Quantum cosmology, minimal length, and holography

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We study the effects of a generalized uncertainty principle on the classical and quantum cosmology of a closed Friedmann universe whose matter content is either a dust or a radiation fluid. More concretely, assuming the existence of a minimal length, we show that the entropy will constitute a Dirac observable. In addition, 't Hooft conjecture on the cosmological holographic principle is also investigated. We describe how this holographic principle is satisfied for large values of a quantum number, n. This occurs when the entropy is computed in terms of the minimal area.

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I. INTRODUCTION

It has been pointed out that our existing theories will break down when applied to small distances or very high energies. In particular, the geometrical continuum beyond a certain limit will no longer be valid. This suggested the development of a scenario based on indivisible units of length. In recent years, the concept of minimal length has been described through algebraic methods, by means of a generalized uncertainty principle, which could be induced by gravitational effects, first proposed by Mead [1]. Moreover, a generalized uncertainty principle (GUP), as presented in theories such as string theory and doubly special relativity, conveys the prediction of a minimal measurable length [2]. A similar feature appears in the polymer quantization in terms of a mass scale [3]. The concept of a subsequent minimal area can, thus, be considered. This then raises the question of whether it can be used in discussions about entropy, namely, involving the holographic principle.

The holographic principle in quantum gravity was first suggested by 't Hooft [4] and later extended to string theory by Susskind [5]. The most radical part of this principle proposes that the degrees of freedom of a spatial region reside not in the bulk but in the boundary. Furthermore, the number of boundary degrees of freedom per Planck area should not be larger than 1. In this context, it is worth reminding the reader of the general assumption that the Bekenstein-Hawking area law applies universally to all cosmic or black hole horizons. On the other hand, it has been shown recently (in Ref. [6]), that there is a derivation for the holographic/conformal-anomaly Friedmann equation. This is of interest because that derivation was obtained by assuming that the effect of GUP on the entropy from the apparent horizon admits a constraint which relates the anomaly coefficient and the GUP parameter.

In this paper we investigate the quantum cosmology of a closed Friedmann-Lemaître-Robertson-Walker (FLRW) universe, filled with either radiation or dust. Our aim in this scenario, by assuming a minimal length, is to determine whether a corresponding minimal area will be of relevance in discussing the holographic principle. The paper is organized as follows. In Sec. II we present the classical setting in the presence of a GUP. Section III provides the quantum cosmological description of our model. Section IV conveys how our model can be used to discuss holographic features. In Sec. V, we summarize our results.

II. CLASSICAL MODEL WITH GUP

Let us start with the line element of the closed homogeneous and isotropic FLRW geometry

$$ds^2 = -N^2(\eta)d\eta^2 + a^2(\eta)d\Omega^2_{(3)}, \tag{1}$$

where $N(\eta)$ is the lapse function, $a(\eta)$ is the scale factor and $d\Omega^2_{(3)}$ is the standard line element for a unit three-sphere. The action functional corresponding to the line element (1) displays in the gravitational and matter sectors (with the latter as perfect fluid) [7]

$$\begin{aligned}\mathcal{S} &= \frac{M_{\rm Pl}^2}{2}\int_{\mathcal{M}}\sqrt{-g}Rd^4x \\ &\quad + M_{\rm Pl}^2\int_{\partial\mathcal{M}}\sqrt{g^{(3)}}Kd^3x - \int_{\mathcal{M}}\sqrt{-g}\rho d^4x \\ &= 6\pi^2M_{\rm Pl}^2\int\left(-\frac{a\dot{a}^2}{N}+Na\right)d\eta - 2\pi^2\int Na^3\rho d\eta, \end{aligned}\tag{2}$$

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in which $M_{\rm Pl}^2 = \frac{1}{8\pi G}$ (where G is the Newton gravitational constant) is the reduced Planck's mass squared in natural units, $\mathcal{M} = I \times S^3$ is the spacetime manifold, $\partial\mathcal{M} = S^3$, R is the Ricci scalar associated to the metric $g_{\mu\nu}$ whose determinant has been denoted by g, ρ is the energy density, K is the trace of the extrinsic curvature of the spacetime boundary and an overdot denotes differentiation with respect to η.

