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QED spectra in the path integral formalism

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Relativistic Hamiltonians, derived from the path integrals, are known to provide a simple and useful formalism for hadron spectroscopy in QCD. The accuracy of this approach is tested using the QED systems, and the calculated spectrum is shown to reproduce exactly that of the Dirac hydrogen atom, while the Breit-Fermi nonrelativistic expansion is obtained using Foldy-Wouthuizen transformation. The calculated positronium spectrum, including spin-dependent terms, coincides with the standard QED perturbation theory to the considered order $O(\alpha^4)$.

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I. INTRODUCTION

The path integral approach to QCD and QED has been actively developed since the first formulation in [1,2] (see [3] for reviews, references, and discussions). The particular line of development is the so-called Fock-Feynnman-Schwinger representation (FFSR) [4–6], where both relativism and gauge invariance are made explicit. In this latter framework one derives the path-integral relativistic Hamiltonian, which is called the relativistic Hamiltonian (RH), originally exploited in a simple form in [7].

Recently a new integral form of the hadron Green's function and a rigorous derivation of the RH were given in [8], and we use this latter form in what follows.

The RH formalism is one of the most powerful methods in QCD, which allows us to predict spectra and wave functions of hadrons using a minimal input: current quark masses, string tension, and $\Lambda_{\rm QCD}$. Therefore, it is very important to check its validity for different systems and the accuracy of results. In the course of the derivation, some approximations have been given, the significance of which can be made clear by comparison with other relativistic approaches. In the case of the one-particle system in an external field, the basic approach is that of the Dirac equation, and one can compare results of two approaches—the path integral Hamiltonian and Dirac equation in external fields, e.g., for the Coulomb case in QED. In the case of the linear potential in QCD, results can be compared with lattice and experimental data.

It is important that the FFSR is derived for the Green's functions, and the RH appears in the kernel in the exponent and depends on additional integration variables, which play the role of virtual particle energies. Therefore, one encounters the problem of the proper definition of the RH as an operator and its excited states. This topic will also be discussed in comparison with the Dirac formalism, the QED perturbation theory for positronium, and relativistic quark models. As a result, we shall estimate the accuracy of approximations made and shall give the scheme of calculations for the ground and excited states, both in one-particle and two-particle systems. As an additional topic, we compare nonrelativistic expansions for the RH with the known Breit-Fermi expansion.

The plan of the paper is as follows. The short derivation of the RH is done in Sec. II. Section III is devoted to the Breit-Fermi expansion of the RH. In Sec. IV the spectrum of the RH for the hydrogen atom is compared with Dirac and Salpeter equations. The case of positronium and the accuracy of the spectrum of the RH is considered in Sec. V. The last section is devoted to the discussion of results and perspectives.

II. DERIVATIVES OF THE RELATIVISTIC HAMILTONIAN

We start with the FFSR for the fermion propagator in the external gauge field A_μ in QED, as well as in QCD in the Euclidean space-time

$$\begin{aligned} S &= (m+D)^{-1} = (m-\hat{D})(m^2-\hat{D}^2)^{-1} \\ &= (m-\hat{D})\int_0^\infty ds e^{-s(m^2-\hat{D}^2)} \\ &= (m-\hat{D})\int_0^\infty ds (D^4z) e^{-K} W_F, \end{aligned} \tag{1}$$

$$\begin{aligned} (D^4z)_{xy} &= \int \frac{d^4p}{(2\pi)^4}\prod_k \frac{d^4\Delta z(k)}{(4\pi\varepsilon)^2} \\ &\quad \times \exp\left[ip\left(\sum_k \Delta z(k) - (x-y)\right)\right], \\ N\varepsilon &= s, \end{aligned} \tag{2}$$

where the kinematic kernel is

$$K = m^2 s + \frac{1}{4}\int_0^s \left(\frac{dz_\mu}{d\tau}\right)^2 d\tau, \tag{3}$$

and the generalized Wilson line is

$$W_F(x,y) = P_A \exp\left(ig \int_y^x A_\mu dz_\mu + g \int_0^s \sigma_{\mu\nu} F_{\mu\nu} d_\tau \right), \tag{4}$$

where P_A is the ordering operator in the case of the non-Abelian field A_μ. Note that matrices γ_μ enter W_F only in the term

$$\sigma_{\mu\nu} F_{\mu\nu} = \begin{pmatrix} \boldsymbol{\sigma}\mathbf{B} & \boldsymbol{\sigma}\mathbf{E}_E \\ \boldsymbol{\sigma}\mathbf{E}_E & \boldsymbol{\sigma}\mathbf{B} \end{pmatrix}, \tag{5}$$

and here $\mathbf{E}_E$ is the Euclidean electric field, which should be replaced by $\mathbf{E}_M \equiv i\mathbf{E}$ in the Minkowskian case. Hence, all connections of large and small Dirac components are provided by the electric field in W_F and the factor $(m - \hat{D})$ in (1).

