Lessons from $f(R, R_c^2, R_m^2, \mathcal{L}_m)$ gravity: Smooth Gauss-Bonnet limit, energy-momentum conservation, and nonminimal coupling

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This paper studies a generic fourth-order theory of gravity with Lagrangian density $f(R, R_c^2, R_m^2, \mathcal{L}_m)$, where R_c^2 and R_m^2 respectively denote the square of the Ricci and Riemann tensors. By considering explicit R^2 dependence and imposing the "coherence condition" $f_{R^2} = f_{R_m^2} = -f_{R_c^2}/4$, the field equations of $f(R, R^2, R_c^2, R_m^2, \mathcal{L}_m)$ gravity can be smoothly reduced to that of $f(R, \mathcal{G}, \mathcal{L}_m)$ generalized Gauss-Bonnet gravity with $\mathcal{G}$ denoting the Gauss-Bonnet invariant. We use Noether's conservation law to study the $f(\mathcal{R}_1, \mathcal{R}_2 \ldots, \mathcal{R}_n, \mathcal{L}_m)$ model with nonminimal coupling between $\mathcal{L}_m$ and Riemannian invariants $\mathcal{R}_i$, and conjecture that the gradient of nonminimal gravitational coupling strength $\nabla^\mu f_{\mathcal{L}_m}$ is the only source for energy-momentum nonconservation. This conjecture is applied to the $f(R, R_c^2, R_m^2, \mathcal{L}_m)$ model, and the equations of continuity and nongeodesic motion of different matter contents are investigated. Finally, the field equation for Lagrangians including the traceless-Ricci square and traceless-Riemann (Weyl) square invariants is derived, the $f(R, R_c^2, R_m^2, \mathcal{L}_m)$ model is compared with the $f(R, R_c^2, R_m^2, T) + 2\kappa\mathcal{L}_m$ model, and consequences of nonminimal coupling for black hole and wormhole physics are considered.

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I. INTRODUCTION

There are two main proposals to explain the accelerated expansion of the Universe [1]. The first assumes the existence of negative-pressure dark energy as a dominant component of the cosmos [2,3]. The second approach seeks viable modifications of both general relativity (GR) and its alternatives [4,5].

Focusing on modifications of GR, the original Lagrangian density can be modified in two ways: (1) extending its dependence on the curvature invariants, and (2) considering nonminimal curvature-matter coupling. The simplest curvature-invariant modification is $f(R) + 2\kappa\mathcal{L}_m$ gravity [5,6] ($\kappa = 8\pi G/c^4 \equiv 8\pi G$ and $c = 1$ hereafter), where the isolated Ricci scalar R in the Hilbert-Einstein action is replaced by the generic function of R. In this case standard energy-momentum conservation $\nabla^\mu T_{\mu\nu} = 0$ continues to hold. Further extensions have introduced dependence on such things as the Gauss-Bonnet invariant $\mathcal{G}$ [4,7] and squares of Ricci and Riemann tensors $\{R_c^2, R_m^2\}$ [8], leading to models with Lagrangian densities like $R + f(\mathcal{G}) + 2\kappa\mathcal{L}_m$, $f(R, \mathcal{G}) + 2\kappa\mathcal{L}_m$ and $R + f(R, R_c^2, R_m^2) + 2\kappa\mathcal{L}_m$. In all these models, the spacetime geometry remains minimally coupled to the matter Lagrangian density $\mathcal{L}_m$.

On the other hand, following the spirit of nonminimal $f(R)\mathcal{L}_d$ coupling in scalar-field dark-energy models [9], for modified theories of gravity an extra term $\lambda\tilde{f}(R)\mathcal{L}_m$ was respectively added to the standard actions of GR and $f(R) + 2\kappa\mathcal{L}_m$ gravity in [10] and [11], which represents nonminimal curvature-matter coupling between R and $\mathcal{L}_m$. These ideas soon attracted a lot of attention in other modifications of GR after the work in [11], and nonminimal coupling was introduced to other gravity models such as generalized Gauss-Bonnet gravity [6,12] with terms like $\lambda f(\mathcal{G})\mathcal{L}_m$. From these initial models, some general consequences of nonminimal coupling were revealed. Most significantly, $\mathcal{L}_m$ enters the gravitational field equation directly, nonminimal coupling violates the equivalence principle, and in general, energy-momentum conservation is violated with nontrivial energy-momentum-curvature transformation. In [13], $f(R, \mathcal{L}_m)$ theory as the most generic extension of GR within the dependence of $\{R, \mathcal{L}_m\}$ was developed, while another type of nonminimal coupling, the $f(R, T) + 2\kappa\mathcal{L}_m$ model, was considered in [14].

In this paper, we consider modifications to GR from both invariant-dependence and nonminimal-coupling aspects, and introduce a new model of generic fourth-order gravity with Lagrangian density $f(R, R_c^2, R_m^2, \mathcal{L}_m)$. This can be regarded as a generalization of the $f(R, \mathcal{L}_m)$ model [13] by adding R_c^2 and R_m^2 dependence, and an extension of the $f(R, R_c^2, R_m^2) + 2\kappa\mathcal{L}_m$ model [8] by allowing nonminimal curvature-matter coupling. Among the fourteen independent algebraic invariants which can be constructed from the Riemann tensor and metric tensor [15,16], besides R we focus on Ricci square R_c^2 and Riemann square (Kretschmann scalar) R_m^2, not only because they are the two simplest square invariants (as opposed to cubic and quartic invariants [16]), but also because they provide a bridge to generalized Gauss-Bonnet theories of gravity [6] and quadratic gravity [17,18]. By studying this model, we hope to get further insights into the effects of nonminimal coupling and dependence on extra curvature invariants.

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This paper is organized as follows. First of all, the field equations for $\mathcal{L} = f(R, R_c^2, R_m^2, \mathcal{L}_m)$ gravity are derived and nonminimal couplings with $\mathcal{L}_m$ and T are compared in Sec. II. In Sec. III, we consider an explicit dependence on R^2, and introduce the condition $f_{R^2} = f_{R_m^2} = -f_{R_c^2}/4$ to smoothly transform $f(R, R^2, R_c^2, R_m^2, \mathcal{L}_m)$ gravity to the generalized Gauss-Bonnet gravity $\mathcal{L} = f(R, \mathcal{G}, \mathcal{L}_m)$; employing $\mathcal{G}$, quadratic gravity is revisited and traceless models like $\mathcal{L} = f(R, R_S^2, \mathcal{C}^2, \mathcal{L}_m)$ are discussed. In Sec. IV, we commit ourselves to understanding the energy-momentum divergence problem associated with $f(R, R_c^2, R_m^2, \mathcal{L}_m)$ gravity and most generic $\mathcal{L} = f(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n, \mathcal{L}_m)$ gravity with nonminimal coupling, as an application of which, the equations of continuity and nongeodesic motion are derived in Sec. V. Finally, in Sec. VI, two implications of nonminimal coupling for black hole physics and wormholes are discussed. In the Appendix generalized energy conditions of $f(R, \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n, \mathcal{L}_m)$ and $f(R, R_c^2, R_m^2, \mathcal{L}_m)$ gravity are considered. Throughout this paper, we adopt the sign convention $R^\alpha_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha_{\delta\beta} - \partial_\delta \Gamma^\alpha_{\gamma\beta} \cdots$ with the metric signature $(-,+++)$, and follow the straightforward metric approach rather than first-order Einstein-Palatini.

II. FIELD EQUATION AND ITS PROPERTIES

A. Action and field equations

The action we propose for a generic fourth-order theory of gravity with possibly nonminimal curvature-matter coupling[1] is

$$\mathcal{S} = \int d^4x \sqrt{-g} f(R, R_c^2, R_m^2, \mathcal{L}_m), \tag{1}$$

where R_c^2 and R_m^2 denote the square of Ricci and Riemann curvature tensor, respectively,

$$R_c^2 := R_{\alpha\beta} R^{\alpha\beta}, \qquad R_m^2 := R_{\alpha\mu\beta\nu} R^{\alpha\mu\beta\nu}. \tag{2}$$

Varying the action [Eq. (1)] with respect to the inverse metric $g^{\mu\nu}$, we get

$$\delta\mathcal{S} = \int d^4x \sqrt{-g} a \Big\{ -\tfrac{1}{2} f g_{\mu\nu} \cdot \delta g^{\mu\nu} + f_R \cdot \delta R + f_{R_c^2} \cdot \delta R_c^2 + f_{R_m^2} \cdot \delta R_m^2 + f_{\mathcal{L}_m} \cdot \delta \mathcal{L}_m \Big\}, \tag{3}$$

where $f_R := \partial f / \partial R$, $f_{R_c^2} := \partial f / \partial R_c^2$, $f_{R_m^2} := \partial f / \partial R_m^2$, and $f_{\mathcal{L}_m} := \partial f / \partial \mathcal{L}_m$. δR_c^2 and δR_m^2 can be reduced into variations of Riemann tensor,

$$\delta R_c^2 = \delta[R_{\alpha\beta} \cdot (g^{\alpha\rho} g^{\beta\sigma} R_{\rho\sigma})] = 2 R_\mu{}^\alpha R_{\alpha\nu} \delta g^{\mu\nu} + 2 R^{\mu\nu} \delta R^\alpha{}_{\mu\alpha\nu}, \tag{4}$$

$$\begin{aligned} \delta R_m^2 &= \delta[R_{\alpha\beta\gamma\epsilon} \cdot (g^{\alpha\rho} g^{\beta\sigma} g^{\gamma\zeta} g^{\epsilon\eta} R_{\rho\sigma\zeta\eta})] \\ &= 4 R_{\mu\alpha\beta\gamma} R_\nu{}^{\alpha\beta\gamma} \cdot \delta g^{\mu\nu} + 2 R^{\alpha\beta\gamma\epsilon} \cdot \\ &\quad \times (R^\rho{}_{\beta\gamma\epsilon} \delta g_{\alpha\rho} + g_{\alpha\rho} \delta R^\rho{}_{\beta\gamma\epsilon}), \end{aligned} \tag{5}$$

while $\delta R^\lambda{}_{\alpha\beta\gamma}$ traces back to $\delta\Gamma^\lambda_{\alpha\beta}$ through the Palatini identity

$$\delta R^\lambda{}_{\alpha\beta\gamma} = \nabla_\beta (\delta \Gamma^\lambda_{\gamma\alpha}) - \nabla_\gamma (\delta \Gamma^\lambda_{\beta\alpha}). \tag{6}$$

Also, as is well known, $\delta\Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\sigma} (\nabla_\alpha \delta g_{\sigma\beta} + \nabla_\beta \delta g_{\sigma\alpha} - \nabla_\sigma \delta g_{\alpha\beta})$ [19,20], and we keep in mind that when raising the indices on $\delta g_{\alpha\beta}$ a minus sign appears: $\delta g_{\alpha\beta} = -g_{\alpha\mu} g_{\beta\nu} \delta g^{\mu\nu}$. Then, Eqs. (4)–(6) yield

$$f_R \cdot \delta R \cong [f_R R_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f_R] \cdot \delta g^{\mu\nu} =: H^{(fR)}_{\mu\nu} \cdot \delta g^{\mu\nu}, \tag{7}$$

$$\begin{aligned} f_{R_c^2} \cdot \delta R_c^2 \cong [& 2 f_{R_c^2} R_\mu{}^\alpha R_{\alpha\nu} - \nabla_\alpha \nabla_\nu (R_\mu{}^\alpha f_{R_c^2}) \\ & - \nabla_\alpha \nabla_\mu (R_\nu{}^\alpha f_{R_c^2}) + \Box (R_{\mu\nu} f_{R_c^2}) \\ & + g_{\mu\nu} \nabla_\alpha \nabla_\beta (R^{\alpha\beta} f_{R_c^2})] \cdot \delta g^{\mu\nu} =: H^{(fR_c^2)}_{\mu\nu} \cdot \delta g^{\mu\nu}, \end{aligned} \tag{8}$$

and

$$\begin{aligned} f_{R_m^2} \cdot \delta R_m^2 \cong [& 2 f_{R_m^2} \cdot R_{\mu\alpha\beta\gamma} R_\nu{}^{\alpha\beta\gamma} + 4 \nabla^\beta \nabla^\alpha (R_{\alpha\mu\beta\nu} f_{R_m^2})] \cdot \delta g^{\mu\nu} \\ & =: H^{(fR_m^2)}_{\mu\nu} \cdot \delta g^{\mu\nu}. \end{aligned} \tag{9}$$

Here, $\Box \equiv \nabla^\alpha \nabla_\alpha$ represents the covariant d'Alembertian, and the symbol $\cong$ denotes an effective equivalence by neglecting a surface integral after integration by parts twice to extract $\{H^{(fR)}_{\mu\nu}, H^{(fR_c^2)}_{\mu\nu}, H^{(fR_m^2)}_{\mu\nu}\}$. Especially, Eq. (9) has utilized the combination $2\nabla^\beta \nabla^\alpha (R_{\alpha\mu\beta\nu} f_{R_m^2}) + 2\nabla^\beta \nabla^\alpha (R_{\alpha\nu\beta\mu} f_{R_m^2}) = 4\nabla^\beta \nabla^\alpha (R_{\alpha\mu\beta\nu} f_{R_m^2})$, where the symmetry of $\nabla^\beta \nabla^\alpha (R_{\alpha\mu\beta\nu} f_{R_m^2})$ under the index switch $\mu \leftrightarrow \nu$ is guaranteed by $\nabla^\beta \nabla^\alpha R_{\alpha\nu\beta\mu} = \nabla^\beta \nabla^\alpha R_{\alpha\mu\beta\nu}$, $\nabla^\alpha \nabla^\beta f_{R_m^2} = \nabla^\beta \nabla^\alpha f_{R_m^2}$ as well as the $\mu \leftrightarrow \nu$ symmetry of its remaining expanded terms. Note that in these equations, total derivatives in individual variations $\{\delta R, \delta R_c^2, \delta R_m^2\}$ are not necessarily pure divergences anymore, because the nontrivial coefficients $\{f_R, f_{R_c^2}, f_{R_m^2}\}$ will be absorbed by the variations into the nonlinear and higher-order-derivative terms in $\{H^{(fR)}_{\mu\nu}, H^{(fR_c^2)}_{\mu\nu}, H^{(fR_m^2)}_{\mu\nu}\}$.

[1] The terms *geometry-matter* coupling and *curvature-matter* coupling are both used in this paper. They are not identical: the former can be either nonminimal or minimal, while the latter by its name is always nonminimal since a curvature invariant contains at least second-order derivative of the metric tensor. Here nonminimal coupling happens between algebraic or differential Riemannian scalar invariants and $\mathcal{L}_m$, so we will mainly use curvature-matter coupling.

In the $f_{\mathcal{L}_m} \cdot \delta\mathcal{L}_m$ term in Eq. (3), we make use of the standard definition of stress-energy-momentum (SEM) density tensor in GR (e.g. [10–14]), which is introduced in accordance with minimal geometry-matter coupling and automatic energy-momentum conservation (for further discussion see Sec. IV A),

$$T_{\mu\nu} \coloneqq \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \tag{10}$$

$$= \mathcal{L}_m g_{\mu\nu} - 2\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}}. \tag{11}$$

The equivalence from Eq. (10) to Eq. (11) is built upon the common assumption that $\mathcal{L}_m$ does not explicitly depend on derivatives of the metric, $\mathcal{L}_m = \mathcal{L}_m(g_{\mu\nu}, \psi_m) \neq \mathcal{L}_m(g_{\mu\nu}, \partial_\alpha g_{\mu\nu}, \psi_m)$ with ψ_m collectively denoting all relevant matter fields.