For a radiation fluid ($p_\gamma = \frac{1}{3}\rho_\gamma$ where the ρ_γ and p_γ are the energy density and pressure associated to the radiation fluid, respectively), redefining the scale factor and lapse function as

$$\begin{cases} a(\eta) = x(\eta) + \frac{M}{12\pi^2 M_{\rm Pl}^2} \coloneqq x - x_0, \\ N(\eta) = 12\pi^2 M_{\rm Pl} a(\eta)\tilde{N}, \end{cases} \tag{3}$$

the total Lagrangian, if we further add a dust fluid that does not interact with the radiation, will be

$$\mathcal{L} = -\frac{1}{2\tilde{N}} M_{\rm Pl}\dot{x}^2 + \frac{\tilde{N}}{2} M_{\rm Pl}\omega^2 x^2 - \mathcal{E}\tilde{N}, \tag{4}$$

where we have employed

$$\begin{cases} \mathcal{E} = \frac{M^2}{2M_{\rm Pl}} + 12\pi^2 \mathcal{N}_\gamma M_{\rm Pl}, \\ \omega = 12\pi^2 M_{\rm Pl}. \end{cases} \tag{5}$$

Moreover, we introduce M and $\mathcal{N}_\gamma$ as

$$\begin{cases} M = \int_{\partial\mathcal{M}} \sqrt{g^{(3)}}\rho_{0m} a_0^3 d^3x, \\ \mathcal{N}_\gamma = \int_{\partial\mathcal{M}} \sqrt{g^{(3)}}\rho_{0\gamma} a_0^4 d^3x. \end{cases} \tag{6}$$

In the above definition, M denotes the total mass of the dust matter. In addition, $\mathcal{N}_\gamma$ could be related to the total entropy of radiation as follows: the energy density ρ_γ, the number density n_γ, the entropy density s_γ and the scale factor are related to temperature, T, via $\rho_\gamma = \frac{\pi^2}{30} g T^4$, $n_\gamma = \frac{\zeta(3)}{\pi^2} g T^3$, $s_\gamma = \frac{4}{3}\frac{\rho_\gamma}{T}$ and $a(\eta) \sim \frac{1}{T}$ [8]. Consequently, we find

$$\mathcal{N}_\gamma = \left(\frac{5 \times 3^5}{2^8 \pi^4 g}\right)^{1/3} (S^{(\gamma)})^{4/3}, \tag{7}$$

where $S^{(\gamma)}$ denotes the corresponding total entropy [9].

The momenta conjugate to x and the primary constraint, which are necessary to construct the Hamiltonian of the model, are given by

$$\begin{cases} \Pi_x = \frac{\partial\mathcal{L}}{\partial\dot{x}} = -\frac{\tilde{N}}{M_{\rm Pl}}\dot{x}, \\ \Pi_{\tilde{N}} = \frac{\partial\mathcal{L}}{\partial\dot{\tilde{N}}} = 0. \end{cases} \tag{8}$$

Therefore, the Hamiltonian corresponding to (4) becomes

$$\mathcal{H} = -\tilde{N}\left[\frac{1}{2M_{\rm Pl}}\Pi_x^2 + \frac{1}{2}M_{\rm Pl}\omega^2 x^2 - \mathcal{E}\right]. \tag{9}$$

From (8) we see that the momentum conjugate to $\tilde{N}$ vanishes, i.e., the Lagrangian of the system is singular. Thus, we have to add it to the Hamiltonian (9) and construct the total Hamiltonian as

$$\mathcal{H}_{\rm T} = -\tilde{N}\left[\frac{1}{2M_{\rm Pl}}\Pi_x^2 + \frac{1}{2}M_{\rm Pl}\omega^2 x^2 - \mathcal{E}\right] + \lambda\Pi_{\tilde{N}}, \tag{10}$$

where λ is a Lagrange multiplier.

During the evolution of the system, the primary constraint should hold; namely, we should have

$$\dot{\Pi}_{\tilde{N}} = \{\Pi_{\tilde{N}}, \mathcal{H}_{\rm T}\} \approx 0, \tag{11}$$

which leads to the secondary (Hamiltonian) constraint as

$$H \coloneqq \frac{1}{2M_{\rm Pl}}\Pi_x^2 + \frac{1}{2}M_{\rm Pl}\omega^2 x^2 - \mathcal{E} \approx 0. \tag{12}$$

We should note that a gauge-fixing condition is required for the constraint (12), in which $\tilde{N} = \text{const.}$ can be a possibility. Thus, by choosing the gauge as $\tilde{N} = 1/\omega$, and reminding the reader that the canonical variables satisfy the Poisson algebra $\{x, \Pi_x\} = 1$, we get the following Hamilton equations of motion:

$$\begin{cases} \dot{x} = -\frac{1}{\omega M_{\rm Pl}}\Pi_x, \\ \dot{\Pi}_x = \omega M_{\rm Pl} x. \end{cases} \tag{13}$$

