}^{\infty}[1-e^{2\pi i n s}] \quad (3.6)$$

Its twenty-fourth power is a modular form of weight 12 which is invariant under the action of group $\mathrm{SL}(2,\mathbb{Z})$ and lies at the heart of reasoning that lead to critical dimension of 26 in bosonic string theory. We refer the interested reader to the article by Atiyah [24] for its many interesting properties. Now making a change of variable $s\to 2i\lambda$ and raising it to power -24, we get

$$\eta(2i\lambda)^{-24} = e^{4\pi\lambda}\prod_{n=1}^{\infty}[1-e^{-4\pi n\lambda}]^{-24}. \quad (3.7)$$

In the path integral notation, we identify the argument of the last exponential in Eq. (3.5) as,

$$\begin{aligned} S_P[X;\lambda] = {} & \frac{-1}{4\pi\alpha'}\sum_{m=-\infty}^{\infty}\frac{2\pi m}{\sinh(2\pi m\lambda)} \\ & \times [(|X_m^i|^2 + |X_m^f|^2)\cosh(2\pi m\lambda) - 2\Re(X_m^i\cdot X_m^{*f})]. \end{aligned} \quad (3.8)$$

Putting together these pieces, the heat kernel Eq. (3.5) takes a simple form

$$K[X_i,X_f;\lambda] = \lambda^{-13}\cdot\eta(2i\lambda)^{-24}\cdot\exp S_P[X;\lambda]. \quad (3.9)$$

The only term that contributes to the spectral dimension in this expression is λ^{-13} in the prefactor. Upon taking the logarithm and differentiating with respect to $\ln\lambda$, all but one term vanish, giving the same answer of $d_s = 26$. See the Appendix for the mathematical details. It is somewhat reassuring that Atick and Witten argue that the effective string theory governing the high-temperature behavior is indeed 26 dimensional. See [25], for a different point of view.

Since string theory involves smooth manifolds, it is expected to have the same number of spectral as well as Hausdorff dimensions. However, in the case of superstring theories, it is interesting to investigate if supersymmetry alters this smoothness of the manifold. This study will be presented elsewhere.

IV. CONCLUSIONS

Within classical physics, the role of spacetime has changed radically over time. In nonrelativistic classical physics, space is an inert background and time is a monotonically increasing variable unaffected by anything and everything. In special relativity, the distinction between space and time disappears. Whereas in general relativity, spacetime is dynamical and plays an utmost central role. Even in quantum physics, the role of spacetime varies from nonrelativistic quantum mechanics to relativistic quantum mechanics to quantum field theory. The role of spacetime in string theory is totally different from that of any other theory. String theory has symmetries which equate spacetimes of different dimensions, geometry and topology. The number of dimensions is fixed by mathematical consistency and there is a provision for reducing the number of dimensions too. Bosonic and fermionic modes "see" different number of spacetime dimensions.

Why do some quantum gravity theories behave like two-dimensional theories at very high energies? Is it a strange coincidence? Or is there some common symmetry among them that is responsible for such a result? Why is it that some other theories of quantum gravity differ from that? Or, could this approach imply the sigma model is inadequate to reveal something novel about the microstructure of spacetime in string theory? Trying to make sense of these differences is worth studying and inquiries in this direction may help us to understand the nature of quantum gravity at a deeper level.

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APPENDIX: LIMITS

We evaluate the limits of the last two terms in the heat kernel expression Eq. (3.9). Following [26], we expand Eq. (3.7) as follows

$$\eta(2i\lambda)^{-24} = e^{4\pi\lambda} + 24 + O(e^{-4\pi\lambda}). \tag{A1}$$

Taking the natural logarithm of this expression with respect to $\ln\lambda$, we have

$$\frac{d}{d\ln\lambda}\ln\left(\eta(2i\lambda)^{-24}\right) = \lambda[4\pi + O(-4\pi)]. \tag{A2}$$

Now taking the limit of this expression, we get

$$\lim_{\lambda\longrightarrow 0}\frac{d}{d\ln\lambda}\ln\left(\eta(2i\lambda)^{-24}\right) = 0. \tag{A3}$$

The last term is tackled in the same way. After some manipulations using hyperbolic identities, we can express Eq. (3.8) as below,

$$S_P[X;\lambda] = \frac{-1}{4\pi\alpha'}\sum_{m=-\infty}^{\infty}\pi m[(|X_m^f| - |X_m^i|)^2\coth(\pi m\lambda) + (|X_m^i| + |X_m^f|)^2\tanh(\pi m\lambda)], \tag{A4}$$

$$\frac{d\ln e^{S_P[X_i,X_f;\lambda]}}{d\ln\lambda} = \frac{\lambda}{4\pi\alpha'}\sum_{m=-\infty}^{\infty}\pi^2 m^2\left[\frac{(|X_m^f| - |X_m^i|)^2}{\sinh^2(\pi m\lambda)} - \frac{(|X_m^f| + |X_m^i|)^2}{\cosh^2(\pi m\lambda)}\right]. \tag{A5}$$

In the limit as $\lambda\longrightarrow 0$ and $(|X_m^f| - |X_m^i|)\longrightarrow 0$, we obtain

$$\lim_{\lambda\longrightarrow 0}\left.\frac{dS_P[X;\lambda]}{d\ln\lambda}\right|_{(|X_m^f|-|X_m^i|)\longrightarrow 0} = 0. \tag{A6}$$

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