# Semi-inclusive charged-current neutrino-nucleus reactions

O. Moreno and T. W. Donnelly

*Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics,*
*Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

J. W. Van Orden

*Department of Physics, Old Dominion University, Norfolk, Virginia 23529, USA and Jefferson Lab,*
*Newport News, Virginia 23606, USA*

W. P. Ford

*Department of Physics, University of Southern Mississippi, Hattiesburg, Mississippi 39406, USA*

The general, universal formalism for semi-inclusive charged-current (anti)neutrino-nucleus reactions is given for studies of any hadronic system, namely, either nuclei or the nucleon itself. The detailed developments are presented with the former in mind and are further specialized to cases where the final-state charged lepton and an ejected nucleon are presumed to be detected. General kinematics for such processes are summarized, and then explicit expressions are developed for the leptonic and hadronic tensors involved and for the corresponding responses according to the usual charge, longitudinal and transverse projections, keeping finite the masses of all particles involved. In the case of the hadronic responses, general symmetry principles are invoked to determine which contributions can occur. Finally, the general leptonic-hadronic tensor contraction is given as well as the cross section for the process.

## I. INTRODUCTION

Many of the ongoing experiments in charge-changing neutrino scattering involve quasielastic scattering from light to medium mass nuclei. An increasing number of these experiments offer the possibility of studying semi-inclusive charge-changing (CC) neutrino or antineutrino reactions, namely, those where a final-state charged lepton and some other particle are presumed to be detected in coincidence. For example, in the ArgoNeuT [1] and MicroBooNE [2] experiments protons, together with muons, are detected in coincidence using argon time projection chambers. Using standard nuclear physics notation such reactions would be denoted $X(\nu_\ell, \ell^- x)$ and $X(\bar{\nu}_\ell, \ell^+ x)$, where $\ell = e$, $\mu$, or $\tau$. Here $x$ can be any kinematically allowed particle, for instance, $\gamma$, a nucleon $N = p$ or $n$, a deuteron $d$ or triton $t$, $^3$He, $\alpha$, fission fragment, $\pi$, $K$, and so on. The target $X$ may be a nucleus or the proton itself. All of these possibilities are contained in the formalism that follows. One should be clear that this notation indicates what is presumed to be detected, not what is actually in the final state. For example, if $x = p$, this means that one proton is definitely in the final state; however, depending on the kinematics chosen for the reaction, there may be many open channels, a proton and a daughter nucleus in some discrete state, two protons and a different nucleus in some discrete state, a proton and a neutron and yet another nucleus in some discrete state, etc. The semi-inclusive cross section is then the sum or integral over all unobserved particles, except the one that is presumed to be detected, in this example a proton. At a level lower, one has the inclusive cross section where all particles for all open channels are to be summed or integrated.

In the rest of the paper, to make things more specific and to explore the present case of interest in the quasielastic regime (CCQE), we focus on the specific case of a nuclear target where a nucleon is the particle that is presumed to be detected $(x = N)$. Nevertheless it should be clear that simply by changing the names of the particles involved, all of the developments can immediately be used in any other semi-inclusive study. Accordingly, we now consider reactions of the type $^A_Z X(\nu_\ell, \ell^- p)^{A-1}_Z Y$, $^A_Z X(\bar{\nu}_\ell, \ell^+ n)^{A-1}_{Z-1} Y$, $^A_Z X(\nu_\ell, \ell^- n)^{A-1}_{Z+1} Y$ and $^A_Z X(\bar{\nu}_\ell, \ell^+ p)^{A-1}_{Z-2} Y$. These are to be viewed in the context of semi-inclusive electron scattering reactions $^A_Z X(e, e'p)^{A-1}_{Z-1} Y$ and $^A_Z X(e, e'n)^{A-1}_Z Y$. In the initial state one has some nucleus $X$ in its ground state with mass number $A$ and charge $Z$, while in the final state one has a nuclear system $Y$ with mass number $A - 1$ and the charges indicated above. The latter daughter nucleus is not presumed to be in its ground state, in general (although this is one possibility when the system is stable to nucleon emission), and may be in some discrete excited state (if any exist), may be a granddaughter nucleus plus two nucleons, and so on. All open channels are to be considered, and we only require that the mass number and charge be as indicated, together with the kinematical information to be discussed in the following section. Note also that of the four neutrino and antineutrino reactions given above, the first two are in some sense

"natural" in that the reactions in the CCQE regime are at least dominated by the basic reactions on nucleons in the target nucleus, namely, $\nu_\ell + n \to \ell^- + p$ and $\bar{\nu}_\ell + p \to \ell^+ + n$, respectively. However, the third and fourth reactions can occur in nuclei. On the one hand, the final states involved are complex interacting many-body states, involving, in general, coupled channels whenever kinematically allowed. There may be several nucleons in the final state, and it is possible that the one with the "wrong" flavor is the one detected. In fact, for some situations there may be no bound state reached for the final nucleus, and one definitely has nucleons of both flavors in the final state. On the other hand, while one certainly has one-body electroweak current operators (those that act on a single nucleon), it is also clear that two-body meson exchange currents (MEC) are present as well. For instance, important contributions to MEC at quasielastic kinematics are diagrams where two nucleons interact with an exchanged $W^\pm$, going through a virtual $\Delta$ which in turn exchanges a pion between the two nucleons, leaving two nucleons in the final state. Take, for example, the third reaction above: if the two initial nucleons are an $nn$ pair in the nuclear ground state, one can absorb the exchanged $W^+$, go through a $\Delta^+$, exchange a $\pi^+$, and have an $np$ pair in the final state where the neutron is the particle detected in the third reaction (and the proton may be the one detected in the first reaction). In the developments presented in the rest of this paper the formalism is general enough to allow for MEC, no assumption is required about which specific reaction is being considered, and only when applying these ideas with particular modeling are the details required. All of the developments are kept relativistic; i.e., no nonrelativistic approximations are made, with one exception which will be discussed later in this paper. All of the formalism may then be used regardless of the energy scale, whether at relatively low energies or, what is more typical, at high energies.

The paper is arranged in the following way: in the next section the required basic kinematics are summarized. Here we assume that the incident neutrino or antineutrino has a given momentum, although in practical situations one usually has to fold the answers with the appropriate neutrino flux. In Sec. III we introduce the general electroweak leptonic and hadronic tensors. We use the notation already employed in studies of electron scattering (see, for instance, Refs. [3,4]), while in Secs. III A and III B the details of these two tensors are further developed. In Sec. IV the tensor contractions and semi-inclusive cross section are presented, and finally, in Sec. V we summarize the results of this study and indicate where we are presently applying the formalism using specific models.

## II. KINEMATICS

To describe the kinematics of the particles involved in the process, we indicate four-vectors with capital letters such as $A^\mu = (A^0, A^1, A^2, A^3)$ and three-vectors with bold-face lowercase letters such as $\mathbf{a}$, with their magnitudes in normal-faced font $a = |\mathbf{a}|$; the metric used, as in [5], yields $A \cdot B = A_\mu B^\mu = A^0 B^0 - \mathbf{a} \cdot \mathbf{b}$ (repeated indices summed).

The incident neutrino (or antineutrino) carries four-momentum $K^\mu = (\varepsilon, \mathbf{k})$, where $\varepsilon = \sqrt{k^2 + m^2}$ is the total energy, $\mathbf{k}$ is the three-momentum, and $m$ is the mass. The outgoing charged lepton has four-momentum $K'^\mu = (\varepsilon', \mathbf{k}')$ and mass $m'$. The spacelike four-momentum of the boson exchanged with the nuclear target is $Q^\mu = (\omega, \mathbf{q})$, with $-Q^2 = |Q^2| = q^2 - \omega^2 \geq 0$. We assume the three-momentum $\mathbf{q}$ to be along the 3-axis so that the incoming and outgoing leptons define the 13-plane (see Fig. 1). By defining the lepton scattering angle $\theta$ (i.e., the angle between $\mathbf{k}$ and $\mathbf{k}'$), the components of the incident and outgoing leptons and exchanged boson four-momenta can be written as

$$
\begin{aligned}
&K^0 = \varepsilon && K'^0 = \varepsilon' && Q^0 = \varepsilon - \varepsilon' = \omega \\
&K^1 = \frac{1}{q} kk' \sin\theta && K'^1 = \frac{1}{q} kk' \sin\theta && Q^1 = 0 \\
&K^2 = 0 && K'^2 = 0 && Q^2 = 0 \\
&K^3 = \frac{1}{q} k(k - k' \cos\theta) && K'^3 = -\frac{1}{q} k'(k' - k \cos\theta) && Q^3 = \sqrt{k^2 + k'^2 - 2kk' \cos\theta} = q.
\end{aligned}
\tag{1}
$$