After some work, Eqs. (3), (7), (8), (9), and (11) eventually give rise to the field equation for $f(R, R_c^2, R_m^2, \mathcal{L}_m)$ gravity:

$$-\frac{1}{2} f g_{\mu\nu} + f_R R_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) f_R + H_{\mu\nu}^{(fR_c^2)} + H_{\mu\nu}^{(fR_m^2)} = \frac{1}{2} f_{\mathcal{L}_m}(T_{\mu\nu} - \mathcal{L}_m g_{\mu\nu}), \tag{12}$$

where $H_{\mu\nu}^{(fR_c^2)}$ and $H_{\mu\nu}^{(fR_m^2)}$ were introduced in Eqs. (8) and (9) to collect all terms arising from R_c^2 and R_m^2 dependence in f,

$$\begin{aligned} H_{\mu\nu}^{(fR_c^2)} + H_{\mu\nu}^{(fR_m^2)} = {} & 2f_{R_c^2} \cdot R_\mu{}^\alpha R_{\alpha\nu} + 2f_{R_m^2} \cdot R_{\mu\alpha\beta\gamma} R_\nu{}^{\alpha\beta\gamma} \\ & - \nabla_\alpha\nabla_\nu(R_\mu{}^\alpha f_{R_c^2}) - \nabla_\alpha\nabla_\mu(R_\nu{}^\alpha f_{R_c^2}) \\ & + \Box(R_{\mu\nu} f_{R_c^2}) + g_{\mu\nu}\nabla_\alpha\nabla_\beta(R^{\alpha\beta} f_{R_c^2}) \\ & + 4\nabla^\beta\nabla^\alpha(R_{\alpha\mu\beta\nu} f_{R_m^2}). \end{aligned} \tag{13}$$

Note that $\{f, f_R, f_{R_c^2}, f_{R_m^2}\}$ herein are all functions of $(R, R_c^2, R_m^2, \mathcal{L}_m)$, and $H_{\mu\nu}^{(fR)} = f_R R_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) f_R$ has been written down directly to facilitate comparison with GR and $f(R) + 2\kappa\mathcal{L}_m$ or $f(R, \mathcal{L}_m)$ gravity. Taking the trace of Eq. (12), the simple algebraic equality $R = -T$ (where $T = g^{\mu\nu}T_{\mu\nu}$) in GR is now generalized to the following differential relation:

$$-2f + f_R R + 2f_{R_c^2} \cdot R_c^2 + 2f_{R_m^2} \cdot R_m^2 + \Box(3f_R + f_{R_c^2} R) + 2\nabla_\alpha\nabla_\beta(R^{\alpha\beta} f_{R_c^2} + 2R^{\alpha\beta} f_{R_m^2}) = f_{\mathcal{L}_m}\left(\frac{1}{2}T - 2\mathcal{L}_m\right). \tag{14}$$

Compared with Einstein's equation $R_{\mu\nu} - R g_{\mu\nu}/2 = \kappa T_{\mu\nu}$ in GR, nonlinear terms and derivatives of the metric up to fourth order have come forth and been encoded into $\{H_{\mu\nu}^{(fR)}, H_{\mu\nu}^{(fR_c^2)}, H_{\mu\nu}^{(fR_m^2)}\}$ on the left-hand side of Eq. (12). On the right-hand side, the matter Lagrangian density $\mathcal{L}_m$ explicitly participates in the field equation as a consequence of the confrontation between nonminimal curvature-matter coupling in $f(R, R_c^2, R_m^2, \mathcal{L}_m)$ and the minimal-coupling definition of $T_{\mu\nu}$ in Eq. (10). Note that not all matter terms have been moved to the right-hand side, because $-\frac{1}{2} f g_{\mu\nu}$ is still $\mathcal{L}_m$ dependent before a concrete $f(R, R_c^2, R_m^2, \mathcal{L}_m)$ model gets specified and rearranged.

Also, $f_{\mathcal{L}_m} = f_{\mathcal{L}_m}(R, R_c^2, R_m^2, \mathcal{L}_m)$ represents the gravitational coupling strength and never vanishes, so in vacuum one has $\mathcal{L}_m = 0$ and $T_{\mu\nu} = 0$, yet $f_{\mathcal{L}_m} \neq 0$. Such a generic coupling strength $f_{\mathcal{L}_m}$ will unavoidably violate Einstein's equivalence principle and the strong equivalence principle unless it reduces to a constant.

B. Field equation under minimal coupling

When the matter content is minimally coupled to the spacetime metric, the coupling coefficient $f_{\mathcal{L}_m}$ reduces to become a constant. In accordance with the gravitational coupling strength in GR, this constant is necessarily equal to Einstein's constant κ (and doubled just for scaling tradition). That is,

$$f_{\mathcal{L}_m} = \text{constant} = 2\kappa, \quad \text{and}$$
$$f(R, R_c^2, R_m^2, \mathcal{L}_m) = \tilde{f}(R, R_c^2, R_m^2) + 2\kappa\mathcal{L}_m. \tag{15}$$

We have neglected the situation when $f_{\mathcal{L}_m}$ is a pointwise scalar field $\phi = \phi(x^\alpha)$, which should be treated as a scalar-tensor theory mixed with metric gravity: in fact, $\phi(x^\alpha)\mathcal{L}_m$ is also a type of nonminimal coupling, but it goes beyond the scope of this paper and will not be discussed here. Under minimal coupling as in Eq. (15), the field equation (12) becomes (with tildes on $\tilde{f}$ omitted)

$$-\frac{1}{2} f g_{\mu\nu} + f_R R_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) f_R + H_{\mu\nu}^{(fR_c^2)} + H_{\mu\nu}^{(fR_m^2)} = \kappa T_{\mu\nu}, \tag{16}$$

which coincides with the result in [8]. The weak field limit of this minimally coupled model has been systematically studied in [21].

C. Two types of nonminimal curvature-matter coupling

Apart from the $\mathcal{L} = f(R, R_c^2, R_m^2, \mathcal{L}_m)$ model under discussion, another type of curvature-matter coupling was introduced in [14] by the $\mathcal{L} = f(R, T) + 2\kappa\mathcal{L}_m$ model, where a curvature invariant was nonminimally coupled to the trace of the SEM tensor $T = g^{\mu\nu}T_{\mu\nu}$ rather than the matter Lagrangian density $\mathcal{L}_m$. In this spirit, we consider the following nonminimally coupled action:

$$\mathcal{S} = \int d^4x\sqrt{-g}\{f(R, R_c^2, R_m^2, T) + 2\kappa\mathcal{L}_m\}. \tag{17}$$

By the standard methods we find that its field equation is

$$-\frac{1}{2}fg_{\mu\nu}+f_R\cdot R_{\mu\nu}+(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)f_R+H^{(fR_c^2)}_{\mu\nu}+H^{(fR_m^2)}_{\mu\nu}=-f_T\cdot(T_{\mu\nu}+\Theta_{\mu\nu})+\kappa T_{\mu\nu},\quad(18)$$

where $\{f,f_R,f_{R_c^2},f_T\}$ are all functions of (R,R_c^2,R_m^2,T), $H^{(fR_c^2)}_{\mu\nu}+H^{(fR_m^2)}_{\mu\nu}$ is given by Eq. (13), $-f_T(T_{\mu\nu}+\Theta_{\mu\nu})$ comes from the T dependence in $f(R,R_c^2,R_m^2,T)$, and

$$\Theta_{\mu\nu}\coloneqq\frac{g^{\alpha\beta}\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}.\quad(19)$$

As will be extensively discussed in Sec. V, for some matter sources $\mathcal{L}_m$ cannot be uniquely specified, and therefore the equations of continuity and motion based on Eq. (12) have to rely on the choice of $\mathcal{L}_m$. In such situations $T_{\mu\nu}$ is easier to set up than $\mathcal{L}_m$, so at first glance, it seems as if the new field equation (18) could avoid the flaws from nonminimal $\mathcal{L}_m$ coupling, at the cost of employing a supplementary matter tensor $\Theta_{\mu\nu}$. However, the definition of $\Theta_{\mu\nu}$ is still based on the relation $T_{\mu\nu}=\mathcal{L}_m g_{\mu\nu}-2\delta\mathcal{L}_m/\delta g^{\mu\nu}$ in Eq. (11), and explicit calculations have revealed that [14]

$$\Theta_{\mu\nu}=-2T_{\mu\nu}+g_{\mu\nu}\mathcal{L}_m-2g^{\alpha\beta}\frac{\partial^2\mathcal{L}_m}{\partial g^{\mu\nu}\partial g^{\alpha\beta}}.\quad(20)$$

Thus, both $\mathcal{L}_m$ and its second-order derivative with respect to the metric are hidden in $\Theta_{\mu\nu}$, and consequently, both $f(R,R_c^2,R_m^2,T)+2\kappa\mathcal{L}_m$ and $f(R,R_c^2,R_m^2,\mathcal{L}_m)$ theories are sensitive to the $\mathcal{L}_m$ in use. The equations of continuity and nongeodesic motion will differ for different choices of $\mathcal{L}_m$ for the same matter sources.

The $\mathcal{L}=f(R,R_c^2,R_m^2,\mathcal{L}_m)$ model and the $\mathcal{L}=f(R,R_c^2,R_m^2,T)+2\kappa\mathcal{L}_m$ model are both reasonable realizations of nonminimal curvature-matter coupling, and in this paper we have adopted the former case as a generalization of the existing $\mathcal{L}=f(R,\mathcal{L}_m)$ [13] and $\mathcal{L}=f(R,R_c^2,R_m^2)+2\kappa\mathcal{L}_m$ [8] theories. Also, it looks redundant and unnecessary to further consider the superposition of nonminimal $\mathcal{L}_m$ and T couplings, which can be depicted by the action

$$\mathcal{S}=\int d^4x\sqrt{-g}f(R,R_c^2,R_m^2,\mathcal{L}_m,T),\quad(21)$$

whose field equation is

$$-\frac{1}{2}fg_{\mu\nu}+f_R\cdot R_{\mu\nu}+(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)f_R+H^{(fR_c^2)}_{\mu\nu}+H^{(fR_m^2)}_{\mu\nu}=\frac{1}{2}f_{\mathcal{L}_m}\cdot(T_{\mu\nu}-\mathcal{L}_m g_{\mu\nu})-f_T\cdot(T_{\mu\nu}+\Theta_{\mu\nu}).\quad(22)$$

Practically it is implicitly assumed in Eq. (21) that nonminimal couplings happen between $(R,R_c^2,R_m^2,\mathcal{L}_m)$ and (R,R_c^2,R_m^2,T) respectively, and there is no matter-matter $\mathcal{L}_m$-T coupling which would cause severe theoretical complexity and physical ambiguity. In fact, $\mathcal{L}_m$ and T are not independent, as Eq. (11) implies that

$$T=g^{\alpha\beta}T_{\alpha\beta}=4\mathcal{L}_m-2g^{\alpha\beta}\frac{\delta\mathcal{L}_m}{\delta g^{\alpha\beta}}.\quad(23)$$

III. R^2 DEPENDENCE, SMOOTH TRANSITION TO GENERALIZED GAUSS-BONNET GRAVITY, AND QUADRATIC GRAVITY

Generalized (Einstein-)Gauss-Bonnet gravity is perhaps the most popular and typical situation in which there is dependence on R and the quadratic invariants $\{R_c^2,R_m^2\}$ [7,22]. However, to the best of our knowledge, there is no demonstration of how generic fourth-order model $f(R,R_c^2,R_m^2,\mathcal{L}_m)$ [or $f(R,R_c^2,R_m^2)+2\kappa\mathcal{L}_m$ model if minimally coupled [8]] may be smoothly reduced into generalized Gauss-Bonnet theories. We tackle this problem by considering an explicit dependence on R^2 in $f(R,R_c^2,R_m^2,\mathcal{L}_m)$ gravity.

A. Two generic R^2-dependent models

Based on the $f(R,R_c^2,R_m^2,\mathcal{L}_m)$ gravity, we consider the following situation with an explicit dependence on R^2:

$$\mathcal{L}=f(R,R^2,R_c^2,R_m^2,\mathcal{L}_m).\quad(24)$$

Here we have formally split the generic R dependence of $f(R,R_c^2,R_m^2,\mathcal{L}_m)$ into an R and R^2 dependence, $f_R\delta R\mapsto f_R\delta R+f_{R^2}\delta R^2$, to lay the foundation for subsequent discussion. However, this $f(R,R^2,R_c^2,R_m^2,\mathcal{L}_m)$ Lagrangian density is not more generic than $f(R,R_c^2,R_m^2,\mathcal{L}_m)$ by one more variable R^2. Absorbing f_{R^2} into $\delta R^2=2R\delta R$ by the replacement $f_R\mapsto 2Rf_{R^2}$ in Eq. (7), we learn that R^2 dependence would contribute to the field equation by

$$f_{R^2}\cdot\delta R^2\cong[2Rf_{R^2}\cdot R_{\mu\nu}+2(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)(R\cdot f_{R^2})]\cdot\delta g^{\mu\nu}\eqqcolon H^{(fR^2)}_{\mu\nu}\cdot\delta g^{\mu\nu},\quad(25)$$

and a resubstitution of $f_R\mapsto f_R+2Rf_{R^2}$ into Eq. (12) directly yields the field equation for $f(R,R^2,R_c^2,R_m^2,\mathcal{L}_m)$ gravity,

$$-\frac{1}{2}fg_{\mu\nu}+f_RR_{\mu\nu}+(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)f_R+H^{(fR^2)}_{\mu\nu}+H^{(fR_c^2)}_{\mu\nu}+H^{(fR_m^2)}_{\mu\nu}=\frac{1}{2}f_{\mathcal{L}_m}(T_{\mu\nu}-\mathcal{L}_m g_{\mu\nu}),\quad(26)$$

where $\{f,f_R,f_{R^2}\}$ and the $\{f_{R_c^2},f_{R_m^2}\}$ in $\{H^{(fR_c^2)}_{\mu\nu}+H^{(fR_m^2)}_{\mu\nu}\}$ are all functions of $(R,R^2,R_c^2,R_m^2,\mathcal{L}_m)$.

Here we have assumed no ambiguity between the R dependence and the R^2 dependence in Eq. (24). To explicitly avoid this problem, one could consider a Lagrangian density of the form

$$\mathcal{L}=\tilde{f}(R)+f(R^2,R_c^2,R_m^2,\mathcal{L}_m). \tag{27}$$

However, potential coupling between R^2 and $\mathcal{L}_m$ can still be turned around and retreated as $R-\mathcal{L}_m$ coupling, so this $\tilde{f}(R)+f(R^2,R_c^2,R_m^2,\mathcal{L}_m)$ model is still equally generic with $f(R,R_c^2,R_m^2,\mathcal{L}_m)$ as well as the $f(R,R^2,R_c^2,R_m^2,\mathcal{L}_m)$ just above. Setting $f\mapsto\tilde{f}+f$ and $f_R\mapsto\tilde{f}_R+2Rf_{R^2}$ in Eq. (12), we get the field equation for Eq. (27),

$$\begin{aligned}-\frac{1}{2}(\tilde{f}\ \ +f)g_{\mu\nu}+\tilde{f}_RR_{\mu\nu}+(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)\tilde{f}_R+H_{\mu\nu}^{(fR^2)}\\+H_{\mu\nu}^{(fR_c^2)}+H_{\mu\nu}^{(fR_m^2)}=\frac{1}{2}f_{\mathcal{L}_m}(T_{\mu\nu}-\mathcal{L}_mg_{\mu\nu}),\end{aligned} \tag{28}$$

where $\tilde{f}_R=\tilde{f}_R(R)$, $f_{R^2}=f_{R^2}(R^2,R_c^2,R_m^2,\mathcal{L}_m)$, and $\{f_{R_c^2},f_{R_m^2}\}$ remain dependent on $(R,R^2,R_c^2,R_m^2,\mathcal{L}_m)$. Moreover, Eq. (28) can instead be obtained from Eq. (26) by the replacement $f_R\mapsto\tilde{f}_R$.

For subsequent investigations, it will be sufficient to just employ the former model $\mathcal{L}=f(R,R^2,R_c^2,R_m^2,\mathcal{L}_m)$ and its field equation (26).

