$$\int d\vec{r}\,\tilde{\rho}(\vec{s},\vec{r};\tau)\tilde{\rho}(\vec{r},\vec{s};\beta-\tau)=\tilde{\rho}(\vec{s},\vec{s};\beta),$$

we obtain

$$C=\beta\int d\vec{s}\tilde{\rho}(\vec{s},\vec{s};\beta)W(\vec{s}). \quad (4.4)$$

From the relation[8]

$$\tilde{\rho}(\vec{r},\vec{r};\beta)\sim(2\pi\beta\hbar^2/m)^{-3/2}|\tilde{\psi}(0,\vec{r})|^2, \quad \beta\to\infty \quad (4.5)$$

with $\tilde{\psi}(0,\vec{r})$ denoting the zero-energy limit of the outgoing scattering solution $\tilde{\psi}(\vec{k},\vec{r})$ of the Schrödinger equation $(E_{\vec{k}}-\tilde{H})\tilde{\psi}=0$, it then follows that

$$C\sim(2\pi\beta\hbar^2/m)^{-1/2}\alpha, \quad \beta\to\infty \quad (4.6)$$

$$\alpha\equiv\frac{m}{2\pi\hbar^2}\int d\vec{r}\,\tilde{\psi}^*(0,\vec{r})W(\vec{r})\tilde{\psi}(0,\vec{r}). \quad (4.7)$$

From (4.6) and (2.4), it follows that $g(C)=-\alpha$. Since it has already been shown that $g(B)=\tilde{A}_0$, and since it is clear from Eq. (2.4) that $g(B+C)=g(B)+g(C)$, we see from Eq. (3.4) that

$$g(Z_B^{(0)})=\tilde{A}_0-\alpha .$$

In view of Eq. (4.3) and the relation

$$\tilde{H}\tilde{\psi}(0,\vec{r})=0,$$

the VUB on A_0 can also be written as

$$g(Z_B^{(0)})=\tilde{A}_0+\frac{m}{2\pi\hbar^2}\int d\vec{r}\,\tilde{\psi}^*(0,\vec{r})H\tilde{\psi}(0,\vec{r}), \quad (4.8)$$

which is the desired result.

---

[1] D. Gelman and L. Spruch, J. Math. Phys. 10, 2240 (1969).
[2] R. P. Feynman, Rev. Mod. Phys. 20, 367 (1948).
[3] L. Spruch and L. Rosenberg, Phys. Rev. 116, 1034 (1959).

[4] All of the terms and notations which we are using were defined in Ref. 1; they will not be repeated here.
[5] A brief discussion of the potential utility of the bounds derived in Ref. 1, which is applicable also to the bound obtained herein, is given in the Introduction of Ref. 1.

[6] S. Servadio, Phys. Rev. A 4, 1256 (1971).

[7] This relation follows from the equation $J(s)=J'(s)$, where J and J' are defined by Eqs. (D1) and (D2) of Ref. 1, by the method exactly analogous to that used in deriving Eq. (D14) from Eq. (D13).
[8] This follows from Eqs. (2.8) and (2.4) of Ref. 1.

PHYSICAL REVIEW D VOLUME 9, NUMBER 4 15 FEBRUARY 1974

Renormalization of gauge theories—unbroken and broken

Benjamin W. Lee*†
National Accelerator Laboratory, Batavia, Illinois 60510
(Received 29 October 1973)

A comprehensive discussion is given of the renormalization of gauge theories, with or without spontaneous breakdown of gauge symmetry. The present discussion makes use of the Ward-Takahashi identities for proper vertices (as opposed to the identities for Green's functions) recently derived. The following features of the present discussion are significant: (1) The present discussion applies to a very wide class of gauge conditions. (2) The present discussion applies to any gauge group and any representation of the scalar fields. (3) The *renormalized S* matrix is shown to be gauge-independent. (4) Dependence of counterterms on the gauge chosen is discussed.

I. INTRODUCTION

In an earlier publication,[1] we have given a derivation of the Ward-Takahashi (WT) identity for the generating functional of proper vertices in non-Abelian gauge theories. Previous discussions on the renormalizability of gauge theories[2-5] were based on the Ward-Takahashi identities for Green's functions.[6,7] The renormalization procedure is usually stated in terms of proper vertices, so that the WT identities for proper vertices would facilitate enormously the discussion of renormalizability.

In this paper we shall reexamine the renormalizability of gauge theories in terms of the WT identities for proper vertices. In addition to rederiving many results of Refs. 2, 3, and 5 (which we shall refer to as LZI, LZII, and LZIV, respectively), we shall add the following elements to our discussions: (1) We shall discuss the renormalizability of gauge theories in a wide class of gauge conditions. The gauge conditions we shall consider are linear in field variables and of dimension 2 or less. Most gauge conditions considered in the literature[4,5,8] are of this kind. (2) We shall consider all possible gauge symmetries based on semi-simple compact Lie groups. Thus, the gauge symmetry G is assumed to be a direct product of n simple groups $G_1\times G_2\times\cdots\times G_n$. Our discussion will apply also to groups which are not completely reducible (i.e., to groups in which the product of two irreducible representations R and R' contains a third R'' more than once). We consider arbitrary representations for scalar fields under G. We shall consider theories consisting of gauge vector bosons and any number of scalar bosons. In an anomaly-free gauge theory[9,10] spinor fields do not present any new problems,[11] and may be treated in much the same way as scalar fields. (3) We shall show that the renormalization procedure leads to the same renormalized S matrix irrespective of the gauge chosen in spontaneously broken gauge theories (SBGT). (4) Dependence of renormalization counterterms and constraints on the gauge chosen is studied.

In our discussion of renormalizability, we shall be deliberately unspecific as to the finite parts of mass-renormalization counterterms and the finite multiplicative factors of renormalization constants, so as to make the discussion free from any specific renormalization conditions one might adopt. There is a price to be paid for this, and it is that one must regularize the theory first in order to give a sensible discussion. We will adopt the gauge-invariant dimensional-regularization method of 't Hooft and Veltman,[12] in the form discussed in Ref. 13.

In Sec. II we review the WT identities for Green's functions and for proper vertices. In Sec. III we discuss the renormalization transformations of field variables and parameters of the theory and study their effects on the WT identities for proper vertices. In Sec. IV we discuss the renormalizability of unbroken gauge theories in the so-called R gauges.[2,3] This discussion supersedes that of LZI. In Sec. V the considerations of Sec. IV are extended to arbitrary linear gauges. Section VI is a discussion of the renormalizability of SBGT in any linear gauges and augments that of LZII. Section VII is an elaboration on the gauge independence of the renormalized S matrix in SBGT.

II. WARD-TAKAHASHI IDENTITIES—A REVIEW

A. Notations

In discussing gauge theories, unless we agree on a highly condensed notation, we are apt to be defeated by the complexities of indices. For this reason, we shall agree to denote all fields by ϕ_i. The index i stands for all attributes of the fields. For the gauge field $b_\mu^\alpha(x)$, i stands for the group index α, the Lorentz index μ, and the space-time variable x; for the scalar field $\psi_a(x)$, i stands for the representation index a and x. Summation and integration over repeated indices shall be understood in this section unless noted otherwise. The infinitesimal local gauge transformations of ϕ_i may be written as

$$\phi_i\to\phi_i'=\phi_i+(\Lambda_i^\alpha+t_{ij}^\alpha\phi_j)\omega_\alpha\,, \tag{2.1}$$

where $\omega_\alpha=\omega_\alpha(x_\alpha)$ is the space-time-dependent parameter of a compact Lie group G. We choose a real basis for ϕ_i so that the matrix $(t^\alpha)_{ij}=t_{ij}^\alpha$ is real antisymmetric. The inhomogeneous term Λ_i^α in (2.1) is of the form

$$\Lambda_i^\alpha=\left(\frac{1}{g}\right)^{\alpha\beta}\partial_\mu\,\delta^4(x-x_\alpha),\ \text{ for }\ \phi_i=b_\mu^\beta(x)$$
$$=0,\ \text{ otherwise} \tag{2.2}$$

where $g_{\alpha\beta}=g_\alpha\delta_{\alpha\beta}$ is the gauge-coupling-constant matrix. If the group G is a direct product of n simple groups $G_1\times G_2\times\cdots\times G_n$, there are in general n gauge coupling constants $g_1, g_2,\ldots,g_n$. Within the same factor group G_i, we have of course $g_\alpha=g_\beta$.