In the laboratory system the incoming nuclear target with mass $M_A^0$ carries four-momentum $P_A^\mu = (M_A^0, 0, 0, 0)$. We assume that the final hadronic state consists of a stripped nucleon and the remaining daughter nucleus with four-momenta $P_N^\mu = (E_N, \mathbf{p}_N)$ and $P_{A-1}^\mu = (E_{A-1}, \mathbf{p}_{A-1})$, respectively. This $A - 1$ daughter system may be in its ground state or in some discrete excited state, it may be an $A - 2$ granddaughter nucleus

plus a nucleon, etc., and it has invariant mass $W_{A-1}$. The only assumption so far is that one nucleon is presumed to be detected and so only final states with one or more nucleons, at least one being of the appropriate flavor, are being considered (see also below). Using the coordinate system introduced above, where $\mathbf{q}$ lies along the $z$ axis and the leptons lie in the 13-plane, the total four-momentum in the hadronic vertex is
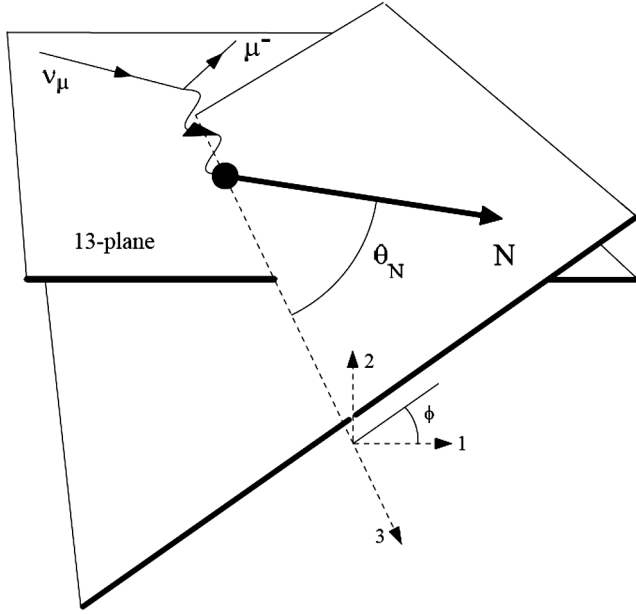
FIG. 1. Kinematics for semi-inclusive neutrino-nucleus reactions.

$$P_{\text{tot}}^{\mu} \equiv Q^{\mu} + P_A^{\mu} = P_N^{\mu} + P_{A-1}^{\mu} = (P_{\text{tot}}^0, 0, 0, P_{\text{tot}}^3), \quad (2)$$

$$P_{\text{tot}}^0 = M_A^0 + \omega = E_N + E_{A-1} \equiv E, \quad (3)$$

$$\mathbf{p}_{\text{tot}} = P_{\text{tot}}^3 \mathbf{u}_3 = q\mathbf{u}_3 = \mathbf{q} = \mathbf{p}_N + \mathbf{p}_{A-1}. \quad (4)$$

Writing out the components of the products' four-momenta one has

$$
\begin{aligned}
P_N^0 &= E_N & P_{A-1}^0 &= E_{A-1} \\
P_N^1 &= p_N \sin\theta_N \cos\phi & P_{A-1}^1 &= -p_{A-1}\sin\theta_{A-1}\cos\phi \\
P_N^2 &= p_N \sin\theta_N \sin\phi & P_{A-1}^2 &= -p_{A-1}\sin\theta_{A-1}\sin\phi \\
P_N^3 &= p_N \cos\theta_N & P_{A-1}^3 &= p_{A-1}\cos\theta_{A-1}, \quad (5)
\end{aligned}
$$

where $\theta_N$ and $\theta_{A-1}$ are the angles of the hadronic products with respect to the 3-axis (direction of $\mathbf{q}$), and $\phi$ is the angle between the plane defined by the nucleon momentum $p_N$ and the momentum transfer $q$ and the leptonic (13) plane. From Eqs. (4) and (5) we have that

$$
\begin{aligned}
\sin\theta_{A-1} &= \frac{1}{p_{A-1}} p_N \sin\theta_N \\
\cos\theta_{A-1} &= \frac{1}{p_{A-1}}(q - p_N \cos\theta_N), \quad (6)
\end{aligned}
$$

and from conservation of energy, Eq. (3), one has $E_{A-1} = E - E_N$, where both products are on shell, i.e., $E_N = \sqrt{p_N^2 + m_N^2}$ and $E_{A-1} = \sqrt{p_{A-1}^2 + W_{A-1}^2}$, where

$W_{A-1}$, as said above, is the invariant mass of the $A-1$ daughter system.

Having set up the basic form for the semi-inclusive cross section, let us next consider the problem in more detail by discussing the general kinematical variables to be used when studying $X(\nu_\ell, \ell^- N)$ and $X(\bar{\nu}_\ell, \ell^+ N)$ reactions in the context of previous studies of $X(e, e'N)$ reactions. We have seen above that the cross section depends on a limited set of kinematic variables. The leptonic variables are those discussed above. The hadronic variables, in contrast, are best transformed into other variables when treating semi-inclusive scattering from nuclei. We see in the following section that the dependences on the azimuthal angle $\phi$ can be made explicit using the general Lorentz structure of the hadronic tensor and so we can leave that variable aside. We have the momentum transfer $q$ and energy transfer $\omega$ from the leptonic side via the exchange of a single $W^\pm$, and so we can use this pair or equivalently $Q^2$ and $\nu$ in other notation, or $Q^2$ and $x \equiv |Q^2|/2m_N\nu$ in still other notation. That leaves us with $p_N$ and $\theta_N$, which are more conveniently transformed into two new variables. While these sets of dynamical variables are, of course, completely usable and indeed natural from an experimental point of view, we see in the following that alternative sets are more convenient when studying the specifics of the cross section in the regime of quasifree scattering.

From three-momentum conservation one has

$$\mathbf{p}_{A-1} = \mathbf{q} - \mathbf{p}_N \equiv -\mathbf{p}, \quad (7)$$

where $\mathbf{p}$ is minus the missing momentum $\mathbf{p}_m$, so that the daughter energy becomes $E_{A-1} = \sqrt{W_{A-1}^2 + p^2}$. This is completely general and, in particular, is not dictated by any specific model for the reaction. Clearly this momentum merely characterizes the split in momentum flow between the detected nucleon and the unobserved daughter nucleus. From energy conservation and using the three-momentum conservation relation, one has

$$
\begin{aligned}
M_A^0 + \omega = E_N + E_{A-1} &= \sqrt{q^2 + p^2 + 2qp\cos\theta_{pq} + m_N^2} \\
&+ \sqrt{p^2 + W_{A-1}^2} \quad (8)
\end{aligned}
$$

with $\theta_{pq}$ being the angle between $\mathbf{p}$ and $\mathbf{q}$. Next we need some energy variable to characterize the degree of excitation of the daughter nucleus. A natural choice is the excitation energy in the rest frame of the recoiling daughter nucleus, $E^* \equiv W_{A-1} - W_{A-1}^0 \geq 0$, where $W_{A-1}$ includes the internal excitation energy of the $A-1$ system while $W_{A-1}^0$ is the smallest possible invariant mass of the $A-1$ and will be the ground state rest mass of this system $M_{A-1}^0$ in most cases. By construction $E^*$ is greater than or equal to zero—and equal to zero when the daughter nucleus is left in its ground state. Using this one can obtain the so-called missing energy

$$E_m = m_N + W_{A-1} - M_A^0 = E_s + E^*, \qquad (9)$$

where $E_s = m_N + W_{A-1}^0 - M_A^0$ is the separation energy (or "$Q$ value" ); another commonly used energy in the problem is defined as the minimum energy needed to separate the nucleus $A$ into a nucleon and the residual nucleus $A-1$ in its ground state. As we see below, we could now use $(E^*, p)$ or $(E_m, p_m)$ in place of $(E_N, \theta_N)$, although it may be shown that still another choice for the energy is preferable for certain purposes other than $E^*$, namely,

$$\mathcal{E} \equiv E_{A-1} - E_{A-1}^0 \geq 0, \qquad (10)$$

where, as before, $E_{A-1} = \sqrt{W_{A-1}^2 + p^2}$ and now also $E_{A-1}^0 = \sqrt{W_{A-1}^0{}^2 + p^2}$. This quantity does not differ much from the excitation energy $E^*$ for $p \ll W_{A-1}^0$, which is typically the case; let us call it the "daughter energy difference," in contrast to the "daughter excitation energy" $E^*$.

Overall, energy conservation yields an equation for $\mathcal{E}$ in terms of $q$, $\omega$, $p$ and the angle $\theta_{pq}$:

$$\mathcal{E} = \omega - E_s + m_N - \sqrt{m_N^2 + p^2 + q^2 + 2pq\cos\theta_{pq}}$$
$$- \sqrt{W_{A-1}^0{}^2 + p^2} + W_{A-1}^0. \qquad (11)$$

Thus there are clear relationships between the sets $(E_N, \theta_N)$ and $(p, \theta)$ and hence $(\mathcal{E}, p)$. Instead of the first set, we now use the last set as a pair of dynamical variables.

With these preliminaries in hand let us discuss the characteristic landscape of the coincidence semi-inclusive cross section as a function of $\mathcal{E}$ and $p$ for fixed $q$ and $\omega$ (and of course fixed $\theta$ and $\phi$). We have not yet required that the kinematic relationships discussed above should be satisfied, and when we do so, we find that only specific regions are accessible. Noting that Eq. (11) yields a curve of $\mathcal{E}$ versus $p$ in the $(\mathcal{E}, p)$ plane for each choice of $\theta_{pq}$, let us see what constraint the requirement that $-1 \leq \cos\theta_{pq} \leq +1$ imposes on the kinematics.