B. Reduced field equation with $f_{R^2}=f_{R_m^2}=-f_{R_c^2}/4$

Now recall that the second Bianchi identity $\nabla_\gamma R_{\alpha\mu\beta\nu}+\nabla_\nu R_{\alpha\mu\gamma\beta}+\nabla_\beta R_{\alpha\mu\nu\gamma}=0$ implies the following simplifications, which rewrite the derivative of a high-rank curvature tensor into that of lower-rank curvature tensors plus nonlinear algebraic terms:

$$\nabla^\alpha R_{\alpha\mu\beta\nu}=\nabla_\beta R_{\mu\nu}-\nabla_\nu R_{\mu\beta}, \tag{29}$$

$$\nabla^\alpha R_{\alpha\beta}=\frac{1}{2}\nabla_\beta R, \tag{30}$$

$$\nabla^\beta\nabla^\alpha R_{\alpha\beta}=\frac{1}{2}\Box R, \tag{31}$$

$$\nabla^\beta\nabla^\alpha R_{\alpha\mu\beta\nu}=\Box R_{\mu\nu}-\frac{1}{2}\nabla_\mu\nabla_\nu R+R_{\alpha\mu\beta\nu}R^{\alpha\beta}-R_\mu{}^\alpha R_{\alpha\nu}, \tag{32}$$

$$\nabla^\alpha\nabla_\mu R_{\alpha\nu}+\nabla^\alpha\nabla_\nu R_{\alpha\mu}=\nabla_\mu\nabla_\nu R-2R_{\alpha\mu\beta\nu}R^{\alpha\beta}+2R_\mu{}^\alpha R_{\alpha\nu}, \tag{33}$$

along with the symmetry $\nabla^\beta\nabla^\alpha R_{\alpha\mu\beta\nu}=\nabla^\beta\nabla^\alpha R_{\alpha\nu\beta\mu}$ and $\nabla^\alpha\nabla_\mu R_{\alpha\nu}+\nabla^\alpha\nabla_\nu R_{\alpha\mu}=2(\Box R_{\mu\nu}-\nabla^\beta\nabla^\alpha R_{\alpha\mu\beta\nu})$. Applying these relations to expand all the second-order covariant derivatives in Eq. (26), it turns out that we have the following theorem

Theorem.—When the coefficients $\{f_{R^2},f_{R_c^2},f_{R_m^2}\}$ satisfy the following proportionality conditions,

$$f_{R^2}=f_{R_m^2}=-\frac{1}{4}f_{R_c^2}=:F, \tag{34}$$

where $F=F(R,R^2,R_c^2,R_m^2,\mathcal{L}_m)$, then the field equation (26) reduces to

$$\begin{aligned}-\frac{1}{2}fg_{\mu\nu}+f_RR_{\mu\nu}+(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)f_R\\+\mathcal{H}_{\mu\nu}^{(F)}=\frac{1}{2}f_{\mathcal{L}_m}(T_{\mu\nu}-\mathcal{L}_mg_{\mu\nu}),\end{aligned} \tag{35}$$

where

$$\begin{aligned}\mathcal{H}_{\mu\nu}^{(F)}:=&2Rf_{R^2}\cdot R_{\mu\nu}-4f_{R_m^2}\cdot R_\mu{}^\alpha R_{\alpha\nu}+(2f_{R_c^2}+4f_{R_m^2})\cdot R_{\alpha\mu\beta\nu}R^{\alpha\beta}\\&+2f_{R_m^2}\cdot R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma}+2R(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)f_{R^2}\\&-R_\mu{}^\alpha\nabla_\alpha\nabla_\nu f_{R_c^2}-R_\nu{}^\alpha\nabla_\alpha\nabla_\mu f_{R_c^2}+R_{\mu\nu}\Box f_{R_c^2}\\&+g_{\mu\nu}\cdot R^{\alpha\beta}\nabla_\alpha\nabla_\beta f_{R_c^2}+4R_{\alpha\mu\beta\nu}\nabla^\beta\nabla^\alpha f_{R_m^2}\\&\times(\text{with }f_{R^2}=f_{R_m^2}=-f_{R_c^2}/4)\\\equiv&2RF\cdot R_{\mu\nu}-4F\cdot R_\mu{}^\alpha R_{\alpha\nu}-4F\cdot R_{\alpha\mu\beta\nu}R^{\alpha\beta}\\&+2F\cdot R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma}+2R(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu)F\\&+4R_\mu{}^\alpha\nabla_\alpha\nabla_\nu F+4R_\nu{}^\alpha\nabla_\alpha\nabla_\mu F-4R_{\mu\nu}\Box F\\&-4g_{\mu\nu}\cdot R^{\alpha\beta}\nabla_\alpha\nabla_\beta F+4R_{\alpha\mu\beta\nu}\nabla^\beta\nabla^\alpha F.\end{aligned} \tag{36}$$

$\mathcal{H}_{\mu\nu}^{(F)}\delta g^{\mu\nu}=f_F\delta F$ and second-order-derivative operators $\{\Box,\nabla_\alpha\nabla_\nu,\text{etc}\}$ only act on the scalar functions $\{f_{R^2},f_{R_c^2},f_{R_m^2}\}$ in contrast to $H_{\mu\nu}^{(fR^2)}+H_{\mu\nu}^{(fR_c^2)}+H_{\mu\nu}^{(fR_m^2)}$ in Eq. (24).[2]

Note that similar techniques have been employed in [23] to finalize the field equation of the dilaton-Gauss-Bonnet model. The simplified field equation (35) after imposing the proportionality condition Eq. (34) to Eq. (26) will serve as a bridge connecting $f(R,R^2,R_c^2,R_m^2,\mathcal{L}_m)$ gravity to generalized Gauss-Bonnet gravity. We refer to the proportionality condition Eq. (34) as the *coherence condition* to highlight the fact that it aligns the behaviors of $\{f_{R^2},f_{R_c^2},f_{R_m^2}\}$, and call F therein the *coherence function*.

C. Generalized Gauss-Bonnet gravity with nonminimal coupling

1. Generic $\mathcal{L}=f(R,\mathcal{G},\mathcal{L}_m)$ model

A nice way to realize the coherence condition Eq. (34) is to let $\{R^2,R_c^2,R_m^2\}$ participate in the action through the well-known Gauss-Bonnet invariant $\mathcal{G}$,

$$\mathcal{G}:=R^2-4R_c^2+R_m^2. \tag{37}$$

In this case, Eq. (24) reduces to become the Lagrangian density of a generalized Gauss-Bonnet gravity model allowing nonminimal curvature-matter coupling,

[2]This is also why we use the denotation $\mathcal{H}_{\mu\nu}^{(F)}$ rather than $H_{\mu\nu}^{(F)}$.

$$\mathcal{L} = f(R, \mathcal{G}, \mathcal{L}_m). \tag{38}$$

Then the proportionality in Eq. (34) is naturally satisfied with the coherence function F recognized as $f_\mathcal{G} \coloneqq \partial f/\partial\mathcal{G}$. Given $F \mapsto f_\mathcal{G}$, Eqs. (36) and (35) give rise to the field equation for $f(R, \mathcal{G}, \mathcal{L}_m)$ gravity right away,

$$-\frac{1}{2} f g_{\mu\nu} + f_R R_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) f_R + \mathcal{H}^{(\mathrm{GB})}_{\mu\nu} = \frac{1}{2} f_{\mathcal{L}_m}(T_{\mu\nu} - \mathcal{L}_m g_{\mu\nu}), \tag{39}$$

where

$$\begin{aligned}\mathcal{H}^{(\mathrm{GB})}_{\mu\nu} \coloneqq\ & 2f_\mathcal{G}\cdot RR_{\mu\nu} - 4f_\mathcal{G}\cdot R_\mu{}^\alpha R_{\alpha\nu} - 4f_\mathcal{G}\cdot R_{\alpha\mu\beta\nu}R^{\alpha\beta} \\ & + 2f_\mathcal{G}\cdot R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} + 2R(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_\mathcal{G} \\ & + 4R_\mu{}^\alpha\nabla_\alpha\nabla_\nu f_\mathcal{G} + 4R_\nu{}^\alpha\nabla_\alpha\nabla_\mu f_\mathcal{G} - 4R_{\mu\nu}\Box f_\mathcal{G} \\ & - 4g_{\mu\nu}\cdot R^{\alpha\beta}\nabla_\alpha\nabla_\beta f_\mathcal{G} + 4R_{\alpha\mu\beta\nu}\nabla^\beta\nabla^\alpha f_\mathcal{G},\end{aligned} \tag{40}$$

and $\{f, f_R, f_\mathcal{G}\}$ are all functions of $(R, \mathcal{G}, \mathcal{L}_m)$, and $\mathcal{H}^{(\mathrm{GB})}_{\mu\nu}\delta g^{\mu\nu} = f_\mathcal{G}\delta\mathcal{G}$.

2. No contributions from pure Gauss-Bonnet term

As for the $\mathcal{G}$ dependence, Eqs. (39) and (40) are best simplified when $f_\mathcal{G} = \lambda =$ constant; that is to say, $\mathcal{G}$ joins $\mathcal{L}$ straightforwardly as a pure Gauss-Bonnet term, with Lagrangian density $\mathcal{L} = f(R, \mathcal{L}_m) + \lambda\mathcal{G}$, for which Eq. (39) gives rise to the field equation [with $f = f(R, \mathcal{L}_m)$, $f_R = f_R(R, \mathcal{L}_m)$]:

$$\begin{aligned}&\lambda\cdot\left(-\frac{1}{2}\mathcal{G}g_{\mu\nu} + 2RR_{\mu\nu} - 4R_\mu{}^\alpha R_{\alpha\nu} - 4R_{\alpha\mu\beta\nu}R^{\alpha\beta} \right.\\ &\quad\left. + 2R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma}\right) - \frac{1}{2} f g_{\mu\nu} + f_R R_{\mu\nu} \\ &\quad + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) f_R = \frac{1}{2} f_{\mathcal{L}_m}(T_{\mu\nu} - \mathcal{L}_m g_{\mu\nu}).\end{aligned} \tag{41}$$

At first glance, it may seem that, after $\mathcal{G}$ decouples from $f(R, \mathcal{G}, \mathcal{L}_m)$ to form a pure term $\lambda\mathcal{G}$, the isolated covariant density $\lambda\sqrt{-g}\mathcal{G}$ would still make a difference to the field equation by the $\lambda\cdot(\cdots)$ term in Eq. (41). This result conflicts our *a priori* anticipation that, since $\mathcal{G}$ is a topological invariant, variation of the Euler-Poincaré topological density $\sqrt{-g}\mathcal{G}$ should not change the gravitational field equation. In fact, by setting $f_{R^2} = f_{R_c^2} = f_{R_m^2} = 1$ in Eqs. (8), (9), and (25), one has

$$\delta R^2/\delta g^{\mu\nu} = 2RR_{\mu\nu} + 2(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R, \tag{42}$$

$$\delta R_c^2/\delta g^{\mu\nu} = 2R_\mu{}^\alpha R_{\alpha\nu} - \nabla_\alpha\nabla_\nu R_\mu{}^\alpha - \nabla_\alpha\nabla_\mu R_\nu{}^\alpha + \Box R_{\mu\nu} + g_{\mu\nu}\cdot\nabla_\alpha\nabla_\beta R^{\alpha\beta}, \tag{43}$$

and

$$\delta R_m^2/\delta g^{\mu\nu} = 2R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} + 4\nabla^\beta\nabla^\alpha R_{\alpha\mu\beta\nu}, \tag{44}$$

which together with the Bianchi implications Eqs. (29)–(33) exactly lead to

$$\delta(\sqrt{-g}\mathcal{G})/\delta g^{\mu\nu} = -\frac{1}{2}\mathcal{G}g_{\mu\nu} + 2RR_{\mu\nu} - 4R_\mu{}^\alpha R_{\alpha\nu} - 4R_{\alpha\mu\beta\nu}R^{\alpha\beta} + 2R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma}. \tag{45}$$

Thus one can recover the term $\lambda\cdot(\cdots)$ in Eq. (41) by directly varying the quadratic invariants comprising $\mathcal{G}$.

However, in four dimensions $\mathcal{G}$ is a most special invariant among all algebraic and differential Riemannian invariants $\mathcal{R} = \mathcal{R}(g_{\alpha\beta}, R_{\alpha\mu\beta\nu}, \nabla_\gamma R_{\alpha\mu\beta\nu}, \ldots, \nabla_{\gamma_1}\nabla_{\gamma_2}, \ldots, \nabla_{\gamma_n}R_{\alpha\mu\beta\nu})$ in the sense that it respects the Bach-Lanczos identity

$$\delta\int dx^4\sqrt{-g}\mathcal{G} \equiv 0, \tag{46}$$

which prevents the Gauss-Bonnet covariant density $\lambda\sqrt{-g}\mathcal{G}$ from contributing to the field equation. This identity can be verified by carrying out the variational derivative [19,24]

$$\frac{\delta(\sqrt{-g}\mathcal{G})}{\delta g^{\mu\nu}} = \frac{\partial(\sqrt{-g}\mathcal{G})}{\partial g^{\mu\nu}} - \partial_\alpha\frac{\partial(\sqrt{-g}\mathcal{G})}{\partial(\partial_\alpha g^{\mu\nu})} + \partial_\alpha\partial_\beta\frac{\partial(\sqrt{-g}\mathcal{G})}{\partial(\partial_\alpha\partial_\beta g^{\mu\nu})} \equiv 0. \tag{47}$$

On the other hand, algebraic identities satisfied by the Riemann tensor also guarantee that $-\frac{1}{2}\mathcal{G}g_{\mu\nu} + 2RR_{\mu\nu} - 4R_\mu{}^\alpha R_{\alpha\nu} - 4R_{\alpha\mu\beta\nu}R^{\alpha\beta} + 2R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} = 0$ [19].

Hence, the $\lambda\cdot(\cdots)$ term in Eq. (41), as a remnant of degrading the generic $f(R, \mathcal{G}, \mathcal{L}_m)$ gravity and all existing generalized Gauss-Bonnet theories, is removable, and Eq. (41) for $\mathcal{L} = f(R, \mathcal{L}_m) + \lambda\mathcal{G}$ gravity finally becomes

$$-\frac{1}{2} f g_{\mu\nu} + f_R R_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu) f_R = \frac{1}{2} f_{\mathcal{L}_m}(T_{\mu\nu} - \mathcal{L}_m g_{\mu\nu}), \tag{48}$$

which coincides with the field equation of $\mathcal{L} = f(R, \mathcal{L}_m)$ gravity [13]. Although a pure Gauss-Bonnet term in the Lagrangian density cannot change the gravitational field equation $\delta(\sqrt{-g}\mathcal{L})/\delta g^{\mu\nu} = 0$, it does join the dynamical equation $\delta(\sqrt{-g}\mathcal{L})/\delta\phi = 0$ when $\mathcal{G}$ is coupled to a scalar field $\phi(x^a)$ (e.g. [23]), and can still cause nontrivial effects in other aspects (e.g. [17]).

3. Recovery of some typical models

$f(R, \mathcal{G}, \mathcal{L}_m)$ is the maximally generalized Gauss-Bonnet gravity when $\{R, \mathcal{G}, \mathcal{L}_m\}$ are the only scalar invariants taken into account, and all existing $(R, \mathcal{G}, \mathcal{L}_m)$-dependent models can be recovered as a specialized $f(R, \mathcal{G}, \mathcal{L}_m)$ gravity. For example, For a detailed review of generalized Gauss-Bonnet gravity, see [6] in which various types of nonminimal coupling are also extensively discussed.

