Partition functions and Casimir energies in higher spin $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$

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Recently, the one-loop free energy of higher spin (HS) theories in Euclidean AdS_{d+1} was calculated and matched with the order N^0 term in the free energy of the large N "vectorial" scalar CFT on the S^d boundary. Here we extend this matching to the boundary theory defined on $S^1 \times S^{d-1}$, where the length of S^1 may be interpreted as the inverse temperature. It has been shown that the large N limit of the partition function on $S^1 \times S^2$ in the $U(N)$ singlet sector of the CFT of N free complex scalars matches the one-loop thermal partition function of the Vasiliev theory in AdS_4, while in the $O(N)$ singlet sector of the CFT of N real scalars it matches the minimal theory containing even spins only. We extend this matching to all dimensions d. We also calculate partition functions for the singlet sectors of free fermion CFTs in various dimensions and match them with appropriately defined higher spin theories, which for $d > 3$ contain massless gauge fields with mixed symmetry. In the zero-temperature case $R \times S^{d-1}$ we calculate the Casimir energy in the scalar or fermionic CFT and match it with the one-loop correction in the global AdS_{d+1}. For any odd-dimensional CFT the Casimir energy must vanish on general grounds, and we show that the HS duals obey this. In the $U(N)$ symmetric case, we exhibit the vanishing of the regularized one-loop Casimir energy of the dual HS theory in AdS_{d+1}. In the minimal HS theory the vacuum energy vanishes for odd d while for even d it is equal to the Casimir energy of a single conformal scalar in $R \times S^{d-1}$ which is again consistent with AdS/CFT, provided the minimal HS coupling constant is $\sim 1/(N-1)$. We demonstrate analogous results for singlet sectors of theories of N Dirac or Majorana fermions. We also discuss extensions to CFTs containing N_f flavors in the fundamental representation of $U(N)$ or $O(N)$.

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I. INTRODUCTION AND SUMMARY

The original AdS/CFT conjectures were made for conformal field theories of $N \times N$ matrices with extended supersymmetry [1–3]. A few years later, a suggestion was made to study AdS/CFT correspondence for simpler field theories where dynamical fields are in the fundamental representation of the $U(N)$ or $O(N)$ symmetry group [4]; for this reason, these theories are often called "vectorial." In these cases, the supersymmetry is not necessary, but it is important that there is an infinite tower of conserved or nearly conserved higher spin (HS) currents that are $U(N)$ or $O(N)$ singlets. Therefore, the dual theories in anti–de Sitter (AdS) must contain the corresponding tower of massless higher spin gauge fields [5]. Theories of this kind have been extensively explored by Vasiliev and others [6–13].

The first explicit vectorial AdS/CFT conjectures were made for the higher spin theories in AdS_4. For the minimal type A theory with even spins only, the conjectured duals were the free or critical $O(N)$ models, with N real scalar fields in the fundamental representation. For the non-minimal type A theory, where all integer spins are present, one instead needs to consider free or critical theories of N complex scalar fields, restricted to the $U(N)$ singlet sector [4]. There also exist type B Vasiliev theories in AdS_4 where the bulk spin 0 field is a pseudoscalar, rather than a scalar. Such theories have been conjectured to be dual to the $U(N)$ or $O(N)$ singlet sector of the theory of N Dirac or Majorana fermions [14,15].

The basic evidence for the initial conjectures involved the matching of the spectra of currents and higher spin gauge fields [4,14,15]. A nice way of summarizing this agreement is to match the CFT partition function on $S^1 \times S^2$ with the corresponding calculation in AdS_4. This was carried out in [16], and a simple explicit formula for the partition function of the $U(N)$ singlet scalar theory was obtained. A crucial ingredient in these calculations is the imposition of the singlet constraint in the CFT of free scalar fields; this was accomplished by integrating over the holonomy of the $U(N)$ gauge field around S^1 [17–19]. The resulting $U(N)$ singlet partition function then becomes the square of the character of the "singleton" representation of $SO(3,2)$, corresponding to the free conformal scalar in $d = 3$. The CFT partition function may then be expanded in characters of the primary fields of spin s and dimension $s + 1$, which correspond to partition functions of gauge fields in AdS_4. Besides providing a nice test of the vectorial

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AdS_4/CFT_3 duality [4], this may be viewed as a modern incarnation of much older ideas [20] (see also [21–27]). The $d = 3$ result of [16] was recently reproduced and also extended to the $O(N)$ singlet sector of N real scalars [28] using the collective field approach [29]. We will further extend these results in several ways, thus obtaining new tests of the higher spin AdS/CFT dualities.

Additional evidence for the vectorial AdS_{d+1}/CFT_d duality for $d = 3$ has been found in [30–39]. Furthermore, evidence has begun accumulating that it is valid for all d [40,41]. On the CFT side, we may consider N complex or real scalar fields in d dimensions and impose the $U(N)$ or $O(N)$ singlet constraint.[1] The corresponding theories in AdS_{d+1}, which involve the tower of totally symmetric tensor spin s gauge fields were formulated in [12]. For arbitrary d, we will study the partition function of such a theory in "thermal" AdS space, which is asymptotic to $S^1 \times S^{d-1}$, and match it with the singlet partition function of the free scalar theory on $S^1 \times S^{d-1}$. This result provides an elegant way of encoding the AdS/CFT matching of the spectra. The thermal free energy on $S^1 \times S^{d-1}$ includes a term linear in the inverse temperature which dominates in the zero-temperature limit. This is related to the Casimir energy of the CFT on $R \times S^{d-1}$, and may be computed by an appropriately regularized sum over the energy spectrum. We will compare this Casimir energy term on the two sides of the duality. In this case, the higher spin theory is defined in the global AdS_{d+1}, which is asymptotic to $R \times S^{d-1}$. For all odd d the Casimir energy must vanish; this is a completely general fact about odd-dimensional CFT related to the absence of anomalies (the theory on $R \times S^{d-1}$ may be obtained from that on R^d via a Weyl transformation). We check this vanishing on the higher spin side by using an appropriate zeta-function regularization of the sum over spins in global AdS_{d+1}. The vanishing of Casimir energy in $d = 3$ was perhaps the reason why it was not emphasized in [16]. However, the vanishing in odd d is not trivial from the AdS point of view because it involves summing over the entire tower of spins. Truncation of the spectrum in AdS_4 to a few low spins, which is commonly performed in "bottom-up" modeling, would generally spoil the cancellation of the Casimir energy. This would violate a possible exact duality to a CFT_3, unless there is another reason for the bulk cancellation, such as supersymmetry (as in [44–46]).

The comparison of the Casimir energies becomes even more interesting for even d, where they are not required to vanish. For the $\mathcal{N} = 4$ $SU(N)$ gauge theory in $d = 4$, the $\mathcal{O}(N^2)$ term in the Casimir energy was reproduced early on using the stress-energy tensor calculation in $AdS_5 \times S^5$ [47]. Due to the cancellation of the total derivative ("D-anomaly") terms in the full expression for the trace anomaly, its Casimir energy is proportional to its a-anomaly coefficient [48,49]. Therefore, the exact AdS/CFT matching of Casimir energy in that case is guaranteed by the a-anomaly matching [50]. In the field theory, the exact result is $a = N^2 - 1$, and the $\mathcal{O}(N^0)$ correction (i.e. the -1 shift) has been studied using the one-loop correction in the type IIB supergravity on the $AdS_5 \times S^5$ background [51]. More recently, additional progress has been made in calculating the $\mathcal{O}(N^0)$ correction to $a - c$ in various $d = 4$ theories, where only contributions of short supermultiplets in AdS_5 need to be included [52,53].

In nonsupersymmetric theories, the Casimir energy is not simply proportional to a due to the presence of the total derivative anomaly terms [48,49]. This makes the comparison of Casimir energies a new check of the vectorial AdS/CFT conjectures, which is independent of the comparison of a anomalies carried out in [39,41]. Unfortunately, due to the lack of information about the form of the classical action, in the higher spin theories there is no known way to calculate the leading, $\mathcal{O}(N)$, terms in the sphere free energies or Casimir energies. So, as in [39,41], we will only compare the terms of order N^0. In the nonminimal Vasiliev theory including all integer spins, we find that the regularized sum in AdS_{d+1} vanishes, in line with the expectation that there is no $\mathcal{O}(N^0)$ correction in the free complex scalar theory on $R \times S^{d-1}$. However, in the minimal theory, which includes even spins only, the regularized sum equals the Casimir energy of a real scalar field. These results are analogous to the recent findings in Euclidean AdS_{d+1}, where the one-loop correction for the minimal theory in AdS_{d+1} was found to be equal to the free energy of a single real scalar on S^d [39,41]. The proposed interpretation of this result is that the bulk coupling constant in the minimal higher spin theory is $G_N \sim 1/(N-1)$, so that the tree level and one-loop terms can add to give the answer which is N times the contribution of a free scalar field. Our new results for Casimir energies in all d provide additional support for this interpretation.

In this paper we also study the vectorial fermionic models on $S^1 \times S^{d-1}$ and match their partition functions and Casimir energies with the corresponding quantities in AdS_{d+1}. Such calculations are quite useful: for $d > 3$ the dual higher spin theory in AdS_{d+1} includes massless gauge fields in mixed symmetry representations [25,27,54–56], in addition to the totally symmetric higher spin fields found in the Vasiliev theories dual to the scalar field theories [12]. The AdS spectrum dual to a fermionic model depends sensitively both on the dimension d and on what type of fermions we consider: Dirac, Majorana or Weyl. These results suggest the existence of a variety of consistent interacting higher spin theories that are dual to fermionic CFTs restricted to singlet sectors.

We start in Sec. II with a brief summary of some standard relations between Casimir energies and partition functions, and then in Sec. III review the expression for the free

[1] Perhaps the constraint can be implemented by coupling the free N-vector theory to an appropriate topological gauge theory, generalizing the idea of coupling to Chern-Simons theory in $d = 3$ [42,43]. For the purposes of this paper, the details of how the singlet constraint is imposed do not seem to matter.

energy of free conformal fields in $S^1 \times S^{d-1}$. In Sec. IV we compute this free energy for a large number N of complex or real scalar or fermion fields in the presence of a singlet constraint. The latter translates into an extra Gaussian averaging over the density of $U(N)$ or $O(N)$ holonomy eigenvalues that leads to a modification of the effective one-particle partition function. The resulting free energy contains an order N Casimir energy term as well as an order N^0 term with nontrivial β-dependence. The scalar free energies are matched onto the corresponding expressions in the dual HS theories in AdS_{d+1} in Sec. V. Section VI contains a similar analysis of the vectorial fermionic CFTs in $d = 2$, 3, 4 and of their higher spin duals. For each admissible type of fermion, we study the quantum numbers of the currents and corresponding gauge fields, and demonstrate the AdS/CFT matching of the Casimir energies and partition functions. In Sec. VII we briefly discuss the HS duals of the CFTs containing N_f fundamental flavors of $U(N)$ or $O(N)$. In the large N limit where N_f is held fixed, we demonstrate the matching of partition functions and Casimir energies with the field theory results.

II. GENERAL BACKGROUND

Given a CFT in d dimensions, in the standard radial quantization picture its states may be described as eigenstates of the Hamiltonian on $R_t \times S^{d-1}$. Given a set of states and ignoring interactions one may then consider, e.g., the corresponding Casimir energy and construct the finite temperature partition function. The same quantities may be computed also on the dual AdS_{d+1} side as the vacuum energy in the global AdS or as the one-loop partition function on a thermal quotient of AdS, i.e. on Euclidean AdS with boundary $S^1_\beta \times S^{d-1}$.

Let us summarize some standard relations (see, e.g., [57]). Given the spectrum of a Hamiltonian H (with eigenvalues ω_n and degeneracies d_n where n is a multi-index) one may consider the "energy" zeta function

$$\zeta_E(z) = \mathrm{tr} H^{-z} = \sum_n \mathrm{d}_n \omega_n^{-z}, \tag{2.1}$$

so that the Casimir or vacuum energy is given by (for fermions one needs to add a minus sign)

$$E_c = \frac{1}{2}\sum_n \mathrm{d}_n \omega_n = \frac{1}{2}\zeta_E(-1). \tag{2.2}$$

One may also define the one-particle or canonical partition function[2]

$$\mathcal{Z}(\beta) = \mathrm{tr} e^{-\beta H} = \sum_n \mathrm{d}_n e^{-\beta\omega_n}. \tag{2.3}$$

It is related to $\zeta_E(z)$ by the Mellin transform

$$\zeta_E(z) = \frac{1}{\Gamma(z)}\int_0^\infty d\beta \beta^{z-1}\mathcal{Z}(\beta), \tag{2.4}$$

i.e. the two functions contain an equivalent amount of information about the spectrum. This is the same as the usual relation between a spectral zeta function for an operator Δ (here $\Delta = H$) and its heat kernel [here $\mathcal{Z}(\beta) = K(\tau) = \mathrm{tr} e^{-\tau\Delta}$ with β playing role of τ]. Note also that a special case of (2.4) is the integral representation for the standard Hurwitz zeta function

$$\zeta(z, a) = \sum_{k=0}^\infty (k+a)^{-z} = \frac{1}{\Gamma(z)}\int_0^\infty d\beta\beta^{z-1}\frac{e^{-a\beta}}{1-e^{-\beta}}. \tag{2.5}$$

The multiparticle or grand canonical partition function, which for bosons is

$$\ln Z(\beta) = \mathrm{tr}\ln(1-e^{-\beta H})^{-1} = -\sum_n d_n \ln(1-e^{-\beta\omega_n}), \tag{2.6}$$

is then directly related to the one-particle one (2.3), with the free energy given by

$$F_\beta = -\ln Z(\beta) = -\sum_{m=1}^\infty \frac{1}{m}\mathcal{Z}(m\beta). \tag{2.7}$$

For fermions

$$\ln Z(\beta) = \mathrm{tr}\ln(1+e^{-\beta H}) = \sum_n d_n \ln(1+e^{-\beta\omega_n}) \tag{2.8}$$

is then directly related to the one-particle one (2.3), with the free energy given by

$$F_\beta = -\ln Z(\beta) = \sum_{m=1}^\infty \frac{(-1)^m}{m}\mathcal{Z}(m\beta). \tag{2.9}$$

Thus the knowledge of the one-particle partition function $\mathcal{Z}(\beta)$ determines the thermodynamic partition function (2.7) as well as the Casimir energy [see (2.2) and (2.4)].