First, consider $\omega$ small (to be specified completely below) and plot the trajectory when $\cos\theta_{pq} = -1$. A curve rising from negative $\mathcal{E}$ to intersect $\mathcal{E} = 0$ at $p = p_{\min} > 0$ which peaks at some value of $p$ and then falls to intersect $\mathcal{E} = 0$ again, this time at $p = p_{\max} > p_{\min}$, is generally obtained. All physically allowable values of $\mathcal{E}$ and $p$ must lie below this curve and, of course, above $\mathcal{E} = 0$. To obtain the other extreme, $\cos\theta_{pq} = +1$, one can simply replace $p$ by $-p$ in Eq. (11); the physically allowable values of $\mathcal{E}$ and $p$ must lie above this curve. For $\omega$ small, no physically allowable values occur near the latter curve and the physical region is completely defined by the $\cos\theta_{pq} = -1$ curve and $\mathcal{E} = 0$. Following past work [6] we call the minimum value of momentum $p_{\min} \equiv -y$ and the maximum value

$p_{\max} \equiv +Y$. The formal definition of $\omega$ small then becomes "$y < 0$." We can set $\mathcal{E} = 0$ in Eq. (11) and solve for $y$ and $Y$, yielding

$$y(q, \omega) = \frac{1}{W_A^2}\left[(M_A^0 + \omega)\sqrt{\Lambda^2 - W_{A-1}^0{}^2 W_A^2} - q\Lambda\right], \quad (12)$$

$$Y(q, \omega) = \frac{1}{W_A^2}\left[(M_A^0 + \omega)\sqrt{\Lambda^2 - W_{A-1}^0{}^2 W_A^2} + q\Lambda\right] \quad (13)$$

with

$$W_A = \sqrt{(M_A^0 + \omega)^2 - q^2}, \qquad (14)$$

$$\Lambda = \frac{1}{2}(W_A^2 + W_{A-1}^0{}^2 - m_N^2). \qquad (15)$$

A useful relationship is the following:

$$M_A^0 + \omega = \sqrt{(q+y)^2 + m_N^2} + \sqrt{y^2 + W_{A-1}^0{}^2}. \qquad (16)$$

Noting that the quasielastic peak occurs—approximately— at the kinematical point where $y = 0$, it is useful to use Eq. (16) to define

$$\omega_{QE} \equiv \omega(y = 0) \equiv \left\{\sqrt{q^2 + m_N^2} - m_N\right\} + E_s$$
$$= |Q_{QE}^2|/2m_N + E_s. \qquad (17)$$

Accordingly, $\omega$ small corresponds to $y < 0$, namely, to $\omega < \omega_{QE}$. Finally, the equation for the upper boundary of the allowed region (i.e., corresponding to $\cos\theta_{pq} = -1$) is given by

$$\mathcal{E}_- = \sqrt{m_N^2 + (q+y)^2} - \sqrt{m_N^2 + (q-p)^2}$$
$$+ \sqrt{W_{A-1}^0{}^2 + y^2} - \sqrt{W_{A-1}^0{}^2 + p^2}. \qquad (18)$$

When the momentum transfer becomes very large, one can show that this goes to the *finite* asymptotic limit

$$\mathcal{E}_- \underset{q\to\infty}{\to} \mathcal{E}_-^\infty = y + p - \left[\sqrt{W_{A-1}^0{}^2 + p^2} - \sqrt{W_{A-1}^0{}^2 + y^2}\right]. \qquad (19)$$

Henceforth, instead of the sets $\{q, \omega, E_N, \theta_N\}$ or $\{Q^2, Q \cdot P_A, P_N \cdot P_A, Q \cdot P_N\}$ we use the set $\{q, y, \mathcal{E}, p\}$ to characterize the general two-arm coincidence cross section. In particular, the response functions to be introduced later are all functions of these four variables together with $\phi$.

In Fig. 2(a) we show families of curves of $\mathcal{E}_-$ versus $p$ for specific values of $q$ and $y < 0$. The physical regions lie below these curves and above $\mathcal{E} = 0$ for the chosen
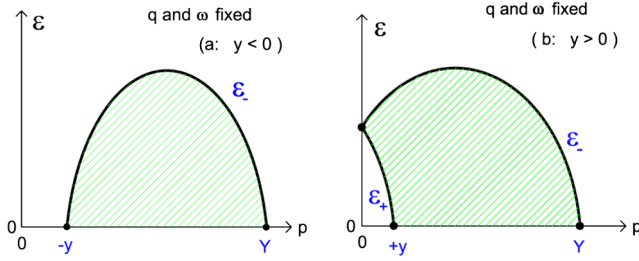
FIG. 2 (color online).   Planes defined by the daughter energy difference $\mathcal{E}$ and the missing momentum $p$, showing the allowed region for semi-inclusive neutrino-nucleus scattering processes. (a) For $y < 0$, i.e., $\omega$ below the quasielastic peak. (b) For $y > 0$, i.e., $\omega$ above the quasielastic peak.

kinematics. Clearly, by imposing these kinematic constraints on the semi-inclusive cross section it is possible to see what features of the dynamics are or are not accessible in the $y < 0$ region. Note that even when $q \to \infty$ only a limited part of the dynamical landscape is accessible. Also note that inclusive scattering corresponds to integrating over the entire accessible region for $q$ and $y$ (or equivalently $\omega$) fixed, and summing over the allowed particle species ($N = p$ and $n$), and correcting for double counting by subtracting the cross section where both a proton and a neutron are detected in coincidence with the charged lepton.

These developments can be extended rather easily to the $\omega$ large region, which becomes equivalent to $y > 0$ and hence to $\omega > \omega_{QE}$. Again the curves of $\mathcal{E}$ versus $p$ when $\cos\theta_{pq} = \pm 1$ define boundaries. The $\cos\theta_{pq} = -1$ curve (namely, $\mathcal{E} = \mathcal{E}_-$ above) is much as before, except that now $p_{\min}$ is negative and so $y \equiv -p_{\min}$ is positive. Reflecting $p \to -p$ to obtain the $\cos\theta_{pq} = +1$ curve from the $\cos\theta_{pq} = -1$ curve as before now yields a nontrivial result: the physically allowable region must lie below the $\cos\theta_{pq} = -1$ curve and above the $\cos\theta_{pq} = +1$ curve, and since the latter lies in the quadrant where $\mathcal{E} \geq 0$ and $p \geq 0$, this provides a new boundary, namely, $\mathcal{E}_-$ obtained from Eq. (18) by changing $p$ to $-p$. In Fig. 2(b) results similar to those in Fig. 2(a) are shown, except now for $y \geq 0$. The physically accessible region in each case lies *above* the lines extending from $p = y$ to the $\mathcal{E}$ axis and *below* the curves extending from the $\mathcal{E}$ axis to peak at some value of $p$ and fall again, eventually intersecting the $\mathcal{E} = 0$ line at $p_{\max} = Y$. Again we see that only specific parts of the semi-inclusive cross section are accessible for these kinematics.

The merit of transforming to the $(\mathcal{E}, p)$ variables is that these are best suited to characterizing the nuclear dynamics. The most important contributions of the semi-inclusive cross section, as studied to some extent via reactions, lie at relatively small values of $\mathcal{E}$, where one typically finds distributions as functions of $p$ that reflect the shell structure of the specific nucleus being studied. For instance, in a simple shell model of the nucleus one sees features that reflect the knockout of nucleons from the valence shell, the next-to-valence shell, etc. These fall relatively rapidly with

increasing $p$. Unfortunately, however, such simple models are not adequate and one also requires overall suppression of these "momentum distributions" by factors of typically 30% via the so-called spectroscopic factors. Also from past studies one knows that some of this "missing strength" is moved to higher values of $\mathcal{E}$, partially through standard nuclear interactions which make both initial and final nuclear states complicated. Said another way, the states involved are undoubtedly not simple single Slater determinants. Also, the NN interaction has both long- and short-range contributions, and especially the latter can promote strength to higher $\mathcal{E}$ and $p$. Something like 20%–30% of the strength is known to reside in this part of the landscape, although the actual amounts are not very well determined. In between the two regions one has other likely issues to deal with, namely, the fact that there are several open channels to be considered and these can conspire via channel coupling to produce the true final many-body state. An example is when a nucleon is ejected from a deep-lying shell model state: for typical kinematics it is also possible to have two or more nucleons ejected, and these channels can couple, yielding a very complex situation. Such issues are very hard to treat, especially in a relativistic context as is required for typical studies of neutrino reactions.

## III. GENERAL ELECTROWEAK TENSORS

The cross section takes on its characteristic form involving the contraction of two second-rank Lorentz tensors, $d\sigma \sim \eta_{\mu\nu} W^{\mu\nu}$, corresponding to the leptonic and the hadronic contributions which are thus factorized and dealt with independently. The leptonic tensor is defined as

$$\eta_{\mu\nu} \equiv 2mm' \overline{\sum_{if}} j_\mu^* j_\nu, \qquad (20)$$

where a factor $4mm'$ (merged here with an additional factor $1/2$) has been included to compensate spinor norms later on, the lepton masses being kept finite until the end of our developments. Its hadronic counterpart is

$$W^{\mu\nu} \equiv \overline{\sum_{if}} J_{fi}^{\mu*}(\mathbf{q}) J_{fi}^\nu(\mathbf{q}), \qquad (21)$$

where the operations $\overline{\sum}_{if}$ in the two cases correspond to sums and averages over the appropriate sets of leptonic quantum numbers (the helicities, in fact) or hadron quantum numbers (helicities or spins, etc.) and integration over all unobserved particles in the final state of the $A - 1$ system for hadrons. It proves useful to decompose both leptonic and hadronic tensors into pieces which are symmetric ($s$) or antisymmetric ($a$) under index interchange $\mu \leftrightarrow \nu$, since in contracting them no symmetric-antisymmetric cross terms are allowed. Both tensors can thus be decomposed as $\eta_{\mu\nu} = \eta_{\mu\nu}^s + \eta_{\mu\nu}^a$ and $W^{\mu\nu} = W_s^{\mu\nu} + W_a^{\mu\nu}$, where the terms are defined as

$$\eta_{\mu\nu}^s = \frac{1}{2}(\eta_{\mu\nu} + \eta_{\nu\mu}); \qquad \eta_{\mu\nu}^a = \frac{1}{2}(\eta_{\mu\nu} - \eta_{\nu\mu});$$

$$W_s^{\mu\nu} = \frac{1}{2}(W^{\mu\nu} + W^{\nu\mu}); \qquad W_a^{\mu\nu} = \frac{1}{2}(W^{\mu\nu} - W^{\nu\mu}). \quad (22)$$