Reference	Lagrangian density	Specialization
[7]	$\frac{R}{2\kappa^2} + f(\mathcal{G}) + \mathcal{L}_m$	$f_R \mapsto 1/(2\kappa^2)$ $f_\mathcal{G} \mapsto f_\mathcal{G}$ $f_{\mathcal{L}_m} \mapsto 1$
[12]	$\frac{R}{2} + \mathcal{L}_m + \lambda f(\mathcal{G})\mathcal{L}_m$	$f_R \mapsto 1/2$ $f_\mathcal{G} \mapsto \lambda\mathcal{L}_m f_\mathcal{G}$ $f_{\mathcal{L}_m} \mapsto 1 + \lambda f(\mathcal{G})$
[12]	$\frac{R}{2} + f(\mathcal{G}) + \mathcal{L}_m + \lambda F(\mathcal{G})\mathcal{L}_m$	$f_R \mapsto 1/2$ $f_\mathcal{G} \mapsto f_\mathcal{G} + \lambda\mathcal{L}_m F_\mathcal{G}$ $f_{\mathcal{L}_m} \mapsto 1 + \lambda F(\mathcal{G})$
[22]	$f(R, \mathcal{G}) + 2\kappa\mathcal{L}_m$	$f_R \mapsto f_R$ $f_\mathcal{G} \mapsto f_\mathcal{G}$ $f_{\mathcal{L}_m} \mapsto 2\kappa$

D. Quadratic gravity

Following the discussion of (generalized) Gauss-Bonnet gravity, we would like to revisit the simplest case with R_c^2 dependence (and R_m^2 dependence), the so-called quadratic gravity (eg. [17]):

$$\begin{aligned}\mathcal{L} &= R + \tilde{a}\cdot R^2 + \tilde{b}\cdot R_c^2 + \tilde{c}\cdot R_m^2 + \tilde{d}\cdot R_S^2 + \tilde{e}\cdot \mathcal{C}^2 + 2\kappa\mathcal{L}_m \\ &= R + (\tilde{a} - \tilde{c} - \tilde{d}/4 - 2\tilde{e}/3)\cdot R^2 \\ &\quad + (\tilde{b} + 4\tilde{c} + \tilde{d} + 2\tilde{e})\cdot R_c^2 + (\tilde{c} + \tilde{e})\cdot\mathcal{G} + 2\kappa\mathcal{L}_m \end{aligned} \tag{49}$$

$$\cong R + a\cdot R^2 + b\cdot R_c^2 + 2\kappa\mathcal{L}_m. \tag{50}$$

The first row is a general linear superposition of some popular quadratic invariants $\{R^2, R_c^2, R_m^2, R_S^2, \mathcal{C}^2\}$ with constant coefficients $\{\tilde{a}, \tilde{b}, \ldots\}$, where $\{R_S^2 = R_c^2 - R^2/4, \mathcal{C}^2 = R_m^2 - 2R_c^2 + R^2/3\}$ respectively denote the square of traceless Ricci tensor and Weyl tensor (see the next subsection). In Eq. (50) the pure Gauss-Bonnet term $(\tilde{c} + \tilde{d})\cdot\mathcal{G}$ has been neglected for reasons indicated above. Substitution of

$$f_R \mapsto 1, \qquad f_{R^2} \mapsto a, \qquad f_{R_c^2} \mapsto b, \qquad f_{R_m^2} \mapsto 0 \quad \text{and} \quad f_{\mathcal{L}_m} \mapsto 2\kappa \tag{51}$$

into Eq. (26) and Eq. (13) yields the field equation for the quadratic Lagrangian density Eq. (50),

$$\begin{aligned}&-\frac{1}{2}(R + a\cdot R^2 + b\cdot R_c^2)g_{\mu\nu} + (1 + 2aR)R_{\mu\nu} \\ &\quad + 2a(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R + H_{\mu\nu}^{(\mathrm{QRc})} = \kappa T_{\mu\nu},\end{aligned} \tag{52}$$

where

$$\begin{aligned}H_{\mu\nu}^{(\mathrm{QRc})} &= b\cdot(2R_\mu{}^\alpha R_{\alpha\nu} - \nabla_\alpha\nabla_\nu R_\mu{}^\alpha - \nabla_\alpha\nabla_\mu R_\nu{}^\alpha \\ &\quad + \Box R_{\mu\nu} + g_{\mu\nu}\nabla_\alpha\nabla_\beta R^{\alpha\beta}).\end{aligned} \tag{53}$$

Moreover, via the Bianchi implications Eq. (31) and Eq. (33), $H_{\mu\nu}^{(\mathrm{QRc})}$ can be rewritten as

$$H_{\mu\nu}^{(\mathrm{QRc})} = b\cdot\left(2R_{\alpha\mu\beta\nu}R^{\alpha\beta} + \left(\frac{1}{2}g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)R + \Box R_{\mu\nu}\right). \tag{54}$$

Using this to rewrite Eq. (52), we obtain the commonly used form of the field equation [17,18].

On the other hand, one can instead drop the Ricci square in favor of the Kretschmann scalar, and accordingly manipulate Eq. (49) via

$$\begin{aligned}\mathcal{L} &= R + (\tilde{a} + \tilde{b}/4 - \tilde{e}/6)\cdot R^2 + (\tilde{b}/4 + \tilde{c} + \tilde{d}/4 + 2\tilde{e})/2\cdot R_m^2 \\ &\quad - (\tilde{b}/4 + \tilde{d}/4 - \tilde{e}/2)\cdot\mathcal{G} + 2\kappa\mathcal{L}_m \\ &\cong R + a\cdot R^2 + b\cdot R_m^2 + 2\kappa\mathcal{L}_m.\end{aligned} \tag{55}$$

Now, substitute $f_R \mapsto 1$, $f_{R^2} \mapsto a$, $f_{R_c^2} \mapsto 0$, $f_{R_m^2} \mapsto b$ and $f_{\mathcal{L}_m} \mapsto 2\kappa$ into Eqs. (26) and (13) to obtain

$$\begin{aligned}&-\frac{1}{2}(R + a\cdot R^2 + b\cdot R_m^2)g_{\mu\nu} + (1 + 2aR)R_{\mu\nu} \\ &\quad + 2b(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)R + H_{\mu\nu}^{(\mathrm{QRm})} = \kappa T_{\mu\nu},\end{aligned} \tag{56}$$

where

$$H_{\mu\nu}^{(\mathrm{QRm})} = b\cdot(2R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} + 4\nabla^\beta\nabla^\alpha R_{\alpha\mu\beta\nu}), \tag{57}$$

and $H_{\mu\nu}^{(\mathrm{QRm})}$ can be recast by the Bianchi property Eq. (33) into

$$\begin{aligned}H_{\mu\nu}^{(\mathrm{QRm})} &= b\cdot(2R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma} + 4R_{\alpha\mu\beta\nu}R^{\alpha\beta} \\ &\quad - 4R_\mu{}^\alpha R_{\alpha\nu} + 4\Box R_{\mu\nu} - 2\nabla_\mu\nabla_\nu R).\end{aligned} \tag{58}$$

E. Field equations with traceless Ricci and Riemann squares

It is worthwhile to mention that, as is well known in Riemann geometry, many other tensors can be built algebraically out of $\{R^2, R_{\alpha\beta}, R_{\alpha\mu\beta\nu}\}$ with their squares recast into $\{R, R_c^2, R_m^2\}$, such as the traceless Ricci tensor, traceless Riemann tensor (Weyl tensor), Schouten tensor, Plebanski tensor, Bel-Robinson tensor, etc. It can be convenient or sometimes preferable for specific purposes to employ these tensors in the field equation, so in this subsection we will take a quick look at how the squares of these tensors in the Lagrangian density contribute to the gravitational field equation. It is unnecessary to exhaust all these tensors here and we will just consider the squares of traceless Ricci tensor and Weyl tensor as an example.

1. Traceless Ricci square

The traceless counterpart of Ricci tensor $S_{\alpha\beta}$ ($g^{\alpha\beta}S_{\alpha\beta} = 0$) and its square (denoted as R_S^2) is

$$S_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{4} R g_{\alpha\beta} \Rightarrow R_S^2 := S_{\alpha\beta} S^{\alpha\beta} = R_c^2 - \frac{1}{4} R^2. \quad (59)$$

Consider $f(\ldots, R_S^2)$ as a generic function of R_S^2, where $\ldots$ collects the dependence on all other possible scalar invariants, and the variation $\delta f(\ldots, R_S^2) = \delta f(\ldots, R_c^2 - R^2/4)$ yields

$$f_{R_S^2} \cdot \delta R_S^2 = f_{R_S^2} \cdot \left(\frac{\partial R_S^2}{\partial R_c^2} \delta R_c^2 + \frac{\partial R_S^2}{\partial R} \delta R \right) = f_{R_S^2} \cdot \left(\delta R_c^2 - \frac{1}{2} R \delta R \right). \quad (60)$$

Absorbing $f_{R_S^2}$ into δR_c^2 by replacing $f_{R_c^2}$ with $f_{R_S^2}$ in Eq. (8), merging $R f_{R_S^2}$ into δR by replacing f_R with $R f_{R_S^2}$ in Eq. (7), and finally replacing all Ricci tensors in $f_{R_S^2} \delta R_c^2$ and $R f_{R_S^2} \delta R$ by their traceless counterparts $R_{\alpha\beta} = S_{\alpha\beta} + R g_{\alpha\beta}/4$, then $f_{R_S^2} \cdot (\delta R_c^2 - \frac{1}{2} R \delta R) = f_{R_S^2} \cdot \delta R_S^2$ becomes

$$\begin{aligned} f_{R_S^2} \cdot \delta R_S^2 = \Big[& 2 f_{R_S^2} S_\mu{}^\alpha S_{\alpha\nu} - \frac{1}{2} R f_{R_S^2} S_{\mu\nu} - \nabla_\alpha \nabla_\nu (S_\mu{}^\alpha f_{R_S^2}) \\ & - \nabla_\alpha \nabla_\mu (S_\nu{}^\alpha f_{R_S^2}) + \Box (S_{\mu\nu} f_{R_S^2}) \\ & + g_{\mu\nu} \nabla_\alpha \nabla_\beta (S^{\alpha\beta} f_{R_S^2}) \Big] \cdot \delta g^{\mu\nu} =: H_{\mu\nu}^{(f R_S^2)} \cdot \delta g^{\mu\nu}, \end{aligned} \quad (61)$$

which is consistent with the field equation in [25]. Thus, for a Lagrangian density dependent on the traceless Ricci square $\mathcal{L} = f(\ldots, R_S^2)$, the contributions of $f_{R_S^2} \cdot \delta R_S^2$ to the field equation is just $H_{\mu\nu}^{(f R_S^2)}$ as in Eq. (61).

2. Weyl square

Being the totally traceless part of the Riemann tensor in the Ricci decomposition, the Weyl conformal tensor $C_{\alpha\beta\gamma\delta}$ ($g^{\alpha\gamma} g^{\beta\delta} C_{\alpha\beta\gamma\delta} = 0$) and its square (denoted as $\mathcal{C}^2$) are

$$\begin{aligned} C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} & + \frac{1}{2} (g_{\alpha\delta} R_{\beta\gamma} - g_{\alpha\gamma} R_{\beta\delta} + g_{\beta\gamma} R_{\alpha\delta} - g_{\beta\delta} R_{\alpha\gamma}) \\ & + \frac{1}{6} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) R \end{aligned} \quad (62)$$

and

$$\begin{aligned} \mathcal{C}^2 := C_{\alpha\mu\beta\nu} C^{\alpha\mu\beta\nu} &= R_m^2 - 2 R_c^2 + \frac{1}{3} R^2 \\ &= R_m^2 - 2 R_S^2 - \frac{1}{6} R^2 = \mathcal{G} + 2 R_c^2 - \frac{2}{3} R^2. \end{aligned} \quad (63)$$

Given a function $f(\ldots, \mathcal{C}^2) = f(\ldots, R_m^2 - 2 R_c^2 + R^2/3) = f(\ldots, R_m^2 - 2 R_S^2 - R^2/6) = f(\ldots, \mathcal{G} + 2 R_c^2 - 2 R^2/3)$, the variation $\delta f(\ldots, \mathcal{C}^2)$ yields

$$\begin{aligned} f_{\mathcal{C}^2} \cdot \delta \mathcal{C}^2 &= f_{\mathcal{C}^2} \cdot \left(\delta R_m^2 - 2 \delta R_c^2 + \frac{2}{3} R \delta R \right) \\ &= f_{\mathcal{C}^2} \cdot \left(\delta R_m^2 - 2 \delta R_S^2 - \frac{1}{3} R \delta R \right) \\ &= f_{\mathcal{C}^2} \cdot \left(\delta \mathcal{G} + 2 \delta R_c^2 - \frac{4}{3} R \delta R \right). \end{aligned} \quad (64)$$

Which of these expressions is most convenient to use will depend on which other Riemann invariants are involved in the Lagrangian density. As such we stop at this stage: the exact expression of $H_{\mu\nu}^{(f\mathcal{C}^2)} \delta g^{\mu\nu} := f_{\mathcal{C}^2} \cdot \delta \mathcal{C}^2$ depends on which expansion we choose for $\mathcal{C}^2$.

IV. NONMINIMAL COUPLING AND ENERGY-MOMENTUM DIVERGENCE

From this section on, we switch our attention to another important aspect of $\mathcal{L} = f(R, R_c^2, R_m^2, \mathcal{L}_m)$ gravity: the stress-energy-momentum-conservation problem. Taking the contravariant derivative of the field equation (12), we find

$$\begin{aligned} f_{\mathcal{L}_m} \nabla^\mu T_{\mu\nu} = & (\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu}) \nabla^\mu f_{\mathcal{L}_m} - f_R \nabla_\nu R - f_{R_c^2} \nabla_\nu R_c^2 \\ & - f_{R_m^2} \nabla_\nu R_m^2 + 2 \nabla^\mu H_{\mu\nu}^{(fR)} + 2 \nabla^\mu H_{\mu\nu}^{(f R_c^2)} \\ & + 2 \nabla^\mu H_{\mu\nu}^{(f R_m^2)}, \end{aligned} \quad (65)$$

where $\{f, f_R, f_{R_c^2}, f_{R_m^2}\}$ remain as functions of the invariants $(R, R_c^2, R_m^2, \mathcal{L}_m)$, and $\{H_{\mu\nu}^{(fR)}, H_{\mu\nu}^{(f R_c^2)}, H_{\mu\nu}^{(f R_m^2)}\}$ have already been concretized in Eqs. (7)–(9). However, despite the extended variable dependence in $f_R(R, R_c^2, R_m^2, \mathcal{L}_m)$ as opposed to $f(R) + 2\kappa \mathcal{L}_m$ gravity, we still have[3]

$$\begin{aligned} \frac{1}{2} (-f_R \nabla_\nu R + 2 \nabla^\mu H_{\mu\nu}^{(fR)}) = & - f_R \nabla^\mu \left(\frac{1}{2} R g_{\mu\nu} \right) \\ & + \nabla^\mu (f_R \cdot R_{\mu\nu}) + (\nabla_\nu \Box - \Box \nabla_\nu) f_R = 0. \end{aligned} \quad (66)$$

It vanishes as a consequence of the contracted Bianchi identity $\nabla^\mu (R_{\mu\nu} - R g_{\mu\nu}/2) = 0$ and the third-order-derivative commutation relation $(\Box \nabla_\nu - \nabla_\nu \Box) f_R = R_{\mu\nu} \nabla^\nu f_R$. Thus, Eq. (65) further reduces to

$$\begin{aligned} f_{\mathcal{L}_m} \nabla^\mu T_{\mu\nu} = & (\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu}) \nabla^\mu f_{\mathcal{L}_m} - f_{R_c^2} \nabla_\nu R_c^2 - f_{R_m^2} \nabla_\nu R_m^2 \\ & + 2 \nabla^\mu H_{\mu\nu}^{(f R_c^2)} + 2 \nabla^\mu H_{\mu\nu}^{(f R_m^2)}, \end{aligned} \quad (67)$$

which constitutes the equation of energy-momentum divergence in $f(R, R_c^2, R_m^2, \mathcal{L}_m)$ gravity. It can be regarded as a

[3] This is actually the stress-energy-momentum conservation condition of $f(R)$ gravity with Lagrangian density $\mathcal{L} = f(R) + 2\kappa \mathcal{L}_m$ and field equation $-f(R) g_{\mu\nu}/2 + f_R R_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f_R = \kappa T_{\mu\nu}$, except that $f_R = f_R(R)$.

generalization of the following divergence equation in $f(R, \mathcal{L}_m)$ gravity [13],

$$\nabla^\mu T_{\mu\nu} = (\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu})\nabla^\mu \ln f_{\mathcal{L}_m}, \tag{68}$$

with $\nabla^\mu \ln f_{\mathcal{L}_m} \equiv f_{\mathcal{L}_m}^{-1}\nabla^\mu f_{\mathcal{L}_m}$, which in turn can be recovered from Eq. (67) by setting $f_{R_c^2} = 0 = f_{R_m^2}$.

In standard GR, $\nabla^\mu T_{\mu\nu} = 0$ is the mathematical expression of conservation of stress-energy momentum. However, for our models it is clear that this does not vanish and so this fundamental conservation law does not hold in the standard form. Then, how do we understand the energy-momentum nonconservation/divergence equation (67)? Is it further reducible and how does it influence the equations of continuity and motion given concrete matter sources? We will investigate these questions in a more generic framework.

A. Automatic energy-momentum conservation under minimal coupling

Consider a generic gravitational Lagrangian $\mathcal{L}_G = f(\mathcal{R})$, where $f(\mathcal{R})$ is an arbitrary function of an $(n+2)$-order algebraic $(n=0)$ or differential $(n \geq 1)$ Riemannian invariant $\mathcal{R}$:

$$\mathcal{R} = \mathcal{R}(g_{\alpha\beta}, R_{\alpha\mu\beta\nu}, \nabla_\gamma R_{\alpha\mu\beta\nu}, \ldots, \nabla_{\gamma_1}\nabla_{\gamma_2}\ldots\nabla_{\gamma_n} R_{\alpha\mu\beta\nu}), \tag{69}$$

so that variational derivative of the covariant density $\sqrt{-g}\mathcal{L}_G$ will lead to a $(2n+4)$-order model of gravity. Such an $\mathcal{L}_G = f(\mathcal{R})$ is still a covariant invariant for which Noether's conservation law would yield [26]

$$\nabla^\mu\left(\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}f(\mathcal{R}))}{\delta g^{\mu\nu}}\right) = 0, \tag{70}$$

which can be expanded into

$$f_\mathcal{R}(\mathcal{R})\cdot\nabla_\nu\mathcal{R} = 2\nabla^\mu H_{\mu\nu}^{(f\mathcal{R})} \quad \text{with} \quad H_{\mu\nu}^{(f\mathcal{R})}\cdot\delta g^{\mu\nu} \coloneqq f_\mathcal{R}\cdot\delta\mathcal{R}, \tag{71}$$

where $H_{\mu\nu}^{(f\mathcal{R})}$ is defined the same way as $\{H_{\mu\nu}^{(fR)}, H_{\mu\nu}^{(fR_c^2)}, H_{\mu\nu}^{(fR_m^2)}\}$ in Eqs. (7)–(9). It absorbs $f_\mathcal{R}$ into $\delta\mathcal{R}$ and collects all nonlinear and higher-order terms generated by $f_\mathcal{R}\cdot\delta\mathcal{R}$.