We have

$$t_{ik}^\alpha(t_{kj}^\beta\phi_j+\Lambda_k^\beta)-t_{ik}^\beta(t_{kj}^\alpha\phi_j+\Lambda_k^\alpha)=f^{\alpha\beta\gamma}(t_{ij}^\gamma\,\phi_j+\Lambda_i^\gamma)\,, \tag{2.3}$$

where $f^{\alpha\beta\gamma}$ is the completely antisymmetric structure constant of the gauge group. The invariance of the Lagrangian under the gauge transformation (2.1) may be formulated as

$$(\Lambda_i^\alpha+t_{ij}^\alpha\phi_j)\,\frac{\delta L[\phi]}{\delta\phi_i}=0\,. \tag{2.4}$$

B. Feynman rules

To quantize the theory, we shall choose a gauge condition linear in ϕ:

$$F_\alpha[\phi]\equiv F_{\alpha i}\,\phi_i=0\,, \tag{2.5}$$

where

$$F_{\alpha i}=(\xi_\alpha)^{1/2}g_{\alpha\beta}\Lambda_i^\beta\quad(\text{not summed over }\alpha),\qquad \text{for }\ \phi_i=b_\mu^\gamma(x)$$
$$=\frac{1}{(\xi_\alpha)^{1/2}}\,c_i^\alpha\quad(\text{not summed over }\alpha),\qquad \text{for }\ \phi_i=\psi_c(x). \tag{2.6}$$

We shall be dealing with the case in which c_i^α is a numerical constant. In Eq. (2.6) ξ_α is a positive numerical parameter which can be varied.

The Feynman rules for constructing Green's functions can be deduced[14] from the effective action L_{eff}:

$$L_{\text{eff}}[\phi,c,c^\dagger]=L[\phi]-\tfrac{1}{2}\{F_\alpha[\phi]\}^2+c_\alpha^\dagger M_{\alpha\beta}[\phi]c_\beta\,, \tag{2.7}$$

where $c^\dagger_\alpha$ and c_β are fictitious anticommuting complex scalar fields which generate the Feynman-DeWitt-Faddeev-Popov ghost[15-17] loops, and $M_{\alpha\beta}[\phi]$ is given by[14]

$$M_{\alpha\beta}[\phi]=\frac{\delta F_\alpha[\phi]}{\delta\phi_i}(\Lambda_i^\beta+t_{ij}^\beta\phi_j)g_\beta\zeta_\alpha^{-1/2}$$

$$=F_{\alpha i}(\Lambda_i^\beta+t_{ij}^\beta\phi_j)g_\beta\zeta_\alpha^{-1/2}$$

$$\text{(not summed over } \alpha,\beta). \quad (2.8)$$

The operator $M_{\alpha\beta}[\phi]$ is in general not Hermitian, so the ghost lines are orientable.

C. Green's functions

We shall be dealing with unrenormalized, but dimensionally regularized[12] quantities (dimension of space-time $d=4-\epsilon$). The generating functional of Green's functions

$$W_F[J]=\int[d\phi\, dc\, dc^\dagger]\exp\{i(L_{\rm eff}[\phi,c,c^\dagger]+J_i\phi_i)\}$$

satisfies the Ward-Takahashi (WT) identity[2,4,6,7]:

$$\left[-F_\alpha\left(\frac{1}{i}\frac{\delta}{\delta J}\right)+J_i\left(\Lambda_i^\beta+t_{ij}^\beta\frac{1}{i}\frac{\delta}{\delta J_j}\right)M^{-1}{}_{\beta\alpha}\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\right]\times W_F[J]=0. \quad (2.9)$$

In (2.9) the quantity

$$M^{-1}{}_{\beta\alpha}\left(\frac{1}{i}\frac{\delta}{\delta J}\right)W_F[J].$$

is the Green's function for the fictitious field c in the presence of the external source J.

An important consequence of Eq. (2.9) is the elucidation of the effects on Green's functions of the change in the gauge-fixing condition (2.5). When F is changed infinitesimally by ΔF, the change induced in W_F may be viewed as a change in the source term:

$$W_{F+\Delta F}[J]=\int[d\phi\, dc\, d\bar c]\ \exp[i(L_{\rm eff}[\phi]+J_i\{\phi_i+(\Lambda_i^\beta+t_{ij}^\beta\phi_j)M^{-1}{}_{\beta\alpha}[\phi]\Delta F_\alpha\})] \quad (2.10)$$

or

$$J_i\phi_i\xrightarrow{F\to F+\Delta F}J_i\{\phi_i+(\Lambda_i^\beta+t_{ij}^\beta\phi_j)M^{-1}{}_{\beta\alpha}[\phi]\Delta F_\alpha\}. \quad (2.11)$$

D. Proper vertices

The generating functional of proper (i.e., single-particle irreducible) vertices $\Gamma[\Phi]$ is given as usual[18] by the Legendre transform:

$$W[J]=\exp\{iZ[J]\},\qquad \Gamma[\Phi]=Z[J]-J_i\Phi_i, \quad (2.12)$$

where

$$\Phi_i=\delta Z[J]/\delta J_i,\qquad J_i=-\delta\Gamma[\Phi]/\delta\Phi_i. \quad (2.13)$$

The expansion coefficients of $\Gamma[\Phi]$ about its minimum $\Phi=v$,

$$0=\delta\Gamma[\Phi]/\delta\Phi_i|_{\Phi_i=v_i}, \quad (2.14)$$

are the proper vertices from which Green's functions are obtained by the tree-diagram construction.

For later use we define $\Delta_{ij}[\Phi]$ by

$$\Delta_{ij}[\Phi]=-\delta\Phi_i/\delta J_j,\qquad \Delta_{ik}[\Phi]\delta^2\Gamma[\Phi]/\delta\Phi_k\delta\Phi_j=\delta_{ij}, \quad (2.15)$$

i.e., $\Delta_{ij}[\Phi]$ are the propagators when the fields ϕ are constrained to have the vacuum expectation values Φ.

It was shown elsewhere[1] that $\Gamma[\Phi]$ satisfies the WT identity:

$$L_{\alpha i}[\Phi]\frac{\delta}{\delta\Phi_i}\Gamma_0[\Phi]=0, \quad (2.16)$$

where

$$\Gamma[\Phi]=\Gamma_0[\Phi]-\tfrac{1}{2}\{F_\alpha[\Phi]\}^2 \quad (2.17)$$

and

$$L_{\alpha i}[\Phi]=\partial_i^\alpha+g_{\alpha\beta}\{t_{ij}^\beta\Phi_j+\gamma_i^\beta[\Phi]\}, \quad (2.18)$$

$$\gamma_i^\alpha[\Phi]=-i\,t_{ij}^\beta\Delta_{jk}[\Phi]G_{\beta\gamma}[\Phi]\frac{\delta}{\delta\Phi_k}G^{-1}{}_{\gamma\alpha}[\Phi]. \quad (2.19)$$

In (2.18) we used the symbol ∂_i^α:

$$\partial_i^\alpha=\delta_{\alpha\beta}\partial_\mu\delta(x_\alpha-x_i),\ \text{for}\ \phi_i=b_\mu^\alpha$$
$$=0,\quad \text{for}\ \phi_i=\psi_\alpha.$$

In (2.19), $G_{\beta\alpha}[\Phi]$ is

$$G_{\beta\alpha}[\Phi]=M^{-1}{}_{\beta\alpha}\left(\Phi+iX\frac{\delta}{\delta\Phi}\right)1 \quad (2.20)$$

and is the generating functional of proper vertices with two ghost lines, so that

$$G^{-1}{}_{\beta\alpha}\equiv G^{-1}{}_{\beta\alpha}[v]$$

is the inverse ghost propagator and

$$\gamma_{\beta\alpha,i}\equiv\frac{\delta G^{-1}{}_{\beta\alpha}}{\delta\Phi_i}[v]$$

is the proper vertex of two ghosts at β and α and the field at i, and so on. Furthermore, it follows from (2.20) that

$$G^{-1}{}_{\beta\alpha}[\Phi]=F_{\beta i}L_{\alpha i}[\Phi]\zeta_\beta{}^{-1/2} \quad \text{(not summed over } \beta\text{)} . \tag{2.21}$$

In Fig. 1, we show a diagrammatic representation of (2.19).