As will be seen, the main differences between the RH and the Dirac equation are as follows:

(1) The RH is a quadratic operator, which stems from the quadratic combination $(m^2 - \hat{D}^2)$, while the Dirac operator is linear in momenta and fields A_μ. This difference can be seen in the resulting nonrelativistic expansion of both operators and eigenvalues, and is cured by the Foldy-Wouthuizen transformation taking into account the factor $(m - \hat{D})$ in (1).

(2) The new element in the relativistic path integral, as compared to its nonrelativistic analog, is the time path in the quantum paths. As shown explicitly in [8], the integration $ds(D^4z)_{xy}$ in Eq. (1) can be written, using the relations

$$s = \frac{T}{2\omega}, \qquad T \equiv |x_4 - y_4|, \qquad d\tau = \frac{dt_E}{2\omega},$$

so that

$$\int ds (D^4 z)_{xy} e^{-K} W_F(x,y) = T \int_0^\infty \frac{d\omega}{2\omega^2} (D^3 z)_{\mathbf{xy}} e^{-K(\omega)} \langle \Phi_z(x,y) \rangle_{\Delta z_4}, \tag{6}$$

where

$$K(\omega) = \int_0^T dt_E \left(\frac{\omega}{2} + \frac{m^2}{2\omega} + \frac{\omega}{2} \left(\frac{d\mathbf{z}}{dt_E} \right)^2 \right), \tag{7}$$

and one can split the time element in (4), $dz_4 \to \Delta z_4 = \Delta t_E + \Delta \tilde{z}_4$, so that Δt_E is a monotonic Euclidean time interval, while $\Delta \tilde{z}_4$ is a stochastic one, with $\sum_{k=1}^N \Delta \tilde{z}_4(k) = 0$. Correspondingly, in the integral $(Dz_4)_{x_4 y_4} = \int \frac{dp_4}{2\pi} \prod_k \frac{d\Delta z_4(k)}{\sqrt{4\pi\varepsilon}} \exp[ip_4(\sum_k \Delta z_4(k) - T)]$ of the Wilson line (4), one can write

$$\langle \Phi_z(x,y) \rangle_{\Delta z_4} = \int (Dz_4)_{x_4 y_4} W_F(x,y) = \overline{W_F(x,y)} \sqrt{\frac{\omega}{2\pi T}} \varphi\{A_\mu\}. \tag{8}$$

Here $\varphi\{A_\mu = 0\} = 1$ and also $\varphi = 1$ for A_μ independent of z_4. As it was argued in [8], the difference $(\varphi\{A\} - 1)$ takes into account the creation of additional particles and hence higher Fock components in the total wave function and higher Fock matrix elements in the Hamiltonian. In terms of one- or two-particle Green's functions, these contributions can be considered as radiative corrections, which are absent in the simplest form of Dirac or Bethe-Salpeter equations. In what follows we shall consider only the minimal Fock component and use the condition $\varphi\{A\} \equiv 1$.

Now the monotonic part $\bar{W}_F$ depends only on 3d trajectories $z_\mu = \{\mathbf{z}(t_E), t_E\}$ and is equal to

$$\bar{W}_F(x,y) = \exp\left\{ \int_0^T dt_E \left[igA_4(t_E) + igA_i \frac{dz_i}{dt_E} + g \frac{\sigma_{\mu\nu} F_{\mu\nu}}{2\omega} \right] \right\}. \tag{9}$$

In what follows we shall test the approximation of smooth trajectories with $\varphi\{A_\mu\} \equiv 1$ and compare the corresponding results with exact calculations of the Dirac equation for the Coulomb potential.

As a result, from (6), (7), and (8) one can write the Hamiltonian for a fermion in the electromagnetic field $\{\mathbf{A}(\mathbf{z},t), A_0(\mathbf{z},t)\}$,

$$H(\omega) = \frac{(\mathbf{p} - e\mathbf{A})^2}{2\omega} + \frac{m^2 + \omega^2}{2\omega} + eA_0 - \frac{e(\boldsymbol{\sigma}\mathbf{B})}{2\omega} - \frac{ie(\boldsymbol{\alpha}\mathbf{E})}{2\omega}, \qquad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}. \tag{10}$$

One can see, that the obtained Hamiltonian contains the parameter ω, which plays the role of the virtual particle energy, to be integrated over in the expression (6) for the Green's function. There are several ways to proceed and get the final spectra, which are discussed in what follows. In the next section we compare H with nonrelativistic expansions of the Dirac equation.

III. NONRELATIVISTIC EXPANSIONS IN RH AND DIRAC EQUATIONS

We start with the Dirac equation for the hydrogenlike atom, where $A_0 = -\frac{Z\alpha}{r}$, and take into account that the exact form of the fermion Green's function (for $\varphi\{A\} \equiv 1$) is

$$G(x,y) = \sqrt{\frac{T}{8\pi}} \int_0^\infty \frac{d\omega}{\omega^{3/2}} (m - \hat{D})_x \langle \mathbf{x} | e^{-H(\omega)T} \mathbf{y} \rangle. \tag{11}$$

In (11) the Hamiltonian was defined in Minkowskian space-time, while the final expression is written for the Euclidean time T (see Appendix in [8] for details of the derivation).