Employing the Hamiltonian constraint (12), it easily leads us to the well-known solution for a closed universe as

$$\begin{cases} a(\eta) = \frac{a_{\rm Max}}{1+\sec\phi}[1 - \sec\phi\cos(\eta + \phi)], \\ a_{\rm Max} \coloneqq \frac{M}{12\pi^2 M_{\rm Pl}^2} + \left(\frac{2\mathcal{E}}{M_{\rm Pl}\omega^2}\right)^{\frac{1}{2}}, \\ \cos\phi \coloneqq \frac{M}{\sqrt{2\mathcal{E}M_{\rm Pl}}}, \end{cases} \tag{14}$$

where $a_{\rm Max}$ represents the maximum radius of the closed universe and we have assumed that the initial singularity occurs at $\eta = 0$.

Let us now investigate the effects, at a classical level, of a deformed Poisson algebra in the presence of a minimal length. We write [10]

$$\begin{cases} \{x, x\} = \{\Pi_x, \Pi_x\} = 0, \\ \{x, \Pi_x\} = 1 + \alpha^2 L_{\rm Pl}^2 \Pi_x^2, \end{cases} \tag{15}$$

where α is a dimensionless constant and $L_{\rm Pl}$ denotes Planck's length in the natural units. It is normally assumed that α is to be of order unity. In this case, the deformation will contribute only to the Planck regime of the Universe, and for this reason, the quantum cosmology of the model will be studied in the next section. A physical length of the order $\alpha L_{\rm Pl}$ is yet unobserved so it cannot exceed the electroweak scale [11], which implies $\frac{1}{\sqrt{8\pi}} \leq \alpha \leq 10^{17}$. As a consequence of the above deformation with Hamiltonian (9), the Hamilton equations become

$$\begin{cases} \dot{x} = -\frac{\tilde{N}}{M_{\rm Pl}}(1+\alpha^2 L_{\rm Pl}^2 \Pi_x^2)\Pi_x, \\ \dot{\Pi}_x = \tilde{N}\omega^2 M_{\rm Pl}(1+\alpha^2 L_{\rm Pl}^2 \Pi_x^2)x. \end{cases} \tag{16}$$

To solve these equations, we relate the Π_x to the new variable y as

$$\Pi_x = \frac{1}{\alpha L_{\rm Pl}} \tan(\alpha L_{\rm Pl} y), \tag{17}$$

which, using the Hamiltonian constraint (12) and the gauge $\tilde{N} = 1/\omega$, gives

$$\begin{cases} a(\eta) = \frac{a_{\rm Max}}{B}\left(1 - \frac{A\cos(\Omega\eta+\phi)}{\sqrt{1+2\mathcal{E}\alpha^2 L_{\rm Pl}\cos^2(\Omega\eta+\phi)}}\right), \\ B := 1 + \Omega^{-1}\sqrt{1+2\alpha^2 L_{\rm Pl}\mathcal{E}\cos^2\phi}\,\sec\phi, \\ A := \sec\phi\sqrt{1+2\alpha^2 L_{\rm Pl}\mathcal{E}\cos^2\phi}, \\ \Omega := \sqrt{1+2\mathcal{E}\alpha^2 L_{\rm Pl}}, \\ \cos\phi := \frac{M}{\sqrt{2\mathcal{E}M_{\rm Pl}}}(1+24\pi^2\alpha^2\mathcal{N}_\gamma)^{-\frac{1}{2}}, \end{cases} \tag{18}$$

and $a_{\rm Max}$ is similar to the nondeformed case defined in (14). If we take the limit $\alpha \to 0$, we find solution (14) which shows that the canonical behavior is recovered in this limit. We immediately obtain from (18) that the Universe reaches its maximum radius at $\eta = \frac{\pi-\phi}{\Omega}$, and it terminates in the big-crunch singularity at $\eta = \frac{2(\pi-\phi)}{\Omega}$.