Clearly one has that $\eta_{\mu\mu}^s = \eta_{\mu\mu}$ and $W_s^{\mu\mu} = W^{\mu\mu}$, whereas $\eta_{\mu\mu}^a = W_a^{\mu\mu} = 0$ (no summation over $\mu$ implied in these expressions). In addition, since each tensor is proportional to the bilinear combinations of the electroweak currents in the forms $\eta_{\mu\nu} \sim j_\mu^* j_\nu$ and $W^{\mu\nu} \sim J^{\mu*} J^\nu$, one has that $\eta_{\mu\nu}^* = \eta_{\nu\mu}$ and $W^{\mu\nu*} = W^{\nu\mu}$, and thus that

$$\eta_{\mu\nu}^s = \mathrm{Re}\eta_{\mu\nu}; \qquad \eta_{\mu\nu}^a = i\mathrm{Im}\eta_{\mu\nu};$$

$$W_s^{\mu\nu} = \mathrm{Re}W^{\mu\nu}; \qquad W_a^{\mu\nu} = i\mathrm{Im}W^{\mu\nu}. \quad (23)$$

Let us begin by defining the following (real) symmetric (no prime) and antisymmetric (prime) hadronic response functions:

$$W^{CC} \equiv \mathrm{Re}W^{00} = W^{00}, \quad (24)$$

$$W^{CL} \equiv 2\mathrm{Re}W^{03} = 2W_s^{03}, \quad (25)$$

$$W^{LL} \equiv \mathrm{Re}W^{33} = W^{33}, \quad (26)$$

$$W^{T} \equiv \mathrm{Re}W^{22} + \mathrm{Re}W^{11} = W^{22} + W^{11}, \quad (27)$$

$$W^{TT} \equiv \mathrm{Re}W^{22} - \mathrm{Re}W^{11} = W^{22} - W^{11}, \quad (28)$$

$$W^{TC} \equiv 2\sqrt{2}\mathrm{Re}W^{01} = 2\sqrt{2}W_s^{01}, \quad (29)$$

$$W^{TL} \equiv 2\sqrt{2}\mathrm{Re}W^{31} = 2\sqrt{2}W_s^{31}, \quad (30)$$

$$W^{\underline{T}T} \equiv 2\mathrm{Re}W^{12} = 2W_s^{12}, \quad (31)$$

$$W^{\underline{T}C} \equiv 2\sqrt{2}\mathrm{Re}W^{02} = 2\sqrt{2}W_s^{02}, \quad (32)$$

$$W^{\underline{T}L} \equiv 2\sqrt{2}\mathrm{Re}W^{32} = 2\sqrt{2}W_s^{32}, \quad (33)$$

$$W^{T'} \equiv -2\mathrm{Im}W^{12} = 2iW_a^{12}, \quad (34)$$

$$W^{TC'} \equiv -2\sqrt{2}\mathrm{Im}W^{02} = 2\sqrt{2}iW_a^{02}, \quad (35)$$

$$W^{TL'} \equiv -2\sqrt{2}\mathrm{Im}W^{32} = 2\sqrt{2}iW_a^{32}, \quad (36)$$

$$W^{\underline{C}L'} \equiv \mathrm{Im}W^{03} = iW_a^{03}, \quad (37)$$

$$W^{\underline{T}C'} \equiv 2\sqrt{2}\mathrm{Im}W^{01} = -2\sqrt{2}iW_a^{01}, \quad (38)$$

$$W^{\underline{T}L'} \equiv 2\sqrt{2}\mathrm{Im}W^{31} = -2\sqrt{2}iW_a^{31}. \quad (39)$$

Here $C$ refers to charge (the $\mu = 0$) projection, $L$ refers to longitudinal (momentum transfer direction, $\mu = 3$) projection and $T$ refers to transverse ($\mu = 1, 2$) projections. Concerning the latter, the meaning of the combinations used above can be elucidated by introducing the spherical components of the transverse projections of the hadronic current, defined as

$$J^{(+1)} = -\frac{1}{\sqrt{2}}(J^1 + iJ^2); \quad J^{(-1)} = \frac{1}{\sqrt{2}}(J^1 - iJ^2); \quad J^{(0)} = J^3,$$
$$(40)$$

or inversely,

$$J^1 = -\frac{1}{\sqrt{2}}(J^{(+1)} - J^{(-1)}); \qquad J^2 = \frac{i}{\sqrt{2}}(J^{(+1)} + J^{(-1)});$$
$$J^3 = J^{(0)}. \quad (41)$$

With these definitions, and using the notation $W^{(mm')}$ for the spherical vector components ($m, m' = \{+1, -1, 0\}$) of the hadronic tensor, one can rewrite the responses that contain transverse projections as

$$W^T \equiv W^{(+1+1)} + W^{(-1-1)}, \quad (42)$$

$$W^{TT} \equiv 2\mathrm{Re}W^{(+1-1)}, \quad (43)$$

$$W^{TL} \equiv -2\mathrm{Re}(W^{(0+1)} - W^{(0-1)}), \quad (44)$$

$$W^{\underline{T}T} \equiv 2\mathrm{Im}W^{(+1-1)}, \quad (45)$$

$$W^{\underline{T}L} \equiv -2\mathrm{Im}(W^{(0+1)} + W^{(0-1)}), \quad (46)$$

$$W^{T'} \equiv W^{(+1+1)} - W^{(-1-1)}, \quad (47)$$

$$W^{TL'} \equiv -2\mathrm{Re}(W^{(0+1)} + W^{(0-1)}), \quad (48)$$

$$W^{\underline{T}L'} \equiv -2\mathrm{Im}(W^{(0+1)} - W^{(0-1)}). \quad (49)$$

It is thus clear that the $T$ response, being an incoherent sum of circularly (or linearly) polarized responses, is the unpolarized transverse response, whereas the $TT$ response contains the information needed to specify the linear polarization information [more clearly seen in Eq. (28)]. The $T'$ response, on the other hand, gives the additional information needed, together with the $T$ response to specify the circular polarization.

Equivalently to the hadronic case, the corresponding symmetric (no prime) and antisymmetric (prime) leptonic quantities may be defined as

$$v_0 \hat{V}_{CC} \equiv \mathrm{Re}\eta_{00} = \eta_{00}, \quad (50)$$

$$v_0 \hat{V}_{CL} \equiv \mathrm{Re}\eta_{03} = \eta_{03}^s, \quad (51)$$

$$v_0 \hat{V}_{LL} \equiv \mathrm{Re}\eta_{33} = \eta_{33}, \quad (52)$$

$$v_0 \hat{V}_T \equiv \frac{1}{2}(\mathrm{Re}\eta_{22} + \mathrm{Re}\eta_{11}) = \frac{1}{2}(\eta_{22} + \eta_{11}), \quad (53)$$

$$v_0 \hat{V}_{TT} \equiv \frac{1}{2}(\mathrm{Re}\eta_{22} - \mathrm{Re}\eta_{11}) = \frac{1}{2}(\eta_{22} - \eta_{11}), \qquad (54)$$

$$v_0 \hat{V}_{TC} \equiv \frac{1}{\sqrt{2}}\mathrm{Re}\eta_{01} = \frac{1}{\sqrt{2}}\eta_{01}^s, \qquad (55)$$

$$v_0 \hat{V}_{TL} \equiv \frac{1}{\sqrt{2}}\mathrm{Re}\eta_{31} = \frac{1}{\sqrt{2}}\eta_{31}^s, \qquad (56)$$

$$v_0 \hat{V}_{\underline{T}T} \equiv \mathrm{Re}\eta_{12} = \eta_{12}^s, \qquad (57)$$

$$v_0 \hat{V}_{\underline{T}C} \equiv \frac{1}{\sqrt{2}}\mathrm{Re}\eta_{02} = \frac{1}{\sqrt{2}}\eta_{02}^s, \qquad (58)$$

$$v_0 \hat{V}_{\underline{T}L} \equiv \frac{1}{\sqrt{2}}\mathrm{Re}\eta_{32} = \frac{1}{\sqrt{2}}\eta_{32}^s, \qquad (59)$$

$$v_0 \hat{V}_{T'} \equiv \mathrm{Im}\eta_{12} = -i\eta_{12}^a, \qquad (60)$$

$$v_0 \hat{V}_{TC'} \equiv \frac{1}{\sqrt{2}}\mathrm{Im}\eta_{02} = -\frac{1}{\sqrt{2}}i\eta_{02}^a, \qquad (61)$$

$$v_0 \hat{V}_{TL'} \equiv \frac{1}{\sqrt{2}}\mathrm{Im}\eta_{32} = -\frac{1}{\sqrt{2}}i\eta_{32}^a, \qquad (62)$$

$$v_0 \hat{V}_{CL'} \equiv -\mathrm{Im}\eta_{03} = i\eta_{03}^a, \qquad (63)$$

$$v_0 \hat{V}_{\underline{T}C'} \equiv -\frac{1}{\sqrt{2}}\mathrm{Im}\eta_{01} = \frac{1}{\sqrt{2}}i\eta_{01}^a, \qquad (64)$$

$$v_0 \hat{V}_{\underline{T}L'} \equiv -\frac{1}{\sqrt{2}}\mathrm{Im}\eta_{31} = \frac{1}{\sqrt{2}}i\eta_{31}^a, \qquad (65)$$

where the overall factor $v_0$ is defined as

$$v_0 \equiv (\varepsilon + \varepsilon')^2 - q^2. \qquad (66)$$

The results found here are completely general; they are simply a convenient rewriting of the original components of the leptonic and hadronic tensors where the projections along the momentum transfer direction $(L)$ and transverse to it provide the organizing principle.