As the next step we consider the "projection operator" $(m-\hat{D})_x$ in the integral (11) and take into account that the derivative $\frac{\partial}{\partial x_\mu}$ is acting on the Wilson line (4), resulting in the following expression (see Appendix 1 of the second reference in [7]),

$$(m-\hat{D}) \to \beta \begin{pmatrix} \omega+m & \boldsymbol{\sigma}\mathbf{p} \\ \boldsymbol{\sigma}\mathbf{p} & \omega-m \end{pmatrix}. \tag{12}$$

At this point one needs to diagonalize the whole expression under the integral (11), which allows us to give the energy eigenvalues of the Hamiltonian H with the account of the lower components of the wave function. In this way one writes

$$(m-\hat{D}) = \beta U^+ \begin{pmatrix} \omega+\sqrt{\mathbf{p}^2+m^2} & 0 \\ 0 & \omega-\sqrt{\mathbf{p}^2+m^2} \end{pmatrix} U, \tag{13}$$

where

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = e^{iS}; \qquad tg2\theta = \frac{\boldsymbol{\sigma}\mathbf{p}}{m},$$
$$S = \frac{i\boldsymbol{\gamma}\mathbf{p}\theta}{p} = -\beta\alpha_2\theta. \tag{14}$$

In a similar way one can write

$$H(\omega) = U^+\tilde{H}(\omega)U, \qquad \tilde{H}(\omega) = e^{iS}H(\omega)e^{-iS}. \tag{15}$$

Our reasoning below and in the next section follows the arguments from the book [9], and $\tilde{H}(\omega)$ can be found as a series (see chapter 2 of [9])

$$\tilde{H}(\omega) = H(\omega) + i[S,H] - \frac{1}{2}[S,[S,H]] - \ldots. \tag{16}$$

As one can see in (14), the series in (16) is in powers of $(\frac{p}{m})$ and gives the higher orders of the nonrelativistic expansion, whereas the first two orders are contained already in $H(\omega)$. Indeed, keeping for simplicity the first three terms in (10), which we denote as $H_0(\omega)$,

$$H_0(\omega) = \frac{(\mathbf{p}-e\mathbf{A})^2}{2\omega} + \frac{m^2+\omega^2}{2\omega} + eA_0, \tag{17}$$

and taking into account that at large T the integration over $d\omega$ in (11) can be done using the minimum of $H_0(\omega)$ in momentum space at some $\omega=\omega_0$, one has

$$\omega_0 = \sqrt{(\mathbf{p}-e\mathbf{A})^2+m^2}, \qquad H_0(\omega_0) = \omega_0 + eA_0. \tag{18}$$

Now the nonrelativistic expansion of ω_0 and $H_0(\omega_0)$ in powers of $(\frac{p}{m})$ yields the first terms of the Breit-Fermi expansion, namely, the so-called Pauli Hamiltonian [10], or to be more precise, its positive energy part. Another root of ω_0, $\omega_0 = -\sqrt{(\mathbf{p}-e\mathbf{A})^2+m^2}$ is out of the integration region, and in the full Minkowskian integral one would obtain instead

$$H_{+/-}(\omega_0) = \beta\sqrt{(\mathbf{p}-e\mathbf{A})^2+m^2-e(\boldsymbol{\sigma}\mathbf{B})} + eA_0. \tag{19}$$

We now turn to the next two terms in (16) and take into account that $\cos\theta \approx 1-\frac{p^2}{8m^2}$, $\sin\theta \approx \frac{\boldsymbol{\sigma}\mathbf{p}}{2m}$, and hence, additional terms from eA_0 and $\beta\boldsymbol{\sigma}\mathbf{E}$ in (10) yield

$$U^+H(\omega)U = \cos\theta eA_0\cos\theta + \cdots = \frac{e\Delta A_0}{8m^2} + \cdots$$
$$= -\frac{e\mathrm{div}\mathbf{E}}{8m^2} + \cdots. \tag{20}$$

In a similar way as in (20) one obtains the full $O(1/m^2)$ form,

$$\tilde{H} = \left(m + \frac{(\mathbf{p}-e\mathbf{A})^2}{2m} - \frac{p^4}{8m^3}\right) + eA_0 - \frac{e}{2m}\boldsymbol{\sigma}\mathbf{B}$$
$$+ \left(-\frac{e}{4m^2}\boldsymbol{\sigma}(\mathbf{E}\times\mathbf{p})\right) - \frac{e}{8m^2}\mathrm{div}\mathbf{E},$$
$$\mathbf{E} = -\nabla A_0. \tag{21}$$

Note, however, that $\mathrm{div}\mathbf{E} \sim \delta^{(3)}(\mathbf{r})$ and the higher in $(\frac{p}{m})$ terms bring about even higher derivatives of the δ function, which makes the evaluation of this Hamiltonian questionable. Therefore, it is more convenient from the beginning to consider the exact solution of the Dirac equation and compare it with the exact eigenvalues of $H(\omega)$ (10), in this way finding the accuracy of approximations made in the path integral method. This is done in the next section.