III. QUANTUM COSMOLOGY WITH MINIMAL LENGTH

At the quantum level, the deformed Poisson algebra (15) is replaced by the following commutation relation between the phase space variables of the minisuperspace:

$$[x, \Pi_x] = i(1+\alpha^2 L_{\rm Pl}^2 \Pi_x^2). \tag{19}$$

Commutation relation (19) provides the minimal length uncertainty relation (MLUR) [12]

$$\Delta x \geq \frac{1}{2}\left(\frac{1}{\Delta\Pi_x} + \alpha^2 L_{\rm Pl}^2 \Delta\Pi_x\right). \tag{20}$$

This MLUR implies the existence of a minimal length

$$\Delta x_{\rm min} = \alpha L_{\rm Pl}, \tag{21}$$

which indicates it is impossible to consider any physical state as the eigenstate of the position operator [13]. Consequently, working with the position representation $|x\rangle$ is impossible. In the presence of GUP, in order to recover the information on the spatial distribution of the quantum system, we would introduce a quasiposition representation, which consists of the projection of the states onto a set of maximally localized states $|\psi_\xi^{ml}\rangle$ [14]. These states are the proper physical states around a position ξ with property $\langle\psi_\xi^{ml}|x|\psi_\xi^{ml}\rangle = \xi$ and $(\Delta x)_{|\psi_\xi^{ml}\rangle} = \Delta x_{\rm min}$. Thus, the quasiposition wave function will be

$$\psi(\xi) = \int_{-\infty}^{\infty} \frac{d\Pi_x e^{\frac{i\xi}{\alpha L_{\rm Pl}}\tan^{-1}(\alpha L_{\rm Pl}\Pi_x)}}{(1+\alpha^2 L_{\rm Pl}^2 \Pi_x^2)^{\frac{3}{2}}}\psi(\Pi_x), \tag{22}$$

which is a generalization of the Fourier transformation. The Hilbert space, in the quasiposition representation, is the space of functions with the usual $\mathbb{L}^2$ norm. On a dense domain of the Hilbert space, the position and momentum operators obeying relation (19) could be represented in momentum space as [15]

$$\begin{cases} \Pi_x = \Pi_x, \\ x = i(1+\alpha^2 L_{\rm Pl}^2 \Pi_x^2)\frac{d}{d\Pi_x}. \end{cases} \tag{23}$$

Hence, the inner product between two arbitrary states on a dense domain will be

$$\langle\varphi|\psi\rangle = \int_{-\infty}^{+\infty} \frac{d\Pi_x}{1+\alpha^2 L_{\rm Pl}^2 \Pi_x^2}\varphi^*(\Pi_x)\psi(\Pi_x). \tag{24}$$

Therefore, the modified Wheeler-DeWitt (WDW) equation in the presence of MLUR (20) is given by

$$\left[-M_{\rm Pl}\omega^2\left((1+\alpha^2 L_{\rm Pl}^2 \Pi_x^2)\frac{d}{d\Pi_x}\right)^2 + \frac{1}{2M_{\rm Pl}}\Pi_x^2\right]\psi = \mathcal{E}\psi. \tag{25}$$

We now proceed using the transformation (17). Hence, the above WDW equation will be changed to the trigonometric Pöchl-Teller (TPT) equation

$$\frac{d^2\psi(z)}{dz^2} + \left(\epsilon - \frac{V}{\cos^2(z)}\right)\psi(z) = 0, \tag{26}$$

where $z = \alpha L_{\rm Pl} y$ and

$$\begin{cases} \epsilon = \frac{1}{(12\pi^2\alpha)^2}\left(\frac{2\mathcal{E}}{M_{\rm Pl}} + \frac{1}{\alpha^2}\right), \\ V = \frac{1}{(12\pi^2\alpha^2)^2}. \end{cases} \tag{27}$$

The normalized eigenfunctions of the TPT equation are given by [16]

$$\psi_n(z) = 2^\nu \Gamma(\nu)\sqrt{\frac{\nu!(n+\nu)\alpha L_{\rm Pl}}{2\pi\Gamma(n+2\nu)}}\cos^\nu(z) C_n^\nu(\sin(z)), \tag{28}$$

where n is an integer, C_n^ν is the Gegenbauer polynomial and

$$\nu := \frac{1}{2}\left(1+\sqrt{1+\frac{1}{36\pi^4\alpha^4}}\right). \tag{29}$$

Moreover, the corresponding eigenvalue of the WDW equation is given by

$$\begin{aligned}\mathcal{E}_n &= 12\pi^2 M_{\rm Pl}\left\{\left(n+\frac{1}{2}\right)\sqrt{1+36\pi^4\alpha^4}\right.\\ &\quad \left.+6\pi^2\alpha^2\left(n^2+n+\frac{1}{2}\right)\right\}\\ &= 72\pi^4\alpha^2 M_{\rm Pl}(n^2+2n\nu+\nu). \end{aligned}\tag{30}$$