III. RENORMALIZATION TRANSFORMATIONS

We shall proceed to the renormalizability of gauge theories on the basis of our master equation (2.16). As discussed in LZII and as we shall discuss in Sec. VI, the renormalizability of the unbroken version of a gauge theory implies the renormalizability of its spontaneously broken version which is obtained, for example, by manipulation of the (renormalized) quadratic coefficients of scalar fields in the Lagrangian. Thus, we shall discuss in Secs. III, IV, and V only the unbroken theory. The Lagrangian of the theory is written as

$$L=-\tfrac{1}{4}(\partial_\mu b_\nu^\alpha-\partial_\nu b_\mu^\alpha+g^{\alpha\delta}f^{\delta\beta\gamma}b_\mu^\beta b_\nu^\gamma)^2 +\tfrac{1}{2}(\partial_\mu\psi_a-t^\alpha_{ab}g^{\alpha\beta}b_\mu^\beta\psi_b)^2-V(\psi,\lambda,M^2+\delta M^2) , \tag{3.1}$$

where V is a G-invariant local quartic polynomial of the scalar fields ψ, λ stands collectively for the coupling constants of scalar self-interactions, and M^2 is the renormalized mass matrix which is assumed to be positive definite. The potential V is bounded from below for all real ψ.

In the following, we shall assume that the potential V is invariant under $\psi\to-\psi$, so that cubic terms in ψ do not appear. This does not cause much loss of generality because the insertion of cubic interactions in a vertex diagram which has the superficial degree of divergence $D=0$ renders it superficially convergent. That is, the divergent parts of gauge boson couplings, quartic scalar couplings, and couplings of scalar and gauge bosons are not affected by cubic interactions of scalar fields. The presence of cubic terms does affect the renormalizations of scalar masses and cubic scalar couplings themselves, but these can be carried out without reference to gauge invariance of the second kind.

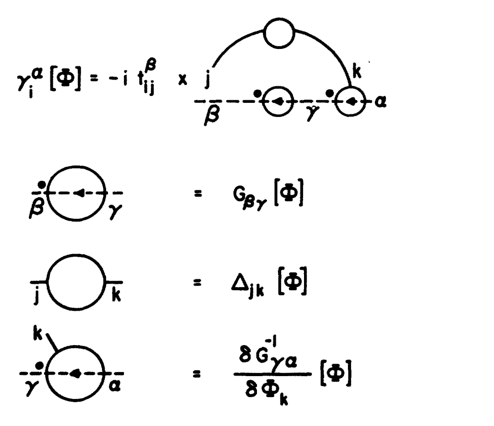

FIG. 1. Diagrammatic representation of $\gamma_i^\alpha[\Phi]$ of Eq. (2.19).

Our task is to show that the derivative $\Gamma[\Phi]$ about its minimum can be rendered finite (i.e., independent of ϵ as $\epsilon\to 0$) by rescaling fields, coupling constants, and $F_{\alpha i}$ in (2.7) (in the following we will suspend the summation-integration convention):

$$\begin{aligned}
&c_\alpha=\tilde Z_\alpha(\epsilon)c_\alpha^r;\; c_\alpha^\dagger=c_\alpha^{r\,\dagger},\\
&\phi_i=Z_i{}^{1/2}(\epsilon)\phi_i^r:\\
&b_\mu^\alpha=Z_\alpha{}^{1/2}(\epsilon)b_\mu^{\alpha(r)};\quad \psi_a=Z_a{}^{1/2}(\epsilon)\psi_a^r ,\\
&g_\alpha=g_\alpha X_\alpha(\epsilon)\tilde Z_\alpha{}^{-1}(\epsilon)Z_\alpha{}^{-1/2}(\epsilon),\\
&\lambda_{abcd}=(\lambda_{abcd}+\delta\lambda_{abcd})[Z_a(\epsilon)Z_b(\epsilon)Z_c(\epsilon)Z_d(\epsilon)]^{-1/2},
\end{aligned} \tag{3.2}$$

and

$$\zeta_\alpha=Z_\alpha{}^{-1}\zeta_\alpha^r;\quad c_a^\alpha=Z_\alpha{}^{-1/2}Z_a{}^{-1/2}c_a^{\alpha(r)},$$

or

$$\begin{aligned}
&F_{\alpha i}=Z_i{}^{-1/2}F^r_{\alpha i} \quad (i \text{ not summed}),\\
&F^r_{\alpha i}=(\zeta_\alpha^r)^{1/2}\delta_{\alpha\beta}\partial_\mu\,\delta^4(x_\alpha-x_i),\ \text{for } \phi_i=b_\mu^\alpha\\
&\qquad=\frac{1}{(\zeta^r_\alpha)^{1/2}}c_a^{\alpha(r)},\ \text{for } \phi_i=\psi_a .
\end{aligned} \tag{3.3}$$

and by choosing $\delta M^2{}_{ab}(\epsilon)$ appropriately. The wave-function renormalization constants Z_i are assumed to satisfy

$$t^\alpha_{ij}Z_j=Z_i t^\alpha_{ij} . \tag{3.4}$$

The c-number fields Φ_i transform covariantly to ϕ_i (and J_i contragrediently) under the transformation of (3.2):

$$\Phi_i=Z_i{}^{1/2}\Phi_i^r,\qquad J_i=Z_i{}^{-1/2}J_i^r . \tag{3.5}$$

Note that the rules of Eqs. (3.2) and (3.3) leave the gauge-fixing term in Γ invariant:

$$F_\alpha[\Phi]=F^r_\alpha[\Phi^r]=F^r_{\alpha i}\Phi_i^r . \tag{3.6}$$

We shall study now how the master equation (2.16) changes under the transformations (3.2) and (3.3). We shall confine ourselves to the so-called R gauges,[2,3] characterized by $c_a^\alpha=0$. In these gauges, the effective action (2.7) is invariant under the G transformations of the first kind, so that we may write, for example,

$$G_{\beta\alpha}[\Phi]g_\alpha = g_\beta G_{\beta\alpha}[\Phi] \quad (\alpha, \beta, \text{not summed})$$
$$G_{\beta\alpha}[\Phi]\tilde{Z}_\alpha^{-1} = \tilde{Z}_\beta^{-1}G_{\beta\alpha}[\Phi]. \tag{3.7}$$

First we observed from (2.15) and (3.5) that

$$\Delta_{ij} = Z_i^{1/2}\Delta^r_{ij}Z_j^{1/2},$$
$$\Delta^r_{ij}[\Phi] = -\delta\Phi^r_i/\delta J^r_j. \tag{3.8}$$

We define $G^r_{\beta\alpha}$ by

$$G_{\beta\alpha}[\Phi] = \tilde{Z}_\beta G^r_{\beta\alpha}[\Phi^r] = G^r_{\beta\alpha}[\Phi^r]\tilde{Z}_\alpha. \tag{3.9}$$