These results can be directly generalized to the situation where $\mathcal{L}_G$ relies on multiple Riemannian invariants, $\mathcal{L}_G = f(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_p) \equiv \mathcal{L}_G(g_{\alpha\beta}, R_{\alpha\mu\beta\nu}, \nabla_\gamma R_{\alpha\mu\beta\nu}, \ldots, \nabla_{\gamma_1}\nabla_{\gamma_2}\ldots\nabla_{\gamma_q} R_{\alpha\mu\beta\nu})$, and we have

$$\sum_i f_{\mathcal{R}_i}\nabla_\nu\mathcal{R}_i = 2\sum_i \nabla^\mu H_{\mu\nu}^{(f\mathcal{R}_i)} \quad \text{with} \quad H_{\mu\nu}^{(f\mathcal{R}_i)}\cdot\delta g^{\mu\nu} \coloneqq f_{\mathcal{R}_i}\cdot\delta\mathcal{R}_i, \tag{72}$$

where $f_{\mathcal{R}_i} = f_{\mathcal{R}_i}(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_p)$, with each $\mathcal{R}_i$ given by Eq. (69) to certain order derivatives of Riemann tensor, and $H_{\mu\nu}^{(f\mathcal{R}_i)} = H_{\mu\nu}^{(f\mathcal{R}_i)}(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_p)$ absorbs $f_{\mathcal{R}_i}$ into $\delta\mathcal{R}_i$.

Since $f(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_p)$ is a purely geometric entity solely dependent on the metric and derivatives of Riemann tensor, Eqs. (71) and (72) arising from Noether's theorem are also called the "generalized (contracted) Bianchi identities" [26,27]. As the simplest example, when $f(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_p) = R$, Eq. (71) or Eq. (72) immediately reproduces the standard contracted Bianchi identity $\nabla^\mu(R_{\mu\nu} - Rg_{\mu\nu}/2) = 0$ which is often used in GR.

On the other hand, for the matter Lagrangian density $\mathcal{L}_m$, Noether's conservation law yields

$$\nabla^\mu\left(\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}\right) = 0 = -\frac{1}{2}\nabla^\mu T_{\mu\nu} \quad \text{with} \quad T_{\mu\nu} \coloneqq \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}, \tag{73}$$

where $T_{\mu\nu}$ is the standard stress-energy-momentum (SEM) tensor as in Eq. (10). This way of defining $T_{\mu\nu}$ from Noether's law therefore naturally guarantees energy-momentum conservation $\nabla^\mu T_{\mu\nu} = 0$. Moreover, in the case of minimal coupling, it is unnecessary to consider a covariant matter density of the form $\sqrt{-g}h(\mathcal{L}_m)$, since $h(\mathcal{L}_m)$ can always be treated as a whole, $h(\mathcal{L}_m) \mapsto \tilde{\mathcal{L}}_m$.

Hence, for a generic Lagrangian density where $\mathcal{L}_m$ is minimally coupled to the spacetime geometry,

$$\mathcal{L} = \mathcal{L}_G + 2\kappa\mathcal{L}_m = f(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_p) + 2\kappa\mathcal{L}_m, \tag{74}$$

and whose field equation arises from extremizing the action or equivalently $\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}} = 0$:

$$-\frac{1}{2}fg_{\mu\nu} + \sum_i H_{\mu\nu}^{(f\mathcal{R}_i)} = \kappa T_{\mu\nu}, \tag{75}$$

the generalized contracted Bianchi identities Eq. (72) for pure geometric $\mathcal{L}_G$ together with the Noether-type definition of $T_{\mu\nu}$ in Eq. (73) yield that contravariant derivatives of the left(geometry)- and right(matter)-hand side of the field equation (75) vanish *independently*.[4] This ensures

[4]Instead of directly starting from Eq. (10), one can consider $T_{\mu\nu}$ from the perspective of diffeomorphism (or gauge) invariance by requiring that the total action $\mathcal{S}_G + \mathcal{S}_m$ be invariant under an arbitrary and infinitesimal active transformation $g_{\mu\nu} \mapsto g_{\mu\nu} + \delta_\zeta g_{\mu\nu} = g_{\mu\nu} + \nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu$, where ζ^μ vanishes at the boundary,

$$\delta\mathcal{S}_m = -\frac{1}{2}\delta\int d^4x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu} = \delta\int d^4x\sqrt{-g}(\nabla^\mu T_{\mu\nu})\zeta^\nu. \tag{76}$$

Now the automatic conservation $\nabla^\mu T_{\mu\nu} = 0$ would become a consequence of the (generalized) Bianchi identities which arise from the diffeomorphism invariance of $\mathcal{S}_G$. Both ways trace back to Noether's law.

automatic fulfillment of energy-momentum conservation in any minimally coupled gravity theories of the form Eqs. (74) and (75), such as $\mathcal{L} = f(R, R_c^2, R_m^2) + 2\kappa\mathcal{L}_m$ gravity and $\mathcal{L} = f(R, \mathcal{G}) + 2\kappa\mathcal{L}_m$ gravity.

B. Divergence of SEM tensor under nonminimal coupling

Now consider a generic Lagrangian density $\mathcal{L} = f(\mathcal{R}_1, \ldots, \mathcal{R}_p, \mathcal{L}_m)$ which allows nonminimal coupling between $\mathcal{L}_m$ and Riemannian invariants $\mathcal{R}_i$. Noether's law yields the following equation for the divergence of the energy-momentum tensor:

$$\nabla^\mu \left(\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g} f(\mathcal{R}_1, \ldots, \mathcal{R}_p, \mathcal{L}_m))}{\delta g^{\mu\nu}} \right) = 0, \tag{77}$$

with expansion

$$f_{\mathcal{L}_m} \nabla^\mu T_{\mu\nu} = (\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu}) \nabla^\mu f_{\mathcal{L}_m} - \sum_i f_{\mathcal{R}_i} \nabla_\nu \mathcal{R}_i + 2 \sum_i \nabla^\mu H_{\mu\nu}^{(f\mathcal{R}_i)}, \tag{78}$$

where $\{f_{\mathcal{L}_m}, f_{\mathcal{R}_i}\}$ are all dependent on $(\mathcal{R}_1 \ldots, \mathcal{R}_p, \mathcal{L}_m)$, and $H_{\mu\nu}^{(f\mathcal{R}_i)} \delta g^{\mu\nu} \coloneqq f_{\mathcal{R}_i} \delta \mathcal{R}_i$ as usual. Note that "conservation" of $\sqrt{-g} f(, \ldots, \mathcal{R}_p, \mathcal{L}_m)$ yields an unavoidable "divergence" term $(\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu}) \nabla^\mu f_{\mathcal{L}_m}$ essentially because of how $T_{\mu\nu}$ was defined; that is to say, for the nonminimally coupled $\mathcal{L} = f(\mathcal{R}_1, \ldots, \mathcal{R}_p, \mathcal{L}_m)$ under discussion, we have continued to use the definition of $T_{\mu\nu}$ from Eq. (73) which was adapted to minimal coupling. Also, for $\mathcal{L} = f(R, \mathcal{R}_1, \ldots, \mathcal{R}_p, \mathcal{L}_m)$ gravity where the first invariant is identified as the Ricci scalar, the same argument as Eq. (66) yields that $-f_R \nabla_\nu R + H_{\mu\nu}^{(fR)} = 0$ for $f_R = f_R(R, \mathcal{R}_1, \ldots, \mathcal{R}_p, \mathcal{L}_m)$.

For the moment, we cannot directly use Eq. (72) to eliminate $-\sum_i f_{\mathcal{R}_i} \nabla_\nu \mathcal{R}_i$ by $2\sum_i \nabla^\mu H_{\mu\nu}^{(f\mathcal{R}_i)}$ in Eq. (78) as they are no longer purely geometric entities. In principle, the coefficient $f_{\mathcal{R}_i} = f_{\mathcal{R}_i}(\mathcal{R}_1, \ldots, \mathcal{R}_p, \mathcal{L}_m)$ allows for arbitrary dependence on $\mathcal{L}_m$, and this complexity gets even further promoted after taking the contravariant derivative of the effective tensor $H_{\mu\nu}^{(f\mathcal{R}_i)}(f_{\mathcal{R}_i})$. Also, note that, for the Lagrangian density $\mathcal{L} = f(R, R_c^2, R_m^2, \mathcal{L}_m)$ and $\mathcal{L} = f(R, \mathcal{L}_m)$, the generic result Eq. (78) soon recovers Eqs. (65) and (68), which were obtained in an alternative way from directly taking contravariant derivatives of their field equation.

As we have already learned, in Eq. (78) the term $(\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu}) \nabla^\mu f_{\mathcal{L}_m}$ originates from the contradiction between the nonminimal $\mathcal{R}_i - \mathcal{L}_m$ coupling and the minimal definition of $T_{\mu\nu}$. However, how can we understand the other divergence terms $-\sum_i f_{\mathcal{R}_i} \nabla_\nu \mathcal{R}_i$ and $2\sum_i \nabla^\mu H_{\mu\nu}^{(f\mathcal{R}_i)}$? Fortunately, investigations of $\mathcal{L} = \tilde{f}(\mathcal{R}) + 2\kappa\mathcal{L}_m + f(\mathcal{R})\mathcal{L}_m$ gravity shed some light on this question.

C. Lessons from $\tilde{f}(\mathcal{R}_i) + 2\kappa\mathcal{L}_m + f(\mathcal{R}_i)\mathcal{L}_m$ model

Now, consider a further specialized model with Lagrangian density

$$\mathcal{L} = \tilde{f}(\mathcal{R}_1, \ldots, \mathcal{R}_p) + 2\kappa\mathcal{L}_m + f(\mathcal{R}_1, \ldots, \mathcal{R}_q) \cdot \mathcal{L}_m. \tag{79}$$

Section IV A has shown us that energy-momentum conservation (divergence freeness) is automatically satisfied for the minimally coupled component $\tilde{f}(\mathcal{R}_1, \ldots, \mathcal{R}_p) + 2\kappa\mathcal{L}_m$, so we just need to concentrate on the nonminimally coupled term $f(\mathcal{R}_1, \ldots, \mathcal{R}_q) \cdot \mathcal{L}_m$. Following the discussion in Sec. IV B just above, treat $f(\mathcal{R}_1, \ldots, \mathcal{R}_q) \cdot \mathcal{L}_m$ as an invariant, so that Noether conservation of the covariant Lagrangian density $\sqrt{-g} f(\mathcal{R}_1, \ldots, \mathcal{R}_q) \cdot \mathcal{L}_m$ yields

$$\nabla^\mu \left(\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g} f(\mathcal{R}_1, \ldots, \mathcal{R}_q) \cdot \mathcal{L}_m)}{\delta g^{\mu\nu}} \right) = 0, \tag{80}$$

which in turn implies that

$$f \nabla^\mu T_{\mu\nu} = (\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu}) \nabla^\mu f - \sum_i f_{\mathcal{R}_i}(\mathcal{R}_1, \ldots, \mathcal{R}_q) \times \cdot \nabla_\nu \mathcal{R}_i + 2 \sum_i \nabla^\mu \left(\frac{\mathcal{L}_m f_{\mathcal{R}_i} \cdot \delta \mathcal{R}_i}{\delta g^{\mu\nu}} \right). \tag{81}$$

Note that in the last term, $\mathcal{L}_m f_{\mathcal{R}_i}(\mathcal{R}_1, \ldots, \mathcal{R}_q) \cdot \delta \mathcal{R}_i$ acts as a unity rather than a triple multiplication and *cannot* be expanded via the product rule when acted upon by ∇^μ: In fact, $\mathcal{L}_m f_{\mathcal{R}_i}(\mathcal{R}_1, \ldots, \mathcal{R}_q) \cdot \delta \mathcal{R}_i \eqqcolon H_{\mu\nu}^{(\mathcal{L}_m f \mathcal{R}_i)} \cdot \delta g^{\mu\nu}$ and thus $\mathcal{L}_m f_{\mathcal{R}_i}$ is merged into $\delta \mathcal{R}_i$.

Now recall that, based on the Petrov and Serge classifications, there are 14 independent algebraic Riemannian invariants $\mathcal{I} = \mathcal{I}(g_{\alpha\beta}, R_{\alpha\mu\beta\nu})$ characterizing a four-dimensional spacetime [15,16], among which nine are of even parity and five are of odd parity, though this minimum set can be slightly expanded after considering the matter content. As a special example of Eq. (81), energy-momentum divergence of the nonminimally coupled Lagrangian $f(\mathcal{I}_1, \ldots, \mathcal{I}_9) \cdot \mathcal{L}_m$ was studied in [28], where $\{\mathcal{I}_1, \ldots, \mathcal{I}_9\}$ refer to the nine parity-even algebraic Riemannian invariants. Explicit calculations of $H_{\mu\nu}^{(\mathcal{L}_m f \mathcal{I}_i)}$ and $\nabla^\mu H_{\mu\nu}^{(\mathcal{L}_m f \mathcal{I}_i)}$ show that [28], for each individual $\mathcal{I}_i$ in $\mathcal{L} = f(\mathcal{I}_i, \mathcal{L}_m)$,

$$-f_{\mathcal{I}_i}(\mathcal{I}_i) \cdot \nabla_\nu \mathcal{I}_i + 2 \nabla^\mu \left(\frac{\mathcal{L}_m f_{\mathcal{I}_i}(\mathcal{I}_i) \cdot \delta \mathcal{I}_i}{\delta g^{\mu\nu}} \right) = 0, \tag{82}$$

and most generally for $f(\mathcal{I}_1, \ldots, \mathcal{I}_9) \cdot \mathcal{L}_m$ with an arbitrary multiple dependence of these nine invariants,

$$-\sum_i f_{\mathcal{I}_i}(\mathcal{I}_1, \ldots, \mathcal{I}_9) \cdot \nabla_\nu \mathcal{I}_i + 2 \sum_i \nabla^\mu \left(\frac{\mathcal{L}_m f_{\mathcal{I}_i}(\mathcal{I}_1, \ldots, \mathcal{I}_9) \cdot \delta \mathcal{I}_i}{\delta g^{\mu\nu}} \right) = 0. \tag{83}$$

Hence, the equation of energy-momentum divergence for $\mathcal{L} = \tilde{f}(\mathcal{I}_1, \ldots, \mathcal{I}_9) + 2\kappa\mathcal{L}_m + f(\mathcal{I}_1, \ldots, \mathcal{I}_9) \cdot \mathcal{L}_m$ gravity finally becomes

$$f(\mathcal{I}_1, \ldots, \mathcal{I}_9) \cdot \nabla^\mu T_{\mu\nu} = (\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu}) \cdot \nabla^\mu f(\mathcal{I}_1, \ldots, \mathcal{I}_9). \tag{84}$$

D. Conjecture for energy-momentum divergence

Now, let us summarize the facts we have confirmed so far:

(1) In the simplest $\mathcal{L} = f(R, \mathcal{L}_m)$ gravity [13], one has $-f_R \nabla_\nu R + 2\nabla^\mu H_{\mu\nu}^{(fR)} = 0$, so R dependence in $\mathcal{L} = f$ makes no contribution and $(\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu})\nabla^\mu f_{\mathcal{L}_m}$ is the only energy-momentum divergence term.

(2) In $\mathcal{L} = f(R, \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_p, \mathcal{L}_m)$ gravity, $-f_R \nabla_\nu R + 2\nabla^\mu H_{\mu\nu}^{(fR)} = 0$ for $f_R = f_R(R, \mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_p, \mathcal{L}_m)$.