Let us now obtain the Dirac observables of the model. According to Dirac [17], the observables of a theory are those quantities which have vanishing commutators with the constraints of theory. In order to retrieve them, we start by finding the symmetries of the WDW equation in the form given by (26). Considering the infinite number of bound states of Eq. (26), the underlying Lie algebra could be expected as its spectrum generating algebra. The lowering and raising operators for the WDW equation in (26) can be built using a factorization type method [18]. To do this, let us start with the WDW Eq. (26), and rewrite it as a bound state stationary Schrödinger equation

$$\begin{cases} h\psi_n = \epsilon_n\psi_n, \\ h := -\frac{d^2}{dz^2} + U(z), \end{cases}\tag{31}$$

where $U(z) := V/\cos^2(z) = \nu(\nu-1)/\cos^2(z)$. Introducing the following first order differential operators [18]

$$a_\nu^\pm := \mp\frac{d}{dz} + W(z;\nu), \tag{32}$$

where $W(z;\nu) = \nu\tan(z)$ is the superpotential, we obtain the first supersymmetric partner Hamiltonian

$$h_+ := a_\nu^+ a_\nu^- = h - \epsilon_0, \tag{33}$$

where $\epsilon_0 = \nu^2$ is the ground state energy eigenvalue. The second supersymmetric partner Hamiltonian is given by

$$h_- := a_\nu^- a_\nu^+. \tag{34}$$

The Hamiltonians $h_\pm$ have the same energy spectrum except the ground state of h_+

$$\begin{cases} h_+\psi_\nu^n = a_\nu^+ a_\nu^-\psi_\nu^n = (\epsilon_n - \nu^2)\psi_\nu^n, \\ h_-\psi_\nu^n = a_\nu^- a_\nu^+\psi_{\nu-1}^n = (\epsilon_n-\nu^2)\psi_{\nu-1}^n. \end{cases}\tag{35}$$

We see that changing the order of operators a_ν^+ and a_ν^- simply leads to the shift of the value of ν. This symmetry is called shape-invariance symmetry [18]. Using shape-invariance symmetry, we can show that ψ_ν^n and $\psi_{\nu-1}^n$ are proportional to $a_\nu^+\psi_{\nu-1}^n$ and $a_\nu^-\psi_\nu^n$, respectively. The shape-invariance condition (35) can be rewritten as

$$a_\nu^- a_\nu^+ = a_{\nu+1}^+ a_{\nu+1}^- + R(\nu), \tag{36}$$

where $R(\nu) = 2\nu + 1$ is independent of z. According to [19], we assume that replacing ν with $\nu+1$ in a given operator, say $\mathcal{O}_\nu$, can be achieved with a similarity transformation T_ν:

$$\begin{cases} T_\nu \mathcal{O}_\nu(z) T_\nu^{-1} = \mathcal{O}_{\nu+1}(z), \\ T_\nu := \exp\left(\frac{\partial}{\partial\nu}\right). \end{cases}\tag{37}$$

The shape-invariant potentials are easy to deal with, if lowering and raising operators are employed, developed originally for the harmonic oscillator. However, as the commutator $[a_\nu^-, a_\nu^+]$ does not yield a constant value, namely,

$$[a_\nu^-, a_\nu^+] = \frac{2\nu}{\cos^2(z)}, \tag{38}$$

the choice of $a_\nu^\pm$ does not work. To establish a suitable algebraic structure, we introduce the following operators [19]:

$$\begin{cases} A := T^\dagger a_\nu^-, \\ A^\dagger := a_\nu^+ T, \end{cases}\tag{39}$$

which lead to

$$[A, A^\dagger] = 1 - 2\nu. \tag{40}$$

The action of these factor operators on normalized eigenfunctions will be

$$\begin{cases} A|\nu,n\rangle = \sqrt{n(2\nu+2+n)}|\nu+2,n-1\rangle, \\ A^\dagger|\nu,n\rangle = \sqrt{(n+1)(2\nu-1+n)}|\nu-2,n+1\rangle. \end{cases}\tag{41}$$