### A. Leptonic tensor

From the definition in Eq. (20) and employing the conventions of [5], we form the general leptonic tensor involving neutrinos and negatively charged leptons—later it is straightforward to extend the results to include antineutrinos and positively charged leptons:

$$\eta_{\mu\nu}(K', K) = mm' \sum_{s,s'} \bar{u}(K, s)(a_V\gamma_\mu + a_A\gamma_\mu\gamma_5)u(K', s')$$
$$\times \bar{u}(K', s')(a_V\gamma_\nu + a_A\gamma_\nu\gamma_5)u(K, s), \qquad (67)$$

which includes the sum over final spin states and the average over initial spin states, the latter implying a factor $1/2$. In the standard model the charged-current vector and axial coupling constants take the values $a_V = 1$ and $a_A = -1$, which yields the usual form of the vertex $\gamma_\mu(1 - \gamma_5)$. Upon eliminating the spinors using traces one finds

$$\eta_{\mu\nu}(K', K) \equiv \frac{1}{4}\{\mathrm{Tr}[a_V\gamma_\mu + a_A\gamma_\mu\gamma_5](K' + m')[a_V\gamma_\nu + a_A\gamma_\nu\gamma_5](K + m)\}$$
$$= \frac{1}{4}\{a_V^2 \mathrm{Tr}[\gamma_\mu(K' + m')\gamma_\nu(K + m)]_{(1)} + a_A^2 \mathrm{Tr}[\gamma_\mu\gamma_5(K' + m')\gamma_\nu\gamma_5(K + m)]_{(2)}$$
$$+ a_V a_A(\mathrm{Tr}[\gamma_\mu(K' + m')\gamma_\nu\gamma_5(K + m)]_{(3)} + \mathrm{Tr}[\gamma_\mu\gamma_5(K' + m')\gamma_\nu(K + m)]_{(4)})\}. \qquad (68)$$

The traces can then be expressed as

$$\frac{1}{4}\mathrm{Tr}[]_{(1)} = K_\mu K'_\nu + K'_\mu K_\nu - g_{\mu\nu}(K \cdot K' - mm'), \qquad (69)$$

$$\frac{1}{4}\mathrm{Tr}[]_{(2)} = K_\mu K'_\nu + K'_\mu K_\nu - g_{\mu\nu}(K \cdot K' + mm'), \qquad (70)$$

$$\frac{1}{4}\mathrm{Tr}[]_{(3)} = \frac{1}{4}\mathrm{Tr}[]_{(4)} = -i\varepsilon_{\mu\nu\alpha\beta}K^\alpha K'^\beta. \qquad (71)$$

Cases (1) and (2) are symmetric under the interchange of $\mu$ with $\nu$, while cases (3) and (4) (the VA-interference terms) are antisymmetric. Note that if studying reactions with an incident or outgoing massless leptons ($m = 0$ or $m' = 0$), then cases (1) and (2) yield the same answer. We introduce the following definitions:

$$\nu \equiv \frac{\omega}{q}, \qquad (72)$$

$$\rho \equiv \frac{|Q^2|}{q^2} = 1 - \nu^2; \qquad \rho' \equiv \frac{q}{\varepsilon + \varepsilon'}, \qquad (73)$$

$$\delta \equiv \frac{m}{\sqrt{|Q^2|}}; \qquad \delta' \equiv \frac{m'}{\sqrt{|Q^2|}}, \qquad (74)$$

$$\tan^2\tilde{\theta}/2 = \frac{|Q^2|}{v_0} = \frac{\rho\rho'^2}{1-\rho'^2}. \tag{75}$$

In terms of the angle $\tilde{\theta}$ the quantities $Q^2$ and $v_0$ [the latter defined in Eq. (66)] can be written as

$$Q^2 = -4\varepsilon\varepsilon'\sin^2\tilde{\theta}/2, \tag{76}$$

$$v_0 = 4\varepsilon\varepsilon'\cos^2\tilde{\theta}/2. \tag{77}$$

Using the previous definitions the components of the leptonic tensor as defined in Eqs. (50)–(65) give rise to the following expressions:

$$\hat{V}_{CC} = \frac{1}{2}\{(a_V^2 + a_A^2) - [a_V^2(\delta-\delta')^2 + a_A^2(\delta+\delta')^2]\tan^2\tilde{\theta}/2\}, \tag{78}$$

$$\hat{V}_{CL} = -\frac{1}{2}(a_V^2 + a_A^2)\left[\nu - \frac{1}{\rho'}(\delta^2 - \delta'^2)\tan^2\tilde{\theta}/2\right], \tag{79}$$

$$\hat{V}_{LL} = \frac{1}{2}\left\{(a_V^2 + a_A^2)\left[\nu^2 - \frac{1}{\rho'}(2\nu - \rho\rho'(\delta^2-\delta'^2))(\delta^2-\delta'^2)\tan^2\tilde{\theta}/2\right] + [a_V^2(\delta-\delta')^2 + a_A^2(\delta+\delta')^2]\tan^2\tilde{\theta}/2\right\}, \tag{80}$$

$$\hat{V}_T = \frac{1}{2}(a_V^2 + a_A^2)\left\{\left[\frac{1}{2}\rho + \tan^2\tilde{\theta}/2\right] + \left(\frac{\nu}{\rho'}(\delta^2-\delta'^2) - \frac{1}{2}\rho(\delta^2-\delta'^2)^2\right)\tan^2\tilde{\theta}/2\right\} - (a_V^2 - a_A^2)\delta\delta'\tan^2\tilde{\theta}/2, \tag{81}$$

$$\hat{V}_{TT} = \frac{1}{2}(a_V^2 + a_A^2)\left\{-\frac{1}{2}\rho + \left[(\delta^2+\delta'^2) - \frac{\nu}{\rho'}(\delta^2-\delta'^2) + \frac{1}{2}\rho(\delta^2-\delta'^2)^2\right]\tan^2\tilde{\theta}/2\right\}, \tag{82}$$

$$\hat{V}_{TC} = -\frac{1}{2}(a_V^2 + a_A^2)\frac{1}{\rho'}\tan\tilde{\theta}/2 \times \left(\frac{1}{2} - \frac{1}{\rho}[(\delta^2+\delta'^2) - \frac{\nu}{\rho'}(\delta^2-\delta'^2) + \frac{1}{2}\rho(\delta^2-\delta'^2)^2]\tan^2\tilde{\theta}/2\right)^{1/2}, \tag{83}$$

$$\hat{V}_{TL} = -(\nu - \rho\rho'(\delta^2-\delta'^2))\hat{V}_{TC}, \tag{84}$$

$$\hat{V}_{\underline{TT}} = 0, \tag{85}$$

$$\hat{V}_{\underline{TC}} = 0, \tag{86}$$

$$\hat{V}_{\underline{TL}} = 0, \tag{87}$$

$$\hat{V}_{T'} = a_V a_A \frac{1}{\rho'}(1 + \nu\rho'(\delta^2-\delta'^2))\tan^2\tilde{\theta}/2, \tag{88}$$

$$\hat{V}_{TC'} = -a_V a_A \tan\tilde{\theta}/2$$
$$\times \left[\frac{1}{2} - \frac{1}{\rho}\left[(\delta^2+\delta'^2) - \frac{\nu}{\rho'}(\delta^2-\delta'^2)\right.\right.$$
$$\left.\left. + \frac{1}{2}\rho(\delta^2-\delta'^2)^2\right]\tan^2\tilde{\theta}/2\right]^{1/2}, \tag{89}$$

$$\hat{V}_{TL'} = -\nu\hat{V}_{TC'}, \tag{90}$$

$$\hat{V}_{\underline{CL'}} = 0, \tag{91}$$

$$\hat{V}_{\underline{TC'}} = 0, \tag{92}$$

$$\hat{V}_{\underline{TL'}} = 0. \tag{93}$$

Within these 16 factors, 10 of them are symmetric and 6 are antisymmetric. Under the conditions in this work, 6 of them vanish, namely, the ones with an underlined subscript (see [4] for processes where they do not); the rest reduce to the following expressions in the extreme relativistic limit (ERL), defined as $\hat{V}_K \xrightarrow{ERL} \frac{1}{2}(a_V^2 + a_A^2)v_K$ for the symmetric ones (no prime) and as $\hat{V}_{K'} \xrightarrow{ERL} a_V a_A v_{K'}$ for the antisymmetric ones (prime):

$$v_{CC} = 1, \tag{94}$$

$$v_{CL} = -\nu, \tag{95}$$

$$v_{LL} = \nu^2, \tag{96}$$

$$v_T = \frac{1}{2}\rho + \tan^2\theta/2, \tag{97}$$

$$v_{TT} = -\frac{1}{2}\rho, \tag{98}$$

$$v_{TC} = -\frac{1}{\sqrt{2}\rho'}\tan\theta/2, \tag{99}$$

$$v_{TL} = -\nu v_{TC}, \tag{100}$$

$$v_{T'} = \tan\theta/2\sqrt{\rho + \tan^2\theta/2}, \tag{101}$$

$$v_{TC'} = -\frac{1}{\sqrt{2}}\tan\theta/2, \tag{102}$$

$$v_{TL'} = -\nu v_{TC'}. \tag{103}$$

It is worth noticing that the following combination is useful when discussing conserved vector current (CVC) terms:

$$\hat{V}_L \equiv \hat{V}_{CC} + 2\nu\hat{V}_{CL} + \nu^2\hat{V}_{LL}$$
$$= \frac{1}{2}(a_V^2 + a_A^2)\left\{\rho^2 + \nu\rho\left[\frac{2}{\rho'} + \nu(\delta^2 - \delta'^2)\right](\delta^2 - \delta'^2)\tan^2\tilde{\theta}/2\right\} - \frac{1}{2}[a_V^2(\delta - \delta')^2 + a_A^2(\delta + \delta')^2]\rho\tan^2\tilde{\theta}/2, \tag{104}$$

whose corresponding ERL factor is $v_L = \rho^2$. Also, the $T$ and $TT$ terms are simply related:

$$\hat{V}_T + \hat{V}_{TT} = \frac{1}{2}\{(a_V^2 + a_A^2) + [a_V^2(\delta - \delta')^2$$
$$+ a_A^2(\delta + \delta')^2]\}\tan^2\tilde{\theta}/2. \tag{105}$$

Finally, one can easily complete the leptonic developments by going to the start and replacing the $u$ spinor by $v$ spinors so that the leptonic tensor for antiparticles can be obtained. The final result is that upon contracting the leptonic and hadronic tensors (see Sec. IV) the VV and AA terms are as above, while the VA interference changes sign.

