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Asymptotically free gauge theories. II *

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Deep-inelastic lepton-hadron scattering is analyzed in asymptotically free gauge theories of the strong interactions. The renormalization-group equations for the coefficients of the twist-two operators in the Wilson expansion are reviewed. A careful treatment of the mixing of operators with identical quantum numbers and dimensions is given. The relevant anomalous dimensions of the twist-two operators are calculated to second order in perturbation theory. These are used to calculate the asymptotic q^2 behavior of the moments of the structure functions. It is shown that the approach to the asymptotic limit is logarithmic, that Bjorken scaling is violated by powers of $\ln(-q^2)$, and that the naive light-cone or parton-model relations for the moments of the structure functions are true asymptotic theorems. A new sum rule for the first moment of F_2, in terms of the energy-momentum tensor, is derived. An example of a function whose moments have roughly the correct asymptotic q^2 behavior is constructed. The q^2 behavior of the structure functions for a given x is discussed.

I. INTRODUCTION

In a recent paper[1] we have constructed a class of gauge theories of the strong interactions, which have the remarkable feature of being "asymptotically free." The primary motivation for this proposal is the evidence that Bjorken scaling requires an asymptotically free theory,[2] that only non-Abelian gauge theories can be asymptotically free,[3] and that indeed many non-Abelian gauge theories are asymptotically free.[4-6] Deep-inelastic scattering is therefore the natural arena in which to test our theories. In this paper we shall discuss in detail the properties of lepton-hadron scattering in asymptotically free gauge theories of the strong interactions.

This paper is a sequel to Ref. 1 and should be read in conjunction with it, although the phenomenological discussion of Sec. III can be understood independently (hereafter Ref. 1 will be referred to as paper I, and the prefix I refers to equations of Ref. 1). The general features of deep-inelastic scattering in asymptotically free theories were already described in paper I. These include the logarithmic approach to scaling, the calculable logarithmic deviations from Bjorken scaling, and the validity of the naive or light-cone parton-model relations.

In Sec. II of this paper, we discuss in some detail the application of renormalization-group techniques to the Wilson expansion. This analysis has appeared in many other places[7-9,2] and is included here for the sake of completeness. In particular we discuss the mixing of operators with the same quantum numbers and dimensions. In gauge theories this mixing is particularly annoying since it would appear that "ghost" operators (i.e., operators involving Feynman-Faddeev-Popov ghost fields[10,11]) mix together with ordinary operators. We argue that this mixing can be ignored. This claim is further substantiated by a calculation, which appears in Appendix A, performed in a gauge which is free of Faddeev-Popov ghosts.

In Sec. III we calculate the anomalous dimensions of the relevant operators in the Wilson expansion.[12] Some of these were already presented in paper I. We derive the asymptotic form for the moments of the structure functions, as well as the various relations and sum rules satisfied by these moments. A sum rule for the first moment of the structure functions, the "energy-momentum-tensor sum rule," is derived. An example of an explicit functional form for the structure functions, with roughly the correct asymptotic behavior, is presented and the general features of this function are discussed.

Section IV contains some concluding remarks.

II. THE RENORMALIZATION-GROUP APPROACH TO THE WILSON EXPANSION

In the deep-inelastic scattering of a lepton off a hadron one measures the Fourier transform of the commutator of electromagnetic or weak currents. We define the standard structure functions as follows:

$$\frac{1}{2\pi}\int dy\, e^{iq\cdot y}\langle p|[J_\mu(a;\tfrac{1}{2}y), J_\nu(b;-\tfrac{1}{2}y)]|p\rangle = \frac{p_\mu p_\nu}{m\nu}F_2^{(a,b)}(\nu,q^2) - \frac{g_{\mu\nu}}{m}F_1^{(a,b)}(\nu,q^2) + i\frac{\epsilon_{\mu\nu\sigma\lambda}p_\sigma q_\lambda}{2m\nu}F_3^{(a,b)}(\nu,q^2)+\cdots, \quad (1)$$

where p is the momentum of the hadron, q is the momentum transfer to the hadrons, $\nu = p\cdot q$, spin averages have been taken, and we have suppressed terms proportional to q_μ and q_ν which give terms proportional to the lepton mass when contracted with the leptonic currents. The label a denotes the SU(3) × SU(3) character of the current. The Bjorken limit corresponds to

$$x = \frac{Q^2}{2\nu} \text{ fixed}, \quad Q^2 = -q^2 \to +\infty \tag{2}$$

and probes the commutator for lightlike values of y.

The structure of the product of local operators at short, and at lightlike, distances is given by Wilson's operator-product expansion. In the case of interest this expansion takes the form

$$\begin{aligned} J_\mu(a;\tfrac{1}{2}y)J_\nu(b;-\tfrac{1}{2}y) = {} & \tfrac{1}{2}g_{\mu\nu}\left(\frac{\partial}{\partial y}\right)^2 \frac{1}{y^2 - i\epsilon y_0}\sum_{n=0}^\infty \sum_i C_{i,1}^{(n)}(a,b;y^2 - i\epsilon y_0)\, O^i_{\mu_1\cdots\mu_n}(0)y^{\mu_1}\cdots y^{\mu_n} \\ & + \frac{1}{y^2 - i\epsilon y_0}\sum_{n=0}^\infty \sum_i C_{i,2}^{(n)}(a,b;y^2 - i\epsilon y_0)\, O^i_{\mu\nu\mu_1\cdots\mu_n}(0)y^{\mu_1}\cdots y^{\mu_n} \\ & + \tfrac{1}{2}i\epsilon_{\mu\nu\sigma\lambda}\frac{\partial}{\partial y^\lambda}\frac{1}{y^2 - i\epsilon y_0}\sum_{n=0}^\infty \sum_i C_{i,3}^{(n)}(a,b;y^2 - i\epsilon y_0)\, O^i_{\sigma\mu_1\cdots\mu_n} y^{\mu_1}\cdots y^{\mu_n} + \cdots . \end{aligned} \tag{3}$$

In this expansion we have neglected, as in Eq. (1), gradient terms (involving $\partial/\partial y^\mu$ or $\partial/\partial y^\nu$). The operator $O^i_{\mu_1\cdots\mu_n}(x)$ has spin k (traceless and symmetric) and dimension $n+2$ (or twist = dimension − spin = 2). The label i denotes the various operators of equal twist which might occur in the expansion. $C_{i,k}^{(n)}$ ($k=1,2,3$) are c-number functions of x^2 and the coupling constants of the theory. They have been normalized to be dimensionless, so that naively (as in free-field theory) they would be nonsingular as $x^2 \to 0$. The three dots refer to other operators of higher twist, whose contribution to the structure functions is suppressed, to any finite order of perturbation theory, by powers of q^2. (In an asymptotically free theory this suppression is guaranteed to all orders in perturbation theory by the vanishing of the anomalous dimensions of all operators at the fixed point $g=0$. One has logarithmic, but not power, corrections to naive scaling.)

The structure functions F_k in the deep-inelastic region are determined by the behavior of $C_{i,k}^{(n)}$ for small x, and by the hadronic matrix elements of the operators O^i

$$\langle p|O^i_{\mu_1\cdots\mu_n}(0)|p\rangle_{\text{spin average}} = i^n \frac{1}{m}p_{\mu_1}\cdots p_{\mu_n} M_n^i \cdots . \tag{4}$$

One cannot simply take the Fourier transform of Eq. (3), since after Fourier transforming it does not converge in the region of interest ($0 \le x \le 1$). Instead the Fourier transforms of $C_{i,k}^{(n)}$ are determined by the *moments* of the structure functions.[13,9]

In fact

$$\begin{aligned} \int_0^1 dx\, x^n F_1^{a,b}(x,q^2) &= \sum_i \tilde{C}_{i,1}^{(n+1)}(a,b;q^2)M_i^{n+1} , \\ \int_0^1 dx\, x^n F_2^{a,b}(x,q^2) &= \sum_i \tilde{C}_{i,2}^{(n)}(a,b;q^2)M_i^{n+2}, \\ \int_0^1 dx\, x^n F_3^{a,b}(x,q^2) &= \sum_i \tilde{C}_{i,3}^{(n)}(a,b;q^2)M_i^{n+1}, \end{aligned} \tag{5}$$

where

$$\tilde{C}_{i,k}^{(n)}(a,b;q^2) = \tfrac{1}{2}i(q^2)^{n+1}\left(-\frac{\partial}{\partial q^2}\right)^n \times \int d^4y\, e^{iq\cdot y}\frac{C_{i,k}^{(n)}(a,b;y^2)}{y^2 - i\epsilon y_0}. \tag{6}$$

In a free-quark model the operators $O^i_{\mu_1\cdots\mu_n}$ are simply

$$O^a_{\mu_1\cdots\mu_n}(x) = S\bar\psi(x)\gamma_{\mu_1}\overleftrightarrow{\partial}_{\mu_2}\cdots\overleftrightarrow{\partial}_{\mu_n}\lambda^a(1\pm\gamma_5)\psi(x) , \tag{7}$$

where S denotes symmetrization of the indices $\mu_1\cdots\mu_n$. In that case the functions $\tilde{C}^{(n)}$ are constants, independent of q^2, which can be determined by the light-cone commutator of the currents (see Ref. 14).