IV. EXACT DIRAC SPECTRUM FROM RH FOR THE HYDROGENLIKE ATOMS

Here we study the energy eigenvalues of hydrogenlike atoms. From (10) the RH is

$$H(\omega) = \frac{\mathbf{p}^2}{2\omega} + \frac{m^2+\omega^2}{2\omega} + eA_0 - \frac{ie(\boldsymbol{\alpha}\mathbf{E})}{2\omega}. \tag{22}$$

At this point one has two possibilities:

(1) To calculate eigenvalues $M_n(\omega)$ of $H(\omega)$ and then find the stationary point ω_0 of $M_n(\omega)$, yielding the

actual eigenvalue $M_n(\omega_0)$. This choice was used in [7] and called "the einbein method."

(2) To define $\omega = \omega_0$ from the condition $\frac{\partial H(\omega)}{\partial \omega}|_{\omega=\omega_1} = 0$, finding in this way the "stationary value" of the Hamiltonian. This brings us to the (generalized) Salpeter equation [11], extensively studied in the framework of RH, e.g., in [12] and in the relativistic quark model [13] (see [14] for reviews). In our case $A_0 = -\frac{Z\alpha}{r}$ and $\mathbf{E} = -\nabla A_0 = \frac{Z\alpha}{r^2}\mathbf{n}$.

We start with the simplest (einbein) procedure for the ground state, solving the equation

$$\left(\frac{\mathbf{p}^2}{2\omega} - \frac{Z\alpha}{r}\right)\psi = \varepsilon\psi, \qquad \varepsilon = -\frac{\omega(Z\alpha)^2}{2n^2}, \quad n = 1, 2.... \tag{23}$$

Inserting ε in (22) and neglecting the last term on the right-hand side, one obtains the expression for the total eigenvalue $M_n(\omega)$:

$$H(\omega)\Psi_n = M_n(\omega)\Psi_n, \qquad M_n(\omega) = \frac{m^2+\omega^2}{2\omega} - \frac{\omega(Z\alpha)^2}{2n^2}. \tag{24}$$

As prescribed by the ω integration in (11), the actual energy eigenvalue $M_n(\omega_0)$ should be obtained from $M_n(\omega)$ by the minimization procedure

$$\left.\frac{\partial M_n(\omega)}{\partial\omega}\right|_{\omega=\omega_0} = 0; \qquad \omega_0 = m\sqrt{1-\left(\frac{Z\alpha}{n}\right)^2} = M_n(\omega_0). \tag{25}$$

This form should be compared with the exact Dirac Hamiltonian eigenvalues M_n^D (see [9]):

$$M_n^D = \frac{m}{\sqrt{1+\left(\frac{Z\alpha}{n-\delta_j}\right)^2}},$$
$$\delta_j = j + \frac{1}{2} - \sqrt{\left(j+\frac{1}{2}\right)^2 - (Z\alpha)^2}. \tag{26}$$

It is remarkable that for the ground state with $n = 1$, $j = \frac{1}{2}$ the einbein approximation gives exactly the same answer, i.e.,

$$M_1(\omega_0) = M_1^D\left(j = \frac{1}{2}\right). \tag{27}$$

However, for higher levels the predictions of (25) and (26) differ by $O(m(Z\alpha)^4)$. Moreover, $M_n(\omega_0)$ does not depend on j. In general, the einbein method gives a reasonable approximation for QCD bound states that are not highly excited [12] but, in principle, does not ensure the orthogonality of different wave functions. To overcome this, we turn to the second possibility—the square root or Salpeter equation.

To this end one keeps in $H(\omega)$ [Eq. (10)] the last term for the hydrogenlike atoms, $\mathbf{A} = 0$, $A_0 = -\frac{Z\alpha}{r}$, and (10) has the form

$$H(\omega) = \frac{\mathbf{p}^2 + m^2 + \omega^2 - ie\alpha\mathbf{E}}{2\omega} - \frac{Z\alpha}{r},$$
$$H\Psi = M(\omega)\Psi. \tag{28}$$

As prescribed in the second (square root or "Salpeter") method, we define ω from the minimum of the kinetic part, written in the momentum space,

$$\left.\frac{\partial H(\omega)}{\partial\omega}\right|_{\omega=\omega_0} = 0; \qquad \omega_0 = \sqrt{\mathbf{p}^2 + m^2 - ie\alpha\mathbf{E}}. \tag{29}$$

Hence $H(\omega_0)$ acquires the form

$$\tilde{H}(\omega_0) = \sqrt{\mathbf{p}^2 + m^2 - ie\alpha\mathbf{E}} - \frac{Z\alpha}{r},$$
$$\tilde{H}(\omega_0)\mathbf{\Psi}_n = \tilde{M}_n(\omega_0)\tilde{\Psi}_n. \tag{30}$$

Notice that in the chiral representation for γ matrices one can write $-ie\alpha\mathbf{E} \to \mp iZ\alpha\frac{\boldsymbol{\sigma}\mathbf{n}}{r^2}$.