### B. Hadronic tensor

Among the various components of the hadronic tensor defined above, only some of them occur, which can be deduced from the general developments of the hadronic tensor as it is constructed from the available four-momenta. The reaction of interest here is semi-inclusive scattering where, as we have seen in Sec. II, at the hadronic vertex one has incoming momentum transfer $Q^\mu$ and the nuclear target momentum $P_A^\mu$. In the final state one has the momentum of the detected nucleon $P_N^\mu$ together with the residual nucleus' momentum which can be eliminated using four-momentum conservation: $P_{A-1}^\mu = Q^\mu + P_A^\mu - P_N^\mu$. Six invariants can be constructed:

$$I_1 \equiv Q^2; \quad I_2 \equiv Q \cdot P_A; \quad I_3 \equiv Q \cdot P_N;$$
$$I_4 \equiv P_A \cdot P_N; \quad I_5 \equiv P_A^2 = M_A^{0\,2}; \quad I_6 \equiv P_N^2 = m_N^2, \tag{106}$$

of which the first four are dynamical variables, whereas the last two are fixed by the target nucleus and nucleon masses. Accordingly, all invariant structure functions depend on the four dynamical invariants $I_i$, $i = 1, ..., 4$. They can be expressed as

$$I_1 = \omega^2 - q^2 < 0, \tag{107}$$

$$I_2 = M_A^0\omega, \tag{108}$$

$$I_3 = \omega E_N - q p_N \cos\theta_N, \tag{109}$$

$$I_4 = M_A^0 E_N. \tag{110}$$

Next one can write symmetric and antisymmetric hadronic tensors as functions of the three independent four-momenta $Q^\mu$, $P_A^\mu$ and $P_N^\mu$. In fact, it proves to be more convenient to introduce projected four-momenta to replace the last two, namely,

$$U^\mu \equiv \frac{1}{M_A^0}\left[P_A^\mu - \left(\frac{Q \cdot P_A}{Q^2}\right)Q^\mu\right], \tag{111}$$

$$V^\mu \equiv \frac{1}{M_N}\left[P_N^\mu - \left(\frac{Q \cdot P_N}{Q^2}\right)Q^\mu\right], \tag{112}$$

where then $Q \cdot U = Q \cdot V = 0$. Also, to keep the dimensions consistent in the developments below, let us introduce a dimensionless four-momentum transfer

$$\tilde{Q}^\mu \equiv \frac{Q^\mu}{\sqrt{|Q^2|}}. \tag{113}$$

The symmetric hadronic tensor may then be written

$$W_s^{\mu\nu} = X_1 g^{\mu\nu} + X_2 \tilde{Q}^\mu\tilde{Q}^\nu + X_3 U^\mu U^\nu + X_4(\tilde{Q}^\mu U^\nu + U^\mu\tilde{Q}^\nu)$$
$$+ X_5 V^\mu V^\nu + X_6(\tilde{Q}^\mu V^\nu + V^\mu\tilde{Q}^\nu) + X_7(U^\mu V^\nu + V^\mu U^\nu), \tag{114}$$

where $X_i$, $i = 1, ..., 7$ are invariant functions of the invariants discussed above. These seven types of terms arise from VV and AA contributions. Likewise, the antisymmetric tensor can be constructed from the basic four-momenta

$$W_a^{\mu\nu} = i\{Y_1(\tilde{Q}^\mu U^\nu - U^\mu \tilde{Q}^\nu) + Y_2(\tilde{Q}^\mu V^\nu - V^\mu \tilde{Q}^\nu)$$
$$+ Y_3(U^\mu V^\nu - V^\mu U^\nu) + Z_1 \varepsilon^{\mu\nu\alpha\beta}\tilde{Q}_\alpha U_\beta + Z_2 \varepsilon^{\mu\nu\alpha\beta}\tilde{Q}_\alpha V_\beta$$
$$+ Z_3 \varepsilon^{\mu\nu\alpha\beta} U_\alpha V_\beta\}, \tag{115}$$

where again $Y_i$ and $Z_i$, $i = 1, \ldots, 3$ are invariant functions of the invariants above. The terms having no $\varepsilon^{\mu\nu\alpha\beta}$, namely, the $Y_i$ terms (as well as the $X_i$ terms, as said above), arise from VV and AA contributions, whereas those with $\varepsilon^{\mu\nu\alpha\beta}$, namely, the $Z_i$ terms, come from VA interferences. Note that for inclusive scattering where one does not have $V^\mu$ as a building block, only terms of the $X_1, X_2, X_3, X_4, Y_1$ and $Z_1$ type can occur.

For a conserved vector current (CVC) situation such as seen here for the VV terms or for purely polar-vector electron scattering, the continuity equation in momentum space requires that

$$Q_\mu (W_s^{\mu\nu})_{VV} = Q_\mu (W_a^{\mu\nu})_{VV} = 0. \tag{116}$$

For the symmetric tensor this contraction removes the terms with $X_3, X_5, Y_3, Z_1$, leaving the conditions

$$(-X_1^{VV} + X_2^{VV})\tilde{Q}^\nu + X_4^{VV} U^\nu + X_6^{VV} V^\nu = 0, \tag{117}$$

$$Y_1^{VV} U^\nu + Y_2^{VV} V^\nu = 0, \tag{118}$$

where no terms with $\varepsilon^{\mu\nu\alpha\beta}$ can occur in a VV situation, i.e., $Z_1^{VV} = Z_2^{VV} = Z_3^{VV} = 0$, as noted above. Since the basic four-momenta are linearly independent of each other, the coefficients above must all be independently zero, namely, $X_1^{VV} - X_2^{VV} = X_4^{VV} = X_6^{VV} = Y_1^{VV} = Y_2^{VV} = 0$. Accordingly, one has

$$(W_s^{\mu\nu})_{VV} = X_1^{VV}\left[g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2}\right] + X_3^{VV} U^\mu U^\nu$$
$$+ X_5^{VV} V^\mu V^\nu + X_7^{VV}(U^\mu V^\nu + V^\mu U^\nu), \tag{119}$$

$$(W_a^{\mu\nu})_{VV} = Y_3^{VV}(U^\mu V^\nu - V^\mu U^\nu). \tag{120}$$

For instance, in semi-inclusive electron scattering the symmetric terms lead to the standard $L$, $T$, $TL$ and $TT$ responses, while the antisymmetric term which becomes accessible with polarized electron scattering yields the $TL'$ response, the so-called 5th response [3,4]. For the other cases, the AA and VA responses, there is no further simplification in general. The resulting number of contributions of each type is summarized in Table I for semi-inclusive and for inclusive scattering, the latter arising from integrating the semi-inclusive contributions. For the semi-inclusive case of interest here, they form the functions $X_i, Y_i, Z_i$ as follows:

TABLE I. Number of electroweak responses in semi-inclusive and inclusive processes, classified according to their properties under spatial inversion (VV, AA, and VA) and index interchange (symmetric and antisymmetric).

| Type | Semi-inclusive | | Inclusive | |
| --- | --- | --- | --- | --- |
| | Sym | A-sym | Sym | A-sym |
| VV | 4 | 1 | 2 | 0 |
| AA | 7 | 3 | 4 | 1 |
| VA | 0 | 3 | 0 | 1 |

$$X_1 = X_1^{VV} + X_1^{AA} \qquad Y_1 = Y_1^{AA}$$
$$X_2 = X_1^{VV} + X_2^{AA} \qquad Y_2 = Y_2^{AA}$$
$$X_3 = X_3^{VV} + X_3^{AA} \qquad Y_3 = Y_3^{VV} + Y_3^{AA}$$
$$X_4 = X_4^{AA} \qquad\qquad Z_1 = Z_1^{VA}$$
$$X_5 = X_5^{VV} + X_5^{AA} \qquad Z_2 = Z_2^{VA}$$
$$X_6 = X_6^{AA} \qquad\qquad Z_3 = Z_3^{VA}$$
$$X_7 = X_7^{VV} + X_7^{AA}. \tag{121}$$