In an interacting theory the functions $\tilde{C}^{(n)}$ will be nontrivial functions of q^2 and the coupling constants of the theory (g). To determine the q^2 dependence of $\tilde{C}^{(n)}$ we employ the renormalization-group equations. Let us outline the derivation of these equations for the Wilson coefficients. Consider the contribution of an operator $O^{(n)}(x)$ to the short-distance expansion of the product of $A(x)$ and $B(x)$ (we suppress all tensor and quantum-number labels):

$$A(x)B(-x) \underset{x^\mu\approx 0}{\approx} C^{(n)}(x^2,g)\,O^{(n)}_{\mu_1\cdots\mu_n}(0)\,x^{\mu_1}\cdots x^{\mu_n}. \tag{8}$$

If $O^{(n)}(x)$ is the dominant operator for short distances (i.e., the operator of smallest dimension, which we assume for the moment to be unique) then the short-distance behavior of $C^{(n)}$ can be determined by calculations performed as if all masses and dimensional coupling constants were zero (for details see paper I). In that case the only dimensional parameter in the theory is the subtraction point μ introduced to perform the required renormalization. A change in μ can be reabsorbed by a change in the coupling constant and the scale of all operators in the theory. If we consider the effect on Eq. (8) of a change in μ we derive that

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+\gamma_A+\gamma_B-\gamma_O\right)C(x^2,g)=0\ , \tag{9}$$

where β is given by Eq. I(4.9), and γ_A (γ_B,γ_O) is the anomalous dimension of the operator A ($B, O^{(n)}$). Thus, for example,

$$\gamma_A(g^2)=\mu\frac{\partial}{\partial\mu}\ln Z_A\bigg|_{g_0,\Lambda\text{ fixed}}\ . \tag{10}$$

Z_A is the renormalization constant of the operator A, where we have assumed that A is multiplicatively renormalizable (this is true for all the operators that control the light-cone behavior of current products).

In the case of interest this implies that the functions $\tilde{C}^{(n)}(q^2/\mu^2,g)$ satisfy

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}-\gamma^n(g)\right)\tilde{C}^{(n)}(q^2/\mu^2,g)=0\ , \tag{11}$$

where again we assume a unique twist-two operator appears in the Wilson expansion of currents, and $\tilde{C}^{(n)}$ is related, by an equation similar to Eq. (6), to the coefficient of the spin-n twist-two operator. Note that the anomalous dimension of conserved or partially conserved currents (by which we mean that the dimension of the divergence of the current is less than four) vanishes for all g.

The solution of this renormalization group equation is expressed in terms of the effective coupling constant $\bar{g}(t,g)$ defined by

$$\frac{d\bar{g}}{dt}=\beta(\bar{g}),\quad \bar{g}(0,g)=g \tag{12}$$

where

$$t=\tfrac{1}{2}\ln(-q^2/\mu^2)\ . \tag{13}$$

In fact

$$\tilde{C}^{(n)}(q^2/\mu^2,g)=\tilde{C}^{(n)}(1,\bar{g}(t,g))\times\exp\left(-\int_0^t dx\,\gamma^n(\bar{g}(x,g))\right)\ . \tag{14}$$

The large-q^2 behavior of $\tilde{C}^{(n)}$ will thus be determined by the large-t behavior of $\bar{g}(t,g)$, which in turn is determined by the fixed points of the renormalization group Eq. (12). In an asymptotically free gauge theory

$$\bar{g}^2(t,g)\underset{t\to\infty}{\sim} b_0^{-1}t^{-1}+b_1b_0^{-3}t^{-2}\ln t+O(1/t^2)\ , \tag{15}$$

where b_0 is given by Eq. I(4.9). The anomalous dimension γ^n will behave for small g like

$$\gamma^n(g)=\gamma^n g^2+O(g^4)\ . \tag{16}$$

Therefore in an asymptotically free theory

$$\tilde{C}^{(n)}(q^2/\mu^2,g)\underset{t\to\infty}{\sim}\text{const}[\ln(-q^2)]^{-a_n}\times[\tilde{C}^{(n)}(1,0)+O(1/\ln(-q^2))]\ , \tag{17}$$

where

$$a_n=\frac{\gamma^n}{2b_0}\ . \tag{18}$$

From this expression we see that the approach to the asymptotic region is logarithmic, i.e., the corrections to the asymptotic form are suppressed by $\ln(-q^2)$. The asymptotic value of $\tilde{C}^{(n)}$ does not exhibit naive (Bjorken) scaling. Instead there are logarithmic deviations, whose magnitude is calculable in second-order perturbation theory. Furthermore the tensor and quantum number *structure* of the operator-product expansion will be that of free-field theory, up to logarithmic corrections, since $\tilde{C}^{(n)}$ is evaluated at $g=0$ on the right-hand side of Eq. (17).

In general there will appear in the Wilson expansion many operators $O_i^{(n)}$ with the same quantum numbers and twist. In that case a given $O_i^{(n)}$ is *not* multiplicatively renormalizable, rather one must take linear combinations of these to obtain operators which are multiplicatively renormalizable and have definite dimensions.

Consider the Wilson expansion

$$J_a(x)J_b(-x)=\sum_{i,k}C^{(n)}_{i,k}(x^2-i\epsilon x_0,g)\times O_i^{(n)}(0)_{\mu_1\cdots\mu_n}x^{\mu_1}\cdots x^{\mu_n}+\ldots, \tag{19}$$

where again we suppress the tensor indices of the currents. The label i runs over the set of operators with spin n and dimension $(n+2)$, and the label k denotes the tensorial and SU(3)×SU(3) structure of the expansion. [In other words k stands for the labels $a,b,1,2,3$ in Eq. (3).] Since the operators $O_i^{(n)}$ can mix, a change in μ will, in general, entail a mixing of the various operators. The Wilson coefficients will obey the renormalization group equation

$$\left(\mu \frac{\partial}{\partial \mu}+\beta(g)\frac{\partial}{\partial g}\right) C_{i,k}^{(n)}=\sum_j \gamma_{ij}^{(n)}(g)\, C_{j,k}^{(n)} . \tag{20}$$

The anomalous dimension is now a matrix

$$(\underline{\gamma}^{(n)}(g))_{ij} \equiv \gamma_{ij}^{(n)}(g) = \left(\underline{Z}_0^{-1}\, \mu \frac{\partial}{\partial \mu} \underline{Z}_0\right)_{ij} \Bigg|_{g_0,\Lambda \text{ fixed}} \tag{21}$$

and $\underline{Z}_0$ is the matrix of wave-function renormalization constants of the operators $O_i^{(n)}$, i.e.,

$$O_{i\,\text{renormalized}}^{(n)} = \sum_j O_{j\,\text{bare}}^{(n)} (\underline{Z}_0^{-1})_{ji} . \tag{22}$$

The solution of Eq. (20) for the Fourier transform of $C_{i,k}^{(n)}$ is again most easily expressed in terms of $\bar{g}(t,g)$

$$\tilde{C}_{i,k}^{(n)}\left(\frac{q^2}{\mu^2}, g\right)=\sum_j \left\{T \exp\left[-\int_0^t \underline{\gamma}^{(n)}(\bar{g}(x,g))\, dx\right]\right\}_{ij} \times \tilde{C}_{j,k}^{(n)}(1, \bar{g}(t,g)) , \tag{23}$$

where T refers to the fact that the exponential is to be t ordered. In an asymptotically free theory $\bar{g}^2$ vanishes for large t like $b_0^{-1} t^{-1}$. The anomalous-dimension matrix $\underline{\gamma}^{(n)}(\bar{g})$ similarly vanishes like $1/t$. We define

$$\underline{\gamma}^{(n)}(g)=\underline{\gamma}^{(n)} g^2+\underline{r}^{(n)}(g) , \tag{24}$$

where $\underline{r}^{(n)}(g)$ is of order g^4, for small g. Then the large-t behavior of $\tilde{C}_{i,k}^{(n)}$ is given by

$$\tilde{C}_{i,k}^{(n)}(q^2/\mu^2, g) \underset{t\to\infty}{\sim} \sum_j \left[\exp\left(\frac{-\ln t}{b_0}\underline{\gamma}^{(n)}\right)\underline{M}^{(n)}\right]_{ij} \times \tilde{C}_{j,k}^{(n)}(1,0) , \tag{25}$$

where

$$\underline{M}^{(n)}=T\exp\left[-\int_0^\infty dt \exp\left(\frac{\ln t}{b_0}\underline{\gamma}^{(n)}\right)\underline{r}^{(n)}(\bar{g}(t,g))\exp\left(-\frac{\ln t}{b_0}\underline{\gamma}^{(n)}\right)\right] . \tag{26}$$