To find the eigenvalues of $\tilde{H}(\omega_0)$ one can write $\sqrt{\mathbf{p}^2 + m^2 - ie\alpha\mathbf{E}}\Psi_n = (\tilde{M}_n + \frac{Z\alpha}{r})\Psi_n$ and multiply it by the Hermitian conjugated equation times β,

$$\Psi_n^*\sqrt{\mathbf{p} + m^2 + ie\alpha\mathbf{E}}\beta\sqrt{p^2 + m^2 - ie\alpha\mathbf{E}}\Psi_n$$
$$= \Psi_n^*\left(\tilde{M}_n + \frac{Z\alpha}{r}\right)\beta\left(\tilde{M}_n + \frac{Z\alpha}{r}\right)\Psi_n, \tag{31}$$

obtaining in this way the Hamiltonian

$$\left\{\mathbf{p}^2 + m^2 \mp iZ\alpha\frac{\boldsymbol{\sigma}\mathbf{n}}{r^2} - \left(\tilde{M}_n + \frac{Z\alpha}{r}\right)^2\right\}\Psi_n = 0. \tag{32}$$

Then, following the same procedure as in [9] for the same Hamiltonian (see the Appendix for details of derivation), one obtains the exact Dirac spectrum (26). In this way we arrived at the Dirac spectrum starting from the square root form (30), using the quadratic expression (32).

However, direct use of the square root form in the x space brings about singularities around zero, as can be seen in the following. Indeed,

$$\sqrt{\mathbf{p}^2+m^2-ie\alpha\mathbf{E}}\,\Psi_n = \left(\tilde{M}_n+\frac{Z\alpha}{r}\right)\Psi_n \to \left(\mathbf{p}+m^2-ie\alpha\mathbf{E}-\left(\tilde{M}_n+\frac{Z\alpha}{r}\right)^2\right)\Psi_n = X\Psi_n \tag{33}$$

with

$$X=\left[\sqrt{\mathbf{p}^2+m^2-ie\alpha\mathbf{E}},\frac{Z\alpha}{r}\right]. \tag{34}$$

One can see that X is a sum of the δ function and its derivatives. These terms can be neglected if one excludes the small region around the origin. It is interesting that to solve Eq. (33) with $X=0$, one can use (32) with $\varepsilon_{\lambda,n}=-\frac{M_n(Z\alpha)^2}{2(n-\delta_j)^2}$, and the resulting equation for $\tilde{M}_n$ is

$$\tilde{M}_n^2=m^2-\frac{\tilde{M}_n^2(Z\alpha)^2}{2(n-\delta_j)^2},\qquad M_n=\frac{m}{\sqrt{1+(\frac{Z\alpha}{n-\delta_j})^2}}, \tag{35}$$

with δ_j given in (25). Therefore, one obtains again the exact spectrum Dirac equation if in the coordinate space one solves the square root equation, excluding the near-zero region.

Notice that the case of the Coulomb potential in the square-root (Salpeter-type) equation was studied analytically in [15], and a singularity in the S-wave radial wave function $R_0(r)\sim(mr)^{-\nu_0}$, $\nu_0\approx 0.086583$ was found there. The spectrum was found in the form ($l=0$)

$$M_{n0}=\frac{2m}{\sqrt{1+\alpha^2/4n^2}}. \tag{36}$$

Note that the term $(-ie\alpha\mathbf{E})$ was not present in [15], and hence δ_j does not enter in (36).

V. TWO-BODY QED HAMILTONIAN FROM THE PATH INTEGRAL

For two-body systems there is no exact formalism to compare with in QCD, since the Bethe-Salpeter equation is not operative with strong nonperturbative forces. In QED one can use standard perturbation theory and the Salpeter equation, which ensure very high accuracy of the results. Our aim in this section is to compare the RH spectrum for two oppositely charged particles (e.g., positronium) with the standard QED calculations. We consider the problem of two charges e_1 and e_2 and write the two-body Green's function without stochastic time contributions (radiative corrections) as in [8,16],

$$G_{e_1e_2}(x,y)=\frac{T}{2\pi}\int_0^\infty\frac{d\omega_1}{\omega_1^{3/2}}\int_0^\infty\frac{d\omega_2}{\omega_2^{3/2}}(D^3z^{(1)}D^3z^{(2)})_{\mathbf{xy}}4\mathrm{tr}Y_\Gamma \times\langle W\rangle\exp(-K_1-K_2), \tag{37}$$

where

$$Y_\Gamma=\frac{1}{4}\Gamma_1(m_1-i\hat{p}_1)\Gamma_2(m_2-i\hat{p}_2), \tag{38}$$

$$K_i=\int_0^T dt_E\left(\frac{\omega_i}{2}+\frac{m_i^2}{2\omega_i}+\frac{\omega_i}{2}\left(\frac{d\mathbf{z}^{(i)}}{dt_E}\right)^2\right). \tag{39}$$