III. PARTITION FUNCTIONS FOR FREE CFTS ON $S^1 \times S^{d-1}$

Let us first consider the partition function of a free conformally coupled scalar

$$F = -\ln Z = \frac{1}{2}\ln\det\Delta_0, \qquad \Delta_0 = -\nabla^2 + \frac{d-2}{4(d-1)}R, \tag{3.1}$$

in (Euclidean) $M^d = R \times S^{d-1}$ and $M^d_\beta = S^1 \times S^{d-1}$ where β is the length of S^1.[3] We shall assume the length of time direction in $R \times S^{d-1}$ to be regularized as $\int dt = \beta \to \infty$, so that the first case may be viewed as the zero-temperature

[2]For simplicity we shall ignore possible chemical potentials.

[3]We shall often set the radius ℓ of S^{d-1} to 1; dependence on it can be restored by rescaling $\beta \to \frac{\beta}{\ell}$, etc.

($\beta^{-1} \to 0$) limit of the second. In general, in $R \times S^{d-1}$ one finds (see, e.g., [58–61])

$$F = F_\infty + F_c, \qquad F_\infty = a_d \ln \Lambda, \qquad F_c = \beta E_c, \quad (3.2)$$

where we separated an *a priori* possible logarithmically divergent term from the vacuum (Casimir) energy E_c of a conformal scalar in the static Einstein universe $R \times S^{d-1}$. The logarithmically divergent part (with Λ standing for the product of a UV cutoff with the scale ℓ) is proportional to the conformal anomaly coefficient a_d which vanishes for odd d.[4] In the present case it vanishes also for even d as for a conformally coupled scalar in a conformally flat space it is proportional to the Euler number density but the latter vanishes for both $R \times S^{d-1}$ and $S^1 \times S^{d-1}$, i.e.

$$a_d = 0. \quad (3.3)$$

The scalar curvature of S^{d-1} is $R = (d-1)(d-2)$ so that the operator in (3.1) is $\Delta_0 = -\partial_t^2 - \nabla^2_{S^{d-1}} + \frac{1}{4}(d-2)^2$. Since the eigenvalues and their degeneracies for a Laplacian $-\nabla^2$ on a sphere of dimension $d-1$ are

$$\lambda_n(S^{d-1}) = n(n+d-2),$$
$$\mathrm{d}_n(S^{d-1}) = \binom{n+d-1}{d-1} - \binom{n+d-3}{d-1} = (2n+d-2)\frac{(n+d-3)!}{(d-2)!n!}, \quad (3.4)$$

the eigenvalues of Δ_0 on $R \times S^{d-1}$ are $\lambda_{w,n} = w^2 + \omega_n^2$, where $\omega_n = n + \frac{1}{2}(d-2)$, $n = 0, 1, 2, \ldots$ and $w \in (-\infty, \infty)$. There is no zero mode for $d > 2$. Integrating $\int_{-\infty}^{\infty} \frac{dw}{2\pi} \ln(w^2 + \omega_n^2)$ over w leads as usual to $F_c = \beta E_c$ with

$$E_c = \frac{1}{2}\sum_{n=0}^{\infty} \mathrm{d}_n \omega_n = \sum_{n=0}^{\infty} \frac{(n+d-3)!}{(d-2)!n!}\left[n + \frac{1}{2}(d-2)\right]^2. \quad (3.5)$$

This is finite if defined using the zeta-function regularization (see, e.g., [61–64]), i.e. by starting as in (2.1)–(2.2) with ω_n as energy eigenvalues, one first computes $\zeta_E(z) \equiv \sum_{n=0}^{\infty} \mathrm{d}_n \omega_n^{-z}$, and then analytically continues to $z = -1$, $E_c = \frac{1}{2}\zeta_E(-1)$.[5]

[4]In (3.2) we include the volume factor $\sim\beta$ into a_d.

[5]Note that if we consider the conformal scalar operator on S^{d-1}, i.e. $\Delta_{0c}(S^{d-1}) = -\nabla^2_{S^{d-1}} + \frac{1}{4}(d-2)^2$, then its eigenvalues are $\lambda_n = \omega_n^2$ with $\omega_n = n + \frac{1}{2}(d-2)$ and the corresponding spectral zeta function is $\zeta_{\Delta_{0c}(S^{d-1})}(z) = \sum_{n=0}^{\infty} d_n \lambda_n^{-z} = \zeta_E(2z)$. In particular, $E_c = \frac{1}{2}\zeta_E(-1) = \frac{1}{2}\zeta_{\Delta_{0c}(S^{d-1})}(-\frac{1}{2})$ [61]. Since the natural spectral parameter is $n + \frac{1}{2}(d-2)$ the vacuum energy E_c can be expressed in terms of the corresponding Hurwitz zeta functions. It can also be computed using an exponential regularization $e^{-\epsilon[n+\frac{1}{2}(d-2)]}$ (dropping all singular terms).

In the case of $M_\beta^d = S^1 \times S^{d-1}$ the eigenvalues of Δ_0 are

$$\lambda_{k,n} = \left(\frac{2\pi k}{\beta}\right)^2 + \omega_n^2, \qquad \omega_n = n + \frac{1}{2}(d-2),$$
$$k = 0, \pm 1, \pm 2, \ldots, \qquad n = 0, 1, 2, \ldots \quad (3.6)$$

One may define the spectral zeta function

$$\zeta_{\Delta_0}(z) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} \mathrm{d}_n \lambda_{k,n}^{-z}, \quad (3.7)$$

in terms of which we have for F in (3.1) (see, e.g., [65,66])

$$F = -\zeta_{\Delta_0}(0) \ln \Lambda - \frac{1}{2}\zeta'_{\Delta_0}(0) = F_\infty + F_c + F_\beta, \quad (3.8)$$

$$F_\infty = a_d \ln \Lambda, \qquad F_c = \beta E_c = \frac{1}{2}\beta \sum_{n=0}^{\infty} \mathrm{d}_n \omega_n, \quad (3.9)$$

$$F_\beta = \sum_{n=0}^{\infty} \mathrm{d}_n \ln(1 - e^{-\beta\omega_n}). \quad (3.10)$$

Here again $a_d = 0$ for a conformally coupled scalar on $S_\beta^1 \times S^{d-1}$ in any d and F_c is the same Casimir energy part with E_c given by (3.5). The nontrivial part of the free energy F_β vanishes in the limit $\beta \to \infty$ when (3.8) reduces to (3.2).

Using the standard Riemann ζ-function regularization [with $\zeta(-2k) = 0$, $\zeta(-2k-1) \neq 0$] one finds for the Casimir energy for $d > 2$ (see, e.g., [61–63])

$$E_c = \sum_{q=0}^{[\frac{d-3}{2}]} c_q \zeta(2q + 1 - d), \quad (3.11)$$

$$d = \text{odd} \geq 3: \quad E_c = 0; \qquad d = \text{even} \geq 4:$$
$$E_c = \sum_{q=0}^{\frac{1}{2}d-2} c_q \zeta(2q + 1 - d), \quad (3.12)$$

where c_q are rational coefficients. Thus E_c is nonvanishing in even d and can be expressed in terms of the Bernoulli numbers (see also below).

We conclude that

$$d = \text{odd} \geq 3: \quad F = F_\beta; \qquad d = \text{even} \geq 4:$$
$$F = \beta E_c + F_\beta, \quad (3.13)$$

where F_β in (3.10) has the following explicit form [cf. (2.3), (2.7)]:

$$F_\beta = -\sum_{m=1}^{\infty} \frac{1}{m} \mathcal{Z}_0(m\beta), \quad (3.14)$$

$$\mathcal{Z}_0(\beta) = \sum_{n=0}^{\infty} \mathrm{d}_n e^{-\beta[n+\frac{1}{2}(d-2)]} = \frac{q^{\frac{d-2}{2}}(1+q)}{(1-q)^{d-1}} = \frac{q^{\frac{d-2}{2}}(1-q^2)}{(1-q)^d}, \qquad q \equiv e^{-\beta}. \tag{3.15}$$

The one-particle partition function $\mathcal{Z}_0(\beta)$ corresponds to the character of the free scalar (Dirac singleton) representation of the conformal group $SO(d,2)$; see for instance [27].

Let us add a few details about the explicit values of the Casimir energy [(3.5) and (3.12)]. For odd $d = 2k+1$ one can rewrite (3.5) as $E_c^{(2k+1)} = \sum_{n=0}^{\infty}\sum_{m=1}^{k} c_m (n+\frac{1}{2})^{2m}$. For example, $E_c^{(3)} = \sum_{n=0}^{\infty}(n+\frac{1}{2})^2$, etc. Since $\sum_{n=0}^{\infty}(n+\frac{1}{2})^{2m} = \zeta(-2m,\frac{1}{2}) = 0$ one concludes that the Casimir energy vanishes for all odd $d = 2k+1$.[6] For even $d = 2k$ one finds

$$E_c^{(2k)} = \sum_{n=1}^{\infty}\sum_{m=1}^{k-1} \tilde{c}_m n^{2m+1}, \tag{3.16}$$

which is equivalent to the expression in (3.12) ($\tilde{c}_m = c_{-m-1+k}$). For example,

$$E_c^{(2)} = \sum_{n=1}^{\infty} n = \zeta(-1) = -\frac{1}{12},$$
$$E_c^{(4)} = \frac{1}{2}\sum_{n=1}^{\infty} n^3 = \frac{1}{2}\zeta(-3) = \frac{1}{240},$$
$$E_c^{(10)} = \frac{1}{40320}[\zeta(-9) - 14\zeta(-7) + 49\zeta(-5) - 36\zeta(-3)] = -\frac{317}{22809600}. \tag{3.17}$$

The same results can be reproduced by introducing a cutoff ($\epsilon \to 0$) with the spectral parameter in the exponent $e^{-\epsilon[n+\frac{1}{2}(d-2)]}$ in the sum in (3.5) and dropping all singular terms in the resulting expression

$$E_c(\epsilon) = \frac{1}{2}\frac{e^{-\frac{1}{2}\epsilon d}}{(1-e^{-\epsilon})^d}[d + (d-2)\cosh\epsilon]. \tag{3.18}$$

Let us also note that an equivalent expression for the Casimir energy (3.5) may also be obtained by a Mellin transform of the one-particle partition function (3.15); see Eqs. (2.2) and (2.4). This gives

$$E_c = \frac{1}{2}\zeta_E(-1),$$
$$\zeta_E(z) = \sum_{n=0}^{\infty}\binom{n+d-2}{d-2}\left[\left(n+\frac{d}{2}-1\right)^{-z} + \left(n+\frac{d}{2}\right)^{-z}\right]. \tag{3.19}$$

In this expression, the parameter z provides a natural analytic regulator, as one can perform the sum in terms of the Hurwitz zeta function and then analytically continue to $z = -1$. One may verify that this expression for E_c gives the same values quoted above. In particular, one can see directly the relation to (3.5) by rearranging the expression for $\zeta_E(z)$ as follows [cf. (3.4)][7]:

$$\zeta_E(z) = \sum_{n=0}^{\infty}\mathrm{d}_n\left[n+\frac{1}{2}(d-2)\right]^{-z},$$
$$\mathrm{d}_n = \binom{n+d-2}{d-2} + \binom{n+d-3}{d-2} = 2\left[n+\frac{1}{2}(d-2)\right]\frac{(n+d-3)!}{(d-2)!n!}. \tag{3.20}$$

It is straightforward to generalize the above analysis to the case of free complex or real fermion theories. First, for a single massless Dirac fermion in $S^1 \times S^{d-1}$ the free energy (2.9) is given by the following analog of the free conformal scalar expressions in (3.14)–(3.15):

$$F_\beta = \sum_{m=1}^{\infty}\frac{(-1)^m}{m}\mathcal{Z}_{\mathrm{F}}(m\beta),$$
$$\mathcal{Z}_{\mathrm{F}} = 2\frac{2^{[\frac{d}{2}]}q^{\frac{d-1}{2}}}{(1-q)^{d-1}}, \qquad q = e^{-\beta}. \tag{3.21}$$

The result for a real (Majorana) fermion or Weyl fermion is half of that,

$$\mathcal{Z}_{\frac{1}{2}}(\beta) = \frac{2^{[\frac{d}{2}]}q^{\frac{d-1}{2}}}{(1-q)^{d-1}}. \tag{3.22}$$

The expression for $\mathcal{Z}_{\mathrm{F}}(\beta)$ was given in [18] (taking the $q \to 1$ limit and comparing it to the real scalar counterpart in (3.15) one checks that it describes the right number of degrees of freedom). $\mathcal{Z}_{\frac{1}{2}}(\beta)$ in (3.22) also has the interpretation of the conformal group character for a complex fermion representation [27].

[6]This agrees with the vanishing of conformal anomaly in odd-dimensional space as there is a relation between a combination of Euler density and total derivative conformal anomaly coefficients, and the Casimir energy of a conformal scalar (or more generally CFT), as discussed in [48,61] and references therein.

[7]One is to change the summation variable in the second term $n \to n' - 1$ with n' running now from 1, and then observe that the $n' = 0$ term in the sum can be added without altering the result.