Thus we see that the large-q^2 behavior of the Wilson coefficients is determined by the eigenvalues of $\underline{\gamma}^{(n)}$. These can be calculated by evaluating the wave-function renormalization matrix $\underline{Z}$ to second order in perturbation theory. The structure of Eq. (25) is more transparent if we write the matrix $\underline{\gamma}^{(n)}$ in terms of its eigenvalues $\gamma_i^{(n)}$

$$\underline{\gamma}^{(n)}=\sum_i \gamma_i^{(n)} \underline{P}^i , \tag{27}$$

where $\underline{P}^i$ are projection matrices

$$\underline{P}^i \underline{P}^j=\delta_{ij}\underline{I} , \qquad \sum_i \underline{P}^i=\underline{I} . \tag{28}$$

Then we have

$$\tilde{C}_{i,k}^{(n)}(q^2/\mu^2, g) \underset{-q^2\to\infty}{\sim} \sum_l [\ln(-q^2)]^{-a_l^{(n)}} \times \sum_j (\underline{P}^l \underline{M}^{(n)})_{ij}\, \tilde{C}_{j,k}^{(n)}(1,0), \tag{29}$$

where

$$a_l^{(n)}=\frac{1}{2b_0}\gamma_l^{(n)} . \tag{30}$$

The operator-product expansion has the form (in momentum space)

$$J_a J_b \approx \sum_l [\ln(-q^2)]^{-a_l^{(n)}} \sum_{i,j} O_i^{(n)} [\underline{P}^l \underline{M}^{(n)}]_{ij} \times \tilde{C}_{j,k}^{(n)}(1,0) . \tag{31}$$

The dominant operator for large q^2 will thus be picked out by the smallest eigenvalue of $\underline{\gamma}^{(n)}$. The coefficient of this leading operator is not, in general, determined. Although the functions $C_{i,k}^{(n)}(1,0)$ are known (equal to their free-field-theory values) the matrix $\underline{M}^{(n)}$ is not. It depends on the (unknown) behavior of γ for large coupling constant. The only case in which the coefficient of the dominant operator is known is when the matrix $\underline{P}^l \underline{M}^{(n)}$ vanishes identically for all g. This will be the case if we are considering the coefficient of a conserved or partially conserved current or the energy-momentum tensor. In that case the (appropriately projected) matrix M equals unity and the coefficient of the current, or energy-momentum tensor is determined. In addition $a_l^{(n)}$ vanishes for these operators. These are the only operators for which one is interested in the numerical value of the coefficients since only these operators have known hadronic matrix elements. Also if a particular tensorial or SU(3) ×SU(3) structure holds for the operator-product expansion when $g=0$, it will also be valid when

$-q^2\to\infty$. This is because the tensorial and SU(3) $\times$SU(3) structure is totally contained in the labels k of $\tilde{C}^{(n)}_{i,k}$ evaluated at $g=0$.

Consequently all current-algebra sum rules (Adler, Gross–Llewellyn Smith, etc.) and relations between moments of the structure functions derived in the parton or naive light-cone models will be true asymptotic theorems in our theories.

III. DEEP-INELASTIC SCATTERING IN ASYMPTOTICALLY FREE GAUGE THEORIES

We now supply the explicit results one obtains by applying the techniques of Sec. II to an asymptotically free gauge theory. In order to ensure sufficient generality we will work with the Lagrangian

$$L_0=-\tfrac{1}{2}\,\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}+\bar\psi(i\nabla\!\!\!\!/-m)\psi\,, \tag{32}$$

where $\nabla_\mu=\partial_\mu+ig\sigma^aB^a_\mu$ is the covariant derivative, the σ^a being the matrices of the representation of the Lie algebra of the gauge group G in the space of fermions. We retain the normalization of Eqs. I(2.3)–(2.4) and omit for brevity the gauge-fixing and ghost terms needed to define (32) properly. For simplicity we assume that G is simple, but allow the fermions to transform according to an arbitrary representation of G.

The class of theories described by (32) includes the case where $G=\mathrm{SU}(3)$ and ψ transforms as a direct sum of three triplets. In this case we may interpret G as the gauge group of "color" and the group which mixes the three G-triplets as ordinary SU(3), so the fermions are just the usual "colored" quarks. As was discussed in paper I, there is some reason to hope that this theory provides a model of hadrons. We will adopt it in the following for illustrative purposes. If $m=0$ we also have ordinary chiral SU(3). Mass terms, and in general symmetry-breaking terms with operator dimension less than four, have no influence on the leading-order effects we discuss in most of this paper. In particular, all symmetries broken by such terms will be reinstated asymptotically, so that for instance all the operators in a given chiral SU(3)$\times$SU(3) multiplet have the same anomalous dimensions.

More complicated models, with charm quarks or a different gauge group (so long as it is non-Abelian), are clearly allowed. It might be interesting to see what the effects of Higgs scalars on the light-cone behavior of asymptotically free theories are (asymptotically free theories with scalar particles were constructed in paper I). One would be faced with some complications due to the presence of at least two dimensionless coupling constants and more involved mixings of lowest-twist operators. We have not carried out this study.

Throughout this paper we shall assume that the gauge group G commutes with the ordinary symmetry group H of the hadronic currents, e.g., chiral SU(3)$\times$SU(3). This excludes models of the Bars-Halpern-Yoshimura type.[15] There are good reasons for this assumption. The spin content of the fundamental charged constituents of the nucleon as measured by the Callan-Gross[16] sum rule points to spin-$\frac{1}{2}$ constituents. A "shielding" mechanism (like that contemplated in paper I) is impossible if one wishes to identify these particles with the observed vector or axial-vector octets, since the shielding mechanism yields only singlets of the gauge group for physical states. The alternative complicated system of Higgs scalars required in Ref. 15 almost certainly destroys asymptotic freedom.

We are interested in the light-cone behavior of the product of weak (by this we mean electromagnetic or truly weak) currents. The operators of twist two appearing in the operator-product expansion of two weak currents near the light cone will be

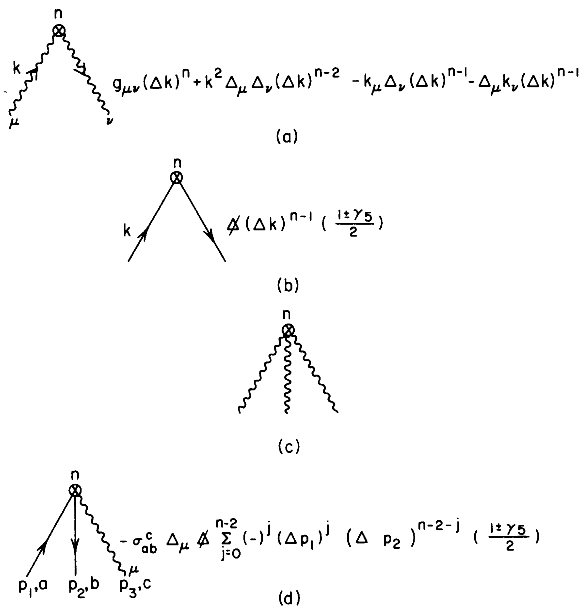
$${}^nO^V_{\mu_1\cdots\mu_n}=i^{n-2}S\,\mathrm{Tr}\,F_{\mu_1\alpha}\nabla_{\mu_2}\cdots\nabla_{\mu_{n-1}}F^\alpha{}_{\mu_n}-\text{trace terms}, \tag{33}$$

$${}^nO^{F\pm,0}_{\mu_1\cdots\mu_n}=\tfrac{1}{2}i^{n-1}S\bar\psi\gamma_{\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_n}(1\pm\gamma_5)\psi-\text{trace terms}, \tag{34}$$

$${}^nO^{F\pm,a}_{\mu_1\cdots\mu_n}=\tfrac{1}{2}i^{n-1}S\bar\psi\gamma_{\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_n}(1\pm\gamma_5)\tfrac{1}{2}\lambda^a\psi-\text{trace terms}, \tag{35}$$

where recall that ∇_μ is the covariant derivative, which is $\partial_\mu+ig\sigma^aB^a_\mu$ acting on fermions and $\partial_\mu+ig\tau^aB^a_\mu$ acting on vectors, while (34) and (35) represent the SU(3)$\times$SU(3) singlet and octet pieces of the fermion operators. The λ^a, $a=1,\ldots,8$, are the standard λ matrices of Gell-Mann. The product of two octet currents will in general also have decimet and 27-plet components; these correspond to higher-twist operators and are therefore suppressed. [The vertices associated with the operators (33)–(35) are given in Fig. 1.]