(3) In $\mathcal{L} = \tilde{f}(\mathcal{I}_1, \ldots, \mathcal{I}_9) + 2\kappa\mathcal{L}_m + f(\mathcal{I}_1, \ldots, \mathcal{I}_9) \cdot \mathcal{L}_m$ gravity [28], one has individually $-f_{\mathcal{I}_i}(\mathcal{I}_i) \cdot \nabla_\nu \mathcal{I}_i + 2\nabla^\mu H_{\mu\nu}^{(\mathcal{L}_m f \mathcal{I} i)} = 0$ and collectively $-\sum_i f_{\mathcal{I}}(\mathcal{I}_i) \cdot \nabla_\nu \mathcal{I}_i + 2\sum_i \nabla^\mu H_{\mu\nu}^{(\mathcal{L}_m f \mathcal{I} i)} = 0$, so $(\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu}) \nabla^\mu f_{\mathcal{L}_m}$ is the only nonconservation term, while $\mathcal{I}_i$ dependence in $f \cdot \mathcal{L}_m$ makes no contribution.

(4) In the case of minimal coupling, all algebraic and differential Riemannian invariants $\mathcal{R}_i$ act equally and indiscriminately in front of Noether's conservation law and generalized Bianchi identities.

Starting with these results, the belief that for the situation of generic nonminimal curvature-matter coupling all Riemannian invariants continue to play equal roles in energy-momentum conservation/divergence leads us to propose the following.

1. Weak conjecture

Consider a Lagrangian density allowing generic nonminimal coupling between the matter density $\mathcal{L}_m$ and Riemannian invariants $\mathcal{R}$,

$$\mathcal{L} = f(\mathcal{R}_1, \mathcal{R}_2 \ldots, \mathcal{R}_n, \mathcal{L}_m), \tag{85}$$

where

$$\mathcal{R}_i = \mathcal{R}_i(g_{\alpha\beta}, R_{\alpha\mu\beta\nu}, \nabla_\gamma R_{\alpha\mu\beta\nu}, \ldots, \nabla_{\gamma_1} \nabla_{\gamma_2} \ldots \nabla_{\gamma_m} R_{\alpha\mu\beta\nu}).$$

Then contributions from the $\mathcal{R}_i$ dependence of $\mathcal{L} = f$ in the Noether-induced divergence equation cancel out collectively,

$$-\sum_i f_{\mathcal{R}_i} \cdot \nabla_\nu \mathcal{R}_i + 2\sum_i \nabla^\mu H_{\mu\nu}^{(f\mathcal{R}_i)} = 0, \tag{86}$$

and the equation of energy-momentum conservation/divergence takes the form[5]

$$f_{\mathcal{L}_m} \cdot \nabla^\mu T_{\mu\nu} = (\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu}) \cdot \nabla^\mu f_{\mathcal{L}_m}, \tag{87}$$

where $H_{\mu\nu}^{(f\mathcal{R}_i)} \coloneqq \frac{f_{\mathcal{R}_i}(\mathcal{R}_1, \ldots, \mathcal{L}_m) \cdot \delta\mathcal{R}_i}{\delta g^{\mu\nu}}$, $f_{\mathcal{R}_i} = f_{\mathcal{R}_i}(\mathcal{R}_1, \ldots, \mathcal{L}_m)$, and $f_{\mathcal{L}_m} = f_{\mathcal{L}_m}(\mathcal{R}_1, \ldots, \mathcal{R}_n, \mathcal{L}_m)$.

Moreover, inspired by the behavior of R in Eq. (66) that $-f_R \nabla_\nu R + 2\nabla^\mu H_{\mu\nu}^{(fR)} = 0$ in spite of $f_R = f_R(R, R_c^2, R_m^2, \mathcal{L}_m)$, we further promote the weak conjecture to the following.

2. Strong conjecture

For every Riemannian invariant $\mathcal{R}_i$ in $\mathcal{L} = f(\mathcal{R}_1, \mathcal{R}_2 \ldots, \mathcal{R}_n, \mathcal{L}_m)$, the divergence terms arising from each $\mathcal{R}_i$ dependence in $\mathcal{L} = f$ cancel out *individually*,

$$-f_{\mathcal{R}_i} \cdot \nabla_\nu \mathcal{R}_i + 2\nabla^\mu H_{\mu\nu}^{(f\mathcal{R}_i)} = 0, \tag{88}$$

and the equation of energy-momentum conservation/divergence remains the same as in Eq. (87),

$$f_{\mathcal{L}_m} \cdot \nabla^\mu T_{\mu\nu} = (\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu}) \cdot \nabla^\mu f_{\mathcal{L}_m}.$$

Specifically, when the possible nonminimal coupling reduces to ordinary minimal coupling, Eq. (85) will be specialized into $\mathcal{L} = f(\mathcal{R}_1, \ldots, \mathcal{R}_n) + 2\kappa\mathcal{L}_m$ as in Eq. (74), so Eqs. (86) and (88) in the weak conjecture are naturally satisfied because of the generalized Bianchi identities Eqs. (71) and (72). Also, if the conjecture were correct, then the generalized Bianchi identities Eqs. (71) and (72) could be generalized again, and they cannot serve as a sufficient condition for judging minimal coupling.

Furthermore, reading left to right the nonconservation equation (87) clearly shows that the energy-momentum divergence is transformed into the gradient of nonminimal gravitational coupling strength $f_{\mathcal{L}_m}$. On the other hand, if the weak or even the strong conjecture were true, does it mean that differences between the set of Riemannian invariants which the Lagrangian density depends on are trivial? The answer is of course no, because the gradient $\nabla^\mu f_{\mathcal{L}_m}$ is superposed by the gradient of $\mathcal{L}_m$ and the gradients of all characteristic Riemannian invariants $\mathcal{R}_i$ used in $\mathcal{L} = f$:

[5]When talking about its nontrivial divergence, $T_{\mu\nu}$ can be effectively understood as the $T_{\mu\nu}^{(\mathrm{NC})}$ which comes from the $\mathcal{L}_m$ under nonminimal coupling; the contribution $T_{\mu\nu}^{(\mathrm{MC})}$ to the total SEM tensor by an isolated (i.e. minimally coupled) covariant matter density $\sqrt{-g}\mathcal{L}_m$ automatically satisfies the standard stress-energy-momentum conservation.

$$f_{\mathcal{L}_m}\nabla^\mu T_{\mu\nu} = (\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu})\cdot \times\left(f_{\mathcal{L}_m\mathcal{L}_m}\cdot\nabla^\mu\mathcal{L}_m + \sum_i f_{\mathcal{L}_m\mathcal{R}_i}\cdot\nabla^\mu\mathcal{R}_i\right), \tag{89}$$

where $f_{\mathcal{L}_m\mathcal{L}_m} = \partial f_{\mathcal{L}_m}/\partial\mathcal{L}_m$, $f_{\mathcal{L}_m\mathcal{R}_i} = \partial f_{\mathcal{L}_m}/\partial\mathcal{R}_i$. Note that, if we adopt Eq. (89) rather than Eq. (87) as the final form of nonconservation equation, the coefficient $(\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu}) = 2\delta\mathcal{L}_m/\delta g^{\mu\nu}$ associated to the divergences $\{\nabla^\mu\mathcal{L}_m, \nabla^\mu\mathcal{R}_i\}$ helps to clarify that they exclusively come from the $\mathcal{L}_m$-dependence in $\mathcal{L} = f$.

Following the weak conjecture, we now formally rewrite the divergence equation (67) for $f(R, R_c^2, R_m^2, \mathcal{L}_m)$ gravity into

$$f_{\mathcal{L}_m}\nabla^\mu T_{\mu\nu} = (\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu})\nabla^\mu f_{\mathcal{L}_m} + \mathcal{E}_\nu, \tag{90}$$

where

$$\mathcal{E}_\nu := -f_{R_c^2}\nabla_\nu R_c^2 - f_{R_m^2}\nabla_\nu R_m^2 + 2\nabla^\mu H_{\mu\nu}^{(fR_c^2)} + 2\nabla^\mu H_{\mu\nu}^{(fR_m^2)}, \tag{91}$$

and $\mathcal{E}_\nu$ is expected to vanish by the weak conjecture, while $\mathcal{E}_\nu \equiv 0$ trivially holds under minimal coupling because of generalized Bianchi identities. Since we have not yet proved that $\mathcal{E}_\nu = 0$, we preserve $\mathcal{E}_\nu$ in the divergence equation (90) and proceed to use it to check the equations of continuity and motion with different matter sources.

V. EQUATIONS OF CONTINUITY AND NONGEODESIC MOTION

Once the matter content in the spacetime is known, Eq. (90) can be concretized in accordance with the particular forms of $T_{\mu\nu}$, which would imply the equations of continuity of the energy-matter content and the equation of (nongeodesic) motion for a test particle.[6] This topic will be studied in this section, and note that $T_{\mu\nu}$ and $\mathcal{L}_m$ will be adapted to the $(-,+++)$ metric signature.

A. Perfect fluid

The stress-energy-momentum (SEM) tensor of a perfect fluid (no internal viscosity, no shear stresses, and zero thermal-conductivity coefficients) with mass-energy density $\rho = \rho(x^\alpha)$, isotropic pressure $P = P(x^\alpha)$ and equation of state $P = w\rho$, is given by [20]

$$\begin{aligned} T_{\mu\nu}^{(\mathrm{PF})} &= (\rho + P)u_\mu u_\nu + P g_{\mu\nu} \\ &= \rho u_\mu u_\nu + P(g_{\mu\nu} + u_\mu u_\nu) \\ &= \rho u_\mu u_\nu + P h_{\mu\nu}, \end{aligned} \tag{92}$$

where u^μ is the four-velocity along the worldline, satisfying $u_\mu u^\mu = -1$ and $u_\mu\nabla_\nu u^\mu = 0$; $h_{\mu\nu}$ is the projected spatial 3-metric, $h_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu$ with inverse $h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$, $h^{\mu\nu}u_\mu = 0$, and $h^{\mu\nu}h_{\mu\nu} = 3$. Substituting Eq. (92) into Eq. (90) and multiplying both sides by u^ν, we get

$$u^\mu\nabla_\mu\rho + (\rho + P)\nabla^\mu u_\mu = -(\mathcal{L}_m + \rho)u^\mu\nabla_\mu \ln f_{\mathcal{L}_m} - f_{\mathcal{L}_m}^{-1}u^\nu\mathcal{E}_\nu, \tag{93}$$

which generalizes the original continuity equation of perfect fluid in GR, $u^\mu\nabla_\mu\rho + (\rho + P)\nabla^\mu u_\mu = 0$.

On the other hand, after putting Eq. (92) back to Eq. (90), use $h^{\xi\nu}$ to project the free index ν, and it follows that

$$(\rho + P)\cdot u^\mu\nabla_\mu u^\xi = -h^{\xi\mu}\cdot\nabla_\mu P + h^{\xi\mu}(\mathcal{L}_m - P)\nabla_\mu \ln f_{\mathcal{L}_m} + f_{\mathcal{L}_m}^{-1}h^{\xi\nu}\mathcal{E}_\nu, \tag{94}$$

where we have employed the properties $h^{\xi\nu}\cdot u_\mu\nabla^\mu u_\nu = g^{\xi\nu}\cdot u_\mu\nabla^\mu u_\nu = u_\mu\nabla^\mu u^\xi$. In general, $\rho + P \neq 0$ (in fact $\rho + P \geq 0$ by all four energy conditions in GR, and equality happens only for matters with large negative pressure). Thus we obtain the following absolute derivative along u^ξ as the equation of motion:

$$\frac{Du^\xi}{D\tau} \equiv \frac{du^\xi}{d\tau} + \Gamma^\xi_{\alpha\beta}u^\alpha u^\beta = a^\xi_{(\mathrm{PF})} + a^\xi_{(f_{\mathcal{L}_m})} + a^\xi_{(\mathcal{E})}, \tag{95}$$

where τ is an affine parameter (e.g. proper time) for the timelike worldline along which $dx^\alpha = u^\alpha d\tau$, and the three proper accelerations are given by

$$\begin{aligned} a^\xi_{(\mathrm{PF})} &\equiv -h^{\xi\mu}\cdot(\rho + P)^{-1}\nabla_\mu P, \\ a^\xi_{(f_{\mathcal{L}_m})} &\equiv -h^{\xi\mu}\cdot(\rho + P)^{-1}(P - \mathcal{L}_m)\nabla_\mu \ln f_{\mathcal{L}_m}, \\ a^\xi_{(\mathcal{E})} &\equiv -h^{\xi\nu}\cdot(\rho + P)^{-1}f_{\mathcal{L}_m}^{-1}\mathcal{E}_\nu. \end{aligned} \tag{96}$$

Thus, three proper accelerations are responsible for the nongeodesic motion. $a^\xi_{(\mathrm{PF})}$ is the standard acceleration from the pressure of fluid as in GR [20], $a^\xi_{(f_{\mathcal{L}_m})}$ comes from the curvature-matter coupling, while $a^\xi_{(\mathcal{E})}$ is a collaborative effect of the $\{R_c^2, R_m^2\}$ dependence in the action and their generic nonminimal coupling to $\mathcal{L}_m$. This is consistent with the result in [11] in the absence of $\{R_c^2, R_m^2\}$. Also, all three accelerations are orthogonal to the worldline with tangent u^ξ, since

[6] The method and discussion in this section are also valid for a generic $\mathcal{L} = f(\mathcal{R}_1, \mathcal{R}_2 \ldots, \mathcal{R}_n, \mathcal{L}_m)$ gravity as in Eq. (86), and we just need to define the effective one-form $\tilde{\mathcal{E}}_\nu = -\sum_i f_{\mathcal{R}_i}(\mathcal{R}_1 \ldots \mathcal{L}_m)\cdot\nabla_\nu\mathcal{R}_i + 2\sum_i\nabla^\mu H_{\mu\nu}^{(f\mathcal{R}_i)}$ in place of the $\mathcal{E}_\nu$ for $f(R, R_c^2, R_m^2, \mathcal{L}_m)$ gravity. Specifically, $\tilde{\mathcal{E}}_\nu \equiv 0$ under minimal coupling, and furthermore $\tilde{\mathcal{E}}_\nu$ vanishes universality if the weak conjecture were correct.

$$a^{\xi}_{(\mathrm{PF})} u_\xi = 0, \qquad a^{\xi}_{(f_{\mathcal{L}_m})} u_\xi = 0, \qquad a^{\xi}_{(\mathcal{E})} u_\xi = 0. \tag{97}$$

Equations (93), (95), and (96) depend on the choice of the perfect-fluid matter Lagrangian density. If $\mathcal{L}_m = -\rho$ [20,29], the continuity equation (93) becomes

$$u^\mu \nabla_\mu \rho + (\rho + P)\nabla_\mu u^\mu = -f^{-1}_{\mathcal{L}_m} u^\nu \mathcal{E}_\nu, \tag{98}$$

which is free from the gradient of the geometry-matter coupling strength $f^{-1}_{\mathcal{L}_m} u^\mu \nabla_\mu f_{\mathcal{L}_m}$, while $a^{\xi}_{(f_{\mathcal{L}_m})}$ reduces to

$$a^{\xi}_{(f_{\mathcal{L}_m})} \equiv -h^{\xi\mu} \cdot \nabla_\mu \ln f_{\mathcal{L}_m}, \tag{99}$$

which does not rely on the equation of state $P = w\rho$.

On the other hand, for the choice $\mathcal{L}_m = P$ [29,30], Eqs. (93) and (96) respectively yield

$$u^\mu \nabla_\mu \rho + (\rho + P)\nabla^\mu u_\mu = -(\rho + P) u^\mu \nabla_\mu \ln f_{\mathcal{L}_m} - f^{-1}_{\mathcal{L}_m} u^\mu \mathcal{E}_\mu, \tag{100}$$

and

$$a^{\xi}_{(f_{\mathcal{L}_m})} \equiv 0. \tag{101}$$

Although the continuity equation (100) looks pretty ordinary, the proper acceleration $a^{\xi}_{(f_{\mathcal{L}_m})}$ vanishes identically for $\mathcal{L}_m = P$ and consequently the nongeodesic motion in the gravitational field of the perfect fluid becomes independent of the gradient of the nonminimal coupling strength $u^\mu \nabla_\mu f_{\mathcal{L}_m}$.