It can be verified that these operators, together with $\tilde{A}|\nu,n\rangle = (1/2-\nu)|\nu,n\rangle$, obey the $su(2)$ Lie algebra

$$[\tilde{A}, A] = -A, [\tilde{A}, A^\dagger] = A^\dagger, [A, A^\dagger] = -2\tilde{A}. \tag{42}$$

Also, based on recursion relations of Gegenbauer polynomials, we can introduce the following three operators [20] associated with the dynamical group $su(1,1)$ of the WDW equation (26):

$$\begin{cases} J_+ := [-\cos(z)\frac{d}{dz} + \sin(z)(\hat{N}+\nu)]\frac{1+\hat{N}+\nu}{\sqrt{(\hat{N}+\nu)(\hat{N}+2\nu)}}, \\ J_- := [\cos(z)\frac{d}{dz} + \sin(z)(\hat{N}+\nu)]\sqrt{\frac{\hat{N}+\nu-1}{\hat{N}+2\nu-1}}, \\ J_0 := \hat{N} + \frac{\nu+1}{2}, \end{cases} \tag{43}$$

where $\hat{N}$ denotes the number operator with the property $\hat{N}|\nu,n\rangle = n|\nu,n\rangle$. The action of the above generators on a set of basis eigenvectors $|\nu,n\rangle$ is given by

$$\begin{cases} J_0|\nu,n\rangle = (-j+n)|\nu,n\rangle, \\ J_-|\nu,n\rangle = \sqrt{n(-2j+n-1)}|\nu,n-1\rangle, \\ J_+|\nu,n\rangle = \sqrt{(n+1)(-2j+n)}|\nu,n+1\rangle, \end{cases} \tag{44}$$

where $j := -(\nu+1)/2 < 0$ denotes the Bargmann index of the dynamical group. The corresponding Casimir operator can be calculated as

$$\begin{cases} J^2 := J_0(J_0-1) - J_+J_-, \\ J^2|\nu,n\rangle = j(j+1)|\nu,n\rangle, \end{cases} \tag{45}$$

with well-known properties

$$[J^2, J_\pm] = 0, \qquad [J^2, J_0] = 0. \tag{46}$$

Hence, a representation of $su(1,1)$ is determined by the Bargmann index and the eigenvectors of the Casimir and J_0. In addition, we find that the Hamiltonian (12) could be written as [20]

$$H = 72\pi^4\alpha M_{\rm Pl}[J_+J_- - (2j+1)J_0] - 3\pi^2\alpha M_{\rm Pl}(j+1)(2j+1) - \mathcal{E}, \tag{47}$$

from which we can conclude that the Casimir operator (45) and J_0 commute with the Hamiltonian

$$[J^2, H] = [J_0, H] = 0. \tag{48}$$

Therefore, J^2 and J_0 leave the physical Hilbert space invariant and we choose them as physical operators of the model.

IV. HOLOGRAPHIC PRINCIPLE AND THE MINIMAL AREA

Let us first concentrate on a radiation dominated very early universe, $(M=0)$. In this case, comparing Eq. (5) and (30) gives

$$\mathcal{N}_\gamma = \left(n+\frac{1}{2}\right)\sqrt{1+36\pi^4\alpha^4} + 6\pi^2\alpha^2\left(n^2+n+\frac{1}{2}\right). \tag{49}$$

Moreover, according to Eq. (7), $\mathcal{N}_\gamma$ is related to the entropy of the radiation fluid. Hence, Eqs. (7), (49) and from the definition of ν, (29), lead us to extract the corresponding entropy of the radiation, $S_n^{(\gamma)}$ (in terms of a minimal surface) as

$$S_n^{(\gamma)} = \left(\frac{4\pi^7 g}{45}\right)^{\frac{1}{4}}\left(\frac{\mathcal{A}_{\rm min}}{4G}\right)^{\frac{3}{4}}(n^2+2n\nu+\nu)^{\frac{3}{4}}, \tag{50}$$

where $\mathcal{A}_{\rm min} = 4\Delta x_{\rm min}^2 = 4\alpha^2 L_{\rm Pl}^2$ is the minimal surface [21]. Therefore, according to Eqs. (44) and (48), the entropy of radiation is a Dirac observable. To obtain a relation between the entropy of radiation and the surface of the apparent horizon, let us retrieve the expectation value of the square of the scale factor. Using relations (3), (17), (23) and (39), we obtain $a(t) = x = i\alpha L_{\rm Pl}\frac{d}{dz} = \frac{i\alpha L_{\rm Pl}}{2}(TA - A^\dagger T^\dagger)$. Hence, the expectation value of the square of the scale factor reads