In addition to the operators ${}^nO^V$, ${}^nO^F$ (we often suppress in the following tensor and H indices) there are composite operators formed from Faddeev-Popov ghosts which may have the same twist and therefore are expected to mix with H singlet operators. Arguments to exclude the ghosts on the basis of gauge invariance are unconvincing because the ghosts do in fact mix with the above operators in off-shell matrix elements,



all momenta flow into vertices

FIG. 1. (a) The vertex for ${}^nO^V$ (order 0). (b) The vertex for ${}^nO^F$ (order 0). (c) The vertex for ${}^nO^V$ (order g). (d) The vertex for ${}^nO^F$ (order g).

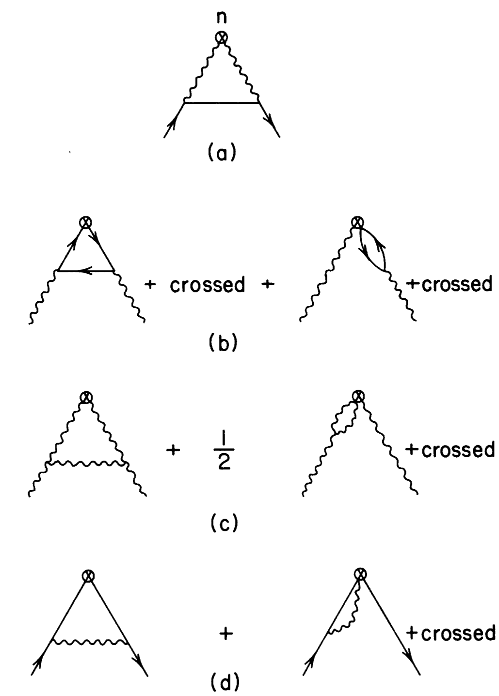


FIG. 2. Graphs for computing $\underline{\gamma}$ (radiative corrections to matrix elements). (a) The matrix element of ${}^nO^V$ between fermion states. (b) The matrix element of ${}^nO^F$ between vector states. (c) The matrix element of ${}^nO^V$ between vector states. (d) The matrix element of ${}^nO^F$ between fermion states.

as is shown by explicit calculation.

It is argued in detail in Appendix A that one can solve this problem by going to a gauge in which no ghosts are present, so that the mixing does not appear. Moreover, the anomalous dimensions of the gauge-invariant operators ${}^nO^V$, ${}^nO^F$ and their mixing are gauge-independent, so they may be calculated in the standard Fermi-type gauges. In the remainder of the text we shall take the results of Appendix A for granted and ignore the ghost mixings.

According to the prescriptions of Sec. II the light-cone behavior of the coefficients in the Wilson expansion, and thereby the high-q^2 behavior of the moments of the structure functions, will be controlled by the renormalization group Eqs. (20). The only missing ingredient in these equations is the matrix $\underline{\gamma}$.

As was briefly mentioned in Sec. II, operators of the same structure (tensor and symmetry properties) and twist will mix, and only certain linear combinations of them will be multiplicatively renormalizable in the usual sense. This phenomenon appears already in Fig. 2(a), where we see that ${}^nO^V$, which in lowest order vanishes between fermion states, acquires (logarithmically divergent) contributions in higher order. In order to compute the $\underline{\gamma}$ matrix of Eq. (21) to second order in g we must according to Eq. (22) express the renormalized operators in terms of the bare ones. This is most easily done by taking matrix elements between vector and fermion states, since the bare operators have trivial matrix elements. We are thus lead to computing the logarithmically divergent pieces of the matrix elements of ${}^nO^V$ and ${}^nO^F$ with the same tensor structure (twist) as ${}^nO^V_{\text{bare}}$ and ${}^nO^F_{\text{bare}}$, between these states. From this we determine the wave-function renormalization matrix and thereby $\underline{\gamma}$. The computations involve evaluating the graphs of Figs. 2(a)–2(d). Because we want to amputate the external propagators there will also be an order-g^2 contribution to the diagonal elements of $\underline{\gamma}$ due to the possibility of self-energy insertions on the propagators, as in Eq. I(5.18).

To summarize the results we introduce the matrix

$$ {}^n\underline{\gamma} = \begin{pmatrix} {}^n\gamma^F_{FF} & {}^n\gamma^V_{FF} \\ {}^n\gamma^F_{VV} & {}^n\gamma^V_{VV} \end{pmatrix}, \tag{36}$$

where ${}^n\gamma^F_{VV}$ is given by $\mu(\partial/\partial\mu)\ln(Z)^F_{VV}$ and so forth.

The results of the computations are (for n = even, $n \geqslant 2$)

$$ {}^n\gamma^V_{VV} = \frac{g^2}{8\pi^2}\left[C_2(G)\left(\tfrac{1}{3} - \frac{4}{n(n-1)} - \frac{4}{(n+1)(n+2)} + 4\sum_2^n \frac{1}{j}\right) + \tfrac{4}{3}\,T(R)\right], \quad (37) $$

$$ {}^n\gamma^F_{FF} = \frac{g^2}{8\pi^2}\left[C_2(R)\left(1 - \frac{2}{n(n+1)} + 4\sum_2^n \frac{1}{j}\right)\right], \quad (38) $$

$$ {}^n\gamma^F_{VV} = \frac{-g^2}{8\pi^2}\,\frac{4(n^2+n+2)}{n(n+1)(n+2)}\,T(R)\,, \quad (39) $$

$$ {}^n\gamma^V_{FF} = \frac{-g^2}{8\pi^2}\,\frac{2(n^2+n+2)}{n(n^2-1)}\,C_2(R)\,, \quad (40) $$

where, as in paper I, $C_2(G)$ is the quadratic Casimir operator of G evaluated on the adjoint representation $C_2(R)$ is the evaluation of the quadratic Casimir operator of G on the irreducible representation R of G to which the fermions belong,[17] and $T(R)$ is the trace of the square of a matrix in the Lie algebra of the representation R. It should be remarked that the mixing terms (39) and (40) vanish for fermion operators which are not singlets under H (we have been suppressing H labels on fermion operators, they should always be understood). The remarkable feature of Eqs. (37)–(38), characteristic of gauge theories,[2] is the $\sum_2^n 1/j \sim \ln(n)$ term. It arises from the graphs of 2(c) and 2(d) with three lines coming out of the vertex, which in turn corresponds to the fact that A_μ occurs with ∂_μ in the covariant derivative.

These values of ${}^n\underline{\gamma}$ satisfy the constraints of positivity (the smallest eigenvalue of ${}^n\underline{\gamma}$ must increase with n), gauge-invariance, and nonrenormalization of the energy-momentum tensor, which provides a check on the calculation (and on the argument of Appendix A).

Having calculated the $\underline{\gamma}$ matrix we can now infer the properties of deep-inelastic scattering according to the methods outlined in Sec. II. First let us review the general features.

1. The approach to the asymptotic region is logarithmic. In other words the leading corrections to the asymptotic forms for the moments of the structure functions will be suppressed by powers of $\ln(-q^2)$. These corrections arise from two sources. First, the fact that the effective coupling $\bar{g}^2(t, g)$ vanishes logarithmically for large t ($\bar{g}^2 \sim 1/t$) means that the order-g^2 corrections to the Wilson coefficients will be suppressed by $1/\ln(-q^2)$ for large $-q^2$. It is hard to estimate how rapidly these corrections vanish in a realistic model, since this will depend on the unknown scale (μ), the actual value of the physical coupling constant, and the large-g^2 behavior of $\beta(g^2)$. The large-g^2 behavior of β is, of course, totally unknown. If, for example, $\beta(\bar{g}^2) = d\bar{g}^2/dt$ were to be linear in $\bar{g}^2$ for large $\bar{g}^2$ then the effective coupling would decrease rapidly (like a power of $-q^2$) from its physical value to small values of $\bar{g}^2$ and only then approach zero logarithmically. In that case one might understand the rapid onset of scaling.