In (37) the function W is the vacuum averaged contour integral over paths of charges e_1 and e_2 in the e.m. field A_μ,

$$W=\left\langle\exp\left(\sum_{k=1,2}\left(e_k^i\int A_\mu(z^k)dz_\mu^k+e_k\int_0^T\frac{dt_E}{2\omega_k}(\sigma_{\mu\nu}F_{\mu\nu})\right)\right)\equiv e^{-\hat{V}_T}\right\rangle. \tag{40}$$

In the case $e_1=-e_2$, W is the gauge invariant QED analogue of the Wilson loop, and below we shall consider this case for simplicity. To get rid of the c.m. motion, one integrates over $d(\mathbf{x}-\mathbf{y})$ and obtains

$$\int d^3(\mathbf{x}-\mathbf{y})G_{e_1e_2}(x,y)=\frac{T}{2\pi}\int_0^\infty\frac{d\omega_1}{\omega_1^{3/2}}\int_0^\infty\frac{d\omega_2}{\omega_2^{3/2}}Y_\Gamma d^3 \times(\mathbf{x}-\mathbf{y})e^{i\mathbf{P}(\mathbf{x}-\mathbf{y})} \times\langle\mathbf{x}|e^{-H(\omega_1,\omega_2,\mathbf{p}_1,\mathbf{p}_2)T}|\mathbf{y}\rangle, \tag{41}$$

$$H(\omega_1,\omega_2,\mathbf{p}_1,\mathbf{p}_2)=\sum_i\frac{\mathbf{p}_i^2+m_i^2+\omega_i^2}{2\omega_i}+\hat{V} =\sum_i\frac{m_i^2+\omega_i^2}{2\omega_i}+\frac{\mathbf{p}^2}{2\tilde{\omega}}+\hat{V}+\frac{\mathbf{P}^2}{2(\omega_1+\omega_2)},\qquad \tilde{\omega}=\frac{\omega_1\omega_2}{\omega_1+\omega_2}. \tag{42}$$

Since the last term on the right-hand side in (42) vanishes, one is left with the c.m. Hamiltonian,

$$H(\omega_1,\omega_2,\mathbf{p})=\sum_{i=1,2}\frac{m_i^2+\omega_i^2}{2\omega_i}+\frac{\mathbf{p}^2}{2\tilde{\omega}}+\hat{V}, \tag{43}$$

where the potential $\hat{V}$ is found from the cluster expansion of the Wilson loop. Keeping only the $O(e^2)$ terms (bilocal correlators), one has (see [16,17] for details)

$$\hat{V}=V_C(r)+\frac{(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2V_4(r)+S_{12}V_3)}{12\omega_1\omega_2} +\left(\frac{\boldsymbol{\sigma}_1\mathbf{L}}{4\omega_1^2}+\frac{\boldsymbol{\sigma}_2\mathbf{L}}{4\omega_2^2}\right)\frac{1}{r}V_0'(r)+\frac{(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\mathbf{L}}{2\omega_1\omega_2}\frac{1}{r}V_2'(r), \tag{44}$$

where

$$V_C(r) = \int_0^r \lambda d\lambda \int_0^\infty d\nu D^{(2)}(\lambda,\nu), \tag{45}$$

$$V_4(r) = \int_{-\infty}^{\infty} d\nu \left(3D^{(2)}(r,\nu) + 2r^2 \frac{\partial D^{(2)}(r,\nu)}{\partial r^2}\right), \tag{46}$$

$$V_3(r) = -r^2 \frac{\partial}{\partial r^2} \int_{-\infty}^{\infty} d\nu D^{(2)}(r,\nu), \tag{47}$$

$$\begin{aligned} V_0'(r) &= r\int_0^\infty d\nu D^{(2)}(r,\nu), \\ V_2'(r) &= r\int_0^\infty d\nu D^{(2)}(r,\nu), \end{aligned} \tag{48}$$

and $D^{(2)}(\lambda,\nu)$ is the quadratic correlator,

$$\begin{aligned} &e^2\langle F_{\mu\nu}(x)F_{\lambda\rho}(y)\rangle \\ &= \frac{1}{2}\left[\frac{\partial}{\partial u_\mu}(u_\lambda\delta_{\nu\rho} - u_\rho\delta_{\lambda\nu}) + \begin{pmatrix}\mu\leftrightarrow\nu\\ \lambda\leftrightarrow\rho\end{pmatrix}\right] D^{(2)}(u), \end{aligned} \tag{49}$$

with $u = x - y$. To the lowest order $D^{(2)}(u)$ is ($e_1 = -e_2 = e$)

$$D^{(2)}(u) = \frac{4\alpha}{\pi u^4}, \qquad \alpha = \frac{e^2}{4\pi}. \tag{50}$$