$$\begin{aligned} \langle a^2\rangle &= \langle x^2\rangle \\ &= \frac{\alpha^2 L_{\rm Pl}^2}{4}\langle\nu,n|(TAA^\dagger T^\dagger + A^\dagger A)|\nu,n\rangle \\ &= \frac{\mathcal{A}_{\rm min}}{8}\left(n^2+2n\nu+\nu+n-\frac{1}{2}\right), \end{aligned} \tag{51}$$

where we have used (41). In the presence of the minimum length, from relation (29), we get $\nu \simeq 1$. Hence for large values of the quantum number, n, we find $\langle a^2\rangle \simeq \mathcal{A}_{\rm min}n^2/8$. On the other hand, the apparent horizon of a FLRW model for the radiation case is given [22] by $R_{\rm ah} = (H^2 + 1/a^2)^{-1/2} = \sqrt{\frac{6\pi^2 M_{\rm Pl}^2}{\mathcal{N}}a^2}$. Inserting this result into Eq. (50), we find, for large values of n, that

$$S_n^{(\gamma)} \simeq \left(\frac{2048\pi^7 g}{45}\right)^{\frac{1}{4}}\left(\frac{\mathcal{A}_{\rm ah}}{4G}\right)^{\frac{3}{4}}, \tag{52}$$

where $\mathcal{A}_{\rm ah} := 4\pi\langle R_{\rm ah}\rangle^2$ denotes the area of the apparent horizon. The above equation is in the form as conjectured by 't Hooft [4]: assuming that the matter occupies a specific volume, then the entropy of that matter will be $S^{(\gamma)} \propto (4G)^{-3/4}\mathcal{A}^{3/4}$ [4], where $\mathcal{A}$ denotes the area of the containing volume.

We now turn to a universe filled with only dust fluid, $(\mathcal{N}_\gamma = 0)$. In this case, comparing Eqs. (5) and (30), we get

$$M^2 = 144\pi^4\alpha^2 M_{\rm Pl}^2(n^2+2n\nu+\nu). \tag{53}$$

In addition, let us discuss the total entropy of the dust content of the universe, by means of investigating the following expression [23] for the entropy of an ideal gas, which consists of N ideal particles in a volume V, namely,

$$S_{\rm (ideal)} = N\ln\left(\frac{V}{N}\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}}e^{\frac{5}{2}}\right), \tag{54}$$

where m is the mass of the particles. For the case of a continuous fluid, let us rewrite (54). To this end, we consider an ideal gas contained within a small volume element dV. The number of particles inside dV is

$$dN = \frac{\rho}{m} dV. \tag{55}$$

Inserting this expression into Eq. (54), the entropy associated with the volume element, in terms of the density of the fluid, can be written as

$$dS^{(\mathrm{dust})} = \frac{\rho_m}{m} \ln \left(\frac{KT^{\frac{3}{2}}}{\rho}\right) dV, \tag{56}$$

where $K = (\frac{m^5 e^5}{2\pi})^{1/2}$ [24]. For a dust dominated universe where $\rho = \rho_0 (a/a_0)^{-3}$ and $T = T_0 (a/a_0)^{-2}$, we get $S^{(\mathrm{dust})} = \ln(KT_0^{3/2}/\rho_0) N$. Let us use the simple approximation

$$S^{(\mathrm{dust})} \simeq N = \frac{M}{m}. \tag{57}$$

Therefore, from Eqs. (53) and (57), we obtain

$$S_n^{(\mathrm{dust})} \simeq 12\pi^2 \alpha \frac{M_{\mathrm{Pl}}}{m} \sqrt{n^2 + 2n\nu + \nu}. \tag{58}$$

For this dust universe, the apparent horizon [22] is $R_{\mathrm{ah}}^2 = \frac{6\pi^2 M_{\mathrm{Pl}}^2}{M} a^3$. Employing the relation associated to the variable x from (3) and reminding the reader that the expectation values of odd powers of x vanish, i.e., $\langle x \rangle = \langle x^3 \rangle = 0$, we obtain $\langle a^3 \rangle = -x_0 [3\langle x^2 \rangle + x_0^2]$. Now, substituting the relation for x_0, which is defined as in (3), gives $\langle a^3 \rangle = \frac{M}{12\pi^2 M_{\mathrm{Pl}}^2} [3\langle x^2 \rangle + x_0^2]$. The terms inside the brackets can be replaced by the expectation value of x^2 from (51) and an expression for x_0^2 as