Then, Eq. (2.16) takes the form

$$L^r_{\alpha i}[\Phi^r]\frac{\delta}{\delta\Phi^r_i}\Gamma^r_0[\Phi^r] = 0, \tag{3.10}$$

where

$$\Gamma^r_0[\Phi^r; g^r, \lambda^r, M^2] = \Gamma_0[\Phi; g, \lambda, M^2 + \delta M^2]$$

and

$$L_{\alpha i}[\Phi] = \tilde{Z}_\alpha^{-1}Z_\alpha^{-1/2}Z_i^{1/2}L^r_{\alpha i}[\Phi^r], \tag{3.11}$$

so that

$$L^r_{\alpha i}[\Phi^r] = \tilde{Z}_\alpha\partial^\alpha_i + g^r_\alpha X_\alpha\left\{\sum_j t^\alpha_{ij}\Phi^r_j + \gamma^{\alpha(r)}_i[\Phi^r]\right\}, \tag{3.12}$$

$$\gamma^{\alpha(r)}_i = -i\sum_{k,j,\beta,\alpha} t^\beta_{ik}\Delta^r_{kj}[\Phi^r]G^r_{\beta\gamma}[\Phi^r]\frac{\delta\{G^{(r)-1}[\Phi^r]\}_{\beta\alpha}}{\delta\Phi^r_j}. \tag{3.13}$$

(Since we have suspended the summation-integration convention, we note the summation and integration over a catch-all index α by $\sum_{\alpha}$.) Thus, from the definition (3.9) and Eq. (2.21) we have

$$\{G^{(r)-1}[\Phi^r]\}_{\beta\alpha} = \sum_i F^r_{\beta i}L^r_{\alpha i}[\Phi^r]\zeta^{(r)-1/2}_\beta, \tag{3.14}$$

or, in the R gauges

$$\{G^{(r)-1}[\Phi^r]\}_{\beta\alpha} = \tilde{Z}_\alpha\partial^2\delta_{\alpha\beta} + g^r_\alpha X_\alpha\sum_i\partial^\alpha_i\left\{\sum_j t^\alpha_{ij}\Phi^r_j + \gamma^{\alpha(r)}_i\right\} \tag{3.15}$$

IV. RENORMALIZABILITY—R GAUGES

We are now in a position to show that all divergences in proper vertices can be eliminated by the rescaling transformations of (3.2) and (3.3). We shall do this first in the R gauges in this section; this demonstration will then be extended to arbitrary linear gauges in Sec. V.

We will develop the perturbation expansion of proper vertices, starting with the unperturbed Lagrangian given by

$$L_0 = -\tfrac{1}{4}\sum_\alpha(\partial_\mu b^{\alpha(r)}_\nu - \partial_\nu b^{\alpha(r)}_\mu)^2 - \tfrac{1}{2}\zeta^r(\partial^\mu b^{\alpha(r)}_\mu)^2 + \sum_\alpha c^{(r)\dagger}_\alpha\partial^2 c^{(r)}_\alpha + \tfrac{1}{2}\left[\sum_a(\partial_\mu\psi^r_a)^2 - \sum_{a,b}\psi^r_a M^2_{ab}\psi^r_b\right]. \tag{4.1}$$

We expand proper vertices by the number of loops a Feynman diagram contains.

Suppose that our basic proposition is true up to the $(n-1)$-loop approximation: i.e., up to this order, it has been shown that all divergences are removed by rescalings of fields and parameters and adjusting mass counterterms in the Lagrangian as discussed in Sec. III. We suppose that we have determined the renormalization constants and counterterms up to the $(n-1)$-loop approximation:

$$\begin{aligned}
Z_i(\epsilon) &\simeq 1 + z_{i1}(\epsilon) + \cdots + z_{i(n-1)}(\epsilon),\\
\tilde{Z}_\alpha(\epsilon) &\simeq 1 + \tilde{z}_{\alpha 1}(\epsilon) + \cdots + \tilde{z}_{\alpha(n-1)}(\epsilon),\\
X_\alpha(\epsilon) &\simeq 1 + x_{\alpha 1}(\epsilon) + \cdots + x_{\alpha(n-1)}(\epsilon),\\
\delta\lambda(\epsilon) &\simeq \delta\lambda_1(\epsilon) + \delta\lambda_2(\epsilon) + \cdots + \delta\lambda_{n-1}(\epsilon),\\
\delta M^2(\epsilon) &\simeq \delta M^2{}_i(\epsilon) + \cdots + \delta M^2{}_{n-1}(\epsilon).
\end{aligned} \tag{4.2}$$

We wish to show that the divergences in the n-loop approximation are also removed by suitably chosen $z_{in}(\epsilon)$, $\tilde{z}_{\alpha n}(\epsilon)$, $x_{\alpha n}(\epsilon)$, $\delta\lambda_n(\epsilon)$, and $\delta M^2{}_n(\epsilon)$. This inductive reasoning makes sense starting from $n=1$, since $n-1=0$ then corresponds to the tree approximation where there are no divergences.

Let us now consider a proper diagram with n loops, and carry out the BPH (Bogoliubov, Parasiuk, and Hepp) renormalization.[19] Since any subdiagram contains at most $(n-1)$ loops, the divergences associated with any subdiagrams are removed by the previously determined counterterms. To make an effective use of Eq. (2.16) we must also construct $L^r_{\alpha i}[\Phi^r]$ up to this order. In this connection we must remark at this point that $\gamma^{\alpha(r)}_i$ defined in (3.13) contains divergent subdiagrams shown in Fig. 2 by a shaded square. Such divergent subgrams arise from a two-particle cut in $\delta(G^{(r)-1})_{\gamma\alpha}/\delta\Phi^r_k$. As we shall establish, the divergence associated with such a subdiagram is removed by X_β. Thus, lower-order terms in X_α multiplying $\gamma^{\alpha(r)}_i$ in (3.12) will remove such divergences, since these subdiagrams contain at most $n-1$ loops.

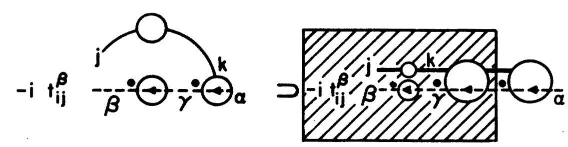

FIG. 2. Divergent subdiagram (shaded area) arising from the insertion of t^β_{ij} in Fig. 1.

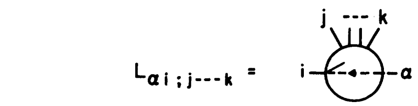

FIG. 3. Definitions of proper vertices. See Eq. (4.3).

In order to obviate infrared divergences, it is prudent to choose as the subtraction point for n-point proper vertices the point at which $p_i^2=a^2$, $p_i\cdot p_j=a^2/(n-1)$. We may write down the proper vertex as the sum of terms, each being the product of scalar function of external momenta and a tensor covariant, which is a polynomial in the components of external momenta carrying available Lorentz indices. Except for the scalar self-masses, all other renormalization parts have $D=0$, or 1. (As we shall see, the self-masses of gauge bosons are purely transverse in the R gauges, and therefore have effectively $D=0$: The latter is also true in other linear gauges, as we show in Sec. V.) Thus only the scalar functions associated with tensor covariants of lowest order are logarithmically divergent. If we expand such a scalar function about the subtraction point, only the first term in the expansion is divergent. Among the vertices derived from $L_{\alpha i}[\Phi]$ only the two- and three-point vertices are linearly and logarithmically divergent, so the same remark applies here too. (The reader is invited to verify this statement and that the four-point vertices derived from $L_{\alpha i}[\Phi]$ are superficially covergent.) Thus when we discuss the relationships among the divergent parts of proper vertices, we need focus only on these terms.