Note that the accurate derivation of the spin-dependent terms, valid both for QCD and QED, taking into account the proper positions of $(m_i - \hat{D}_i)$ terms, is done in [16]. In the QED case substituting $D^{(2)}$ from (50), one obtains the familiar results [$S_{12} = \frac{1}{4}(3\boldsymbol{\sigma}_1\mathbf{n}\boldsymbol{\sigma}_2\mathbf{n} - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)$]

$$\begin{aligned} &V_C(r) = -\frac{\alpha}{r}, \qquad \frac{1}{r}V_0' = \frac{1}{r}V_2' = \frac{\alpha}{r^3}; \qquad V_3 = \frac{3\alpha}{r^3}, \\ &V_4 = 8\pi\alpha\delta^{(3)}(\mathbf{r}). \end{aligned} \tag{51}$$

These expressions coincide with the corresponding nonrelativistic spin-dependent potentials, when $\omega_i = m_i$, but in our case (44) and (51) are applicable in the relativistic case to the order $O(\alpha^5)$. Note that in the case of positronium the additional term in $\hat{V}$ appears due to the annihilation diagram, which in the nonrelativistic limit is

$$V_5 = \frac{\pi\alpha}{2m^2}(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 + 3). \tag{52}$$

One can now proceed as in (28) and (29), but treating all terms in $\hat{V}$ (44) as a perturbation, except for $V_C(r)$, and for $m_1 = m_2 = m$, $\Delta\hat{V} \equiv \hat{V} - V_C(r)$ one obtains

$$\tilde{H}_{e,-e} = 2\sqrt{\mathbf{p}^2 + m^2} - \frac{\alpha}{r} + \Delta\hat{V} \equiv H_0 + \Delta\hat{V}. \tag{53}$$

Again, as in (32), for $\Psi_n^{(0)}$, $H_0\Psi_n^{(0)} = M_n^{(0)}\Psi_n^{(0)}$, one has

$$\left\{4(\mathbf{p}^2 + m^2) - \left(M_n^{(0)} + \frac{\alpha}{r}\right)^2\right\}\Psi_n^{(0)} = 0, \tag{54}$$

and the analog of the angular operator $\hat{N}^2$ (see the Appendix) is now diagonal with eigenvalues $\lambda(\lambda+1) = L(L+1) - \frac{\alpha^2}{4}$, yielding the eigenvalues $\varepsilon_n = -\frac{M_n^{(0)}\alpha^2}{8\tilde{n}^2}$, $\tilde{n} = n - \delta_L$, with

$$\delta_L = L - \sqrt{\left(L + \frac{1}{2}\right)^2 - \frac{\alpha^2}{4}} + \frac{1}{2}. \tag{55}$$

Finally one obtains for $M_n^{(0)}$,

$$M_n^{(0)} = \frac{2m}{\sqrt{1 + \frac{\alpha^2}{4\tilde{n}^2}}}. \tag{56}$$

The expansion in α^2 produces the expected result,

$$M_n^{(0)} = 2m - \frac{\alpha^2 m}{4\tilde{n}^2} + \cdots \approx 2m - \frac{\alpha^2 m}{4n^2} + O(\alpha^4). \tag{57}$$

At this point we can compare the accuracy of our expressions (56) with the account of the potentials V_4, V_5 in (50) and (51) to the results of QED perturbation theory for the orthopositronium $(1^3S_1 - 2^3S_1)$ interval ΔE (see reviews [18,19] for results and discussions). From [18], Table V, one obtains, in perturbation theory, $\Delta E_{\rm PT} = \Delta E_{\rm PT}(\alpha^2) + \Delta E_{\rm PT}(\alpha^4) + \Delta E_{\rm PT}(\alpha^n, n \geq 5)$, where

$$\Delta E_{\rm PT}(\alpha^2) = 1.2336907351 \times 10^9 \text{ MHz}, \tag{58}$$

$$\Delta E_{\rm PT}(\alpha^4) = -82.0056 \times 10^3 \text{ MHz}, \tag{59}$$

$$\Delta E_{\rm PT}(\alpha^5) = -1.5014 \times 10^3 \text{ MHz}. \tag{60}$$

At the same time our Eq. (56) contributes the same amount in the order $O(\alpha^2)$, $\Delta E_{\rm RH}(\alpha^2) = \Delta E_{\rm PT}(\alpha^2)$, while in $O(\alpha^4)$ its contribution from $M_n^{(0)}$ is $\Delta E'_{\rm RH}(\alpha^4) = 23.9515582 \times 10^3$ MHz, and from the potentials V_4, V_5 one obtains $\Delta E''_{\rm RH}(\alpha^4) = -102.1933153 \times 10^3$ MHz, so that the total contribution in the order $O(\alpha^4)$ is

$$\begin{aligned} \Delta E_{\rm RH}(\alpha^4) &\equiv \Delta E'_{\rm RH}(\alpha^4) + \Delta E''_{\rm RH}(\alpha^4) \\ &= -78.2417571 \times 10^3 \text{ MHz}, \end{aligned} \tag{61}$$

which should be compared to $\Delta E_{\rm PT}(\alpha^4)$, Eq. (59). One can see that the difference between these numbers is of the order $O(10^{-6})$ of the total result for ΔE, and is in the realm of the $O(\alpha^5)$ corrections. Note, also, that the relativistic $O(\alpha^4)$ corrections coming from the square root expression (56) are of vital importance for the resulting accuracy. In this way we have proved that the square root of the

two-body Hamiltonian (53) is able to provide high accuracy for the positronium spectrum.