The corresponding Casimir energy in the Majorana or Weyl case is [cf. (2.4), (3.5), (3.19)][8]

$$E_c = -\frac{1}{2}\sum_n \mathrm{d}_n\omega_n = -\frac{1}{2}\zeta_E(-1),$$
$$\zeta_E(z) = \sum_{n=0}^{\infty} 2^{[\frac{d}{2}]}\frac{(n+d-2)!}{(d-2)!n!}\left[n+\frac{1}{2}(d-1)\right]^{-z}. \qquad (3.23)$$

As in the scalar case (3.12), this is zero for odd d and nonzero for even d; one finds, for instance, $E_c = -\frac{1}{24}, \frac{17}{960}, -\frac{367}{48384}$ in $d = 2, 4, 6$ respectively.[9]

IV. FREE CFT PARTITION FUNCTIONS ON $S^1 \times S^{d-1}$ WITH SINGLET CONSTRAINTS

In the context of AdS/CFT duality [4] we are interested in a conformal scalar partition function with an extra singlet constraint. As found in [16,28], in the case of N complex scalars transforming in the fundamental representation of $U(N)$, taking the large N limit and imposing the singlet constraint one effectively gets instead of (3.15) the *square* of the one-particle partition function

$$\mathcal{Z}_{U(N)}(\beta) = [\mathcal{Z}_0(\beta)]^2 = \frac{q^{d-2}(1+q)^2}{(1-q)^{2(d-1)}}. \qquad (4.1)$$

Below we shall first review the derivation of this result in [16] (which was based on [17–19]) streamlining the argument and extending it to any dimension $d \geq 3$. We shall then generalize it to the real scalar $O(N)$ case as this will allow us to compare to the minimal HS theory free energy in thermal AdS_{d+1}. In the real case we shall find that[10]

$$\mathcal{Z}_{O(N)}(\beta) = \frac{1}{2}[\mathcal{Z}_0(\beta)]^2 + \frac{1}{2}\mathcal{Z}_0(2\beta)$$
$$= \frac{1}{2}\frac{q^{d-2}(1+q)^2}{(1-q)^{2d-2}} + \frac{1}{2}\frac{q^{d-2}(1+q^2)}{(1-q^2)^{d-1}}. \qquad (4.2)$$

Similarly, in the complex and real fermion cases we shall find

$$\mathcal{Z}^{\rm ferm}_{U(N)}(\beta) = [\mathcal{Z}_{\frac{1}{2}}(\beta)]^2,$$
$$\mathcal{Z}^{\rm ferm}_{O(N)}(\beta) = \frac{1}{2}[\mathcal{Z}_{\frac{1}{2}}(\beta)]^2 - \frac{1}{2}\mathcal{Z}_{\frac{1}{2}}(2\beta), \qquad (4.3)$$

where $\mathcal{Z}_{\frac{1}{2}}(\beta)$ is given in (3.22).

One may also consider free theories with several fundamental flavors. For instance, we can start with NN_f free complex scalars ϕ^{ia}, $i = 1, \ldots, N$, $a = 1, \ldots, N_f$, and impose the $U(N)$ singlet constraint, similarly for fermionic theories. Of course, one can also consider NN_f real fields with the $O(N)$ singlet constraint. For such theories, the Casimir term is simply $F_c = NN_f\beta E_c$, where E_c is the Casimir energy of a single scalar or fermion. On the other hand, the one-particle partition functions that contribute to the nontrivial part of the free energy now take the form (assuming N_f is fixed in the large N limit)

$$\mathcal{Z}^{N_f}_{U(N)}(\beta) = N_f^2[\mathcal{Z}_0(\beta)]^2,$$
$$\mathcal{Z}^{N_f}_{O(N)}(\beta) = \frac{N_f^2}{2}[\mathcal{Z}_0(\beta)]^2 + \frac{N_f}{2}\mathcal{Z}_0(2\beta) \qquad (4.4)$$

for the scalar theories, and

$$\mathcal{Z}^{N_f-{\rm ferm}}_{U(N)}(\beta) = N_f^2[\mathcal{Z}_{\frac{1}{2}}(\beta)]^2,$$
$$\mathcal{Z}^{N_f-{\rm ferm}}_{O(N)}(\beta) = \frac{N_f^2}{2}[\mathcal{Z}_{\frac{1}{2}}(\beta)]^2 - \frac{N_f}{2}\mathcal{Z}_{\frac{1}{2}}(2\beta) \qquad (4.5)$$

for the fermion ones. These theories are dual to versions of Vasiliev higher spin theory with $U(N_f)$ or $O(N_f)$ bulk gauge symmetry [11,67]. In Sec. VII we will show how the above thermal partition functions, as well as the Casimir energies, are reproduced by the sums over higher spin fields in AdS_{d+1}.

A. Complex scalar case

Starting with N complex scalars of fundamental representation of $U(N)$, to ensure the singlet condition one may couple them to a constant flat connection $A_0 = U^{-1}\partial_0 U$ with $U \in U(N)$ and integrate over U. The resulting scalar operator will have eigenvalues as in (3.6) but with k shifted by phases α_i of the eigenvalues $e^{i\alpha_i}$ $(i = 1, \ldots, N)$ of the holonomy matrix, i.e. $\lambda_{k,n} = (\frac{2\pi k+\alpha_i}{\beta})^2 + \omega_n^2$. The resulting scalar determinant is then to be integrated over α_i with the standard $U(N)$ invariant measure given by the Van der Monde determinant, $[dU] = \prod_{k=1}^N d\alpha_k \prod_{i\neq j=1}^N |e^{i\alpha_i} - e^{i\alpha_j}|$ (see, e.g., [68]). As a result, the singlet partition function $\hat{Z}$ is the following modification of Z in (3.1) and (3.8) (for $2N$ real scalars) [16,19]:

$$\hat{Z} = e^{-\hat{F}}, \qquad \hat{F} = \hat{F}_c + \hat{F}_\beta,$$
$$e^{-\hat{F}_\beta} = \int \prod_{k=1}^N d\alpha_k e^{-\tilde{F}_\beta(\alpha_1,\ldots,\alpha_N)}, \qquad (4.6)$$

---

[8] The expression (3.23) agrees, of course, with the spectrum of the Dirac operator $-\nabla^2 + \frac{1}{4}R$ on S^{d-1}: the eigenvalues are $\lambda_n = [n + \frac{1}{2}(d-1)]^2 = \omega_n^2$ $(n = 0, 1, 2, \ldots.)$ and their degeneracy is twice $\mathrm{d}_n = 2^{[\frac{d}{2}]}\binom{n+d-2}{d-2}$ [there are two $(n \pm \frac{1}{2}, \frac{1}{2}, 0, \ldots)$ representations]. For Majorana fermions these are to be counted with normalization $\frac{1}{2}$ relative to a real scalar contribution, leading to $E_c = -\frac{1}{2}\sum_{n=0}^{\infty}\mathrm{d}_n\omega_n$ equivalent to (3.23).

[9] This agrees, e.g., with the standard values of the Majorana fermion Casimir energy in $d = 2$ (i.e. $-\frac{c}{12}, c = \frac{1}{2}$) and also with the value of the Dirac fermion Casimir energy in $d = 4$: $2 \times \frac{17}{960}$ [61].

[10] The $d = 3$ case of this expression was found in [28] using a collective field theory approach to vectorial duality.

$$\tilde{F}_\beta = -\frac{1}{2}\sum_{i\neq j=1}^{N} \ln \sin^2\frac{\alpha_i-\alpha_j}{2} + 2\sum_{i=1}^{N} f_\beta(\alpha_i), \qquad (4.7)$$

$$f_\beta(\alpha) = \sum_{m=1}^{\infty} c_m(\beta)\cos(m\alpha), \qquad c_m(\beta) = -\frac{1}{m}\mathcal{Z}_0(m\beta). \qquad (4.8)$$

Here $\mathcal{Z}_0(\beta)$ is the same one-particle partition function as in (3.15) ("one-letter" partition function of [18]) so that in the formal limit of $\alpha_i \to 0$ we get $\tilde{F}_\beta$ reducing to $2N$ times free energy of a real scalar F_β in (3.14). The trivial (not sensitive to α_i averaging) Casimir part $\hat{F}_c$ is the same as in (3.5) and (3.9) up to the $2N$ factor

$$\hat{F}_c = 2N\beta E_c. \qquad (4.9)$$

In the large N limit $\hat{F}_c$ scaling as N should match the contribution of the classical higher spin action in the AdS_{d+1} bulk. At the same time, the nontrivial part of $\hat{F}_\beta$ which will happen to scale as N^0 due to extra averaging over α_k [16] should thus be matched with the one-loop partition function of HS theory in thermal AdS_{d+1}.

Considering the large N limit one introduces as usual the eigenvalue density $\rho(\alpha), \alpha \in (-\pi,\pi)$ and replaces the integral over α_i by the path integral over the perdiodic field $\rho(\alpha)$ defined on a unit circle with the action

$$\tilde{F}_\beta(\rho) = N^2\int d\alpha d\alpha' K(\alpha-\alpha')\rho(\alpha)\rho(\alpha') + 2N\int d\alpha \rho(\alpha) f_\beta(\alpha), \qquad (4.10)$$

$$K(\alpha) = -\frac{1}{2}\ln(2-2\cos\alpha),$$
$$f_\beta(\alpha) = \sum_{m=1}^{\infty} c_m(\beta)\cos(m\alpha). \qquad (4.11)$$

Note that the fact the second term in (4.11) scales as N is because the matter is in the fundamental representation. Then, in the large N limit and as long as the temperature is parametrically smaller than a power of N [16], the saddle point solution for the eigenvalue density takes the form $\rho(\alpha) = \frac{1}{2\pi} + \frac{1}{N}\tilde{\rho}(\alpha)$, where $\tilde{\rho}(\alpha)$ does not contain a constant part.[11] An important point is that the constant part of $\rho(\alpha)$ does not couple to the source f_β (which does not contain a zero mode term) so that it can be effectively projected out without changing the nontrivial β-dependent part of the result. This allows us, in particular, to ignore the constant part of K.[12] Then doing the formal Gaussian path integral over periodic nonconstant $\rho(\alpha)$ gives an N-independent result for $\hat{F}_\beta$ in (4.6)

$$\hat{F}_\beta = -\int d\alpha d\alpha' K^{-1}(\alpha-\alpha') f_\beta(\alpha) f_\beta(\alpha'). \qquad (4.12)$$

The kernel K and its inverse K^{-1} have simple Fourier expansions,[13]

$$K(\alpha) = \sum_{m=1}^{\infty}\frac{1}{m}\cos(m\alpha), \qquad K^{-1}(\alpha) = \frac{1}{\pi^2}\sum_{m=1}^{\infty} m\cos(m\alpha). \qquad (4.13)$$

We conclude that (4.12) is given by

$$\hat{F}_\beta = -\sum_{m=1}^{\infty} m[c_m(\beta)]^2, \qquad (4.14)$$

where $c_m(\beta)$ was defined in (4.8), or explicitly [16]

$$\hat{F}_\beta = -\sum_{m=1}^{\infty}\frac{1}{m}\mathcal{Z}_{U(N)}(m\beta), \qquad \mathcal{Z}_{U(N)}(\beta) = [\mathcal{Z}_0(\beta)]^2. \qquad (4.15)$$

Compared to the "unprojected" free energy of a single real scalar in (3.14) here one gets the second power of the one-particle partition function $\mathcal{Z}_0$ factor in (3.15) and (4.1). As we have seen above, this squaring of $\mathcal{Z}_0$ has its origin in the Gaussian averaging over the density of the large N distribution of the eigenvalues of the holonomy matrix. In Sec. V, we will see that this result precisely matches the one-loop partition function of the higher spin theory in thermal AdS. Note that an important difference compared to the Yang-Mills case [18] is that in these vectorial models there is no phase transition at temperatures $T \sim 1$ (i.e. temperatures of order of the AdS scale). A phase transition only occurs at much higher (Planck scale) temperatures $T \sim N^{\frac{1}{d-1}}$ [16], where the calculation above breaks down [see the discussion below Eq. (4.11)].

B. Real scalar case

Let us now repeat the above discussion in the case of N real scalars transforming as a fundamental representation of $O(N)$. Since we are interested only in the large N limit we may choose N to be even, $N = 2\mathrm{N}$.[14] An orthogonal $N \times N$

[11] The constant mode of ρ ensures that it satisfies the normalization condition $\int_{-\pi}^{\pi} d\alpha \rho(\alpha) = 1$.

[12] Note that up to the factor of $\frac{1}{\pi}$ the kernel K is the same as the restriction of the Neumann function on a unit disc to its boundary.

[13] We used the identity $\ln(1+b^2-2b\cos\alpha) = -2\sum_{m=1}^{\infty}\frac{b^m}{m}\cos(m\alpha)$. Note also that the delta-function on nonconstant functions on a circle is $\bar{\delta}(\alpha) = \frac{1}{\pi}\sum_{m=1}^{\infty}\cos(m\alpha)$ with $\int d\alpha \cos(m\alpha)\cos(n\alpha) = \pi\delta_{mn}$.