$$x_0^2 = \frac{\mathcal{A}_{\min}}{4} (n^2 + 2n\nu + \nu), \tag{59}$$

where we have used (3), $\alpha^2 = (\mathcal{A}_{\min} M_{\mathrm{Pl}}^2)/4$ and the quantization condition (53). Consequently, using Eqs. (51) and (59) in the last expression associated to the $\langle a^3 \rangle$, and finally substituting it into the relation of the apparent horizon, we find

$$\begin{aligned} \langle R_{\mathrm{ah}}^2 \rangle &= \frac{1}{2} [3\langle x^2 \rangle + x_0^2] \\ &= \frac{\mathcal{A}_{\min}}{16} \left[5(n^2 + 2n\nu + \nu) + 3\left(n - \frac{1}{2}\right) \right]. \end{aligned} \tag{60}$$

Therefore, at the presence of minimal length and for large values of the quantum number n, by comparing (58) and (60) we obtain

$$S^{(\mathrm{dust})} = \frac{12\pi}{\sqrt{10}} \frac{M_{\mathrm{Pl}}}{m} \left(\frac{\mathcal{A}_{\mathrm{ah}}}{4G}\right)^{\frac{1}{2}}, \tag{61}$$

which implies that the 't Hooft holographic conjecture is satisfied for a universe filled with a dust fluid. Let us, in particular, consider the specific case where primordial black holes (PBHs) constitute the sole content of the dust fluid [25]. In this case, Eq. (53) gives us the total entropy of the PBHs: if we assume each PBH to have the same mass, m_{bh}, then we have $M = Z m_{\mathrm{bh}}$, where Z denotes the total number of PBHs. The area of the event horizon of a Schwarzschild black hole is given by $\mathcal{A}_{\mathrm{eh}} = 16\pi G^2 m_{\mathrm{bh}}^2$ and consequently we obtain $M^2 = 4\pi Z^2 \mathcal{A}_{\mathrm{eh}} M_{\mathrm{Pl}}^4$, assuming each PBH to be of a Schwarzschild type. Therefore, from Eq. (53) we can obtain the following relations between the PBH event horizon area and the minimal surface area:

$$\mathcal{A}_{\mathrm{eh}} \simeq \frac{9\pi^3}{Z^2} \mathcal{A}_{\min} (n^2 + 2n\nu + \nu). \tag{62}$$

Hence, the event horizons of PBHs are Dirac observables, with the event horizon being related to the quantum number n. Using the Bekenstein-Hawking formula $S^{(\mathrm{bh})} = \frac{\mathcal{A}_{\mathrm{eh}}}{4G}$, we obtain the total entropy of the PBHs, $(ZS^{(\mathrm{bh})})$, as

$$S_n^{(\mathrm{PBHs})} = \frac{9\pi^3}{Z} \frac{\mathcal{A}_{\min}}{4G} (n^2 + 2n\nu + \nu). \tag{63}$$

Therefore, for large values of quantum number n, using (60) we obtain

$$S^{(\mathrm{PBHs})} = \frac{81\pi^3}{7Z} \left(\frac{\mathcal{A}_{\mathrm{ah}}}{4G}\right), \tag{64}$$

which is in agreement with 't Hooft holographic conjecture [4].

V. CONCLUSIONS

In this paper we studied the effects of a deformed Heisenberg algebra in terms of a MLUR in a closed quantum FRLW model, whose matter is either a fluid of radiation or dust. Quantum cosmologies with a perfect fluid matter content were investigated, e.g., in [26]. In particular, the case of a dust or radiation dominated quantum universe was studied in [27].

Our main result is that the extended dynamical group of the model, $su(1,1)$, admits a minimal area retrieved from a MLUR, in the form of a Dirac observable. It reasonably agrees with the cosmological holographic principle, in the case of a large quantum number, n. We are aware that our results are obtained within a very simple as well as restricted setting. Nevertheless, we think they are intriguing and provide motivation for subsequent research works. Possible extensions to test the relation among a GUP, a minimal surface being subsequently obtained (constituting a Dirac observable) and the cosmological holography may include

(i) Considering other perfect fluids besides radiation and dust.
(ii) Including instead, e.g., scalar fields.
(iii) Considering a Bianchi IX geometry.
(iv) Exploring string features by means of a broader gravitational sector.

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