VI. DISCUSSION OF RESULTS

We have calculated the spectrum of the hydrogenlike atoms in QED, using our RH, derived in the framework of the path integral. This spectrum exactly coincides with the spectrum of the Dirac equation.

It was shown above that in the first approach (the einbein approximation), where the eigenvalues are functions of virtual energy ω, one obtains a reasonable result for the relativistic ground state energy; however, for higher eigenvalues corrections are of the order of $(Z\alpha)^4$.

At the same time, the second approach, where the virtual energy is defined on the operator level, provides the square-root-type Hamiltonian, which yields the exact Dirac spectrum. In this way our results support the so-called Salpeter approach in the relativistic quark models, which was so successful in predicting hadronic states [12,13,17,20]. However, in QCD the string correction needs to be taken into account to provide orbital and radial Regge trajectories [12] in good agreement with experiment.

We have also shown how the Breit-Fermi nonrelativistic expansion is obtained from our RH, when Foldy-Wouthuizen transformation is applied.

Finally, the case of two oppositely charged particles was considered, and all interaction terms, including spin-dependent ones, were derived and included in the resulting Hamiltonian. The latter contains both kinematic relativistic effects and lowest order dynamic effects, and our formalism allows us to distinguish between two contributions. A short comparison to the standard QED perturbation results is done for the $(2^3S_1 - 1^3S_1)$ energy interval of positronium, showing a good accuracy of the RH for the positronium spectrum.

Summarizing these results, one can consider RH as a reliable tool for the studies in QED as well as of hadronic properties in QCD with the proper comparison with lattice and experimental results.

Another important line of development is the theory of QED systems in a strong magnetic field, where the RH approach was formulated in [8,16]; a new phenomenon of the magnetic focusing can be found in [21].

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APPENDIX: EXPLICIT SOLUTION OF EQ. (33)

Following [9] we write Eq. (32) in the form

$$\left\{-\Delta_r + \frac{\hat{N}^2}{r^2} - \frac{2Z\alpha}{r}M_n - (M_n^2 - m^2)\right\}\psi_n = 0, \quad \text{(A1)}$$

where $\hat{N}^2$ in the chiral representation for the matrices α_i in the term $(-i\alpha\mathbf{E})$ is written in the diagonal form as $\mp\frac{iZ\alpha(\boldsymbol{\sigma}\mathbf{n})}{r^2}$. For the total angular momentum $\mathbf{J} = \mathbf{L} + \frac{\boldsymbol{\sigma}}{2}$ with eigenvalues $j = \frac{1}{2}, \frac{3}{2}, \ldots$, one can define $\hat{N}^2$ as the matrix in the states $l_\pm = j \pm \frac{1}{2}$, which has the form

$$\hat{N}^2 = \begin{pmatrix} l_+(l_+ + 1) - (Z\alpha)^2 & \mp iZ\alpha \\ \mp iZ\alpha & l_-(l_- + 1) - (Z\alpha)^2 \end{pmatrix}. \quad \text{(A2)}$$

The eigenvalues of $\hat{N}^2$ are found from (A2) to be

$$\hat{N}^2 = \lambda(\lambda + 1),$$
$$\lambda = \sqrt{\left(j + \frac{1}{2}\right)^2 - (Z\alpha)^2} - 1, \quad \sqrt{\left(j + \frac{1}{2}\right)^2 - (Z\alpha)^2}, \quad \text{(A3)}$$

and writing $\lambda = (j \pm \frac{1}{2}) - \delta_j$, one can define the radial quantum number n_r, pertinent to Δ_r, $n_r = 0, 1, 2, \ldots$, and the solution of the reduced Coulomb problem [the first three terms in (A1)] is

$$\varepsilon_n = -\frac{(Z\alpha)^2 M_n}{2\tilde{n}^2}, \quad \text{(A4)}$$

where
$\tilde{n} = n_r + \lambda + 1 = n_r + j \pm \frac{1}{2} + 1 - \delta_j = n - \delta_j$, $n = 1, 2, \ldots$.

Finally, from (A1) one finds that $M_n^2 - m^2 = 2M_n\varepsilon_n$, or

$$M_n = \frac{m}{\sqrt{1 + \frac{(Z\alpha)^2}{(n-\delta_j)^2}}}. \quad \text{(A5)}$$

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