Additional logarithmic corrections occur when more than one operator can contribute to a given moment. This occurs in our theories for the H singlet component of the structure functions. In that case, as explained in Sec. II, the leading asymptotic behavior of the moments will be given by the lowest eigenvalue of the $\underline{\gamma}$ matrix. However, the other eigenvectors of $\underline{\gamma}$ will also contribute terms, which will be suppressed by some (in general noninteger) power of $\ln(-q^2)$. Thus a generic structure function $F(x, q^2)$ will satisfy

$$ \int_0^1 dx\, x^n F(x, q^2) \underset{q^2\to-\infty}{\sim} \sum_i (\ln - q^2)^{-a_i^n} \times F_i^n(\ln(-q^2))\cdots\,, \quad (41) $$

where the a_i^n are proportional to the eigenvalues of $\underline{\gamma}$, and the F's approach constants (at an unknown rate) as $q^2 \to -\infty$.

2. Bjorken scaling is violated by finite powers of logarithms. These logarithmic violations are readily calculated in terms of the $\underline{\gamma}$ matrix evaluated in second-order perturbation theory. Here we must distinguish between the strong symmetry group singlet and nonsinglet pieces of the structure functions. The latter are easy to analyze since there is only one operator, that given in Eq. (35), that contributes. The relevant anomalous dimension is given in Eq. (38). It is, of course, common to all the SU(3)×SU(3) operators that appear in the operator-product expansion. Therefore if $F^{\rm NS}(x,q^2)$ stands for the nonsinglet piece of F_2, xF_1, or xF_3 (independent of the quantum numbers of the currents) we have that

$$ \int_0^1 dx\, x^n F^{\rm NS}(x, Q^2) \underset{Q^2\to\infty}{\sim} C_{\rm NS}^{(n)} (\ln Q^2)^{-A^{\rm NS}_{n+2}}\,, \quad (42) $$

where

$$ A_n^{\rm NS} = \frac{3C_2(R)}{22C_2(G) - 8T(R)}\left(1 - \frac{2}{n(n+1)} + 4\sum_{k=2}^n \frac{1}{k}\right) \quad (43) $$

and

$$ Q_2 = -q^2\,. $$

In the "red, white, and blue" quark model [$H = {\rm SU}(3)$], we have

$$ \begin{aligned} C_2(G) &= 3,\\ C_2(R) &= \tfrac{4}{3},\\ T(R) &= \tfrac{3}{2}\,. \end{aligned} \quad (44) $$

The numerical value of these coefficients is small, i.e., $A_2^{NS}=\frac{16}{81}$, $A_4^{NS}=0.36$, etc. An excellent interpolation formula, accurate to 1% already for $n=2$, is

$$A_n^{NS}\simeq 0.296\ \ln(n)-0.051\ . \tag{45}$$

In treating the singlet piece of the structure functions we must take into account the mixing of the vector-meson and fermionic operators. This mixing occurs for H-singlet operators of both normal and abnormal parity. In the case of abnormal parity the fermion operator mixes with a vector-meson operator given by Eq. (33) with one $F_{\mu\nu}$ replaced by its dual. These operators contribute to the parity-violating structure function F_3. The appropriate $\underline{\gamma}$ matrix in this case will be treated in a subsequent publication. Here we will only deal with H-singlet normal parity operators. Consequently we must diagonalize the $\underline{\gamma}$ matrix evaluated above. For $n=2$ this matrix has the form

$$\underline{\gamma}^{(2)}=\frac{g^2}{8\pi^2}\begin{pmatrix}\frac{8}{3}\ C_2(R) & -\frac{8}{3}C_2(R)\\ \frac{4}{3}\ T(R) & \frac{4}{3}\ T(R)\end{pmatrix}. \tag{46}$$

Thus the smallest eigenvalue is zero, so that the corresponding moments will in fact scale. If $F^S(x,q^2)$ stands for the singlet pieces of F_2 or xF_1, then

$$\int_0^1 dx\,x^n\,F^S(x,Q^2)\underset{Q^2\to\infty}{\sim} C_S^{(n)}\,(\ln Q^2)^{-A^S_{n+2}}\ , \tag{47}$$

where

$$A_n^S=\frac{1}{2g^2b_0}\{{}^n\gamma^F_{FF}+{}^n\gamma^V_{VV} - [({}^n\gamma^F_{FF}-{}^n\gamma^V_{VV})^2+4\,{}^n\gamma^{Fn}_{VV}\,\gamma^V_{FF}]^{1/2}\}\ . \tag{48}$$

Since the off-diagonal matrix elements of $\underline{\gamma}$ vanish rapidly (like $1/n$), we have to a very good approximation (1% for $n=4$) that

$$A_2^S=0\ ,\qquad A_n^S=A_n^{NS}-O\left(\frac{1}{n^2\ln n}\right),\quad n>2\ . \tag{49}$$

The fact that $A_n^S<A_n^{NS}$ is a simple consequence of positivity; however, we note that the difference between these coefficients vanishes rapidly as n increases. Already for $n=4$ it is less than 1%, so that one can hardly differentiate between the single and nonsinglet parts of the structure functions (except of course in the behavior of the first moment).

In asymptotically free gauge theories there are then two stages in the approach to the asymptotic region. In the first stage, when we have reached what might be called the approximate scaling region, the effective coupling constant becomes small and the one-loop approximation to the renormalization-group equation becomes valid. In this region we have scaling up to finite powers of logarithms as previously computed. The rate of approach to this region is not determined by the methods of this paper. In the second stage, which might be called the true asymptotic region, quantities suppressed by powers of logarithms become effectively zero, and the sum rules (energy-momentum and parton-model sum rules) mentioned below become valid.

3. *Sum rules* and relations between moments of the structure functions which follow from the tensorial and SU(3)×SU(3) structure of the free-quark model are true asymptotic theorems in our theories [if $H=$ SU(3)]. This transpires because we have chosen the strong gauge group G to commute with H [say SU(3)]. Therefore the vector mesons are neutral with respect to the SU(3)×SU(3) charges, and the coefficients $C_V^{(n)}$ vanish when $g=0$. Thus the tensorial and SU(3)×SU(3) structure of the Wilson expansion for large $-q^2$ will be identical to that of free-field theory.

The Adler sum rule[18] is of course valid for all q^2. The Gross–Llewellyn Smith sum rule[19] holds, since for this moment the appropriate anomalous dimension vanishes for all g. It, however, is approached logarithmically, i.e.,

$$\int_0^1 dx[F_3^{\nu p}(x,q^2)+F_3^{\nu n}(x,q^2)]=-6\,F(q^2)\ ,\qquad F(q^2)\underset{q^2\to-\infty}{\sim}1+O\left(\frac{1}{\ln(-q^2)}\right)\ . \tag{50}$$

The Llewellyn Smith relation[20] holds for individual moments

$$\int_0^1 dx\,x^n\{6[F_2^{ep}(x,q^2)-F_2^{en}(x,q^2)] - x[F_3^{\nu p}(x,q^2)-F_3^{\nu n}(x,q^2)]\}\approx O\left(\frac{1}{\ln(-q^2)}\right)\ , \tag{51}$$

as well as the various inequalities between *moments* of the structure functions discussed by Llewellyn Smith[20] and Nachtmann.[21]

All these are relations or sum rules for the *moments* of the structure functions. As will be shown below it does not necessarily follow that the relations are valid for the structure functions themselves. Thus the quantity in parentheses in Eq. (51) does not necessarily vanish for a given x like $1/\ln(-q^2)$ as $q^2\to-\infty$.

Similarly the Callan-Gross relation holds for moments of the structure function. In other words the ratio of the moments of $F_L = F_2 - 2xF_1$ and F_2 vanishes logarithmically for large $-q^2$:

$$\frac{\int_0^1 dx\, x^n F_L(x, q^2)}{\int_0^1 dx\, x^n F_2(x, q^2)} \underset{q^2 \to -\infty}{\sim} O\left(\frac{1}{\ln(-q^2)}\right) . \quad (52)$$

However this does not necessarily imply that $F_L(x, q^2)/F_2(x, q^2)$ vanishes like $1/\ln(-q^2)$ for fixed x.