Equation (3.10) contains all possible relationships among proper vertices which follow from gauge invariance of the second kind. To make use of this equation it is convenient to resort to a diagrammatic approach. We shall represent (we will drop the superscript r; all quantities and equations are renormalized ones)

$$\frac{\delta^{(n)}L_{\alpha i}[\Phi]}{\delta\Phi_j\cdots\delta\Phi_k}\bigg|_{\Phi=v}\equiv L_{\alpha i;j\cdots k} \tag{4.3}$$

as in Fig. 3, and

$$\frac{\delta^{(n)}\Gamma[\Phi]}{\delta\Phi_i\cdots\delta\Phi_j}\bigg|_{\Phi=v}\equiv \Gamma_{i\cdots j} \tag{4.4}$$

as in Fig. 4 (in the present case, $v=0$). We shall use the notation

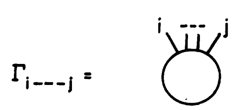

FIG. 4. Definitions of proper vertices. See Eq. (4.4).

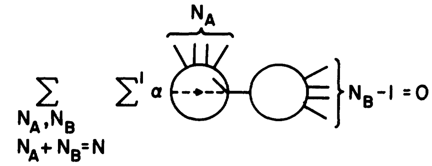

FIG. 5. The WT identity for proper vertices. $\sum'$ means summation over all partitions of N_A+N_B-1 external lines into two groups of N_A and N_B-1 members each.

$\Phi_i=B_\mu^\alpha$ as $\phi_i=b_\mu^\alpha$,

$\Phi_i=\Psi_a$ as $\phi_i=\psi_a$.

Equation (3.10) may be represented diagrammatically as in Fig. 5.

In examing (3.10) in the n-loop approximation, the following simplification will be noted. In the n-loop approximation, we have

$$\{L_{\alpha i}\}_0\frac{\delta}{\delta\Phi_i}\{\Gamma_0\}_n+\{L_{\alpha i}\}_n\frac{\delta}{\delta\Phi_i}\{\Gamma_0\}_0 = -\{L_{\alpha i}\}_1\frac{\delta}{\delta\Phi_i}\{\Gamma_0\}_{n-1}+\cdots, \tag{4.5}$$

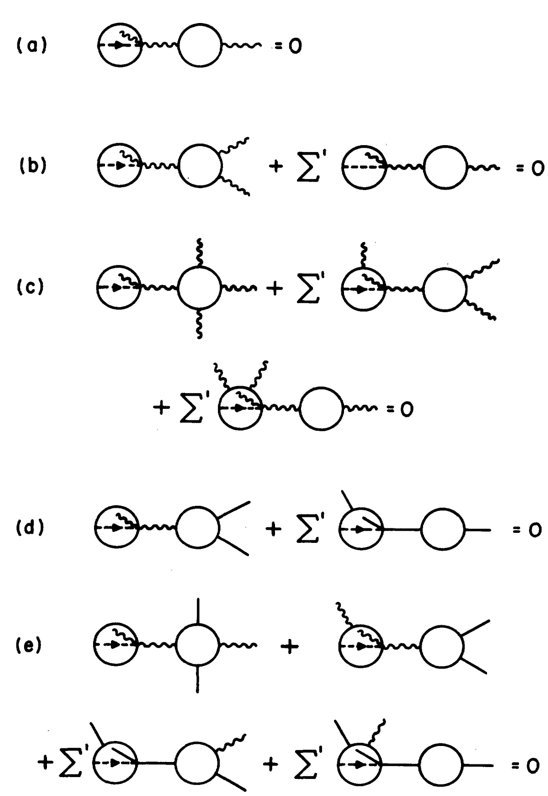

FIG. 6. The WT identities for renormalization parts in the R gauge.

where $\{\ \}_m$ denotes the quantity evaluated in the m-loop approximation. Since the right-hand side of (4.5) involves only quantities with less than n loops, it is finite by the induction hypothesis. Thus, we have, denoting by $\{\ \}^{\text{div}}$ the divergent part,

$$\{L_{\alpha i}\}_0 \frac{\delta}{\delta\Phi_i}\{\Gamma_0\}_n^{\text{div}}+\{L_{\alpha i}\}_n^{\text{div}}\frac{\delta}{\delta\Phi_i}\{\Gamma_0\}_0=\text{finite}, \quad (4.6)$$

which means that the left-hand side is independent of ϵ as $\epsilon\to 0$. By differentiating (4.6) with respect to Φ N times and setting $\Phi=0$, we obtain equations connecting $\{\Gamma_{i\ldots j}\}_n^{\text{div}}$ and $\{L_{\alpha i;j\ldots k}\}_n^{\text{div}}$.

We need consider five equations which follow from (3.10). These equations are shown in Figs. 6(a)–6(e). We distinguish here the vector-boson lines (wiggly lines) and scalar-boson lines (straight lines). Note that any vertex with an odd number of scalar lines vanishes in the R gauges.

(a) Let $\delta_{\alpha\beta}\Gamma_{0\mu\nu}(p)$ be the Fourier transform of $\delta^2\Gamma_0/\delta B_\mu^\alpha\delta B_\nu^\beta|_0$ and $\delta_{\alpha\beta}p_\mu\Lambda^\alpha(p^2)$ the Fourier transform of $L_{\alpha i}[\Phi=0]$ for $\phi_i=b_\mu^\beta$. Then we have

$$\delta_{\alpha\beta}\Lambda^\alpha(p^2)p^\mu\Gamma_{0\mu\nu}^\beta(p)=0, \quad (4.7)$$

which shows that $\Gamma_{\mu\nu}(p)$ must be transverse:

$$\Gamma_{0\mu\nu}^\alpha(p)=(g_{\mu\nu}p^2-p_\mu p_\nu)\pi_0^\alpha(p^2). \quad (4.8)$$

Both $\{\pi_0^\alpha(p^2)\}_n^{\text{div}}$ and $\{\Lambda^\alpha(p^2)\}_n^{\text{div}}$ are made finite by wave-function renormalizations, i.e., by counterterms $z_{\alpha n}$ and $\tilde{z}_{\alpha n}$, because

$$\{\Lambda^\alpha(p^2)\}_n^{\text{div}}=\Lambda^\alpha(-a^2)+\tilde{z}_{\alpha n}\,,\ \text{etc.}$$

(b) Equation (4.6) corresponding to Fig. 6(b) is

$$p^\lambda\{\Lambda^\alpha(p^2)\}_n^{\text{div}}\{\Gamma_{\lambda\mu\nu}^{\alpha\beta\gamma}(p,q,r)\}_0+p^\lambda\{\Lambda^\alpha(p^2)\}_0\{\Gamma_{\lambda\mu\nu}^{\alpha\beta\gamma}(p,q,r)\}_n^{\text{div}}$$
$$+\sum{}'\{L_{\alpha(\gamma\sigma);(\beta\mu)}(p,r;q)\}_n^{\text{div}}(g_{\sigma\nu}r^2-r_\sigma r_\nu)+\sum{}'\{L_{\alpha(\gamma\sigma);(\beta\mu)}(p,r;q)\}_0(g_{\sigma\nu}r^2-r_\sigma r_\nu)\{\pi^2(r^2)\}_n^{\text{div}}=\text{finite}$$
$$(\alpha \text{ and } \gamma \text{ not summed}),\quad p+q+r=0, \quad (4.9)$$

where the quantities $\Gamma_{\lambda\mu\nu}^{\alpha\beta\nu}(p,q,r)$ and $L_{\alpha(\beta\mu);(\gamma\nu)}(p,q;r)$ are defined in Fig. 7, and $\sum'$ means sum over the terms gotten by the interchange $(q,\mu,\beta)\longleftrightarrow(r,\nu,\gamma)$.