Upon using the kinematic variables in the laboratory system discussed in Sec. II, in particular, Eqs. (72), 73), together with the following definitions:

$$\eta_T \equiv \frac{p_N}{m_N} \sin\theta_N, \tag{122}$$

$$H \equiv \frac{1}{m_N}[E_N - \nu p_N \cos\theta_N], \tag{123}$$

the hadronic response functions defined in Sec. III can be written as

$$W_s^{CC} = \frac{1}{\rho^2}\{\rho^2 X_1 + \rho\nu^2 X_2 + X_3 + 2\sqrt{\rho}\nu X_4 + H^2 X_5$$
$$+ 2\sqrt{\rho}\nu H X_6 + 2H X_7\}, \tag{124}$$

$$W_s^{CL} = \frac{2\nu}{\rho^2}\left\{\rho X_2 + X_3 + \sqrt{\rho}\left(\frac{1}{\nu} + \nu\right)X_4\right.$$
$$\left. + H^2 X_5 + \sqrt{\rho}\left(\frac{1}{\nu} + \nu\right)H X_6 + 2H X_7\right\}, \tag{125}$$

$$W_s^{LL} = \frac{1}{\rho^2}\{-\rho^2 X_1 + \rho X_2 + \nu^2 X_3 + 2\sqrt{\rho}\nu X_4$$
$$+ \nu^2 H^2 X_5 + 2\sqrt{\rho}\nu H X_6 + 2\nu^2 H X_7\}, \tag{126}$$

$$W_s^T = -2X_1 + X_5 \eta_T^2, \tag{127}$$

$$W_s^{TT} = -X_5 \eta_T^2 \cos 2\phi, \tag{128}$$

$$W_s^{TC} = \frac{2\sqrt{2}}{\rho}\eta_T\{HX_5 + \sqrt{\rho}\nu X_6 + X_7\}\cos\phi, \quad (129)$$

$$W_s^{TL} = \frac{2\sqrt{2}}{\rho}\eta_T\{\nu HX_5 + \sqrt{\rho}X_6 + \nu X_7\}\cos\phi, \quad (130)$$

$$W_s^{\underline{TT}} = X_5\eta_T^2\sin 2\phi, \quad (131)$$

$$W_s^{\underline{TC}} = \frac{2\sqrt{2}}{\rho}\eta_T\{HX_5 + \sqrt{\rho}\nu X_6 + X_7\}\sin\phi, \quad (132)$$

$$W_s^{\underline{TL}} = \frac{2\sqrt{2}}{\rho}\eta_T\{\nu HX_5 + \sqrt{\rho}X_6 + \nu X_7\}\sin\phi, \quad (133)$$

$$W_a^{T'} = \frac{1}{\sqrt{\rho}}\{Z_1 + HZ_2\}, \quad (134)$$

$$W_a^{TC'} = \frac{2\sqrt{2}}{\rho}\eta_T\{-(\sqrt{\rho}\nu Y_2 + Y_3)\sin\phi \\ + (\sqrt{\rho}Z_2 + \nu Z_3)\cos\phi\}, \quad (135)$$

$$W_a^{TL'} = \frac{2\sqrt{2}}{\rho}\eta_T\{-(\sqrt{\rho}Y_2 + \nu Y_3)\sin\phi + (\sqrt{\rho}\nu Z_2 + Z_3)\cos\phi\}, \quad (136)$$

$$W_a^{CL'} = -\frac{1}{\sqrt{\rho}}\{Y_1 + HY_2\}, \quad (137)$$

$$W_a^{\underline{TC'}} = \frac{2\sqrt{2}}{\rho}\eta_T\{(\sqrt{\rho}\nu Y_2 + Y_3)\cos\phi + (\sqrt{\rho}Z_2 + \nu Z_3)\sin\phi\}, \quad (138)$$

$$W_a^{\underline{TL'}} = \frac{2\sqrt{2}}{\rho}\eta_T\{(\sqrt{\rho}Y_2 + \nu Y_3)\cos\phi + (\sqrt{\rho}\nu Z_2 + Z_3)\sin\phi\}. \quad (139)$$

Note how the explicit dependence on the azimuthal angle $\phi$ emerges: one has pairs of symmetric contributions, namely, $TT\leftrightarrow\underline{TT}$, $TC\leftrightarrow\underline{TC}$, and $TL\leftrightarrow\underline{TL}$, where a cosine is replaced by a sine, as well as pairs of antisymmetric contributions, namely, $TC'\leftrightarrow\underline{TC}'$ and $TL'\leftrightarrow\underline{TL}'$, where a rotation is involved. Also note that, while these constitute the complete set of semi-inclusive responses, in fact none of the underlined cases enter when combined with the leptonic factors obtained above, since the latter are all zero [see Eqs. (78)–(93)].

## IV. CONTRACTION OF TENSORS AND CROSS SECTION

The contraction of the leptonic and the hadronic tensors arises from the application of standard Feynman rules to the evaluation of the cross section of the process under study here; it is an invariant, taking the same form in the laboratory, in the center of momentum, or in any other system of reference. As mentioned in Sec. III, the symmetric and the antisymmetric components of the leptonic and the hadronic tensors can be contracted separately since no cross terms are allowed:

$$v_0\mathcal{F}_\chi^2 \equiv \eta_{\mu\nu}W^{\mu\nu} = \eta_{\mu\nu}^s W_s^{\mu\nu} + \chi\eta_{\mu\nu}^a W_a^{\mu\nu}, \quad (140)$$

where $\chi = 1$ for incident neutrinos, as obtained in Sec. III A, and $\chi = -1$ for antineutrinos, as can be easily shown with the same formalism but using antiparticle spinors $v$ in Eq. (67). In Cartesian components the symmetric and the antisymmetric contractions above yield

$$\eta_{\mu\nu}^s W_s^{\mu\nu} = \eta_{00}^s W_s^{00} + 2\eta_{03}^s W_s^{03} + \eta_{33}^s W_s^{33} + \eta_{11}^s W_s^{11} + \eta_{22}^s W_s^{22} \\ + 2\eta_{01}^s W_s^{01} + 2\eta_{31}^s W_s^{31} + 2\eta_{02}^s W_s^{02} \\ + 2\eta_{32}^s W_s^{32} + 2\eta_{12}^s W_s^{12}, \quad (141)$$

$$\eta_{\mu\nu}^a W_a^{\mu\nu} = 2\eta_{03}^a W_a^{03} + 2\eta_{01}^a W_a^{01} + 2\eta_{31}^a W_a^{31} \\ + 2\eta_{02}^a W_a^{02} + 2\eta_{32}^a W_a^{32} + 2\eta_{12}^a W_a^{12}, \quad (142)$$

which, according to the developments of Sec. III, can be expressed as

$$\eta_{\mu\nu}^s W_s^{\mu\nu} = \text{Re}\eta_{00}\text{Re}W^{00} + 2\text{Re}\eta_{03}\text{Re}W^{03} + \text{Re}\eta_{33}\text{Re}W^{33} \\ + \text{Re}\eta_{11}\text{Re}W^{11} + \text{Re}\eta_{22}\text{Re}W^{22} + 2\text{Re}\eta_{01}\text{Re}W^{01} \\ + 2\text{Re}\eta_{31}\text{Re}W^{31} + 2\text{Re}\eta_{02}\text{Re}W^{02} + 2\text{Re}\eta_{32}\text{Re}W^{32} \\ + 2\text{Re}\eta_{12}\text{Re}W^{12}, \quad (143)$$

$$-\eta_{\mu\nu}^a W_a^{\mu\nu} = 2\text{Im}\eta_{03}\text{Im}W^{03} + 2\text{Im}\eta_{01}\text{Im}W^{01} + 2\text{Im}\eta_{31}\text{Im}W^{31} \\ + 2\text{Im}\eta_{02}\text{Im}W^{02} + 2\text{Im}\eta_{32}\text{Im}W^{32} + 2\text{Im}\eta_{12}\text{Im}W^{12}. \quad (144)$$

Finally, in terms of projections with respect to the momentum transfer direction, the contractions read

$$\eta_{\mu\nu}^s W_s^{\mu\nu} = v_0\{[\hat{V}_{CC}W^{CC} + \hat{V}_{CL}W^{CL} + \hat{V}_{LL}W^{LL} \\ + \hat{V}_T W^T + \hat{V}_{TT}W^{TT} + \hat{V}_{TC}W^{TC} + \hat{V}_{TL}W^{TL}] \\ + [\hat{V}_{\underline{TT}}W^{\underline{TT}} + \hat{V}_{\underline{TC}}W^{\underline{TC}} + \hat{V}_{\underline{TL}}W^{\underline{TL}}]\}, \quad (145)$$

$$\eta_{\mu\nu}^a W_a^{\mu\nu} = v_0\{[\hat{V}_{T'}W^{T'} + \hat{V}_{TC'}W^{TC'} + \hat{V}_{TL'}W^{TL'}] \\ + [\hat{V}_{\underline{CL'}}W^{\underline{CL'}} + \hat{V}_{\underline{TC'}}W^{\underline{TC'}} + \hat{V}_{\underline{TL'}}W^{\underline{TL'}}]\}, \quad (146)$$

where the hadronic responses contain all the VV, AA, and VA terms applicable to each of them, as shown in Eqs. (121).

In any of the above representations the symmetric contraction involves 10 terms and the antisymmetric one

involves 6 terms, for an expected total of 16 terms. From the tensor contractions above, the matrix element of the process is [see definition of the leptonic tensor in Eq. (67)]

$$|\mathcal{M}_\chi|^2 = \frac{G^2 \cos^2 \theta_c v_0}{2mm'} \mathcal{F}_\chi^2, \qquad (147)$$

where $G = 1.166 \times 10^{-5}$ GeV$^{-2}$ is the coupling constant of the weak interaction, $\cos \theta_c = 0.974$ with $\theta_c$ the Cabibbo

angle accounting for the misalignment between the strong and the weak hadronic eigenstates, $v_0$ was defined in Eq. (66), and, as said above, $\chi = +1$ for neutrino and $\chi = -1$ for antineutrino scattering.