Finally we can derive, in our models, a new sum rule which is related to the matrix element of the energy-momentum tensor.[22] The singlet piece of the Wilson expansion will contain the energy-momentum tensor $\theta_{\mu\nu}$ with vanishing anomalous dimension. According to the discussion in Sec. II this will appear, for large $-q^2$, in the form

$$(O_F^{(2)}, O_V^{(2)})\underline{P}^{(2)} \begin{pmatrix} C_F(1, 0) \\ 0 \end{pmatrix}, \quad (53)$$

where we have set $\underline{P}^{(2)}\underline{M}^{(2)} = \underline{P}^{(2)}$ (since the anomalous dimension of the energy-momentum tensor vanishes), C_F is the coefficient of the fermionic part of $\theta_{\mu\nu}$ $(O_F^{(2)})$, and the coefficient of the vector contribution to $\theta_{\mu\nu}$ $(O_V^{(2)})$ vanishes when $\bar{g} = 0$. The projection matrix $\underline{P}^{(2)}$ is given from Eq. (46) as

$$\underline{P}^{(2)} = \frac{1}{2C_2(R) + T(R)} \begin{pmatrix} T(R) & 2C_2(R) \\ T(R) & 2C_2(R) \end{pmatrix} \quad (54)$$

so that the coefficient of $\theta_{\mu\nu} = \theta_F^{(2)} + \theta_V^{(2)}$ is given, for large $-q^2$, by

$$\frac{T(R)}{2C_2(R) + T(R)} C_F(1, 0) . \quad (55)$$

The net effect of the mixing of operators is to multiply the free-field theory value by r, where

$$r = \frac{T(R)}{2C_2(R) + T(R)}$$

$= \frac{9}{25}$ for the "red, white, and blue" quark model. (56)

The contribution of the energy-momentum tensor to the commutator of two vector (or axial-vector) currents is thus (for a quark model)

$$[J_\mu^a(x), J_\nu^b(-x)] = (\tfrac{1}{3}\,\mathrm{Tr}\,\lambda^a\lambda^b)\frac{-i}{2\pi}\,\delta'(x^2) r \times [x^\lambda \theta_{\mu\lambda}(0) x_\nu + x^\lambda \theta_{\nu\lambda} x_\mu - g_{\mu\nu} x^\alpha x^\lambda \theta_{\alpha\lambda}] . \quad (57)$$

As a consequence the singlet piece of F_2 for electroproduction will satisfy

$$\int_0^1 dx\, F^e_{2,\,\mathrm{singlet}}(x, q^2) \underset{q^2 \to -\infty}{\longrightarrow} \langle Q^2 \rangle r , \quad (58)$$

where $\langle Q^2 \rangle$ is the average quark charge. In the "red, white, and blue" quark model this has the value $\frac{2}{25}$. This sum rule holds for any hadronic target. However, the corrections to it, arising from the nonsinglet operators, might be very large. They vanish for infinite q^2 but at a rate governed by $A_2^{\mathrm{NS}} = 0.2$ (in the "red, white, and blue" quark model).

Similarly in the case of neutrino or antineutrino the singlet contribution to $F_2^{(\nu,\bar{\nu})}$ satisfies (setting the Cabibbo angle equal to zero) in the quark model

$$\int_0^1 dx\, F^{(\nu,\bar{\nu})}_{2,\,\mathrm{singlet}}(x, q^2) \underset{q^2 \to -\infty}{\longrightarrow} \tfrac{2}{3} r . \quad (59)$$

To improve the convergence one can take linear combinations of structure functions to obtain a pure SU(3) singlet:

$$\int_0^1 dx [6(F_2^{ep} + F_2^{en}) - (F_2^{\nu p} + F_2^{\nu n})] \underset{q^2 \to -\infty}{\longrightarrow} \tfrac{4}{3} r . \quad (60)$$

The value of this sum rule for "red, white, and blue" quarks is 0.48, whereas experimentally one has 0.72 ± 0.28.

We have determined the large-$(-q^2)$ behavior of the moments of the structure functions. What can one say about the q^2 behavior of the functions themselves? It is useful to construct an example of a function with roughly the correct anomalous dimensions. Consider the nonsinglet piece of $F_2^{\mathrm{NS}}(x, q^2)$ for electroproduction. It satisfies Eq. (42). If the constants $C_{\mathrm{NS}}^{(n)}$ were known then one could construct $F_2^{\mathrm{NS}}(x, q^2)$. Since these constants are not known there exist many functions which satisfy Eq. (42). Let us approximate A_n^{NS} by its asymptotic form, $A_n^{\mathrm{NS}} = \alpha \ln(n) - \beta$. For the "red, white, and blue" quark model $\alpha = 0.296$, $\beta = 0.051$. Then we have to find a function $F^{\mathrm{NS}}(x, q^2)$ which satisfies

$$\lim_{q^2 \to -\infty} \int_0^1 dx\, x^n F(x, q^2) = C^{(n)} e^{+\beta L}(n+2)^{-\alpha L},$$

$$L = \ln\ln(-q^2) . \quad (61)$$

In addition we shall impose Regge behavior, in the sense that for fixed q^2 we demand that $F(x, q^2)$ approach a constant as $x \to 0$.

A solution of this is provided by

$$F(x, q^2) = \frac{e^{\beta L}}{\Gamma(\alpha L + 1)} \int_c^{c - \ln x} dy\, e^{-y} y^{\alpha L} , \quad (62)$$

where c is an arbitrary constant. This solves Eq. (61) with

$$C^{(n)} = \frac{e^{c(n+1)}}{(n+1)(n+2)} .$$

One can easily construct additional solutions by multiplying Eq. (62) by a polynomial in x. This function has Regge behavior, i.e.,

$$F(0, q^2) = \frac{e^{\beta L}}{\Gamma(\alpha L + 1)} \int_c^\infty dy\, e^{-y}\, y^{\alpha L} \underset{q^2 \to -\infty}{\sim} [\ln(-q^2)]^\beta \,. \quad (63)$$

At fixed $x \neq 0, 1$ it behaves, for large $-q^2$, like

$$F(x, q^2) \approx \left(\frac{(c - \ln x)}{\alpha L}\right)^{\alpha L} e^{\beta L} \,. \quad (64)$$

Accordingly $F(x, q^2)$ will (a) for any $x \neq 0, 1$, vanish faster than any power of $\ln(-q^2)$ for large enough $-q^2$ (in fact $F(x, q^2) < [\ln(-q^2)]^{-\ln\ln\ln(-q^2)}$), and will (b) increase for intermediate values of $-q^2$ for sufficiently small x and decrease for x close to unity. The transition point, at which F does not change as $-q^2$ increases, is roughly $x \approx 1/[\ln(-q^2)]^p$, where p is some positive constant. (c) Finally, $F(x, q^2)$ will show largest deviations from scaling in the vicinity of $x = 1$.

Such behavior is more general than this particular example. Specifically (a) is a simple consequence of positivity,

$$\begin{aligned} F(x, q^2) &\simeq \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} dy\, F(y, q^2) \\ &\leq \frac{1}{2\epsilon(x-\epsilon)^n} \int_0^1 dy\, y^n\, F(y, q^2) \\ &\underset{q^2 \to -\infty}{\sim} F_n [\ln(-q^2)]^{-A_n}, \end{aligned} \quad (65)$$

and the fact that $A_n \to \ln n$ as $n \to \infty$.

This example also illustrates the rather small variation in q^2 of the asymptotic form of the structure functions. To test for deviations from scaling one will require large variations of q^2 and measurements in the vicinity of $x = 1$. One might expect a 50% variation for $x \approx 0.9$ as $-q^2$ increases from 10 to 50 BeV^2.

This example further illustrates that two different solutions of Eq. (61) might have quite different q^2 behavior for a given value of x.[23] Consider the solution F' to Eq. (61) which is given by Eq. (62) with c replaced by $c' < c$. Then, for large $-q^2$,

$$\frac{F(x, q^2)}{F(x, q^2)} \approx [\ln(-q^2)]^{\ln(c/c')} \geq \left(\frac{c - nx}{c' - nx}\right)^L \geq 1 \,.$$

By making the ratio c/c' large we can arrange for F/F' to increase like a large power of $\ln(-q^2)$ for x near unity.

As a consequence the parton-model relations for moments of the structure functions, for example Eq. (51), do not imply that these relations are satisfied for the structure functions themselves. Also the fact that the moments of the longitudinal structure functions decrease like $1/\ln(-q^2)$ relative to the moments of the transverse structure function [Eq. (52)] does not imply that $R(q^2, x) = F_L(q^2, x)/F_2(q^2, x)$ decreases logarithmically for all x. Indeed it might very well increase for x close to unity. Without additional input the only reliable predictions one can make are with respect to the q^2 behavior of the moments.