[14] We expect that the $1/N$ expansion for $O(N)$ should not be sensitive to whether N is even or odd. See for instance [69] for an explicit example of the large N expansion of $SO(N)$ Chern-Simons theory.

matrix can be put into a canonical form with N diagonal 2×2 blocks $\binom{\cos\alpha\ -\sin\alpha}{\sin\alpha\ \ \cos\alpha}$ which can be further diagonalized to $\binom{e^{i\alpha}\ 0}{0\ e^{-i\alpha}}$. This may be formally viewed as a special $U(N)$ case where N eigenvalues α_i are chosen as $(\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \ldots, \alpha_{\rm N}, -\alpha_{\rm N})$. Then the analog of $\tilde{F}_\beta$ in (4.7) becomes (here we use $r, s = 1, 2, \ldots, {\rm N}$ to label α_i from different 2-planes)

$$\tilde{F}_\beta = -\frac{1}{2}\sum_{r\neq s=1}^{\rm N} \ln\sin^2\frac{\alpha_r-\alpha_s}{2} - \frac{1}{2}\sum_{r,s=1}^{\rm N}\sin^2\frac{\alpha_r+\alpha_s}{2} + \frac{1}{2}\sum_{r=1}^{\rm N}\ln\sin^2\alpha_r + 2\sum_{r=1}^{\rm N} f_\beta(\alpha_r), \tag{4.16}$$

where $f_\beta(\alpha_r)$ is given by the same expression as in (4.8) [that for $\alpha_r = 0$ the last term becomes $N f_\beta(0)$ or N times F_β in (3.14) as it should be for N real scalars]. The Casimir energy term is the same as in (4.9) with $N\to{\rm N}$. In the large N limit (4.16) is then replaced by

$$\tilde{F}_\beta(\rho) = {\rm N}^2\int d\alpha d\alpha' {\rm K}(\alpha,\alpha')\rho(\alpha)\rho(\alpha') + 2{\rm N}\int d\alpha\rho(\alpha){\rm k}(\alpha) + 2{\rm N}\int d\alpha\rho(\alpha) f_\beta(\alpha), \tag{4.17}$$

$$\begin{aligned} {\rm K}(\alpha,\alpha') &= -\frac{1}{2}\ \ln([2-2\cos(\alpha-\alpha')][2-2\cos(\alpha+\alpha')]) \\ &= 2\sum_{m=1}^{\infty}\frac{1}{m}\cos(m\alpha)\cos(m\alpha'), \\ {\rm k}(\alpha) &= \frac{1}{4}\ \ln(2-2\cos 2\alpha) = -\sum_{m=1}^{\infty}\frac{1}{2m}\cos(2m\alpha). \end{aligned} \tag{4.18}$$

Since $f_\beta(\alpha)$ in (4.8) contains only $\cos m\alpha$ modes in its expansion, the constant and $\sin m\alpha$ modes of a generic periodic function $\rho(\alpha) = a_0 + \sum_{m=1}^\infty (a_m \cos m\alpha + b_n \sin m\alpha)$ do not couple to the β-dependent source, i.e. we may restrict the integration to even nonconstant functions $\rho(\alpha) = \sum_{m=1}^\infty a_m \cos m\alpha$ (this allows us, in particular, to ignore constant terms in K and K). Then the Gaussian integration gives again an order N^0 term [cf. (4.12)]

$$\hat{F}_\beta = -\int d\alpha d\alpha' {\rm K}^{-1}(\alpha,\alpha') j(\alpha) j(\alpha'), \tag{4.19}$$

$${\rm K}^{-1}(\alpha,\alpha') = \frac{1}{2\pi^2}\sum_{m=1}^\infty m\cos(m\alpha)\cos(m\alpha'), \tag{4.20}$$

$$\begin{aligned} j &= f_\beta(\alpha) + {\rm k}(\alpha) \\ &= \sum_{m=1,3,5,\ldots}^\infty c_m\cos(m\alpha) + \sum_{m=2,4,6,\ldots}^\infty c'_m \cos(m\alpha), \end{aligned} \tag{4.21}$$

$$\begin{aligned} c_m &= -\frac{1}{m}\mathcal{Z}_0(m\beta), \\ c'_m &= c_m - \frac{1}{m} = -\frac{1}{m}[\mathcal{Z}_0(m\beta)+1], \end{aligned} \tag{4.22}$$

where we used (4.8). As a result, we find in the real scalar case (omitting the β-independent constant)

$$\begin{aligned} \hat{F}_\beta &= -\sum_{m=1}^\infty \frac{1}{m}\mathcal{Z}_{O(N)}(m\beta), \\ \mathcal{Z}_{O(N)}(\beta) &= \frac{1}{2}[\mathcal{Z}_0(\beta)]^2 + \frac{1}{2}\mathcal{Z}_0(2\beta). \end{aligned} \tag{4.23}$$

The second term in $\mathcal{Z}_{O(N)}(\beta)$ originates from the cross-term between the source ${\rm k}(\alpha)$ coming from the measure and the β-dependent source $f_\beta(\alpha)$. The explicit form $\mathcal{Z}_{O(N)}(\beta)$ was already given in (4.2).

If the case of N_f free complex or real scalar flavors in the fundamental of $U(N)/O(N)$, with N_f fixed in the large N limit, the only modification is that the last term in (4.10) and (4.17) acquires an extra factor of N_f. Then the same calculation as described above readily leads to the results in (4.4) for complex or real scalars.

C. Fermionic theories

In the $U(N)$ invariant case of N Dirac fermions the singlet constraint is again implemented by averaging the Dirac operator determinant over the $U(N)$ holonomy eigenvalues [16,19]. One difference compared to the scalar case in (4.6) is that now we will have the Casimir part of free energy (4.9) replaced by its fermion analog; also, the nontrivial β-dependent term f_β in (4.8) will be replaced by

$$\begin{aligned} f_\beta(\alpha) &= \sum_{m=1}^\infty c_m(\beta)\cos(m\alpha), \\ c_m(\beta) &= -\frac{(-1)^{m+1}}{m}\mathcal{Z}_{\frac{1}{2}}(m\beta), \end{aligned} \tag{4.24}$$

where $\mathcal{Z}_{\frac{1}{2}}$ was defined in (3.22), i.e. one is to replace the real scalar one-particle partition function $\mathcal{Z}_0$ in (4.8) by the real or Weyl fermion partition function $\mathcal{Z}_{\frac{1}{2}}$ [and add an extra $(-1)^{m+1}$ factor].[15]

The rest of the argument is unchanged, so we again end up with (4.14), now with c_m given in (4.24), i.e.

[15]The normalization can be checked by considering the $U(1)$ case when the free energy of a single complex scalar or $F_\beta = -2\sum_{m=1}^\infty \frac{1}{m}\mathcal{Z}_0(m\beta)$ [cf. (3.14)] should be replaced by free energy of a single Dirac fermion, or $F_\beta = -2\sum_{m=1}^\infty \frac{(-1)^{m+1}}{m}\mathcal{Z}_{\frac{1}{2}}(m\beta)$ [cf. (3.21)–(3.22)]. An extra factor of 2 in Eq. (24) of [16] appears to be a misprint.

$$\hat{F}_\beta = -\sum_{m=1}^{\infty} \frac{1}{m} \mathcal{Z}^{\text{ferm}}_{U(N)}(m\beta),$$
$$\mathcal{Z}^{\text{ferm}}_{U(N)}(\beta) = [\mathcal{Z}_{\frac{1}{2}}(\beta)]^2. \tag{4.25}$$

In the $O(N)$ singlet sector of N Majorana fermions, the starting point is (4.16) with the same function f_β as in (4.24). Doing the obvious replacement in (4.22), i.e. $c_m = \frac{(-1)^m}{m}\mathcal{Z}_{\frac{1}{2}}(m\beta)$, $c'_m = c_m - \frac{1}{m}$ (so that $c'_{2m} = \frac{1}{2m}[\mathcal{Z}_{\frac{1}{2}}(2m\beta) - 1]$) we finish with the following analog of (4.23):

$$\hat{F}_\beta = -\sum_{m=1}^{\infty} \frac{1}{m} \mathcal{Z}^{\text{ferm}}_{O(N)}(m\beta),$$
$$\mathcal{Z}^{\text{ferm}}_{O(N)}(\beta) = \frac{1}{2}[\mathcal{Z}_{\frac{1}{2}}(\beta)]^2 - \frac{1}{2}\mathcal{Z}_{\frac{1}{2}}(2\beta). \tag{4.26}$$

We shall see in Sec. VI how the partition functions in (4.25) and (4.26) are reproduced on the dual AdS higher spin theory side.

While the above calculation was presented in the case of a single fundamental fermion, it is straightforward to generalize it to the case of N_f flavors ($N_f \ll N$). One simply includes an extra factor of N_f in the free fermion one-particle partition function (4.24), and performing the Gaussian integral over the eigenvalue density immediately leads to the results quoted in (4.5).

V. HIGHER SPIN PARTITION FUNCTION IN AdS_{d+1} WITH $S^1 \times S^{d-1}$ BOUNDARY

Our aim in this section will be to compare the singlet sector scalar CFT free energies, calculated above, with their counterparts for higher spin theories in thermal AdS_{d+1}. The HS free energy can be found by summing the individual massless spin s contributions[16]:

$$F = \sum_s F^{(s)}, \qquad F^{(s)} = -\ln Z_s,$$
$$Z_s = \left(\frac{\det[-\nabla^2 + (s-1)(s+d-2)]_{s-1\perp}}{\det[-\nabla^2 - s + (s-2)(s+d-2)]_{s\perp}}\right)^{1/2}, \tag{5.1}$$

where the operators are defined on symmetric traceless transverse tensors [73,74].[17]

On general grounds, for a quantum field in AdS_{d+1} with boundary $S^1_\beta \times S^{d-1}$ one expects the one-loop free energy to have the following structure [cf. (3.8)]:

$$F = F_0 + F_\beta, \qquad F_0 = \beta \bar{F}_0 = F_\infty + F_c,$$
$$F_\infty = a_{d+1} \ln \Lambda, \tag{5.2}$$

where F_0 is linear in β (i.e. proportional to the volume) while the part F_β with nontrivial β dependence is finite and vanishes in the zero-temperature limit $\beta \to \infty$. We have split F_0 into a possible UV logarithmically divergent part, and a finite part F_c (power divergences are assumed to be regularized away).

The coefficient a_{d+1} of the UV divergent term vanishes automatically if $d+1$ is odd. For even $d+1$ it is given by an integral of the corresponding local Seeley coefficient, which, in the case of AdS_{d+1}, is proportional to the product of the volume factor and $\frac{1}{2}(d+1)$ power of the constant curvature. Since this curvature factor is the same for any β or regardless of the topology of the boundary, the dependence of $a^{(s)}_{d+1}$ on the spin s should be the same as that found in [39,41] for the case of Euclidean AdS_{d+1}, i.e. the hyperboloid H^{d+1}, whose boundary is S^d. In particular, it was shown in [39,41] (for various values of d) that the total anomaly coefficient $\sum_s a^{(s)}_{d+1}$ vanishes after summing over spins (assuming the zeta-function regularization of the sum), so that there are no logarithmic UV divergences in the standard or minimal HS theory. Thus in what follows we shall set $F_\infty = 0$.

The problem of computing the β-dependent part F_β of the one-loop free energy in thermal AdS_{d+1} can be approached from the Hamiltonian point of view [44,57], using group-theoretic considerations to determine the energy spectrum [44] of a spin s field in global AdS_{d+1} with reflective boundary conditions [75,76]. An equivalent result for F_β is found in the path integral approach by starting with the heat kernel for the hyperboloid H^{d+1} [77–79] and using the method of images to find its counterpart for thermal AdS_{d+1} viewed as a quotient H^{d+1}/Z (see [80,81] for the AdS_3 case and [73,82] for the general case).[18] In this heat kernel approach F_0 in (5.2) is the zero-mode part of the sum over the images, and it is thus natural to identify it with $\hat{F}_0 = \frac{\mathrm{Vol}(H^{d+1}/Z)}{\mathrm{Vol}(H^{d+1})} F(H^{d+1})$, where $F(H^{d+1})$ is the free energy on H^{d+1}. This $\hat{F}_0$ requires a proper definition or regularization (cf. [80]) and was not studied in [82].[19]

At the same time, the expected correspondence with the free energy (3.8) of the boundary CFT in $S^1 \times S^{d-1}$ suggests that the finite part of F_0 should be closely related to the vacuum or Casimir energy of the corresponding fields in AdS_{d+1}. As this relation appears to be obscure in the H^{d+1}/Z construction let us discuss an alternative approach to justify it.

[16]Here we consider free massless totally symmetric higher spins with the Lagrangian originally found by Fronsdal in AdS_4 [70]. An extension of Fronsdal's work to higher dimensions was carried out in [71,72].

[17]We set the AdS scale to 1. The energies (or scaling dimensions) of the corresponding representations are $\Delta = e_0 = s + d - 2$ for the physical field [71] and $\Delta = e_0 = s + d - 1$ for the ghost one.

[18]Ref. [82] used, in fact, an analytic continuation of a heat kernel of a quotient of a sphere S^{d+1}.

[19]For a related recent discussion of $\hat{F}_0$ in the case of a massive scalar in AdS_2 and AdS_4 see [83].

Let us recall that starting with the Euclidean AdS_{d+1}, realized as a hyperboloid $x_{d+1}^2 - x_0^2 - x_i x_i = 1$ in $R^{1,d+1}$, one may choose different sets of coordinates (see, e.g., [84]). For example, one may set $x_{d+1} = \cosh\xi$, $x_m = (x_0, x_i) = \sinh\xi n_m$ where $n_m n_m = 1$, getting the H^{d+1} metric $ds_\xi^2 = d\xi^2 + \sinh^2\xi d\Omega_d$ with S^d as its boundary. One may also choose the coordinates as $x_{d+1} = \cosh\rho\cosh t$, $x_0 = \cosh\rho\sinh t$, $x_i = \sinh\rho n_i$, $n_i n_i = 1$ obtaining the Euclidean continuation of the global AdS_{d+1} metric, i.e. $ds_t^2 = \cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_{d-1}$. Compactifying t on a circle of length β gives thermal AdS_{d+1} metric $ds_{t,\beta}^2$ with $S_\beta^1 \times S^{d-1}$ as the boundary. A direct computation of the full expression for the determinant of a scalar Laplacian in the case of AdS_{d+1} with ds_t^2 or $ds_{t,\beta}^2$ metric did not seem to appear in the literature. In particular, it is not obvious how a finite part F_c of the $\beta\to\infty$ limit of F computed for $ds_{t,\beta}^2$ will match the expression found in [77,79] in the case of the hyperboloid ds_ξ^2.

The AdS_{d+1} metric ds_t^2 may be written also as $ds_t^2 = \frac{1}{\cos^2\theta}(dt^2 + d\theta^2 + \sin^2\theta d\Omega_{d-1})$, where $\theta\in[0,\frac{\pi}{2})$ with the boundary $R\times S^{d-1}$ at $\theta = \frac{\pi}{2}$. Thus, it is conformal to a half of the Einstein universe $R\times S^d$. Indeed, there is a close correspondence [85] between the spectrum of the energy operator in AdS_{d+1} and the spectrum of the Laplacian in the half of the Einstein universe with reflective boundary conditions (Dirichlet or Neumann) at the equator of S^d [75,76]. In particular, the Casimir energies in AdS_4 (as defined in [44]) and in $R\times S^3$ are the same up to a factor [85]. This implies also that this energy spectrum determines the nontrivial part F_β of the free energy,[20] in agreement with its alternative derivations in [86] (from direct evaluation of the scalar stress tensor in AdS) and in [74,80,82] (from the H^{d+1}/Z construction of the heat kernel).[21]

It remains then to understand the relation between the F_0 part of free energy in AdS_{d+1} and in the conformally related Einstein universe. Given an ultrastatic space-time $R\times M^d$ with a Euclidean metric $ds^2 = dt^2 + \tilde g_{ij}(x)dx^i dx^j$, one can readily show that, up to a standard logarithmically divergent term proportional to $\zeta(0)$ [65,88], the corresponding free energy or $-\ln Z$ is given by the Casimir energy term $F_c = \beta E_c$. Here $\beta\to\infty$ is the length of the time interval and $E_c = \frac{1}{2}\sum_n d_n\omega_n$, with ω_n^2 being the eigenvalues of the Laplacian restricted to M^d (cf. Sec. II). For a conformally related static space-time $ds^2 = g_{00}(x)dt^2 + g_{ij}(x)dx^i dx^j$ the full expression for the free energy will contain, in addition to F_c, extra g_{00}-dependent local terms reflecting the required conformal rescaling by $g_{00}(x)$ [58,89] (see also, e.g., [78,90]).