We then evaluate the coincidence cross section of the processes $^AX(\nu_\ell, \ell^-N)^{A-1}Y$ or $^AX(\bar\nu_\ell, \ell^+N)^{A-1}Y$ in the laboratory system [see [4] for the procedures for the analogous case of $(e, e'N)$ reactions]. Using standard Feynman rules we get for the cross section,

$$d\sigma_\chi = \frac{G^2 \cos^2 \theta_c}{2(2\pi)^5} \frac{m_N W_{A-1} v_0}{k\varepsilon' E_N E_{A-1}} \mathcal{F}_\chi^2 d^3\mathbf{k}' d^3\mathbf{p_N} d^3\mathbf{p_{A-1}} \delta^4(K + P_A - K' - P_{A-1} - P_N). \qquad (148)$$

This form is exact in the cases where the $A - 1$ system is in a bound ground state or a long-lived excited state. In other cases this form assumes that the wave function of the $A - 1$ system can be factorized into center-of-mass and relative wave functions, which is not, in general, true for relativistic wave functions. However, since the momenta available to the $A - 1$ system will generally be of the order of the Fermi momentum and the masses of the undetected fragments will tend to be large, the nuclear system will generally be treated nonrelativistically and the factorization of the wave function will then be exact. Upon integration over the unobserved residual daughter nucleus momentum $\mathbf{p_{A-1}}$ and energy $E_{A-1}$, one gets

$$\frac{d\sigma_\chi}{dk' d\Omega_{k'} d\Omega_{p_N}} = \frac{G^2 \cos^2 \theta_c}{2(2\pi)^5} \frac{m_N W_{A-1}}{M_A^0} \frac{p_N k'^2 v_0}{k\varepsilon' F_{\text{rec}}} \mathcal{F}_\chi^2, \quad (149)$$

where $W_{A-1}$ is defined so that $f \equiv 0$, with

$$f = \varepsilon + M_A^0 - \varepsilon' - (p_N^2 + m_N^2)^{1/2}$$
$$- (q^2 + p_N^2 - 2qp_N \cos \theta_N + W_{A-1}^2)^{1/2}. \qquad (150)$$

This equation is a rewriting of the energy conservation condition stated in Eq. (8). From the function $f$ one also obtains the recoil factor $F_{\text{rec}}$ as

$$F_{\text{rec}} = \frac{E_N E_{A-1}}{M_A^0 p_N} \left| \frac{\partial f}{\partial p_N} \right| = \left| 1 + \frac{\omega p_N - q E_N \cos \theta_N}{M_A^0 p_N} \right|. \quad (151)$$

When ERL applies, the cross section in Eq. (149) becomes

$$\frac{d\sigma_{\chi[\text{ERL}]}}{d\varepsilon' d\Omega_{k'} d\Omega_{p_N}} = \frac{G^2 \cos^2 \theta_c}{16\pi^5} \frac{m_N W_{A-1}}{M_A^0} \frac{p_N \varepsilon'^2 \cos^2(\theta/2)}{F_{\text{rec}}} \mathcal{F}_\chi^2. \qquad (152)$$

## V. CONCLUSIONS

In this study we have presented the general formalism for *semi-inclusive* charged-current neutrino-nucleus reactions, i.e., those processes where neutrinos (antineutrinos)

interact with a nuclear target, and in addition to the final-state lepton (antilepton), one assumes that some other particle is also detected in coincidence. Such processes are called semi-inclusive reactions to contrast them to inclusive reactions where only the final-state lepton is detected. The features summarized below highlight the generality of this formalism. We note the following:

(i) The masses of the incoming and outgoing leptons are kept; viz., no extreme relativistic limit has been invoked. Although for typical kinematical situations the impact is limited when considering scattering of active neutrinos with production of electrons or muons, it becomes relevant for tau production, and it can also be easily extended to study massive sterile neutrino interactions with nuclei.

(ii) The scattering of both neutrinos and antineutrinos is considered, differing just in the sign of the antisymmetric tensor contraction contribution to the matrix element of the process.

(iii) No assumptions are made on the hadronic target, on the particle emitted and detected in coincidence, or on the state of the residual, undetected hadronic system after the emission. In particular, the latter can be in an excited bound state or be partially or totally unbound, as long as charge and baryon numbers are conserved.

(iv) The detailed characterization of the semi-inclusive neutrino cross section is organized in a form that makes it easy to understand as a straightforward generalization of the well-known formalism for inclusive [3] and semi-inclusive [4] electron scattering cross sections, as well as for inclusive neutrino reactions [7]. Indeed, the purely vector semi-inclusive neutrino responses are the same as the corresponding isovector electron scattering responses, viz., because of CVC. Two forms are given for the general response structure of the cross sections, one in terms of chargelike, longitudinal and transverse projections of the electroweak current (the Ws of Secs. III and IV) and another in terms of invariant structure functions (the Xs, Ys and Zs of Sec. III).

(v) Using the basic symmetries in the problem (angular momentum, parity and four-momentum conservation) we have isolated the general dependences on the azimuthal angle $\phi$. For instance, even without detailed modeling, one can see how specific interference terms in the response change sign when going from $\phi = 0$ to $\phi = \pi$. One should be clear that such interference contributions are intrinsic to the basic semi-inclusive electroweak reaction and must be modeled. They are not, for instance, present for inclusive reactions, and indeed, the modeling typically used in studies of the latter are often quite inadequate when studying semi-inclusive scattering.

(vi) The general semi-inclusive response is organized into symmetric and antisymmetric contributions, and contributions that are purely vector (VV), purely axial-vector (AA) and VA interferences. For such processes, of the 16 possible response functions, the 6 underlined contributions [see Eqs. (145), (146)] do not enter for CC$\nu$ reactions, leaving 10 distinct contributions to the semi-inclusive cross section. These in turn are built from the 17 invariant structure functions introduced in Eqs. (114) and (115) (note that the term containing $Y_1$ does not contribute for CC$\nu$ reactions). In contrast, there are only 5 distinct contributions to the inclusive cross section.

(vii) Furthermore, the semi-inclusive responses are all functions of 4 kinematic variables, whereas the inclusive ones depend on only 2 kinematic variables. Of course, complete integrations over two of the variables in the former yield either zero for some of the interference responses or yield their inclusive counterparts.

(viii) Ultimately, when specific models are considered and when the neutrino fluxes commonly employed when comparing with experiment are taken into consideration, it will be necessary to integrate over the neutrino energies involved with the fluxes as weighting factors. Note, however, that this does not at all mean that one reverts to the inclusive responses. In fact, those integrations can be cast as line integrals in the $(\mathcal{E}, p)$ plane, which are not simply related to the complete integrations in that plane that would yield the inclusive responses. Indeed, such integrations leave averaged responses that depend on 3 kinematic variables and the interference responses do not integrate to zero.

(ix) Accordingly, the demands being placed on modeling the coincidence reactions are much greater. Where crude models such as the relativistic Fermi gas model may be acceptable for studies of inclusive scattering (to the extent that errors of perhaps 30% are viewed as acceptable), for semi-inclusive studies many of the models being employed are certainly inapplicable, since they are incapable of predicting even roughly the correct $(\mathcal{E}, p)$ dependence of the cross section.

Furthermore, neutral-current neutrino weak interactions can also be described by the formalism in this work upon integration over the outgoing neutrino variables. This inclusive $u$ channel results in nonvanishing responses in general, in contrast to inclusive $t$-channel reactions where integration over the momentum of the ejected particle (of course, consistent with four-momentum conservation) causes the responses dependent on the angle $\phi$ to vanish (see the discussion in [8]).

As stated above, the formalism has been kept entirely general and any type of coincidence reaction can be represented in terms of the response functions introduced in this work. However, to make the formalism clearer, we have focused on the case where the particle detected in coincidence with the final-state muon is a nucleon. In fact, in practical situations this is likely to be a proton so that the semi-inclusive reactions of interest will typically be of the type ${}^A_Z X(\nu_\ell, \ell^- p){}^{A-1}_Z Y$ and ${}^A_Z X(\bar{\nu}_\ell, \ell^+ p){}^{A-1}_{Z-2} Y$. A general differential cross section is given, from which a variety of integrations can be performed; we do so over the residual daughter nucleus variables, assuming that the incoming neutrino energy is known, to produce a differential cross section suitable for Monte Carlo generators. In practical situations, however, the energy of the incoming neutrinos lies within a rather wide range, connecting to a variety of possible dynamic regimes in the nuclear target. This is the reason why we introduce in this work the excitation energy and the momentum of the residual system as hadronic kinematic variables. For given (measured) conditions such as the final lepton and emitted nucleon momenta (both magnitude and direction, or angles), a range of incoming neutrino energies translates into a curve in the $(\mathcal{E}, p)$ plane that reveals which nuclear dynamics are most relevant for the process, for instance multinucleon versus one-nucleon emission. Some care has been taken in providing the interconnections between the "experimental" kinematic variables (energies and momenta of the detected particles) and the "nuclear" kinematic variables, $p$ and $\mathcal{E}$, since the response of the nucleus is a rapidly varying function of the latter.

Our plan for work already in progress is to study specific reactions involving particular nuclei. In doing so, it is clearly essential to understand where the dominant regions in the $(\mathcal{E}, p)$ plane lie to be able to predict the semi-inclusive (and also inclusive) neutrino cross sections with sufficient confidence.

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