IV. CONCLUSIONS

A crucial test of asymptotically free gauge theories of the strong interactions is the verification of the q^2 behavior of the moments of the structure functions. In the best of all worlds one could confront an infinite number of these moments with the predicted asymptotic forms (which are determined solely by the gauge group and the fermion representation). An additional test is provided by the energy-momentum sum rule derived above.

In reality, of course, it will be very hard to determine the q^2 behavior of the moments. More than likely the most practical place to look for violations of Bjorken scaling is in the vicinity of threshold ($x \approx 1$). There we expect to see the structure functions decreasing like ever-increasing powers of $\ln(-q^2)$.

Whether the picture described in the paper is consistent with experiment is an open question. The fact that scaling appears to have set in already for rather small values of q^2 is neither explained nor contradicted by our theories. The rate of approach to asymptopia must be determined by nonperturbative methods. This problem, as well as the understanding of the low-energy and on-mass-shell behavior of the theory, requires the development of new theoretical techniques.

APPENDIX A: PROBLEM OF GHOST MIXING

One is tempted to argue that only manifestly gauge-invariant operators appear in the Wilson expansion of the product of two gauge-invariant operators. However, in the case of composite operators constructed from Faddeev-Popov ghost fields it is not clear how to formulate the criterion of gauge invariance. It does not seem that the ghost fields have any simple transformation properties under gauge transformations (in particular, one can choose gauges in which the ghosts are entirely absent). Moreover we will show in a simple example that operators constructed from ghost fields "mix" in a highly nontrivial way with the manifestly gauge-invariant operators of lowest twist which control the Bjorken limit, in the sense that the γ matrix appearing in the renormalization-group equations for these operators has nonvanishing off-diagonal entries. This appendix is devoted first to showing that a nontrivial problem is involved here, and second to describing that argument we used to convince ourselves that the calculations given in the main test do indeed give the correct asymptotic behavior for products of currents in the Bjorken limit. We argue that the anomalous dimensions (renormalization constants) of manifestly gauge-invariant operators are independent of the gauge in which they are computed, so since there exist gauges in which no Faddeev-Popov ghosts are present, the correct values are obtained in gauges with ghosts by ignoring the ghost mixing.

An instructive example of a gauge field theory is quantum electrodynamics with the unusual gauge choice[24]

$$\partial_\mu A^\mu - \tfrac{1}{2}\alpha g A_\mu A^\mu = 0\,, \tag{A1}$$

where g is the coupling constant and α is a gauge parameter. The Feynman rules for this theory are given in Ref. 24. Notice the presence of Faddeev-Popov ghosts.

For simplicity we consider the theory without fermions. In this case we are of course just dealing with free-field theory. The manifestly gauge-invariant operators of lowest twist are

$$^{V}O^{(n)}_{\mu_1\cdots\mu_n} = S F_{\mu_1\alpha}\overleftrightarrow{\partial}_{\mu_2}\cdots\overleftrightarrow{\partial}_{\mu_{n-1}} F_{\mu_n}{}^{\alpha}\,, \tag{A2}$$

where S denotes symmetrization with respect to the μ's. To compute γ_V, the anomalous dimension of the vector field, to order g^2 we need only consider the logarithmically divergent parts of the two self-energy graphs with vector and ghost loops. The result is[25]

$$\gamma_V = 0\,. \tag{A3}$$

We can now compute $^{n}\gamma^{V}_{VV}$, the diagonal term in the γ matrix for the operator $^{V}O^{(n)}$, from the radiative corrections to the vertex shown in Fig. 3(a). The result is

$$\gamma^{V}_{VV} = 0\,. \tag{A4}$$

To see whether these operators mix with the ghost operators we need only check that the anomalous dimension for $^{V}O^{(n)}$, sandwiched between ghost states γ^{V}_{GG}, does not vanish. A calculation exactly like the one just described for the vectors involving the diagrams of Fig. 4(a) gives

$$\gamma_G = \frac{-\alpha^2 g^2}{16\pi^2}\,, \quad \gamma^{V}_{GG} = \frac{-1}{n(n-1)}\,\frac{\alpha^2 g^2}{16\pi^2}\,. \tag{A5}$$

Notice for $\alpha = 0$ we are in ordinary Landau gauge, and of course no ghost mixing can occur (there are no ghosts).

This shows there is an operator that mixes with $^{V}O^{(n)}$ which to lowest order (g^0) is given by (ρ is the ghost field)

$$^{G}O^{(n)}_{\mu_1\cdots\mu_n}(x) = S\rho^*(x)\overleftrightarrow{\partial}_{\mu_1}\cdots\overleftrightarrow{\partial}_{\mu_n}\rho(x) + O(g)\,. \tag{A6}$$

In addition there might be order-g terms in which one of the derivatives is replaced by gA_μ, arising from the diagrams in Fig. 4(b).

To complete the picture we notice that γ^{G}_{VV}, the anomalous dimension of the ghost operator sandwiched between vector states, does not vanish. This is readily seen by noticing that only Fig. 5(a) can give a term of the structure $g_{\mu\nu}\ln\Lambda^2$ (notations are as in Sec. III). So we know that if we failed to consider other operators the γ matrix

$$^{n}\gamma = \begin{pmatrix} ^{n}\gamma^{V}_{VV} & ^{n}\gamma^{G}_{VV} \\ ^{n}\gamma^{V}_{GG} & ^{n}\gamma^{G}_{GG} \end{pmatrix} \tag{A7}$$

could not have zero eigenvalues ($^{n}\gamma^{V}_{VV} = 0$, $^{n}\gamma^{G}_{VV} \neq 0$, $^{n}\gamma^{V}_{GG} \neq 0$, so $\det\gamma \neq 0$). However, we know that we are dealing with free-field theory, so there must be an operator with zero anomalous dimensions. Evidently there are even more operators which mix, and there is a complicated cancellation mechanism among many operators which are not individually gauge-invariant.

To verify that the same phenomenon occurs in

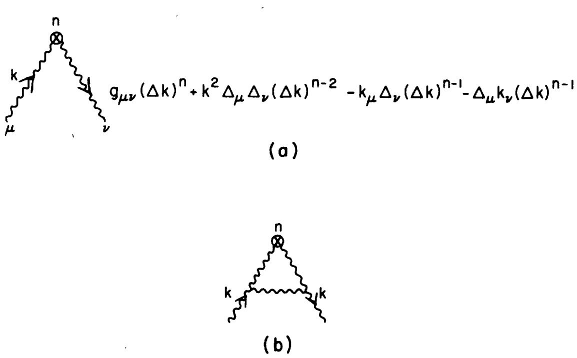


FIG. 3. Graphs for computing γ^{V}_{VV} in quantum electrodynamics in the gauge of Eq. (A1). (a) The vertex for $^{V}O^{(n)}$. (b) The radiative correction to the matrix element of $^{V}O^{(n)}$ between vector states.

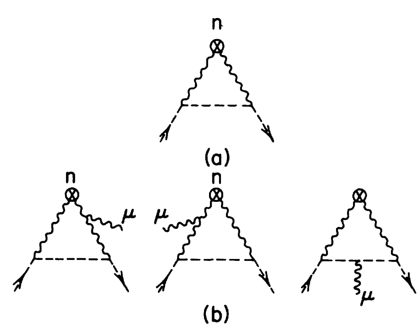


FIG. 4. Mixing of vector and ghost operators. (a) The zero-order contributions to $^{G}O^{(n)}$. (b) The order-g contributions to $^{G}O^{(n)}$.

Yang-Mills theory we will work in a gauge where there are no Faddeev-Popov ghosts.[26] Our gauge is specified by

$$n^\mu A_\mu^a = 0\,, \tag{A8}$$

where n is a fixed vector and a is the group index. (This gauge condition is not Lorentz-invariant.) The absence of Faddeev-Popov ghosts is a consequence of the fact that, under a gauge transformation

$$\delta A_\mu(x) = \partial_\mu \chi(x) - g[\chi(x), A_\mu(x)],$$

$\delta(n^\mu A_\mu^a)$ is independent of A for fields satisfying (A8).