The first and the last terms on the right of (4.9) are finite by the renormalizations performed in (a), so we have

$$p^\lambda\{\Gamma_{\lambda\mu c}^{\alpha\beta\gamma}(p,g,r)\}_n^{\text{div}}+\sum{}'\{L_{\alpha(\gamma\sigma);(\beta\mu)}(p,r;q)\}_n^{\text{div}}\times(g_{\sigma\nu}r^2-r_\sigma r_\nu)=\text{finite}\,. \quad (4.10)$$

From Lorentz invariance and a remark made previously, we have

$$\{\Gamma_{\lambda\mu\nu}^{\alpha\beta\gamma}(p,q,r)\}_r^{\text{div}}=g_{\lambda\mu}(ap_\nu+bq_\nu+cr_\nu)$$
$$=g_{\mu\nu}(a'p_\lambda+b'q_\lambda+cr_\lambda)$$
$$=g_{\nu\lambda}(a''p_\mu+b''q_\mu+c''r_\mu)\,, \quad (4.11)$$

where $a=a^{\alpha\beta\gamma}(\epsilon)$, etc., are constants and

$$\{L_{\alpha(\beta\mu);(\gamma\nu)}(p,q,r)\}_n^{\text{div}}=g_{\mu\nu}L_{\alpha\beta\gamma}(\epsilon)\,. \quad (4.12)$$

We ask under what circumstances $p^\lambda\{\Gamma_{\lambda\mu\nu}^{\alpha\beta\gamma}(p,q,r)\}$ can be a linear combination of $(g_{\mu\nu}r^2-r_\mu r_\nu)$ and $(g_{\mu\nu}q^2-q_\mu q_\nu)$. It turns out that it can happen only if

$$\{\Gamma_{\lambda\mu\nu}^{\alpha\beta\gamma}(p,q,r)\}_n^{\text{div}}=A_{\alpha\beta\gamma}(\epsilon)[g_{\lambda\mu}(p-q)_\nu+g_{\mu\nu}(q-r)_\lambda+g_{\nu\lambda}(r-p)_\mu]$$
$$+B_{\alpha\beta\gamma}(\epsilon)(g_{\lambda\mu}p_\nu-g_{\nu\lambda}p_\mu)\,. \quad (4.13)$$

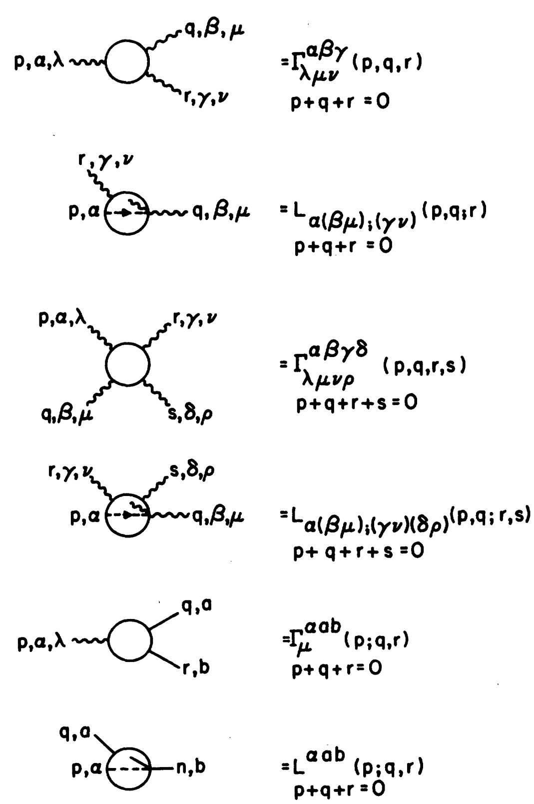

FIG. 7. Definitions of vertices appearing in Fig. 6.

Now the Bose symmetry applied to the three-point boson vertex tells us that

$$B_{\alpha\beta\gamma}(\epsilon)=0$$

and

$$A_{\alpha\beta\gamma}(\epsilon)=-A_{\beta\alpha\gamma}(\epsilon)=-A_{\alpha\gamma\beta}(\epsilon)\,, \qquad (4.14)$$

i.e., $A_{\alpha\beta\gamma}$ is completely antisymmetric in α, β, and γ. Furthermore, Eq. (4.10) shows that we can adjust the finite part of $A_{\alpha\beta\gamma}(\epsilon)$ so that

$$A_{\alpha\beta\gamma}(\epsilon)=L_{\alpha\beta\gamma}(\epsilon)\,. \qquad (4.15)$$

(c) Consider now the relation depicted in Fig. 6(c). In the equation which deals with divergent parts, a derivative of (4.6), the last term on the left-hand side of Fig. 6(c) does not contribute because

$$\{L_{\alpha(\beta\mu);(\gamma\nu)(\delta\rho)}\}_n$$

is not a renormalization part for $n\geq 1$ and $=0$ for $n=0$. Since

$$\{L_{\alpha(\beta\mu);(\gamma\nu)}\}_0=f_{\alpha\beta\gamma}g_{\mu\nu}\,,$$

$$\{\Gamma^{\alpha\beta\gamma}_{\lambda\mu\nu}(p,q,r)\}_0=f_{\alpha\beta\gamma}[g_{\lambda\mu}(p-q)_\nu+g_{\mu\nu}(q-r)_\lambda+g_{\nu\lambda}(r-p)_\mu]\,, \qquad (4.16)$$

we have

$$f_{\alpha\beta\epsilon}\{\Gamma^{\epsilon\gamma\delta}_{\lambda\mu\nu}(p+q,r,s)\}^{\text{div}}_n+L_{\alpha\beta\epsilon}\{\Gamma^{\epsilon\gamma\delta}_{\lambda\mu\nu}(p+q,r,s)\}_0+[\text{cyclic permutations of }(q,\beta,\lambda),\ (r,\gamma,\mu),\ (s,\delta,\nu)]+p^\sigma\{\Gamma^{\alpha\beta\gamma\delta}_{\sigma\lambda\mu\sigma}(p,q,r,s)\}^{\text{div}}_n=\text{finite}. \qquad (4.17)$$

From Lorentz invariance and power-counting arguments, we have

$$\{\Gamma^{\alpha\beta\gamma\delta}_{\sigma\lambda\mu\nu}\}^{\text{div}}_n=C^{\alpha\beta\gamma\delta}(\epsilon)g_{\sigma\lambda}g_{\mu\nu}+\cdots\,.$$

In (4.17), the term $\{\Lambda^\alpha\}^{\text{div}}_n\{\Gamma^{\alpha\beta\gamma\delta}_{\sigma\lambda\mu\nu}\}_0$ does not appear because the first factor is finite the wave-function renormalization $\tilde z_{\alpha n}$. Consider the terms proportional to $(q-r)_\nu$ in Eq. (4.17), taking into account (4.14), and (4.15). The last term on the left-hand side of (4.17) does not contribute. Equation (4.17) tells us that the finite part of $L_{\alpha\beta\gamma}$ may be adjusted so that

$$([f^\beta,L^\gamma]-[f^\gamma,L^\beta])_{\alpha\delta}=(f^\delta L^\beta+L^\delta f^\beta)_{\alpha\gamma}\,,$$

where $(f^\beta)_{\alpha\gamma}=f_{\alpha\beta\gamma}$, $(L^\beta)_{\alpha\gamma}=L_{\alpha\beta\gamma}(\epsilon)$. Since

$$[f^\alpha,L^\beta]=f_{\alpha\beta\gamma}L^\gamma\,,$$

as follows from the fact that $L_{\alpha\beta\gamma}$ is a G-invariant tensor operator, we have

$$f_{\beta\gamma\epsilon}L_{\epsilon\alpha\delta}=f_{\alpha\delta\epsilon}L_{\epsilon\beta\gamma}\,. \qquad (4.18)$$

By multiplying Eq. (4.18) by $f_{\beta\gamma\omega}$ and summing over β and γ, we find that

$$L_{\alpha\beta\gamma}(\epsilon)=D_\alpha(\epsilon)f_{\alpha\beta\gamma}\ (\alpha\text{ not summed}), \qquad (4.19)$$

where

$$D_\alpha(\epsilon)=E_\alpha(\epsilon)/C_2\,,$$

$$\sum_{\beta,\gamma}f_{\alpha\beta\gamma}f_{\delta\beta\gamma}=\delta_{\alpha\delta}C_2\,,$$

$$\sum_{\beta,\gamma}f_{\alpha\beta\gamma}L_{\delta\beta\gamma}(\epsilon)=\delta_{\alpha\delta}E_\alpha(\epsilon)\,.$$

[Equation (4.14) is sufficient to establish (4.19) for well-known groups such as SU(2) and SU(3), but the point of this demonstration is to avoid too much reliance on group theory.]