These extra terms are linear in β, i.e. not changing the F_β part in the finite temperature case. These terms are similar [91] to the "integrated conformal anomaly" terms found for conformally invariant matter fields. They should be closely related to the $\zeta(0)$-type terms that contribute to the F_∞ part of F in (5.2) and should thus vanish like $\sum_s a_{d+1}^{(s)}$ after one sums over the spins. For that reason in what follows we shall assume that the $\beta\to\infty$ limit of the UV finite part of the free energy (5.2) in thermal AdS_{d+1} has indeed the interpretation of the Casimir energy term, i.e. $F_c = \beta E_c$. As we shall see below, this is fully consistent with the AdS/CFT correspondence.

A. Temperature-dependent part of the free energy

Let us first discuss the temperature-dependent part, F_β, of the free energy of higher spin theories and then turn to the Casimir part in the next subsection.

The expression for F_β of the totally symmetric massless spin s field is [73,82]

$$F_\beta^{(s)} = -\sum_{m=1}^{\infty}\frac{1}{m}\mathcal{Z}_s(m\beta), \tag{5.3}$$

$$\mathcal{Z}_s(\beta) = \frac{q^{s+d-2}}{(1-q)^d}(\mathrm{d}_s - \mathrm{d}_{s-1}q), \qquad q\equiv e^{-\beta} \tag{5.4}$$

$$\mathrm{d}_s = (2s+d-2)\frac{(s+d-3)!}{(d-2)!s!}. \tag{5.5}$$

Here d_s is the number of symmetric traceless rank s tensors in d dimensions or dimension of $(s,0,0,\ldots)$ representation of the "little" group $SO(d)$ (i.e. $\mathrm{d}_s|_{d=3} = 2s+1$, $\mathrm{d}_s|_{d=4} = (s+1)^2$ etc.). Note that d_s is exactly the same as the degeneracy of eigenvalues of the scalar Laplacian on S^{d-1} if we replace the angular momentum quantum number n in d_n in (3.4) by the spin s.

$\mathcal{Z}_s(\beta)$ in (5.4) thus has an interpretation of the corresponding one-particle partition function [cf. (2.3) and (2.7)]. From the point of view of d-dimensional CFT, $\mathcal{Z}_s$ is the character of the representation of $SO(d,2)$ containing the spin s primary field of dimension $\Delta = s+d-2$ and its descendants [27,57]. The explicit results for $d = 2,3,4$ are

$$d=2:\quad \mathcal{Z}_{s>1}(\beta) = \frac{2q^s - 2q^{s+1}}{(1-q)^2} = \frac{2q^s}{(1-q)}, \tag{5.6}$$

[20]Note that for conformally invariant fields F_β is always the same in conformally related static spaces [58].

[21]It should be noted that while one may expect the vacuum energy to scale as volume of global AdS space (which should factorize as AdS is a homogeneous space) this is actually in contradiction with reflective energy-conserving boundry conditions (appropriate for finite temperature set up) which lead to discrete spectrum of the Laplacian (see [87] for a discussion in the AdS_2 case). We expect that under an appropriate regularization, the large β limit of the stress energy computation in [86] should reproduce the "non-extensive" expression for the total AdS vacuum energy as a sum over global energy eigenvaluaes (2.2) found in the Hamiltonian approach [44,57]. One possibility may be to do the integration over the radial AdS direction for finite β and then take the limit $\beta\to\infty$ in the result.

$$d=3: \quad \mathcal{Z}_{s>0}(\beta)=\frac{q^{s+1}}{(1-q)^3}[2s+1-(2s-1)q], \quad (5.7)$$

$$d=4: \quad \mathcal{Z}_s(\beta)=\frac{q^{s+2}}{(1-q)^4}[(s+1)^2-s^2q]. \quad (5.8)$$

The low spin cases in $d=2,3$ are special. For the spin 0 primary field of general dimension Δ,

$$\mathcal{Z}_0^{(\Delta)}=\frac{q^\Delta}{(1-q)^d}, \quad (5.9)$$

since no ghosts need to be subtracted (this ghost term vanishes automatically for $s=0$ for all $d>3$). In AdS_3 there are two possibilities for $s=1$. For the Maxwell action, which was conjectured to be relevant to the $d=2$ scalar theory [41], $\mathcal{Z}_1^{\mathrm{Maxwell}}=\frac{2q-q^2}{(1-q)^2}$; for the Chern-Simons action, which is relevant to the $d=2$ fermionic theory, $\mathcal{Z}_1^{\mathrm{CS}}=\frac{2q-2q^2}{(1-q)^2}$.

Putting these elements together, we find the total free energy of the standard Vasiliev theory in AdS_{d+1} where each integer spin is counted once [12]. Including the $s=0$ contribution with $\Delta=d-2$, and $s=1$ Maxwell theory for $d=2$, we find that in all dimensions $d\geq 2$

$$F_\beta=\sum_{s=0}^{\infty}F_\beta^{(s)}=-\sum_{m=1}^{\infty}\frac{1}{m}\mathcal{Z}(m\beta), \quad (5.10)$$

$$\mathcal{Z}(\beta)=\mathcal{Z}_0^{(d-2)}+\sum_{s=1}^{\infty}\mathcal{Z}_s(\beta)=\frac{q^{d-2}(1+q)^2}{(1-q)^{2d-2}}. \quad (5.11)$$

Comparing to (4.15), (4.1) and (3.15) we conclude that this is exactly the same as the order N^0 term in the large N limit of the free energy of a complex $U(N)$ scalar in $S^1\times S^{d-1}$ with the singlet condition imposed. This is thus a generalization to all dimensions $d\geq 2$ of the matching found in [16] (and also, in the collective field theory approach, in [28]) in the $d=3$ case. This matching is a consistency check that the boundary and the bulk theories have the same spectrum of states: indeed, the free spectra determine both the one-loop term in the β-dependent bulk theory free energy and also the leading order N^0 term in the boundary theory free energy.

The identity (5.11) also has an interpretation as an expansion of the CFT partition function in terms of characters $\mathcal{Z}_s$ of the conformal group. This expansion is completely determined by the spectrum of primary fields and by the conformal symmetry. For example, if we consider the large N limit of the $d=3$ critical $U(N)$ model, then the dimension of the singlet scalar primary operator $\bar\phi_i\phi^i$ changes from 1 to 2, while the dimensions of singlet higher spin primaries remain the same as in the free theory.[22] Therefore, the partition function of such a large N CFT must have the form[23]

$$\mathcal{Z}_{\mathrm{crit}}(\beta)=\frac{q^2}{(1-q)^3}+\sum_{s=1}^{\infty}\frac{q^{s+1}}{(1-q)^3}\times[2s+1-(2s-1)q]. \quad (5.12)$$

This is equal to the one-particle partition function of the Vasiliev theory in AdS_4 with the $\Delta=2$ boundary condition for the bulk scalar. Therefore, once again, the AdS/CFT agreement of the partition functions follows from the conformal symmetry and the agreement of the spectra.

Let us note that the HS partition function corresponding to (5.10)–(5.11) may be rewritten as[24]

$$\begin{aligned}F_\beta=-\ln Z_\beta&=\sum_{n=1}^{\infty}\binom{n+2d-4}{2d-3}\ln[(1-q^{n+d-3})(1-q^{n+d-2})^2(1-q^{n+d-1})]\\&=\ln[(1-q^{d-2})(1-q^{d-1})^{2d}]+\sum_{n=1}^{\infty}C_n\ln(1-q^{n+d-1}),\\C_n&=\binom{n+2d-4}{2d-3}+2\binom{n+2d-3}{2d-3}+\binom{n+2d-2}{2d-3}=\frac{(n+2d-2)!}{(2d-3)!(n+1)!}[4n(n+2d-2)+2d(2d-3)]. \quad (5.13)\end{aligned}$$

This generalizes to any d the expressions given for $d=3,4,6$ in [73].

In the minimal Vasiliev theory in AdS_{d+1}, which should be dual to the $O(N)$ singlet sector of the d-dimensional real scalar theory, one is to sum over all even spins only. Then instead of (5.10)–(5.11) one finds from (5.4)–(5.5)

[22] In the critical vector model, the anomalous dimension of the spin s currents is of order $1/N$.

[23] While this is guaranteed on general grounds, it would be nice to give a direct path integral derivation.

[24] One is to use that $\sum_{m=1}^{\infty}\frac{1}{m}\frac{q^{ma}}{(1-q^m)^b}=-\sum_{n=1}^{\infty}\binom{n+b-2}{b-1}\times\ln(1-q^{n+a-1})$ since $(1-x)^{-b}=\sum_{n=1}^{\infty}\binom{n+b-2}{b-1}x^{n-1}$.

$$F_{\beta_{\min}} = \sum_{s=0,2,4,\ldots}^{\infty} F_{\beta}^{(s)} = -\sum_{m=1}^{\infty} \frac{1}{m} \mathcal{Z}_{\min}(m\beta), \qquad (5.14)$$

$$\mathcal{Z}_{\min}(\beta) = \mathcal{Z}_0^{(d-2)} + \sum_{s=2,4,\ldots}^{\infty} \mathcal{Z}_s(\beta) = \frac{1}{2}\frac{q^{d-2}(1+q)^2}{(1-q)^{2d-2}} + \frac{1}{2}\frac{q^{d-2}(1+q^2)}{(1-q^2)^{d-1}}. \qquad (5.15)$$

This nicely matches the order N^0 term (4.23) and (4.2) in the free energy of the $O(N)$ singlet CFT.[25]

B. Casimir part of the free energy

Since we have already matched the N^0 part of dual scalar free energy, the Casimir energy part of the HS free energy in (4.6) should vanish after the summation over spins. The Casimir part of the scalar free energy (4.9) scales as N and thus should be compared to the classical order N part of the HS free energy.

More precisely, the above should apply to the standard HS theory dual to $U(N)$ complex scalar theory. In the $O(N)$ real scalar theory there is a subtlety noticed in [39]: the matching should work provided the classical HS coupling constant is not N but $N-1$ (cf. also [28]). In this case the Casimir energy of the minimal HS theory should not vanish but should match the Casimir energy of a single real conformal scalar in $R\times S^{d-1}$, i.e. (3.5). In other words, we should have the $N\beta E_c$ term in the free energy of the boundary theory matching the sum of the classical $(N-1)\beta E_c$ term plus the one-loop βE_c term in the minimal HS theory. We shall indeed confirm this below.

The Casimir part $F_c = \beta E_c$ is the same as in the case of the global AdS_{d+1} with boundary $R\times S^{d-1}$ (with time interval regularized by $\beta\to\infty$). It is defined by the spectrum of the Hamiltonian associated to the global AdS time. Equivalently, it can also be determined [e.g., via (2.2) and (2.4)] from the one-particle HS partition functions (5.11) and (5.15) found above. Explicitly, in the standard HS theory we should get

$$E_c = \frac{1}{2}\zeta_E(-1), \qquad \zeta_E(z) = \frac{1}{\Gamma(z)}\int_0^{\infty} d\beta\beta^{z-1}\mathcal{Z}(\beta), \qquad (5.16)$$

$$\mathcal{Z}(\beta) = \frac{e^{-(d-2)\beta}(1+e^{-\beta})^2}{(1-e^{-\beta})^{2d-2}} = \frac{\cosh^2\frac{\beta}{2}}{4^{d-2}(\sinh^2\frac{\beta}{2})^{d-1}}. \qquad (5.17)$$

Using that

$$(1-q)^{-b} = \sum_{n=1}^{\infty}\binom{n+b-2}{b-1} q^{n-1}, \qquad \frac{1}{\Gamma(z)}\int_0^{\infty} d\beta\beta^{z-1}e^{-a\beta} = a^{-z}, \qquad (5.18)$$

this gives for a general d[26]

$$\zeta_E(z) = \sum_{n=1}^{\infty}\binom{n+2d-4}{2d-3}\left[\frac{1}{(n+d-3)^z} + \frac{2}{(n+d-2)^z} + \frac{1}{(n+d-1)^z}\right] = \sum_{n=1}^{\infty}\frac{b_n(d)}{(n+d-3)^z}, \qquad (5.19)$$

$$b_n(d) = \binom{n+2d-4}{2d-3} + 2\binom{n+2d-5}{2d-3} + \binom{n+2d-6}{2d-3} = \frac{(n+2d-6)!}{(n-1)!(2d-3)!}[4n^2 + 8(d-3)n + 4d^2 - 22d + 32]. \qquad (5.20)$$

One can then compute E_c in (5.16), i.e. $E_c = \frac{1}{2}\sum_{n=1}^{\infty} b_n(d)(n+d-3)$, using the standard Riemann ζ-function regularization, finding that the standard HS theory vacuum energy in AdS_{d+1} vanishes for any d,

$$E_c = \frac{1}{2}\zeta_E(-1) = \sum_{s=0}^{\infty} E_{c,s} = 0. \qquad (5.21)$$

For example, for the HS theory in AdS_4, AdS_5 and AdS_7 we get

[25] Reference [28] also checked this matching in the $d=3$ case using the collective field formalism.