As shown in [31], both $\mathcal{L}_m = P$ and $\mathcal{L}_m = -\rho$ are correct matter densities and both lead to the SEM tensor given in Eq. (92). Differences of physical effects only occur in the situation of nonminimal coupling, where $\mathcal{L}_m$ becomes a direct and explicit input in the energy-momentum divergence equation. In fact, as for the matter Lagrangian density $\mathcal{L}_m$ for a perfect fluid, one can also adopt the following ansatz:

$$\begin{aligned}\mathcal{L}_m &= (a\rho + bP) \cdot g^{\alpha\beta} u_\alpha u_\beta + (c\rho + dP) \cdot g^{\alpha\beta} g_{\alpha\beta} \\ &= (4c - a)\rho + (4d - b)P.\end{aligned} \tag{102}$$

Applying this to Eq. (11), the equality with Eq. (92) yields $a = -1/2 = b$ and $c = -1/4 = -d$, so

$$\begin{aligned}\mathcal{L}_m &= \left(-\frac{1}{2}\rho - \frac{1}{2}P\right) \cdot g^{\alpha\beta} u_\alpha u_\beta + \left(-\frac{1}{4}\rho + \frac{1}{4}P\right) \cdot g^{\alpha\beta} g_{\alpha\beta} \\ &= -\frac{1}{2}\rho + \frac{3}{2}P.\end{aligned} \tag{103}$$

This density makes Eqs. (93), (95), and (96) act normally, losing the aforementioned extraordinary properties associated with $\mathcal{L}_m = -\rho$ and $\mathcal{L}_m = P$.

B. (Timelike) dust

The (timelike) dust source with mass-energy density ρ has SEM tensor [20,30]

$$T^{(\mathrm{Dust})}_{\mu\nu} = \rho u_\mu u_\nu, \tag{104}$$

where $u_\mu = g_{\mu\nu} u^\nu$ with u^ν being the tangent vector field along the worldline of a timelike dust particle. One can still introduce the spatial metric $h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$ orthogonal to u^μ, with $\{u_\mu, h_{\mu\nu}\}$ sharing all those properties as in the case of perfect fluid, so dust acts just like a perfect fluid with zero pressure, $P = 0$. Substituting Eq. (104) back into Eq. (90) and multiplying by u^ν on both sides yields

$$u^\mu \nabla_\mu \rho + \rho \nabla^\mu u_\mu = -(\mathcal{L}_m + \rho) u^\nu \nabla_\nu \ln f_{\mathcal{L}_m} - f^{-1}_{\mathcal{L}_m} u^\nu \mathcal{E}_\nu, \tag{105}$$

which modifies the continuity equation of dust $\nabla_\mu(\rho u^\mu) = 0$ in GR. Meanwhile, projection of the free index ν by $h^{\xi\nu}$ in $\nabla^\mu T^{(\mathrm{Dust})}_{\mu\nu}$ gives rise to the modified equation of motion,

$$\frac{Du^\xi}{D\tau} \equiv \frac{du^\xi}{d\tau} + \Gamma^\xi_{\alpha\beta} u^\alpha u^\beta = \hat{a}^{\xi}_{(f_{\mathcal{L}_m})} + \hat{a}^{\xi}_{(\mathcal{E})}, \tag{106}$$

where

$$\begin{aligned}\hat{a}^{\xi}_{(f_{\mathcal{L}_m})} &\equiv h^{\xi\mu} \cdot \rho^{-1} \mathcal{L}_m \nabla_\mu \ln f_{\mathcal{L}_m}, \\ \hat{a}^{\xi}_{(\mathcal{E})} &\equiv -h^{\xi\nu} \cdot \rho^{-1} f^{-1}_{\mathcal{L}_m} \mathcal{E}_\nu.\end{aligned} \tag{107}$$

Being pressureless, the dust inherits just the two extra accelerations $\hat{a}^{\xi}_{(f_{\mathcal{L}_m})}$ and $\hat{a}^{\xi}_{(\mathcal{E})}$, and both remain orthogonal to the worldline with tangent u^ξ,

$$\hat{a}^{\xi}_{(f_{\mathcal{L}_m})} u_\xi = 0, \qquad \hat{a}^{\xi}_{(\mathcal{E})} u_\xi = 0. \tag{108}$$

C. Null dust

The SEM tensor for null dust with energy density ϱ is (e.g. [30])

$$T^{(\mathrm{ND})}_{\mu\nu} = \varrho \ell_\mu \ell_\nu, \tag{109}$$

where $\ell_\mu = g_{\mu\nu} \ell^\nu$ with ℓ^ν being the tangent vector field along the worldline of a null dust particle, $\ell_\mu \ell^\mu = 0$. $T^{(\mathrm{ND})}_{\mu\nu}$ together with the energy-momentum divergence equation (90) yields

$$\begin{aligned}&\ell_\nu \ell^\mu \nabla_\mu \varrho + \varrho \ell^\mu \nabla_\mu \ell_\nu + \varrho \ell_\nu \nabla_\mu \ell^\mu \\ &\quad = (\mathcal{L}_m g_{\mu\nu} - \varrho \ell_\mu \ell_\nu) \nabla^\mu \ln f_{\mathcal{L}_m} + f^{-1}_{\mathcal{L}_m} \mathcal{E}_\nu.\end{aligned} \tag{110}$$

Multiplying both sides with ℓ^ν, $\ell^\nu \ell_\nu = 0$, $\ell_\nu \nabla_\mu \ell^\nu = 0$, we obtain the following constraint:

$$f_{\mathcal{L}_m}\ell^\nu\nabla_\nu f_{\mathcal{L}_m} = -\ell^\nu\mathcal{E}_\nu. \tag{111}$$

Now, introduce an auxiliary null vector field n^μ as null normal to ℓ^μ such that $n^\mu n_\mu = 0$, $\ell^\mu n_\mu = -1$, which induces the two-dimensional spatial metric $g_{\mu\nu} = -\ell_\mu n_\nu - n_\mu\ell_\nu + q_{\mu\nu}$, satisfying the conditions

$$q_{\mu\nu}q^{\mu\nu} = 2, \qquad q_{\mu\nu}\ell^\nu = 0 = q_{\mu\nu}n^\nu, \qquad \ell^\alpha\nabla_\alpha q_{\mu\nu} = 0. \tag{112}$$

Multiplying Eq. (110) by n^ν, and with $n^\nu\nabla_\mu\ell_\nu = -\ell^\nu\nabla_\mu n_\nu$, we get the continuity equation

$$\begin{aligned}&\ell^\mu\nabla_\mu\varrho + \varrho\nabla_\mu\ell^\mu + \varrho\ell^\nu\ell^\mu\nabla_\mu n_\nu\\&= -(\mathcal{L}_m n^\mu + \varrho\ell^\mu)\nabla_\mu\ln f_{\mathcal{L}_m} - f_{\mathcal{L}_m}^{-1}n^\nu\mathcal{E}_\nu,\end{aligned} \tag{113}$$

while projecting Eq. (110) with $h^{\xi\nu}$ gives rise to the equation of motion along ℓ^ξ,

$$\varrho\ell^\mu\nabla_\mu\ell^\xi = \varrho\ell^\xi\ell^\nu\ell^\mu\nabla_\mu n_\nu + h^{\xi\nu}\mathcal{L}_m\nabla_\nu\ln f_{\mathcal{L}_m} + f_{\mathcal{L}_m}^{-1}h^{\xi\nu}\mathcal{E}_\nu, \tag{114}$$

$$\frac{D\ell^\xi}{D\lambda} \equiv \frac{d\ell^\xi}{d\lambda} + \Gamma^\xi_{\alpha\beta}\ell^\alpha\ell^\beta = \check{a}^\xi_{(\mathrm{ND})} + \check{a}^\xi_{(f_{\mathcal{L}_m})} + \check{a}^\xi_{(\mathcal{E})}, \tag{115}$$

where λ is an affine parameter for the null worldline along which $dx^\alpha = \ell^\alpha d\xi$, and the three proper accelerations are respectively

$$\begin{cases}\check{a}^\xi_{(\mathrm{ND})} \equiv \ell^\xi\ell^\nu\ell^\mu\nabla_\mu n_\nu,\\ \check{a}^\xi_{(f_{\mathcal{L}_m})} \equiv h^{\xi\mu}\cdot\varrho^{-1}\mathcal{L}_m\nabla_\nu\ln f_{\mathcal{L}_m},\\ \check{a}^\xi_{(\mathcal{E})} \equiv h^{\xi\nu}\cdot\varrho^{-1}f_{\mathcal{L}_m}^{-1}\mathcal{E}_\nu.\end{cases} \tag{116}$$

As we can see, compared with timelike dust, one more proper acceleration $\check{a}^\xi_{(\mathrm{ND})}$ shows up in the case of null dust, and we will refer to it the *affine* acceleration or *inaffinity* acceleration.

D. Scalar field

The matter Lagrangian density and SEM tensor of a massive scalar field $\phi(x^\alpha)$ with mass m in a potential $V(\phi)$ are respectively given by

$$\begin{aligned}\mathcal{L}_m &= -\frac{1}{2}(\nabla_\alpha\phi\nabla^\alpha\phi + m^2\phi^2) + V(\phi),\\ T_{\mu\nu} &= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla_\alpha\phi\nabla^\alpha\phi + m^2\phi^2 - 2V(\phi)),\end{aligned} \tag{117}$$

thus $\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu} = -\nabla_\mu\phi\nabla_\nu\phi$. For the ν component, the equations of continuity and motion are both given by

$$(\Box\phi - m^2\phi + V_\phi)\cdot\nabla_\nu\phi = -\nabla_\nu\phi\cdot\nabla_\mu\phi\nabla^\mu\ln f_{\mathcal{L}_m} + f_{\mathcal{L}_m}^{-1}\mathcal{E}_\nu. \tag{118}$$

Specifically, by setting $V(\phi) = 0$ and under minimal coupling ($f_{\mathcal{L}_m} =$ constant, $\mathcal{E}_\nu = 0$), we get

$$\Box\phi - m^2\phi = 0, \tag{119}$$

which is the standard covariant Klein-Gordon equation for spin-zero particles in GR.

VI. FURTHER PHYSICAL IMPLICATIONS OF NONMINIMAL COUPLING

We have seen that under nonminimal curvature-matter coupling, the divergence of the standard SEM density tensor is equal to the gradient of the coupling strength $\nabla^\mu f_{\mathcal{L}_m}$ which, in general, will be nonvanishing. As such, the usual energy-momentum conservation laws for particular matter fields will be modified as compared to the corresponding fields in general relativity. At the same time, as discussed in the Appendix, nonminimal coupling also affects the energy conditions. The standard energy conditions of general relativity are phrased in terms of the stress-energy tensor and require positive energies (null and strong) and causal flows of matter (dominant). However, in applications these conditions are generally used to constrain the Riemann tensor and so the allowed geometries of spacetime and structures like singularities or horizons. For standard general relativity the two approaches are essentially equivalent but for modified gravity they are not: if the Einstein equations are modified then the bounds on the Ricci tensor that achieve the desired effects generally do not translate into the usual restrictions on the stress-energy momentum. Thus one is faced with a choice: either keep the standard GR results and give up the usual energy conditions or keep the usual energy conditions but lose those results.

In this section we consider some immediate physical consequences of this choice. All of these are consequences of the Raychaudhuri equations for null and timelike geodesic congruences and so the difference between the standard energy conditions and those needed to enforce the focusing theorems is crucial to these discussions. These are considered in some detail in the Appendix and in the following $T^{(\mathrm{eff})}_{\mu\nu}$ refers to an effective stress-energy tensor for which the standard form of the energy conditions will leave those theorems intact.

A. Black hole physics

Many results in black hole physics follow from understanding a black hole horizon as a congruence of null geodesics whose evolution is governed by the (twist-free) Raychaudhuri equation:

$$\frac{d\theta_{(\ell)}}{d\lambda} = \kappa_{(\ell)}\theta_{(\ell)} - \frac{1}{2}\theta_{(\ell)}^2 - \sigma_{\mu\nu}^{(\ell)}\sigma_{(\ell)}^{\mu\nu} - R_{\mu\nu}\ell^\mu\ell^\nu, \tag{120}$$

where $\ell^\mu = (\frac{\partial}{\partial\lambda})^\mu$ is a null tangent to the horizon, and $\kappa_{(\ell)}$, $\theta_{(\ell)}$ and $\sigma_{\mu\nu}^{(\ell)}$ are respectively the associated acceleration/inaffinity, expansion and shear.

The second law of black hole mechanics follows from this equation along with the requirement that the congruence of null curves that rules the event horizon have no future end points (see, for example, the discussion [20]). Now choosing an affine parametrization for the congruence $\kappa_{(\ell)} = 0$ it is straightforward to see that the right-hand side of (120) is nonpositive as long as $R_{\mu\nu}\ell^\mu\ell^\nu \geq 0$. In standard GR this follows from the null energy condition: $T_{\mu\nu}\ell^\mu\ell^\nu \geq 0$. It then almost immediately follows that $\theta_{(\ell)}$ must be everywhere non-negative. Else $\theta_{(\ell)} \to -\infty$ and the congruence focuses. However, for modified gravity we will usually lose the equivalence $T_{\mu\nu}\ell^\mu\ell^\nu \geq 0 \Leftrightarrow R_{\mu\nu}\ell^\mu\ell^\nu \geq 0$ and so we will be faced with a modified area increase theorem if we require the standard energy conditions.

By similar arguments, again involving the null Raychaudhuri equation, the energy conditions play a crucial role in the theorems that require trapped surfaces to be contained in black holes and singularities to lie in their causal future [20]. Thus for black hole physics, modifications of the energy conditions are a serious business which can affect core results and intuitions.

B. Wormholes

On the other hand, for those interested in faster-than-light travel changing the energy conditions would be a boon. Introducing the nonminimal gravitational coupling strength $f_{\mathcal{L}_m}$ brings new flexibility and the possibility of supporting wormholes, as shown in [32] and [33] for a $\lambda R \cdot \mathcal{L}_m$ coupling term. More generally for the $\mathcal{L} = f(R, \mathcal{R}_1, \ldots, \mathcal{R}_n, \mathcal{L}_m)$ gravity, based on the generalized null and weak energy conditions developed in the Appendix, it proves possible to defocus null and timelike congruences and form wormholes by violating these generalized conditions, while having the standard energy conditions in GR [20] maintained to exclude the need for exotic matters. It also leads to an extra constraint $f_{\mathcal{L}_m}/f_R \geq 0$ as in Eq. (A9).

From Eq. (A10) in the Appendix, for a null congruence ℓ^μ, one can maintain the standard null energy condition $T_{\mu\nu}\ell^\mu\ell^\nu \geq 0$ while violating $T_{\mu\nu}^{(\mathrm{eff})}\ell^\mu\ell^\nu \leq 0$ (and so evade the focusing theorems) if

$$0 \leq T_{\mu\nu}\ell^\mu\ell^\nu \leq 2f_{\mathcal{L}_m}^{-1}\left(\sum_i H_{\mu\nu}^{(f\mathcal{R}i)}\ell^\mu\ell^\nu - \ell^\nu\ell^\mu\nabla_\mu\nabla_\nu f_R\right). \tag{121}$$

Similarly for a timelike congruence, one has $T_{\mu\nu}u^\mu u^\nu \geq 0$ while $T_{\mu\nu}^{(\mathrm{eff})}u^\mu u^\nu \leq 0$, and Eq. (A11) leads to

$$0 \leq T_{\mu\nu}u^\mu u^\nu \leq -\mathcal{L}_m + f_{\mathcal{L}_m}^{-1}\Bigg(f - Rf_R + 2\sum_i H_{\mu\nu}^{(f\mathcal{R}i)}u^\mu u^\nu - 2(u^\mu u^\nu\nabla_\mu\nabla_\nu + \Box)f_R\Bigg). \tag{122}$$

Specifically for $\mathcal{L} = f(R, R_c^2, R_m^2, \mathcal{L}_m)$ gravity, these two conditions are concretized as

$$0 \leq T_{\mu\nu}\ell^\mu\ell^\nu \leq 2f_{\mathcal{L}_m}^{-1}(H_{\mu\nu}^{(fR_c^2)}\ell^\mu\ell^\nu + H_{\mu\nu}^{(fR_m^2)}\ell^\mu\ell^\nu - \ell^\nu\ell^\mu\nabla_\mu\nabla_\nu f_R) \tag{123}$$

and

$$0 \leq T_{\mu\nu}u^\mu u^\nu \leq -\mathcal{L}_m + f_{\mathcal{L}_m}^{-1}(f - Rf_R + 2H_{\mu\nu}^{(fR_c^2)}u^\mu u^\nu + 2H_{\mu\nu}^{(fR_m^2)}u^\mu u^\nu - 2(u^\mu u^\nu\nabla_\mu\nabla_\nu + \Box)f_R), \tag{124}$$

where $\{H_{\mu\nu}^{(fR_c^2)}, H_{\mu\nu}^{(fR_m^2)}\}$ have been given in Eqs. (8) and (9).