Since $D_\alpha(\epsilon)$ is of the form

$$D_\alpha(\epsilon)=D'_\alpha(\epsilon)+x_{\alpha n}(\epsilon)\,, \qquad (4.20)$$

we can choose $x_{\alpha n}$ to cancel the divergent part of D'_α: $L_{\alpha\beta\gamma}(\epsilon)$ is made independent of ϵ as $\epsilon\to 0$ by so doing, and so is $\{\Gamma^{\alpha\beta\gamma}_{\lambda\mu\nu}\}^{\text{div}}_n$, by (4.13)–(4.15).

Actually, Eq. (4.17) contains information on $\{\Gamma^{\alpha\beta\gamma\delta}_{\lambda\mu\nu\rho}\}^{\text{div}}_n$ as well. However, it is not necessary to dwell on it here.

(d) First of all, the inverse scalar propagators are made finite by $\delta M^2(\epsilon)$ and $z_a(\epsilon)$. Now look at Fig. 6(d). The treatment of this relation proceeds in much the same way as that of Fig. 6(b). If we define (see Fig. 7 for the definitions of $\Gamma^{\alpha ab}_\mu$ and $L^{\alpha ab}$)

$$\{\Gamma^{\alpha ab}_\mu(p;q,r)\}^{\text{div}}_n=(q-r)_\mu T^\alpha_{ab}(\epsilon)\,,$$

$$\{L^{\alpha ab}(p;q,r)\}^{\text{div}}_n=S^\alpha_{ab}(\epsilon)\,,$$

where T^α_{ab} and M^α_{ab} are divergent constants, we find that

$$T^\alpha_{ab}(\epsilon)=S^\alpha_{ab}(\epsilon)\,, \qquad (4.21)$$

$$S^\alpha_{ab}=-S^\alpha_{ab}\,, \qquad (4.22)$$

and

$$[S^\alpha,M^2]_{ab}=0\,, \qquad (4.23)$$

where M^2 is the scalar mass-squared matrix.

(e) The relation depicted in Fig. 6(e) can be processed similarly to (c) above. Making use of (4.21), (4.22), and

$$\{\Gamma^{\alpha ab}_\mu(p;q,r)\}_0=(q-r)_\mu t^\alpha_{ab}\,,$$

$$\{L^{\alpha ab}(p;q,r)\}_0=t^\alpha_{ab}\,,$$

$$[t^\alpha,S^\beta]_{ab}=f_{\alpha\beta\gamma}S^\gamma_{ab}\,,$$

one finds that

$$S^{\alpha}_{ab}(\epsilon)=D_{\alpha}(\epsilon)t^{\alpha}_{ab} \quad (\alpha \text{ not summed}). \tag{4.24}$$

Thus the choice of $x_{\alpha n}(\epsilon)$ made in (c) above [see Eq. (4.20) *et seq.*] will make both $\{\Gamma^{\alpha ab}_{\mu}\}_n$ and $\{L^{\alpha ab}\}_n$ finite.

Let us summarize the results so far. We have shown that by suitable choices of $\delta M^2(\epsilon)$, $Z_i(\epsilon) =\{Z_{\alpha}(\epsilon), Z_a(\epsilon)\}$, $\tilde{Z}_{\alpha}(\epsilon)$, and $X_{\alpha}(\epsilon)$, all two-point and three-point vertices derived from $\Gamma_0[\Phi]$ and $L_{\alpha i}[\Phi]$ can be made finite. More importantly, since only the two- and three-point vertices of $L_{\alpha i}[\Phi]$ are renormalization parts, $\{L^r_{\alpha i}[\Phi^r]\}_n$ is made finite by the above counterterms. (We shall restore the superscript r for "renormalized" from here.)

Therefore, from (4.6), we find that

$$\sum_i \{L^r_{\alpha i}[\Phi^r]\}_0 \frac{\delta}{\delta \Phi^r_i}\{\Gamma^r_0[\Phi^r]\}^{\text{div}}_n=0, \tag{4.25}$$

where, from (3.12),

$$\{L^r_{\alpha i}[\Phi^r]\}_0=\partial^{\alpha}_i+g^r_{\alpha}\sum_j t^{\alpha}_{ij}\Phi^r_j . \tag{4.26}$$

Since $\{\Gamma^r_0[\Phi^r]\}^{\text{div}}_n$ must be a local functional, at most quartic in Φ^r, Eq. (4.25) can be solved. The solution is

$$\begin{aligned}\{\Gamma^r_0[\Phi^r]\}^{\text{div}}_n = \int d^4x\{&-\tfrac{1}{4}A(\partial_{\mu}B^{\alpha(r)}_{\nu}-\partial_{\nu}B^{\alpha(r)}_{\mu}\\ &+g^r_{\alpha}f_{\alpha\beta\gamma}B^{\beta(r)}_{\mu}B^{\gamma(r)}_{\nu})^2\\ &+\tfrac{1}{2}B(\partial_{\mu}\Psi^r_a-t^{\alpha}_{ab}B^{\alpha(r)}_{\mu}\Psi^r_{\alpha})^2\\ &-V'[\Psi^r]\},\end{aligned} \tag{4.27}$$

where V' is a G-invariant quartic polynomial in Ψ^r, where coefficients depend in general on ϵ. After the renormalizations outlined in (a)–(d) above,

$$A=B=\frac{\delta^2 V'}{\delta\Psi_a\delta\Psi_b}[0]=0,$$

so the remaining divergences lie in the quartic couplings. But these divergent quartic couplings are G-invariant, so that the set $\{\delta\lambda(\epsilon)\}$ which contains all possible quartic couplings will eliminate these divergences.

We have shown that the scale transformations (3.2) make $\Gamma^r_0[\Phi_r]$ finite in each order of loopwise perturbation theory in the R gauges. Further, from (2.17) and (3.6), we find that

$$\Gamma^r[\Phi_r]=\Gamma^r_0[\Phi^r]-\tfrac{1}{2}\{F^r_{\alpha}[\Phi^r]\}^2$$

is finite in this gauge.

V. RENORMALIZABILITY—LINEAR GAUGES

We shall now extend the discussion of Sec. IV to arbitrary linear gauges discussed in Sec. II. First notice that

$$\begin{aligned}L_{\text{eff}}(\zeta,c)-L_{\text{eff}}(\zeta,0)=-\tfrac{1}{2}\Big[&\sum_{\alpha,a}2\partial^{\mu}b^{\alpha}_{\mu}c^{\alpha}_a\psi_a\\ &+\sum_a\frac{1}{\zeta_{\alpha}}\Big(\sum_a c^{\alpha}_a\psi_a\Big)^2\Big]\\ &+\sum_{\alpha,\beta}c^{\dagger}_{\alpha}\Big(\frac{1}{\zeta_{\alpha}}\sum_{a,b}c^{\alpha}_a t^{\beta}_{ab}\psi_b\Big)c_{\beta},\end{aligned} \tag{5.1}$$

where $L_{\text{eff}}(\zeta,c)$ is the effective Lagrangian considered as a function of gauge-fixing parameters ζ_{α} and c^{α}_a. We note here that c^{α}_a is of the form

$$c^{\alpha}_a=t^{\alpha}_{ab}A^{\alpha}_b \quad (\alpha \text{ not summed}), \tag{5.2}$$

where A^{α}_a is a constant vector in the space of a, and $A^{\alpha}_a=A^{\beta}_a$ if α and β belong to the same factor group.