[26] Here [as also earlier in (3.20)] we have shifted the summation variable and noted that one can restore the lower value of the summation interval due to vanishing of the coefficients of the second and third terms at $n = 1, 2$.

$$
\begin{aligned}
d=3:\quad \zeta_E(z) &= \sum_{n=1}^{\infty}\frac{1}{6}n(n+1)(n+2)[n^{-z}+2(n+1)^{-z}+(n+2)^{-z}]\\
&= \sum_{n=1}^{\infty}\frac{1}{3}(2n^2+1)n^{-z} = \frac{1}{3}[2\zeta(z-3)+\zeta(z-1)],\\
d=4:\quad \zeta_E(z) &= \sum_{n=1}^{\infty}\frac{4}{5!}n(n+1)(n+2)(n^2+2n+2)(n+1)^{-z} = \frac{1}{30}[\zeta(z-5)-\zeta(z-1)],\\
d=6:\quad \zeta_E(z) &= \sum_{n=1}^{\infty}\frac{4}{9!}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n^2+6n+11)(n+3)^{-z}\\
&= \frac{4}{9!}[\zeta(z-9)-12\zeta(z-7)+21\zeta(z-5)+62\zeta(z-3)-72\zeta(z-1)]. \qquad (5.22)
\end{aligned}
$$

These expressions vanish at $z=-1$ due to $\zeta(-2n)=0$. Equivalently, one may use an exponential cutoff $e^{-\epsilon(n+d-3)}$ with the spectral parameter $(n+d-3)$ appearing in (5.19). Then the sum in (5.19) can be done exactly at $z=-1$, giving for the regularized vacuum energy (2.2) [cf. (3.18)]

$$
\begin{aligned}
E_c(\epsilon) &= \frac{1}{2}\zeta_E(-1;\epsilon)\\
&= \frac{4e^{-\epsilon d}}{(1-e^{-\epsilon})^{2d}}[d+(d-2)\cosh\epsilon]\sinh\epsilon. \qquad (5.23)
\end{aligned}
$$

Expanding in $\epsilon\to 0$ and subtracting the singular $\frac{1}{\epsilon^k}$ terms one finds that the finite part in (5.23) is always zero.

One can see the reason for this vanishing of the Casimir energy directly from (5.16). Since $\Gamma(-1)=\infty$, the result for E_c can be nonzero only if the remaining integral over β has a pole at $z=-1$. The pole cannot appear since the partition function $\mathcal{Z}(\beta)$ appearing in the integrand of (5.16) is even in β, i.e. contains only even powers of β in its small β expansion: $\mathcal{Z}(\beta)=4\beta^{-2(d-1)}[1+\frac{1}{12}(d-4)\beta^2+\ldots]$.

Let us now repeat the above analysis in the case of the minimal HS theory with the one-particle partition function in (5.15). Here we get the following analog of (5.16):

$$
E_c^{\min} = \frac{1}{2}\zeta_E^{\min}(-1), \qquad \zeta_E^{\min}(z) = \frac{1}{2}\zeta_E(z)+\delta\zeta(z), \quad (5.24)
$$

$$
\delta\zeta(z) \equiv \frac{1}{2}\frac{1}{\Gamma(z)}\int_0^\infty d\beta\beta^{z-1}\frac{e^{-(d-2)\beta}(1+e^{-2\beta})}{(1-e^{-2\beta})^{d-1}}, \quad (5.25)
$$

where $\zeta_E(z)$ in (5.24) is the standard HS function given by (5.16) and (5.19) which vanishes at $z=-1$ (5.21) as discussed above. Using (5.18) we find [cf. (5.19)]

$$
\delta\zeta(z) = \frac{1}{2}\sum_{n=1}^{\infty}\binom{n+d-3}{d-2}[(2n+d-4)^{-z}+(2n+d-2)^{-z}] = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(n+d-4)!}{(d-2)!(n-1)!}(2n+d-4)^{1-z}. \qquad (5.26)
$$

Thus we get

$$
E_c^{\min} = \frac{1}{2}\delta\zeta(-1) = \frac{1}{4}\sum_{n=1}^{\infty}\frac{(n+d-4)!}{(d-2)!(n-1)!}(2n+d-4)^2 = \sum_{n=0}^{\infty}\frac{(n+d-3)!}{(d-2)!n!}\left[n+\frac{1}{2}(d-2)\right]^2. \qquad (5.27)
$$

Comparing this expression to (3.5) we observe that it is exactly the same as the Casimir energy E_c of a single real conformal scalar in $R\times S^{d-1}$ given in (3.12). We conclude that, as already mentioned above, this is consistent with the $N\to N-1$ shift in identification of the coupling constant in the minimal HS theory–$O(N)$ real scalar duality [39,41].

The equivalence between the scalar Casimir energy in $R\times S^{d-1}$ (3.5) and the minimal HS theory Casimir energy in AdS_{d+1} (5.27) is seen at the level of formal series so the equality of the resulting finite expressions requires the use of the same (zeta-function) regularization on both sides of the duality.

While the Casimir energy (5.16) of the standard HS theory vanishes in AdS_{d+1} for any value of d, the Casimir energy of the minimal HS theory vanishes only for odd d, i.e. in AdS_4, AdS_6 etc. It is nonvanishing for even d, i.e. in AdS_5, AdS_7 etc. [see (3.12)]. This is to be compared with the well-known vanishing of vacuum energies of $\mathcal{N}>4$ extended gauged supergravities in AdS_4 [44]

and of each Kaluza-Klein level of the massive spectrum of 11-dimensional supergravity compactified on S^7 [45,46].[27]

The expressions for the HS theory vacuum energies in (5.21) and (5.27) were found above by first doing the formally convergent sum over spins s for fixed β under the integral in (5.16) and then regularizing the sum over n. If instead we first found the standard (zeta-function regularized) expressions for the Casimir energies of each spin s field and then summed over spin we would get a divergent series that would require a zeta-function regularization, now of the sum over s. While the cancellation of vacuum energy in supergravities happened due to a large amount of supersymmetry, in the HS theory it may be viewed as being due to a special (zeta-function) definition of the formally divergent sum over spins—a definition that should be consistent with the underlying symmetries of HS theory.

To further illustrate the role of the regularization of the sum over spins (already emphasized earlier in the case of the partition function in the Euclidean AdS_{d+1} with S^d boundary in [39,41,92]) below we shall consider explicitly the individual spin s contributions to the Casimir energy for some particular values of dimension d.

C. Casimir energies of individual higher spin fields in AdS_{d+1}

The vacuum energy for a given massless spin s field in AdS_{d+1} can be found for a general d by using the expression for $\mathcal{Z}_s(\beta)$ from (5.4) in the representation (2.4) for the corresponding energy zeta function

$$\zeta_{E,s}(z) = \frac{1}{\Gamma(z)}\int_0^\infty d\beta\beta^{z-1}\frac{e^{-(s+d-2)\beta}}{(1-e^{-\beta})^d}(\mathrm{d}_s - \mathrm{d}_{s-1}e^{-\beta}) = \sum_{n=1}^\infty \binom{n+d-2}{d-1}[\mathrm{d}_s(n+s+d-3)^{-z} - \mathrm{d}_{s-1}(n+s+d-2)^{-z}], \tag{5.28}$$

$$E_{c,s} = \frac{1}{2}\zeta_{E,s}(-1) = \frac{1}{2}\sum_{n=1}^\infty \binom{n+d-2}{d-1}[\mathrm{d}_s(n+s+d-3) - \mathrm{d}_{s-1}(n+s+d-2)]. \tag{5.29}$$

The expression for d_s was given in (5.5) and we used again the relations in (5.18).

For example, for a scalar $s=0$ in (5.1) (with the operator $-\nabla^2 - 2d + 4$) we have $\mathrm{d}_0 = 1$ and thus[28]

$$E_{c,s=0} = \frac{1}{2}\zeta_{E,s=0}(-1) = \frac{1}{2}\sum_{n=0}^\infty \frac{(n+d-1)!}{(d-1)!n!}(n+d-2). \tag{5.30}$$

This expression for the vacuum energy of a scalar (with dimension $\Delta = e_0 = d-2$) in AdS_{d+1} is similar but different from the one (3.5) for the vacuum energy of a conformal scalar in $R\times S^d$ or in $R\times S^{d-1}$.[29]

Let us consider cases of few low values of the boundary dimension d. Let us start with $d=2$ or the AdS_3 case ($\mathrm{d}_s = 2, s>1$)

$$d=2:\ \zeta_{E,s>1}(z) = 2\zeta(z,s),$$
$$E_{c,s>1} = \zeta(-1,s) = -\frac{1}{12}[1+6s(s-1)], \tag{5.31}$$

where $\zeta(z,a) = \sum_{r=0}^\infty (r+a)^{-z}$ is the standard Hurwitz zeta function. The above formula is applicable for $s\geq 2$ while $s=0,1$ are special cases that need to be discussed separately [41]. Note that in the $d=2$ case the Casimir energy coefficient is directly related to the conformal anomaly [61] but this will not be true in general.

For $d=3$ or AdS_4 one finds

$$\begin{aligned} d=3:& \\ \zeta_{E,s}(z) &= \sum_{n=1}^\infty \frac{1}{2}n(n+1)[(2s+1)(n+s)^{-z} \\ &\quad - (2s-1)(n+s+1)^{-z}] \\ &= \sum_{n=1}^\infty n(n+2s)(n+s)^{-z} \\ &= \zeta(z-2,s+1) - s^2\zeta(z,s+1), \end{aligned} \tag{5.32}$$

$$E_{c,s>0} = \frac{1}{2}\zeta_{E,s>0}(-1) = \frac{1}{8}s^4 - \frac{1}{12}s^2 + \frac{1}{240},$$
$$E_{c,0} = \frac{1}{480}. \tag{5.33}$$

For completeness, let us recall that the computation of the vacuum energies for massless higher spin fields in AdS_4

[27] The computation of the vacuum energy of individual fields in [44] still required, of course, the use of the standard zeta-function regularization of the sum over radial quantum number n [as, e.g., in the scalar case on the sphere in (3.11)].

[28] In the scalar case one should drop the second ghost term in the bracket in (5.28) but the general expression (5.28) applies also for the $s=0$ case as $\mathrm{d}_{s-1} = \frac{(2s+d-4)(s+d-4)!}{(d-2)!(s-1)!}$ vanishes automatically for $s=0$ if $d>3$.

[29] For example, for $d=3$ Eq. (5.30) gives $E_{c,s=0} = \frac{1}{4}\sum_{n=1}^\infty (n+1)n^2 = \frac{1}{4}\zeta(-3) = \frac{1}{480}$ while for a conformal scalar in $R\times S^3$ one finds from (3.5) that $E_c = \frac{1}{2}\zeta(-3) = \frac{1}{240}$ and for $R\times S^2$ one has $E_c = 0$.

was originally discussed in the context of extended gauged supergravities [44], starting directly from the energy spectrum $E_n = \omega_n$ for massless spin $s = 0, \frac{1}{2}, \ldots, 2$ fields (assuming reflective boundary conditions at infinity giving discrete energy spectrum [75,76]). Explicitly, for a massless spin $s > 0$ field corresponding to $SO(2,d) = SO(2,3)$ representation $(e_0, s) = (s+1, s)$ with lowest energy or dimension $\Delta = e_0 = s+1$ one has[30] $\omega_{k,j} = k + j + 1$,, $\mathrm{d}_{k,j} = 2j+1$ where $k = 0, 1, 2, \ldots$ and $j = s, s+1, s+2, \ldots$. This leads to the following expression for the corresponding energy zeta function [44]

$$d=3:$$

$$\zeta_{E,s}(z) = \sum_{k=0}^{\infty}\sum_{j=s}^{\infty}(2j+1)(k+j+1)^{-z} = \sum_{r=0}^{\infty}(r+1)(r+2s+1)(r+s+1)^{-z}. \tag{5.34}$$

This is equivalent to (5.32) and thus gives the same expression for the vacuum energy as in (5.33).

Note that $E_{c,s>0}$ in (5.33) formally applies also for $s=0$ if the scalar is assumed to be complex, i.e. carries the same number (two) of degrees of freedom as all other massless spin s fields in $d+1=4$.[31] The expression in (5.33) is true also for half-integer spins and thus one can directly apply (5.33) to compute the vacuum energies of extended four-dimensional supergravity theories using the supermultiplet sum rules [93] $\sum_s (-1)^{2s} d(s) s^p = 0$, $p < \mathcal{N} = 1, \ldots, 8$ where $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$ and $d(s)$ are multiplicities of the spin s fields. One then finds that the vacuum energy vanishes in $\mathcal{N} > 4$ extended gauged supergravities [44]. The vanishing of vacuum energy was found also at each level of the massive KK spectrum of 11-dimensional supergravity compactified on S^7 [45,46].

We can now see directly that similar cancellation of vacuum energy happens also in the purely bosonic HS theory assuming the sum over all spins is zeta-function regularized (as suggested in [39,41,92]):

$$d=3:$$

$$(E_c)_{\mathrm{HS}} = E_{c,0} + \sum_{s=1}^{\infty} E_{c,s} = \frac{1}{480} + \sum_{s=1}^{\infty}\left(\frac{1}{8}s^4 - \frac{1}{12}s^2 + \frac{1}{240}\right) = 0, \tag{5.35}$$

where we used that $\zeta(0) = -\frac{1}{2}$,, $\zeta(-2) = \zeta(-4) = 0$. Such cancellation happens also in the minimal HS theory in AdS_4 where one sums over even spins only. As discussed above, this is consistent (in agreement with AdS/CFT) with the vanishing (3.12) of the Casimir energy of a conformal scalar in $d=3$, i.e. in $R \times S^2$.

Let us note that the expression appearing in the vanishing of vacuum energy in (5.35) is similar but not identical to the one found for the vanishing coefficient of the UV logarithmically divergent part $(\ln Z_{\mathrm{HS}})_\infty = -a_{\mathrm{HS}} \ln \Lambda$ of the partition function of HS theory in the Euclidean AdS_4 with S^3 as the boundary [39],

$$a_{\mathrm{HS}} = \frac{1}{360} + \sum_{s=1}^{\infty}\left(\frac{5}{24}s^4 - \frac{1}{24}s^2 + \frac{1}{180}\right) = 0. \tag{5.36}$$

In general, for $d \geq 3$ the spin-dependent coefficients in the Casimir energy and in the coefficient of the UV divergence in AdS_{d+1} appear to be different.