Moreover, Eqs. (121) and (122) indicate that in the case without dependence on Riemannian invariants beyond R, i.e. $\mathcal{L} = f(R, \mathcal{L}_m)$, a wormhole can be solely supported by the nonminimal-coupling effect if

$$0 \leq T_{\mu\nu}\ell^\mu\ell^\nu \leq -2f_{\mathcal{L}_m}^{-1}\ell^\nu\ell^\mu\nabla_\mu\nabla_\nu f_R \quad \text{and} \tag{125}$$

$$0 \leq T_{\mu\nu}u^\mu u^\nu \leq -\mathcal{L}_m + f_{\mathcal{L}_m}^{-1}(f - Rf_R - 2(u^\mu u^\nu\nabla_\mu\nabla_\nu + \Box)f_R). \tag{126}$$

For example, let $\mathcal{L} = f(R, \mathcal{L}_m) = R + 2\kappa\mathcal{L}_m + \lambda R\mathcal{L}_m$, and the field equation (48) becomes

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \lambda\cdot(\mathcal{L}_m R_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\mathcal{L}_m) = \left(\kappa + \frac{1}{2}\lambda R\right)T_{\mu\nu}. \tag{127}$$

To have a quick realization of Eq. (125), we further assume $\lambda = 1$, $T_{\mu\nu} = \mathrm{diag}[-\rho(r), P(r), P(r), P(r)]$, $\mathcal{L}_m = P(r)$ (recall Sec. V A), and adopt the following simplest wormhole metric:

$$ds^2 = -dt^2 + dr^2 + (r^2 + L^2)\cdot(d\theta^2 + \sin^2\theta d\phi^2), \tag{128}$$

with minimum throat scale L and outgoing radial null vector field $\ell^\mu\partial_\mu = (-1, 1, 0, 0)$. Then the condition Eq. (125) reduces to become

$$0 \leq -\rho + 3P \leq \left(1 + \frac{r^2}{L^2}\right)\partial_r\partial_r P, \tag{129}$$

which clearly shows that the standard null energy condition remains valid while spatial inhomogeneity of the pressure $\partial_r \partial_r P$ supports the wormhole.

Finally, note that it remains to be carefully checked whether solutions exist that meet these conditions.

VII. CONCLUSIONS

In this paper, we have derived the field equation for $\mathcal{L} = f(R, R_c^2, R_m^2, \mathcal{L}_m)$ fourth-order gravity allowing for participation of the Ricci square R_c^2 and Riemann square R_m^2 in the Lagrangian density and nonminimal coupling between the curvature invariants and $\mathcal{L}_m$ as compared to GR. It turned out that $\mathcal{L}_m$ appears explicitly in the field equation because of confrontation between the nonminimal coupling and the traditional minimal definition of the SEM tensor $T_{\mu\nu}$. When $f_{\mathcal{L}_m} = \text{constant} = 2\kappa$, we recover the minimally coupled $\mathcal{L} = f(R, R_c^2, R_m^2) + 2\kappa\mathcal{L}_m$ model. Also, we have shown that both the curvature-$\mathcal{L}_m$ nonminimal coupling and the curvature-T coupling are sensitive to the concrete forms of $\mathcal{L}_m$.

Secondly, by considering an explicit R^2 dependence, we have found the smooth transition from $f(R, R_c^2, R_m^2, \mathcal{L}_m)$ gravity to the $\mathcal{L} = f(R, \mathcal{G}, \mathcal{L}_m)$ generalized Gauss-Bonnet gravity by imposing the coherence condition $f_{R^2} = f_{R_m^2} = -f_{R_c^2}/4$. When $f(R, \mathcal{G}, \mathcal{L}_m)$ reduces to the case $f(R, \mathcal{L}_m) + \lambda\mathcal{G}$ where $\mathcal{G}$ appears as a pure Gauss-Bonnet term, an extra term $\lambda(-\frac{1}{2}\mathcal{G}g_{\mu\nu} + 2RR_{\mu\nu} - 4R_\mu{}^\alpha R_{\alpha\nu} - 4R_{\alpha\mu\beta\nu}R^{\alpha\beta} + 2R_{\mu\alpha\beta\gamma}R_\nu{}^{\alpha\beta\gamma})$ is left behind in the field equation representing the contribution from the covariant density $\lambda\sqrt{-g}\mathcal{G}$. We have shown that this term actually vanishes and thus $\lambda\mathcal{G}$ makes no difference to the gravitational field equation.

After studying the Gauss-Bonnet limit of $f(R, R_c^2, R_m^2, \mathcal{L}_m)$ gravity, we moved on to more generic theories focusing on how the standard stress-energy-momentum conservation equation $\nabla^\mu T_{\mu\nu} = 0$ in GR is violated. Under minimal coupling with $\mathcal{L} = f(\mathcal{R}_1, \ldots, \mathcal{R}_p) + 2\kappa\mathcal{L}_m$, we commented that the generalized Bianchi identities and the Noether-induced definition of SEM tensor lead to automatic energy-momentum conservation. Under nonminimal coupling with $\mathcal{L} = f(\mathcal{R}_1, \ldots, \mathcal{R}_p, \mathcal{L}_m)$, we have proposed a weak conjecture and a strong one which state that the gradient of the nonminimal gravitational coupling strength $\nabla^\mu f_{\mathcal{L}_m}$ is the only divergence term balancing $f_{\mathcal{L}_m}\nabla^\mu T_{\mu\nu}$, while contributions from $\mathcal{R}_i$ dependence in the divergence equation all cancel out. Using the energy-momentum nonconservation equation specialized for $f(R, R_c^2, R_m^2, \mathcal{L}_m)$ gravity, we have derived the equations of continuity and nongeodesic motion in the matter sources for perfect fluids, (timelike) dust, null dust, and massive scalar fields. These equations directly generalize those in $f(\mathcal{R}_1, \ldots, \mathcal{R}_p, \mathcal{L}_m)$ gravity.

Also, within $f(\mathcal{R}_1, \ldots, \mathcal{R}_p, \mathcal{L}_m)$ gravity, we have considered some implications of nonminimal coupling and $\mathcal{R}_i$ dependence for black hole and wormhole physics. Moreover, it is expected that the $\mathcal{L} = f(R, R_c^2, R_m^2, \mathcal{L}_m)$ model can provide many more possibilities to realize the late-time phase transition from cosmic deceleration to acceleration, and the energy-momentum nonconservation relation $f_{\mathcal{L}_m} \cdot \nabla^\mu T_{\mu\nu} = (\mathcal{L}_m g_{\mu\nu} - T_{\mu\nu})\nabla^\mu f_{\mathcal{L}_m}$ under nonminimal coupling can cause interesting consequences in early-era cosmic evolution and compact astrophysical objects if it is effective as a high-energy phenomenon. These topics will be extensively investigated in prospective studies.

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APPENDIX: GENERALIZED ENERGY CONDITIONS FOR $f(R, \mathcal{R}_1, \ldots, \mathcal{R}_n, \mathcal{L}_m)$ GRAVITY

For the generic $\mathcal{L} = f(R, \mathcal{R}_1, \ldots, \mathcal{R}_n, \mathcal{L}_m)$ gravity introduced in Sec. IV, the variational principle or equivalently $\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}} = 0$ yields the field equation,

$$-\frac{1}{2}fg_{\mu\nu} + f_R R_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R + \sum_i H_{\mu\nu}^{(f\mathcal{R}_i)} = \frac{1}{2}f_{\mathcal{L}_m} \cdot (T_{\mu\nu} - \mathcal{L}_m g_{\mu\nu}), \quad \text{(A1)}$$

where $H_{\mu\nu}^{(f\mathcal{R}_i)} \cdot \delta g^{\mu\nu} \coloneqq f_{\mathcal{R}_i} \cdot \delta\mathcal{R}_i$. An immediate and very useful implication of this field equation is a group of generalized null, weak, strong and dominant energy conditions (abbreviated into NEC, WEC, SEC and DEC respectively), which has been employed in Sec. VI B in studying effects of nonminimal coupling in supporting wormholes.

Recall that in a (region of) spacetime filled by a null or a timelike congruence, the expansion rate along the null tangent ℓ^μ or the timelike tangent u^μ is given by the respective Raychaudhuri equation [20]:

$$\ell^\mu\nabla_\mu\theta_{(\ell)} = \frac{d\theta_{(\ell)}}{d\lambda} = \kappa_{(\ell)}\theta_{(\ell)} - \frac{1}{2}\theta_{(\ell)}^2 - \sigma_{\mu\nu}^{(\ell)}\sigma_{(\ell)}^{\mu\nu} + \omega_{\mu\nu}^{(\ell)}\omega_{(\ell)}^{\mu\nu} - R_{\mu\nu}\ell^\mu\ell^\nu \quad \text{(A2)}$$

and

$$u^\mu\nabla_\mu\theta_{(u)} = \frac{d\theta_{(u)}}{d\tau} = \kappa_{(u)}\theta_{(u)} - \frac{1}{3}\theta_{(u)}^2 - \sigma_{\mu\nu}^{(u)}\sigma_{(u)}^{\mu\nu} + \omega_{\mu\nu}^{(u)}\omega_{(u)}^{\mu\nu} - R_{\mu\nu}u^\mu u^\nu. \quad \text{(A3)}$$

Under affine parametrizations one has $\kappa_{(\ell)} = 0 = \kappa_{(u)}$, for hypersurface-orthogonal congruences the twist vanishes $\omega_{\mu\nu}\omega^{\mu\nu} = 0$, and the shear as a spatial tensor ($\sigma_{\mu\nu}^{(\ell)}\ell^\mu = 0$,

$\sigma^{(u)}_{\mu\nu} u^\mu = 0$) always satisfies $\sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0$. Thus, to ensure $d\theta_{(\ell)}/d\lambda \leq 0$ and $d\theta_{(u)}/d\tau \leq 0$ under all conditions so that "gravity always gravitates" and the congruence focuses, the following geometric non-negativity conditions should hold:

$$R_{\mu\nu}\ell^\mu\ell^\nu \geq 0(\mathrm{NEC}), \qquad R_{\mu\nu}u^\mu u^\nu \geq 0(\mathrm{SEC}). \tag{A4}$$

On the other hand, the field (12) can be recast into a compact GR form:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T^{(\mathrm{eff})}_{\mu\nu}$$
$$R = -\kappa T^{(\mathrm{eff})} \quad \text{and}$$
$$R_{\mu\nu} = \kappa\left(T^{(\mathrm{eff})}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{(\mathrm{eff})}\right), \tag{A5}$$

where all terms beyond GR ($G_{\mu\nu} = \kappa T_{\mu\nu}$) in Eq. (A1) have been packed into the effective SEM tensor $T^{(\mathrm{eff})}_{\mu\nu}$,

$$T^{(\mathrm{eff})}_{\mu\nu} = \frac{1}{2\kappa}\frac{f_{\mathcal{L}_m}}{f_R}(T_{\mu\nu} - \mathcal{L}_m g_{\mu\nu}) + \frac{1}{2\kappa}\frac{f_{\mathcal{L}_m}}{f_R}\Big((f - Rf_R)g_{\mu\nu} + 2(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f_R - 2\sum_i H^{(f\mathcal{R}i)}_{\mu\nu}\Big). \tag{A6}$$

The purely *geometric* conditions Eq. (A4) can be translated into *matter* non-negativity conditions through Eq. (A5),

$$T^{(\mathrm{eff})}_{\mu\nu}\ell^\mu\ell^\nu \geq 0(\mathrm{NEC}),$$
$$T^{(\mathrm{eff})}_{\mu\nu}u^\mu u^\nu \geq \frac{1}{2}T^{(\mathrm{eff})}u_\mu u^\mu(\mathrm{SEC}),$$
$$T^{(\mathrm{eff})}_{\mu\nu}u^\mu u^\nu \geq 0(\mathrm{WEC}), \tag{A7}$$

where $u_\mu u^\mu = -1$ in SEC for the signature $(-,+++)$ used in this paper. Then the generalized NEC in Eq. (A7) expands as

$$\frac{f_{\mathcal{L}_m}}{f_R}T_{\mu\nu}\ell^\mu\ell^\nu + \frac{2}{f_R}\left(\ell^\nu\ell^\mu\nabla_\mu\nabla_\nu f_R - \sum_i H^{(f\mathcal{R}i)}_{\mu\nu}\ell^\mu\ell^\nu\right) \geq 0 \tag{A8}$$

(for $\kappa > 0$) which is the simplest one with $\mathcal{L}_m$ absent. Now, consider a special situation where $f_R = \text{constant}$ and $H^{(f\mathcal{R}i)}_{\mu\nu} = 0$ (i.e. dropping all dependence on $\mathcal{R}_i$ in f), so Eq. (A8) reduces to $(f_{\mathcal{L}_m}/f_R)\cdot T_{\mu\nu}\ell^\mu\ell^\nu \geq 0$; since $T_{\mu\nu}\ell^\mu\ell^\nu \geq 0$ due to the standard NEC in GR, which continues to hold here as exotic matters are unfavored, we obtain an extra constraint,

$$\frac{f_{\mathcal{L}_m}}{f_R} \geq 0, \tag{A9}$$

with which Eq. (A8) becomes

$$T_{\mu\nu}\ell^\mu\ell^\nu + 2f^{-1}_{\mathcal{L}_m}\left(\ell^\nu\ell^\mu\nabla_\mu\nabla_\nu f_R - \sum_i H^{(f\mathcal{R}i)}_{\mu\nu}\ell^\mu\ell^\nu\right) \geq 0, \tag{A10}$$

and the WEC in Eq. (A7) can be expanded into

$$T_{\mu\nu}u^\mu u^\nu + f^{-1}_{\mathcal{L}_m}\Big(Rf_R - f + 2(u^\mu u^\nu\nabla_\mu\nabla_\nu + \Box)f_R - 2\sum_i H^{(f\mathcal{R}i)}_{\mu\nu}u^\mu u^\nu\Big) + \mathcal{L}_m \geq 0. \tag{A11}$$

In general, the pointwise nonminimal coupling strength $f_{\mathcal{L}_m}$ can take either positive or negative values. However, recall that within $f(R) + 2\kappa\mathcal{L}_m$ gravity, physically viable models specializing $f(R)$ should satisfy $f_R > 0$ and $f_{RR} > 0$ [5]; if this were still true in $f(R, \mathcal{R}_1, \ldots, \mathcal{R}_n, \mathcal{L}_m)$ gravity, we would get $f_{\mathcal{L}_m} > 0$ by the extra constraint Eq. (A9), which would be in strong agreement with the case of minimal coupling when $f_{\mathcal{L}_m} = 2\kappa > 0$.

Applying Eqs. (A6), (A10), and (A11) to the Lagrangian density $\mathcal{L} = f(R, R_c^2, R_m^2, \mathcal{L}_m)$, we immediately obtain

$$T^{(\mathrm{eff})}_{\mu\nu} = \frac{1}{2\kappa}\frac{f_{\mathcal{L}_m}}{f_R}(T_{\mu\nu} - \mathcal{L}_m g_{\mu\nu}) + \frac{1}{2\kappa}\frac{f_{\mathcal{L}_m}}{f_R}\times((f - Rf_R)g_{\mu\nu} + 2(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f_R - 2H^{(fR_c^2)}_{\mu\nu} - 2H^{(fR_m^2)}_{\mu\nu}). \tag{A12}$$

as the effective SEM tensor for $f(R, R_c^2, R_m^2, \mathcal{L}_m)$ gravity. Then relative to the standard SEM tensor the generalized null and weak energy conditions respectively become

$$T_{\mu\nu}\ell^\mu\ell^\nu + 2f^{-1}_{\mathcal{L}_m}(\ell^\nu\ell^\mu\nabla_\mu\nabla_\nu f_R - H^{(fR_c^2)}_{\mu\nu}\ell^\mu\ell^\nu - H^{(fR_m^2)}_{\mu\nu}\ell^\mu\ell^\nu) \geq 0 \tag{A13}$$

and

$$T_{\mu\nu}u^\mu u^\nu + f^{-1}_{\mathcal{L}_m}(Rf_R - f + 2(u^\mu u^\nu\nabla_\mu\nabla_\nu + \Box)f_R - 2H^{(fR_c^2)}_{\mu\nu}u^\mu u^\nu - 2H^{(fR_m^2)}_{\mu\nu}u^\mu u^\nu) + \mathcal{L}_m \geq 0, \tag{A14}$$

where $\{H^{(fR_c^2)}_{\mu\nu}, H^{(fR_m^2)}_{\mu\nu}\}$ have been given in Eqs. (8) and (9).

Also, with Eq. (A6) one can directly obtain the concrete forms SEC and DEC for $\mathcal{L} = f(R, \mathcal{R}_1, \ldots, \mathcal{R}_n, \mathcal{L}_m)$ gravity, which however will not be listed here.

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