In order to implement gauge conditions like (A8) we add a term $-(1/2\alpha)(n^\mu A_\mu)^2$ to the Lagrangian. This leads to the standard Feynman rules for Yang-Mills theories but without the ghosts and with the propagator modified to be

$$D_{ab}^{\mu\nu}(p) = \frac{i\delta_{ab}}{p^2}\left(-g^{\mu\nu} + \frac{\alpha p^2 - n^2}{(n\cdot p)^2}p^\mu p^\nu + \frac{n^\mu p^\nu + p^\mu n^\nu}{n\cdot p}\right). \tag{A9}$$

The theory in this gauge is not renormalizable by power counting if $\alpha \neq 0$. From now on we will always assume that $\alpha = 0$ and simplify further by taking n^μ lightlike, $n^2=0$. Then the propagator is

$$D_{ab}^{\mu\nu}(p) = \frac{i\delta_{ab}}{p^2}\left(-g_{\mu\nu} + \frac{n_\mu p_\nu + p_\mu n_\nu}{n\cdot p}\right). \tag{A10}$$

The calculation of γ_V and ${}^n\gamma_{VV}^V$ is straightforward, following the pattern of Sec. III. The same diagrams [Fig. 2(c)] are involved, except that there is no ghost-loop contribution to the self-energy. The results are

$$\gamma_V = -(g^2/8\pi^2)C_2(G)\tfrac{11}{6}, \tag{A11}$$

$${}^n\gamma_{VV}^V = (g^2/8\pi^2)C_2(G) \times\left(\frac{1}{3} - \frac{4}{n(n-1)} - \frac{4}{(n+1)(n+2)} + 4\sum_2^n \frac{1}{j}\right). \tag{A12}$$

In the calculations we have taken $nk = n\Delta = 0$ (k is the *external* momentum). It turns out that the dangerous-looking denominators in (A10) all cancel from internal propagators, at least in our calculation of logarithmically divergent pieces.

Although γ_V is certainly gauge-dependent, the ${}^n\gamma_{VV}^V$ are not.[27] In fact the ${}^n\gamma_{VV}^V$ have the same values in the ghost-free gauges as in the Fermi-type gauges. In the ghost-free gauges the operator-product expansion for gauge-invariant operators takes a simple form. In particular there appear no ghost operators. The anomalous dimensions of the operators appearing in it can, however, be computed in any convenient gauge. This demonstrates why the calculations given in the text are relevant for determining asymptotic behavior in the Bjorken limit, despite the apparent problem of ghost mixing. The same apparent problem arises even in quantum electrodynamics in the unusual gauge (A1), but here, of course, we *know* the correct answer, and it is correctly given by the arguments of this appendix.

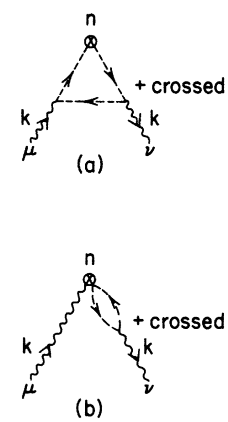


FIG. 5. Contributions to ${}^n\gamma_{VV}^G$. The coefficients of $g_{\mu\nu}$ arise from diagram (a).

APPENDIX B: SAMPLE CALCULATION, ${}^n\gamma_{FF}^F$

In this appendix we will give details of the calculation of ${}^n\gamma_{FF}^F$. The techniques used here should enable the reader to duplicate without too much difficulty most of the numerical results in this paper. The calculation will also clarify the origin of the $\sum_2^n 1/j$ terms in the anomalous dimensions which are characteristic of gauge theories and give pronounced violations of canonical scaling as $n\to\infty$.

The key tool in these calculations is the angular average integral

$$\frac{1}{2\pi^2}\int d\Omega\, k_{\mu_1}\cdots k_{\mu_n} = \frac{2}{(n+2)!!}\sum_{\text{pairs}} g_{\mu_{i(1)}\mu_{i(2)}}\cdots g_{\mu_{i(n-1)}\mu_{i(n)}}, \tag{B1}$$

where on the left-hand side we have the average over the unit sphere in Euclidean 4-space, while on the right-hand side the sum runs over all possible ways of grouping the μ's into pairs. This formula holds for even n; for odd n the left-hand side vanishes. The formula is readily proved by induction.

Let us evaluate the first graph of Fig. 2(d). (Δ is an arbitrary vector which is contracted into all free indices of the fermion tensor.) In the Feynman gauge it is, after some simple Dirac algebra,

$$ig^2C_2(R)\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p^2)^2(p-k)^2}\times[2p^2\not\Delta-4(p\cdot\Delta)\not\Delta](\Delta\cdot p)^{n-1}. \quad \text{(B2)}$$

We are interested in the (logarithmically divergent) coefficient of $(\Delta\cdot k)^{n-1}\not\Delta$ in the expansion of this integral. Throwing away other terms, we write symbolically

$$c_1(\Delta\cdot k)^{n-1}\not\Delta\ln\Lambda^2=ig^2C_2(R)\times\int\frac{d^4p}{(2\pi)^4}\frac{2p^2\not\Delta-4(p\cdot\Delta)\not p}{(p^2)^2(p-k)^2}(\Delta\cdot p)^{n-1}, \quad \text{(B3)}$$

where Λ is an ultraviolet cutoff. Differentiating (B3) on both sides $n-1$ times with respect to k and keeping only logarithmically divergent parts gives

$$c_1(n-1)!\Delta_{\epsilon_1}\cdots\Delta_{\epsilon_{n-1}}\not\Delta\ln\Lambda^2=ig^2C_2(R)\int\frac{d^4p}{(2\pi)^4}\frac{2^{n-1}(n-1)!}{(p^2)^{n+2}}p_{\epsilon_1}\cdots p_{\epsilon_{n-1}}(\Delta\cdot p)^{n-1}[2p^2\not\Delta-4(p\cdot\Delta)\not p]. \quad \text{(B4)}$$

Now rotating into Euclidean space and using (B1) gives

$$c_1=\frac{-g^2}{16\pi^2}C_2(R)\left(\frac{2}{n(n+1)}\right). \quad \text{(B5)}$$

The second graph of Fig. 2(d) gives

$$ig^2C_2(R)\int\frac{d^4p}{(2\pi)^4}\frac{(2p\cdot\Delta)\not\Delta}{p^2(p-k)^2}\sum_{j=0}^{n-2}(\Delta\cdot p)^j(\Delta\cdot k)^{n-2-j}. \quad \text{(B6)}$$

The reader who has worked through the previous example carefully should be able to do this integral "by inspection"; the answer is

$$c_2=\frac{g^2}{16\pi^2}C_2(R)2\sum_{j=2}^{n}\frac{1}{j} \quad \text{(B7)}$$

with the obvious definition of c_2.

The third graph gives the same number as the second.

Putting these results together gives

$${}^n\gamma_{FF}^F-2\gamma_F=\frac{g^2}{8\pi^2}\left(-\frac{2}{n(n+1)}+4\sum_{2}^{n}\frac{1}{j}\right)$$

and with our old result γ_F [Eq. I (4.19)] we get the announced result for ${}^n\gamma_{FF}^F$.

In this calculation it was clear that the origin of the terms in ${}^n\gamma$ growing logarithmically with n arose from the replacement of ordinary by covariant derivitives. In terms of the "hand-waving argument" of Ref. 2, it appears that covariant derivatives do not "separate" in space the fields they are sandwiched between.

The method outlined here is a very powerful one for calculations of anomalous dimensions at the one-loop level. Other methods have also been proposed for this purpose.[28] It is doubtful that any method of comparable simplicity exists for higher-order calculations.

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Eigenvalues of the Faddeev equations with repulsive two-body interactions*

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The attractive (positive) and repulsive (negative) linear eigenvalues of the Lippmann-Schwinger equation can become arbitrarily small in absolute value for sufficiently strong interactions. In this paper we show that this cannot be the case for the Faddeev equations. Three-body eigenvalues arising from the repulsive part of the two-body interaction must have a certain minimum absolute value or the Faddeev equations can predict physical bound states held together by this part of the interaction. It is shown that this limitation upon the eigenvalues arises from the suppression in the T matrix of the repulsive part of the two-body potential.

The linear eigenvalues of the Faddeev equations have been of considerable interest recently due to their importance in iterative methods for determining the three-body binding energy[1] and in studies of the three-body D function.[2] In this paper we study the nature of the linear-eigenvalue spectrum of the Faddeev equations in the presence of two-body interactions which contain a repulsive part. Our primary result is that, because the repulsion in the two-body potential is strongly suppressed in the two-body T matrix, the Faddeev eigenvalues arising from the repulsion in the interaction have a certain minimum absolute value. Physical arguments also dictate this behavior, for otherwise the Faddeev equations would predict bound states held together by the repulsive part of the interaction.

The eigenvalue spectrum

For three identical particles the linear-eigenvalue problem can be expressed as[3]