Finally, let us consider the AdS_5 or $d=4$ case of (5.28)–(5.29),

$$d=4: \qquad \zeta_{E,s}(z) = \sum_{n=1}^{\infty}\frac{1}{6}n(n+1)(n+2)[(s+1)^2(n+s+1)^{-z} - s^2(n+s+2)^{-z}]$$
$$= \sum_{n=1}^{\infty}\frac{1}{6}n(n+1)[(2s+1)n + 3s^2 + 4s + 2](n+s+1)^{-z}$$
$$= \frac{1}{6}[(2s+1)\zeta(z-3, s+2) - 3s(s+1)\zeta(z-2, s+2) - (2s+1)\zeta(z-1, s+2) + s(s^3 + 2s^2 + 2s + 1)\zeta(z, s+2)], \tag{5.37}$$

[30]For the spin 0 case $\omega_{k,j} = k + j + e_0$,, $\mathrm{d}_{k,j} = 2j+1$ where $k = 0, 2, 4, \ldots$,, $j = 0, 1, 2, \ldots$ and $e_0 = 1$ (in the conformal coupling case) or $e_0 = 2$ (in the standard massless case).

[31]One can check directly that the vacuum energy corresponding to a real scalar in either (1,0) or (2,0) representations is [45,57] $E_{c,s=0} = \frac{1}{480}$. In general, for a real scalar field in representation $(e_0, 0)$ one has [45]
$\zeta_{E,s=0}(z; e_0) = \sum_{k,j=0}^{\infty}(2j+1)(e_0 + 2k + j + 1)^{-z} = \frac{1}{2}[\zeta(z-2, e_0) + (3-2e_0)\zeta(z-1, e_0) + (e_0-2)(e_0-1)\zeta(z, e_0)]$
so that $\zeta_{E,s=0}(-1; e_0) = -\frac{1}{24}e_0^4 + \frac{1}{4}e_0^3 - \frac{1}{2}e_0^2 + \frac{3}{8}e_0 - \frac{19}{240}$, giving $\frac{1}{2}\zeta_{E,0}(-1; 2) = \frac{1}{2}\zeta_{E,0}(-1; 1) = \frac{1}{480}$.

$$E_{c,s} = \frac{1}{2}\zeta_{E,s}(-1) = -\frac{1}{1440}s(s+1)[18s^2(s+1)^2 - 14s(s+1) - 11]. \tag{5.38}$$

For example, for low spin $s = 0, 1, 2$ fields (with $e_0 = s + 2$) this expression reproduces the values of the Casimir energies $(0, -\frac{11}{240}, -\frac{553}{240})$ found in [57]. Summing $E_{c,s}$ over $s = 0, 1, 2, \ldots$ should be done again using an appropriate spectral zeta-function regularization as in [41], or, equivalently, introducing a cutoff function $e^{-\epsilon[s+\frac{1}{2}(d-3)]} = e^{-\epsilon(s+\frac{1}{2})}$ and dropping all singular terms in the limit $\epsilon \to 0$. As one readily checks, this gives

$$\sum_{s=1}^{\infty} E_{c,s} e^{-\epsilon(s+\frac{1}{2})}|_{\epsilon\to 0,\mathrm{fin}} = 0. \tag{5.39}$$

This is in agreement with our earlier general result (5.21) obtained directly from the total (summed over spin) partition function (5.17) and using the standard zeta-function regularization of the sum over n in (5.19) and (5.21).[32] Similarly, one can sum up the individual Casimir energies $E_{c,s}$ over even spins only, corresponding to the minimal HS theories in AdS_{d+1}, and in all d the result is equal to the Casimir energy of a real conformal scalar on $R \times S^{d-1}$, in agreement with the computation of Sec. V B.

VI. MATCHING FERMIONIC CFTS WITH HIGHER SPINE THEORIES

Having checked the higher spin AdS/CFT correspondence for singlet sectors of free scalar field theories on $S^1 \times S^{d-1}$, we proceed to analogous checks for similar fermionic theories. More precisely, we will consider the $U(N)$ singlet sector of the theory of N free Dirac fermions or the $O(N)$ singlet sector of the theory of N free Majorana fermions on $S^1 \times S^{d-1}$ and compare them with appropriately defined higher spin theories in AdS_{d+1}. We will explicitly discuss $d = 2$, 3, 4, but extensions to higher d should not be difficult. These checks are interesting because the spectra of higher spin currents in such fermionic theories are more complicated than in the scalar theories. Correspondingly, the dual higher spin description of the fermionic theories generally involves massless gauge fields in more general representations than the fully symmetric ones [25,27].

A. $d = 2$

Let us first discuss $d = 2$ fermionic CFTs. We may start with N massless free Dirac fermions and impose the $SU(N)$ singlet condition. This may be accomplished by gauging the $SU(N)$ symmetry and then adding the Wess-Zumino-Witten term for the $SU(N)$ gauge field $A_\mu = ig\partial_\mu g^{-1}$ with a coefficient k. In the limit $k \to \infty$ we expect to find the free fermion theory restricted to the $SU(N)$ singlet sector.[33] Alternatively, we may start with N massless free Majorana fermions and impose the $O(N)$ singlet constraint by similarly gauging the $O(N)$ symmetry.

In $d = 2$ CFTs, the Casimir energy on $R \times S^1$ is completely determined by the central charge: $E = -\frac{1}{12}c$. Therefore, the $\mathrm{AdS}_3/\mathrm{CFT}_2$ matching of Casimir energies is equivalent to matching of the central charge c. For $U(N)$ and $O(N)$ singlet scalar theories, the central charge matching was carried out in [41]. It was found that the sum of higher spin one-loop contributions vanishes in the theory of all integer spins, while in the theory of even spins it equals 1, which is the central charge of a real scalar field.

Just as in the scalar cases, we find that the $d = 2$ $U(N)$ and $O(N)$ singlet fermionic theories contain conserved currents of spin $s > 1$, and the dual theories in AdS_3 contain corresponding massless gauge fields. The contribution of such fields to the one-loop central charge is as in (5.31) [41]

$$c_s^{(1)} = 1 + 6s(s-1), \qquad s \geq 2. \tag{6.1}$$

In carrying out the matching for the fermionic theories, we find subtle but important differences from the scalar case that affect the fields of spin 1 and 0. For the theory of N Dirac fermions, the spin 1 current is $\bar\psi_i\gamma^\mu\psi^i$, and it generates the standard Kac-Moody algebra. Correspondingly, the dual vector field in AdS_3 has the Chern-Simons action [98]. The contribution of the Chern-Simons field is $c_{\mathrm{CS}}^{(1)} = 1$; this can be deduced from the central charge of the current algebra in the dual theory or can be found from a direct calculation in AdS_3. In contrast, in the $d = 2$ scalar CFT the vector current does not satisfy the standard Kac-Moody algebra. It is plausible to conjecture that in this case the $s = 1$ gauge field in AdS_3 has the Maxwell action [41], and its contribution to the central charge is $c_{\mathrm{Maxwell}}^{(1)} = \frac{1}{2}$. There are two spin 0 operators of dimension $\Delta = 1$ in the fermionic theory: a scalar, $\bar\psi_i\psi^i$, and a pseudoscalar, $\bar\psi_i\gamma_3\psi^i$. Therefore, the dual theory in AdS_3 must contain a complex scalar field with $m^2 = -1$ which is right at the Breitenlohner-Freedman bound. In general, the contribution of a real scalar field to one-loop central charge is $c_0^{(1)}(\Delta) = -\frac{1}{2}(\Delta - 1)^3$. Therefore, the

[32]Note again that the 6th order polynomial in s in (5.38) is similar but not equivalent to the coefficient of the logarithmic infrared divergence in the massless higher spin partition function in AdS_5 [92] that also vanishes when summed over spins with a zeta-function regularization [41].

[33]Such a construction is similar to the Gaberdiel-Gopakumar conjectures [94–96] which involve coset CFTs in $d = 2$. The $\lambda \to 0$ limit of the coset CFT used in [94–96] is simply the singlet sector of the CFT of N free Dirac fermions, but with the $U(1)$ current $\bar\psi_i\gamma^\mu\psi^i$ removed [95,97].

complex scalar field makes no contribution to the central charge, while the Chern-Simons vector does, and their total contribution is 1.

Let us compare this with the AdS_3 theory dual to the $U(N)$ symmetric scalar model. Such a theory contains one $m=0$ scalar, which is dual to the $\Delta=0$ operator $\bar{\phi}_i\phi^i$; it contributes $c_0^{(1)}=\frac{1}{2}$. As suggested in [41], it also contains a Maxwell field. Thus, the $s=0$, 1 fields in the $U(N)$ singlet fermionic model make the same combined contribution to the central charge as in the scalar model. The $s>1$ fields work in the same way in the fermionic and scalar models; therefore, the cancellation occurs in both models.

Indeed, in the $U(N)$ invariant fermionic theory, the total one-loop correction to central charge is $c_{\rm CS}^{(1)}+\sum_{s=2}^{\infty}c_s^{(1)}$. Using the zeta-function regularization for the sum, we see that this vanishes. In the $O(N)$ invariant fermionic theory, the total contribution is the regularized sum of $c_s^{(1)}$ over positive even spins. This equals $\frac{1}{2}$, in agreement with the central charge of a single Majorana fermion. This is consistent with the proposed identification of the coupling constant, $G_N\sim 1/(N-1)$, in the bulk dual of $O(N)$ singlet models.

As a further test of the spectra of the AdS_3 theories dual to the $d=2$ fermionic CFTs, we consider the calculation of the thermal free energy. According to (3.22), in $d=2$ one half of the free Dirac fermion one-particle free energy is[34]

$$\mathcal{Z}_{\frac{1}{2}}(\beta)=\frac{2q^{\frac{1}{2}}}{1-q}. \tag{6.2}$$

According to (4.25), matching of the thermal free energies requires that the square of this partition function equals the sum of the one-particle partition functions in AdS_3. Indeed, for the theory dual to the $U(N)$ singlet sector of N free Dirac fermions, we find

$$\frac{2q}{(1-q)^2}+\sum_{s=1}^{\infty}\left[\frac{2q^s}{(1-q)^2}-\frac{2q^{s+1}}{(1-q)^2}\right]=\frac{4q}{(1-q)^2}$$
$$=[\mathcal{Z}_{\frac{1}{2}}(\beta)]^2. \tag{6.3}$$

The first term here is the contribution of the two $\Delta=1$ scalars in AdS_3, and the sum corresponds to the contribution of all gauge fields with $s\geq 1$ [98], including the Chern-Simons gauge field dual to the spin 1 current $\bar{\psi}_i\gamma_\mu\psi^i$. This matching is essentially a special case of certain identities for a product of characters of the conformal group [27].

Now let us consider the minimal higher spin theory in AdS_3, which is dual to the $O(N)$ singlet sector of N free Majorana fermions. Since for Majorana fermions $\bar{\psi}_i\gamma_3\psi^i$ vanishes, this theory has only one real bulk scalar dual to the $\Delta=1$ operator $\bar{\psi}_i\psi^i$. It also contains massless gauge fields of positive even spins. Therefore, the sum over one-particle partition functions in AdS_3 is

$$\frac{q}{(1-q)^2}+2\sum_{s=2,4,\ldots}^{\infty}\frac{q^s}{1-q}=\frac{q+3q^2}{(1-q)^2(1+q)}$$
$$=\frac{1}{2}[\mathcal{Z}_{\frac{1}{2}}(\beta)]^2-\frac{1}{2}\mathcal{Z}_{\frac{1}{2}}(2\beta), \tag{6.4}$$

and we find agreement with the field theory result (4.26).

B. $d=3$

Next, let us consider the $d=3$ fermionic duality conjectures of [14,15]. In the case of N free Dirac fermions restricted to the $U(N)$ singlet sector, the "single trace" spectrum includes a unique pseudoscalar operator $\bar{\psi}_i\psi^i$ which has dimension 2, and a set of totally symmetric higher spin currents, one for each integer spin. The matching of the thermal partition function on $S^1\times S^2$ follows from the identity [27,42]

$$\frac{q^2}{(1-q)^3}+\sum_{s=1}^{\infty}\left[(2s+1)\frac{q^{s+1}}{(1-q)^3}-(2s-1)\frac{q^{s+2}}{(1-q)^3}\right]$$
$$=\frac{4q^2}{(1-q)^4}=[\mathcal{Z}_{\frac{1}{2}}(\beta)]^2, \tag{6.5}$$

where $\mathcal{Z}_{\frac{1}{2}}(\beta)=\frac{2q}{(1-q)^2}$ is half of the free Dirac fermion one-particle partition function in $d=3$ [see (3.21), (3.22)]. Note that this is the same as the expression (5.12) for the critical scalar theory. This is because at large N the spectrum of the free fermion theory is the same as the one of the critical scalar theory, where the $s=0$ operator $\phi^i\phi^i$ has dimension $\Delta=2+\mathcal{O}(1/N)$, as opposed to $\Delta=1$ at the free fixed point.[35] Analogously, in the minimal theory with even spins only, one has

$$\frac{q^2}{(1-q)^3}+\sum_{s=2,4,\ldots}^{\infty}\left[(2s+1)\frac{q^{s+1}}{(1-q)^3}-(2s-1)\frac{q^{s+2}}{(1-q)^3}\right]$$
$$=\frac{1}{2}[\mathcal{Z}_{\frac{1}{2}}(\beta)]^2-\frac{1}{2}\mathcal{Z}_{\frac{1}{2}}(2\beta), \tag{6.6}$$

in agreement with (4.26).

One may also consider the large N interacting Gross-Neveu model, where the scalar operator has dimension 1 instead of 2. The same $s=0$ operator dimensions appear in

[34] To recall, in general dimensions, the character of the free fermion representation of the conformal group is given by [18,27] $\mathcal{Z}_{\frac{1}{2}}(\beta)=\frac{1}{2}Z_{\rm F}(\beta)=n_F\frac{q^{\frac{d-1}{2}}}{(1-q)^{d-1}}$, where $n_F=2^{\frac{d}{2}-1}$ for a Weyl fermion in even dimensions, and $n_F=2^{\frac{d-1}{2}}$ for a Dirac fermion in odd dimensions. A Dirac fermion in even dimensions can be decomposed as the sum of left and right Weyl spinors.

[35] One difference between the free fermion and critical scalar spectrum is that in the former the $s=0$ operator is a pseudoscalar. However, this difference does not affect the calculation at this order.

the Wilson-Fisher and free scalar models and correspond to the two different boundary conditions for the $m^2 = -2$ scalar in AdS_4. As noted in Sec. V C, for either choice of scalar operator dimension, $E_{c,0} = \frac{1}{480}$. The spectrum of the $s > 0$ currents in the $U(N)$ fermionic models (dual to type B Vasiliev theory) is the same as in the $U(N)$ scalar models (dual to type A Vasiliev theory), and their zeta-function regularized contribution to Casimir energy is $-\frac{1}{480}$, properly canceling the $s = 0$ contribution. Similarly, the cancellation of the Casimir energy in the $O(N)$ fermionic models (dual to minimal type B theory) is exactly the same as in the $O(N)$ scalar models (dual to minimal type A theory).

C. $d = 4$

It is interesting to look at the higher dimensional free fermion theories, as in this case the dual higher spin theory contains new higher spin representations besides the totally symmetric ones [25,27]. As an explicit example, let us consider the $d = 4$ theory of N free Dirac fermions restricted to the $U(N)$ singlet sector. The single trace primary operators in this theory consist of two $\Delta = 3$ scalar operators,

$$\mathcal{O} = \bar{\psi}_i \psi^i, \qquad \tilde{\mathcal{O}} = \bar{\psi}_i \gamma_5 \psi^i, \tag{6.7}$$

two sets of totally symmetric higher spin currents, schematically [54–56]

$$J_{\mu_1\cdots\mu_s} = \bar{\psi}_i \gamma_{(\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_s)}\psi^i + \ldots,$$
$$\tilde{J}_{\mu_1\cdots\mu_s} = \bar{\psi}_i \gamma_5 \gamma_{(\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_s)}\psi^i + \ldots, \qquad s \geq 1 \tag{6.8}$$

and a tower of mixed symmetry higher spin operators of the schematic form

$$B_{\mu_1\cdots\mu_s,\nu} = \bar{\psi}_i \gamma_{\nu(\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_s)}\psi^i + \ldots, \qquad s \geq 1. \tag{6.9}$$

The latter operators have the symmetries of the Young tableaux with s boxes in the first row and one box in the second row. This set of primary operators is dual to two AdS_5 scalar fields with $m^2 = -3$, two towers of totally symmetric higher spin gauge fields and a tower of mixed symmetry fields corresponding to (6.9). In particular, at $s = 1$ we have a massive antisymmetric tensor field dual to the operator $\bar{\psi}_i \gamma_{\mu\nu} \psi^i$. The agreement between single trace primaries and single particle states in AdS_5 can again be seen by computing the thermal partition function on $S^1 \times S^3$. Representations of the $d = 4$ conformal group can be labeled by $(\Delta; j_1, j_2)$, where Δ is the conformal dimension and j_1, j_2 the $SU(2) \times SU(2)$ spins. In this notation, the mixed symmetry operators (6.9) for a given s correspond to the sum of representations

$$\left(s+2; \frac{s+1}{2}, \frac{s-1}{2}\right) \oplus \left(s+2; \frac{s-1}{2}, \frac{s+1}{2}\right), \tag{6.10}$$

and the corresponding character, or one-particle partition function, is [27]

$$\mathcal{Z}_s^{\text{mixed}}(\beta) = 2\frac{q^{s+2}}{(1-q)^4}[s(s+2) - q(s^2-1)]. \tag{6.11}$$

It would be interesting to derive this directly in AdS by computing the heat kernel and one-loop determinant for the mixed symmetry fields in the bulk. Putting this together with the scalar and totally symmetric higher spin contributions and summing over spins one gets

$$\frac{2q^3}{(1-q)^4} + 2\sum_{s=1}^{\infty}\frac{q^{s+2}}{(1-q)^4}[(s+1)^2 - qs^2] + 2\sum_{s=1}^{\infty}\frac{q^{s+2}}{(1-q)^4}[s(s+2) - q(s^2-1)] = \frac{16q^3}{(1-q)^6} = [\mathcal{Z}_{\frac{1}{2}}(\beta)]^2, \tag{6.12}$$

which indeed agrees with (4.25) and the form of the free fermion character (3.21) in $d = 4$. This is another example of an identity between a product of characters of two singleton representations of the conformal group, and the character of the corresponding direct sum of higher spin representations, generalizing the Flato-Fronsdal relation discovered in $d = 3$ [20].[36]

By using (2.2) and (2.4), the knowledge of $\mathcal{Z}(\beta)$ for each representation is enough to determine the corresponding Casimir energies of the AdS_5 fields. For the $\Delta = 3$ scalar field, we get

$$\zeta_{E,0}^{\Delta=3}(z) = \sum_{n=1}^{\infty}\frac{1}{6}n(n+1)(n+2)(n+2)^{-z}, \tag{6.13}$$

which yields

$$E_{c,0}^{\Delta=3} = \frac{1}{2}\zeta_{E,0}^{\Delta=3}(-1) = -\frac{1}{480}. \tag{6.14}$$

The Casimir energy for the totally symmetric higher spins in AdS_5 was already computed in (5.37)–(5.38). Its regularized sum over all integer spins vanishes (5.39). For the mixed symmetry fields, from (6.11) we get [using the same regularization of the sum over spins as in (5.39)]

[36]In $d = 4$ one may consider not only the $j = 0$ and $j = \frac{1}{2}$ singletons [25,99,100] (i.e. free massless scalar and spinor fields of boundary CFT), but also $j = 1$ and higher spin ones [101]. In such cases, one finds similar relations between characters [27]. The $d = 4$ CFT corresponding to the $j = 1$ singleton has N free Maxwell fields restricted to the $O(N)$ singlet sector [102]. Similar theories may be considered in higher dimensions. For example, in $d = 6$, one may study the $O(N)$ singlet sector of N free antisymmetric tensor fields; this theory has a higher spin AdS_7 dual.

$$\zeta_{E,s}^{\text{mixed}}(z) = \sum_{n=1}^{\infty} \frac{1}{6} n(n+1)(n+2)[s(s+2)(n+s+1)^{-z} - (s^2-1)(n+s+2)^{-z}],$$
$$E_{c,s}^{\text{mixed}} = \frac{1}{2}\zeta_{E,s}^{\text{mixed}}(-1) = \frac{1}{720}[-3 - 19s(s+1) + 44s^2(s+1)^2 - 18s^3(s+1)^3],$$
$$E_c^{\text{mixed}} = \sum_{s=1}^{\infty} \frac{1}{720}[-3 - 19s(s+1) + 44s^2(s+1)^2 - 18s^3(s+1)^3]e^{-\epsilon(s+\frac{1}{2})}|_{\epsilon\to 0,\text{fin}} = \frac{1}{240}. \quad (6.15)$$

Thus the total one-loop bulk Casimir energy is

$$E_c = 2\times\left(-\frac{1}{480}\right) + 2\times 0 + \frac{1}{240} = 0. \quad (6.16)$$

This is in agreement with the expected vanishing of order N^0 correction to the Casimir energy of N free Dirac fermions. Note that in this section we have chosen to compute the total Casimir energy by summing up the individual Casimir energies of each bulk field with a suitable regulator [41]. Equivalently, one can obtain the same result by first summing over spins the one-particle partition functions, and then performing the Mellin transform (2.4), as described in Sec. V B for the scalar theories.

Analogously, we can consider the theory of N Majorana fermions, restricted to the $O(N)$ singlet sector as discussed in Sec. IV C. The spectrum of operators is a projection of the one described above for the Dirac case. Given any two Majorana fermions χ_1, χ_2, one has the identities

$$\bar{\chi}_1\chi_2 = \bar{\chi}_2\chi_1, \qquad \bar{\chi}_1\gamma_5\chi_2 = \bar{\chi}_2\gamma_5\chi_1, \quad (6.17)$$

$$\bar{\chi}_1\gamma_\mu\chi_2 = -\bar{\chi}_2\gamma_\mu\chi_1, \qquad \bar{\chi}_1\gamma_\mu\gamma_5\chi_2 = \bar{\chi}_2\gamma_\mu\gamma_5\chi_1, \quad (6.18)$$

$$\bar{\chi}_1\gamma_{\mu\nu}\chi_2 = -\bar{\chi}_2\gamma_{\mu\nu}\chi_1. \quad (6.19)$$

The identities in the first line imply that both $\Delta = 3$ scalar operators are present in the Majorana theory. On the other hand, using the identities in the second line and integration by parts, one can see that the totally symmetric operators $J_{\mu_1\cdots\mu_s}$ with odd spins and the "axial" $\tilde{J}_{\mu_1\cdots\mu_s}$ with even spins are projected out (they are total derivatives). This leaves effectively a single tower of totally symmetric higher spins of all integer s. Finally, the identity in the last line implies that the mixed symmetry operators $B_{\mu_1\cdots\mu_s,\nu}$ with odd spin are projected out. Then the total one-loop bulk Casimir energy is

$$E_{c\times\text{min}} = 2\times\left(-\frac{1}{480}\right) + 0 + \sum_{s=2,4,\ldots}^{\infty} E_{c,s}^{\text{mixed}} e^{-\epsilon(s+\frac{1}{2})}|_{\epsilon\to 0,\text{fin}}$$
$$= \frac{17}{960}. \quad (6.20)$$

This is precisely equal to the Casimir energy of a single Majorana fermion in $d = 4$, in agreement with the shift $N \to N-1$ in the HS coupling which we observe in the real theories in all dimensions.

We can also consider the thermal partition function of this free real fermion theory. The sum over one-particle partition functions of the bulk AdS_5 fields yields

$$\frac{2q^3}{(1-q)^4} + \sum_{s=1}^{\infty}\frac{q^{s+2}}{(1-q)^4}[(s+1)^2 - qs^2]$$
$$+2\sum_{s=2,4,\ldots}^{\infty}\frac{q^{s+2}}{(1-q)^4}[s(s+2) - q(s^2-1)]$$
$$= \frac{8q^3}{(1-q)^6} - \frac{2q^3}{(1-q^2)^3} = \frac{1}{2}[\mathcal{Z}_{\frac{1}{2}}(\beta)]^2 - \frac{1}{2}\mathcal{Z}_{\frac{1}{2}}(2\beta). \quad (6.21)$$

This is indeed in perfect agreement with the expression for the corresponding real free fermion partition function on $S^1\times S^3$ with the $O(N)$ singlet constraint found in Sec. IV B (4.26).

Finally, let us note that in $d = 4$ we could also consider the free theory of N complex Weyl fermions in the $U(N)$ singlet sector. In this theory, the $U(N)$ invariant operators form a single tower of totally symmetric currents with all integer spins $s \geq 1$. In particular, there is no scalar operator and no mixed symmetry operators.[37] The one-loop Casimir energy in the bulk then vanishes due to (5.39).

VII. HIGHER SPIN DUALS OF THEORIES WITH N_f FLAVORS

It is straightforward to generalize the above calculations to the case of free theories with N_f scalars or fermions in the fundamental representation of $U(N)$ or $O(N)$. To be concrete, let us consider NN_f free complex scalars in the $U(N)$ singlet sector. The spectrum of single trace primaries is then given by

$$\mathcal{O}^a{}_b = \bar{\phi}_{ib}\phi^{ia}, \qquad (J_{(s)})^a{}_b \sim \bar{\phi}_{ib}\partial^s\phi^{ia},$$
$$a, b = 1, \ldots, N_f. \quad (7.1)$$

[37]For a Weyl spinor ψ^i_α, one can construct a Lorentz scalar by contracting the $SU(2)$ index with $\psi^{\alpha,i}$. However, the corresponding object is not $U(N)$ invariant because ψ^i_α and $\psi^{\alpha,i}$ are both in the fundamental of $U(N)$.

The dual HS theory should then be a version of Vasiliev theory where all fields are promoted to matrices carrying the $U(N_f)$ indices [11]. As usual, the global $U(N_f)$ symmetry of the CFT becomes a $U(N_f)$ gauge symmetry in the bulk. At the free field level we simply have N_f^2 copies of each field, and the calculations described in Sec. VA readily lead to the result quoted in (4.4) for the $U(N)$ case.

The situation is slightly more interesting in the $O(N)$ case. For $N_f = 1$, recall that all odd spin currents are projected out in this case because the scalar field is real. However, for general N_f it is not difficult to see that there are $N_f(N_f+1)/2$ even spin operators and $N_f(N_f-1)/2$ odd spin ones, corresponding to symmetric or antisymmetric combinations of the flavor indices. Then the sum over the HS one-particle partition functions (5.4) gives

$$\frac{N_f(N_f+1)}{2}\sum_{s=0,2,4,\ldots}\mathcal{Z}_s(\beta)+\frac{N_f(N_f-1)}{2}\sum_{s=1,3,5,\ldots}\mathcal{Z}_s(\beta)$$
$$=\frac{N_f^2}{2}[\mathcal{Z}_0(\beta)]^2+\frac{N_f}{2}\mathcal{Z}_0(2\beta), \qquad (7.2)$$

in agreement with (4.4). Similarly, one can analyze the dual of the fermionic theories with N_f complex or real flavors, and the result for the corresponding higher spin sums is readily seen to agree with (4.5).

Let us also briefly comment on the matching of the Casimir energy. In this case, the CFT predicts that the Casimir term in the thermal free energy should simply be $F_c = NN_f\beta E_c$, with E_c the Casimir energy of a single free field. In the HS dual of the $U(N)$ theories, the results for $N_f = 1$ immediately imply that the sum of one-loop Casimir energies vanishes. So, assuming that F_c is entirely reproduced by the classical bulk calculation (which we do not address here), we would get a result consistent with the duality. For the $O(N)$ theories, on the other hand, one finds a nonvanishing one-loop Casimir energy. For instance, in the HS dual of the free scalar theories, the sum over one-loop bulk Casimir energies gives, in any dimension,[38]

$$\frac{N_f(N_f+1)}{2}\sum_{s=0,2,\ldots}E_{c,s}+\frac{N_f(N_f-1)}{2}\sum_{s=1,3,\ldots}E_{c,s}$$
$$=N_f E_c^{\rm scalar}, \qquad (7.3)$$

where $E_c^{\rm scalar}$ is the Casimir energy of a single conformal scalar. Then agreement with the duality again requires the same shift $N \to N-1$ in the map between N and the bulk coupling constant that we observed in the case $N_f = 1$, i.e. $G_{\rm bulk}^{-1} \sim N-1$. The same result can be seen to apply to the $O(N)$ singlet sectors of real fermionic theories for general N_f.

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[38]This result follows from the fact that for $N_f = 1$ the sum over all spins vanishes, and the sum over even spins gives the Casimir energy of a single conformal scalar.

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