Equation (5.1) tells us that the difference between the effective Lagrangians in the R gauge and the general linear gauge for the same ζ_{α} is a sum of terms of lower dimensions (≤ 3). It follows from this observation that the insertion of vertices that appear on the right-hand side of (5.1) in a vertex diagram of $D=0$ will make the diagram superficially convergent. This means that the counterterms (Z_i-1), $(\tilde{Z}_{\alpha}-1)$, $(X_{\alpha}-1)$, and $\delta\lambda$ defined in Sec. IV [for $L_{\text{eff}}(\zeta,0)$] will render finite these vertices.

Thus our task is to show that the divergences in vertices of lower dimensions are either absent, or, if present, may be removed by a gauge-invariant manipulation. The possible candidates for divergent vertices of lower dimensions are the b^2_{μ}, $\psi\partial^{\mu}b_{\mu}$, ψ^3, ψ^2, and $c^{\dagger}c\psi$ vertices and the ψ-vacuum transition. Note that the invariance $\psi\to-\psi$ is broken by terms on the right-hand side of (5.1).

The fact that ψ can develop nonvanishing vacuum expectation values, even when $M^2>0$, in a general linear gauge is of importance here. These vacuum expectation values v arise from loops, and must be determined from the solutions of

$$\left.\frac{\delta\Gamma^r[\Phi^r]}{\delta\Phi^r_i}\right|_{\psi^r=v,B^r_{\mu}=0}=0. \tag{5.3}$$

Proper vertices in the general linear gauge are given by variational derivatives of $\Gamma^r[\Phi^r]$ with respect to Φ^r evaluated at $\Phi^r=v$. Alternatively, the proper vertices may be obtained by writing

$$\psi^r_a=\psi'^{(r)}_a+v_a \tag{5.4}$$

and defining the c-number fields $\Psi'^{(r)}_a$ as the expectation values of $\psi'^{(r)}_a$ in the presence of the external sources $J^{(r)}$, and expanding $\Gamma^r[\Phi]$ about $\Psi'^{(r)}_a=0$ and $B^{\alpha(r)}_{\mu}=0$. The quantity v_a in (5.4) is to be determined by the condition that ψ'_a not have vacuum expectation values. In perturbation theory

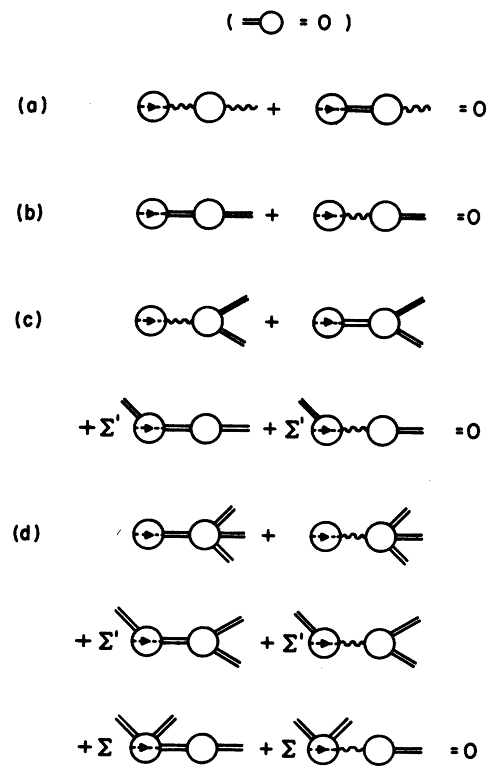

FIG. 8. The WT identities for additional renormalization parts in linear gauges.

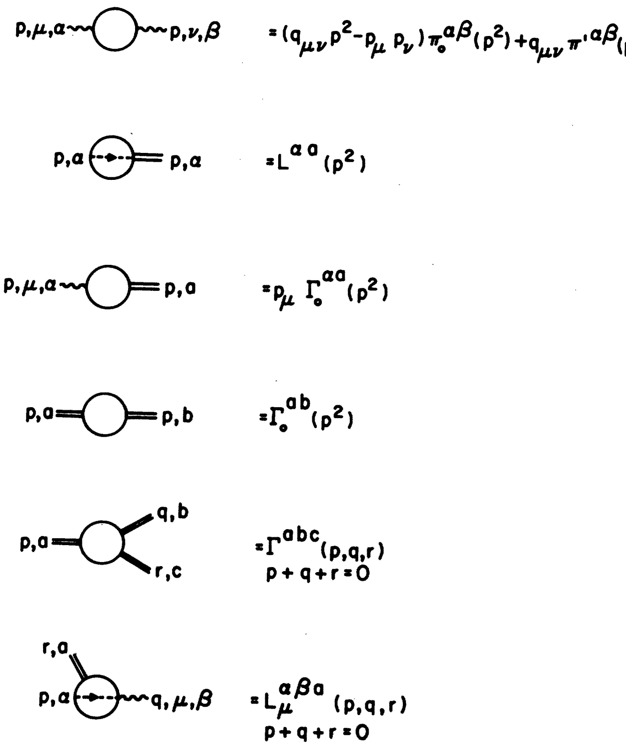

FIG. 9. Definitions of vertices appearing in Fig. 8.

$$v_a = xv_{a1} + x^2 v_{a2} + \cdots ,$$

where x is a fictitious expansion parameter ($x=1$) of the loopwise perturbation expansion. When Eq. (5.4) is substituted in the effective Lagrangian, there will emerge a number of new terms. One of them is a linear term in ψ_a' with a coefficient which is a function of v_a. This term must serve as the counterterm to cancel ψ_a' vacuum diagrams (the so-called tadpole diagrams). This requirement will fix v_{an}. There will also appear quadratic and cubic terms in ψ' by this substitution.

We shall proceed inductively as in Sec. IV: We shall assume that up to the $(n-1)$-loop approximation Z_i, $\tilde{Z}_\alpha$, X_α, $\delta\lambda$ (as determined in Sec. IV), $\delta M^2(\xi^r, c^r)$ (satisfying

$$[\delta M^2(\xi^r, c^r), t^\alpha]_{ab} = 0, \tag{5.5}$$

determined up to this order), and

$$v_a = v_{a1} + v_{a2} + \cdots + v_{a(n-1)} , \tag{5.6}$$

remove divergence from renormalization parts, and we shall show, based on (4.6), that suitable choices of n-loop counterterms do the same in the n-loop approximation. To determine the renormalization parts of dimensions ≤ 3, we look at the four equations of Fig. 8. The double lines in Figs. 8 and 9 refer to external $\Psi'^{(r)}$ lines. The relevant vertices that appear in Fig. 8 are defined in Fig. 9.

(a) Since $T^{\alpha a}(p^2)$ and $L_{\alpha a}$ vanish in the tree approximation, we have

$$\{\pi'^{\alpha\beta}(p^2)\}_n^{\text{div}} = 0 . \tag{5.7}$$

(b) We have

$$\{L^{\alpha a}\}_n^{\text{div}}(p^2 - M^2)^{ab} + p^2\{\Gamma_0^{\alpha b}\}_n^{\text{div}} = 0 . \tag{5.8}$$

Consider, now the limit $p^2 \to 0$: We learn that

$$\lim_{p^2 \to 0} \{L^{\alpha a}(p^2)\}_n^{\text{div}} = p^2 f^{\alpha\beta}(p^2) \tag{5.9}$$

and $f^{\alpha\beta}(p^2)$ is convergent, because $L^{\alpha a}(p^2)$ has superficially $D=0$. Further, Eq. (5.8) tells us that $\{\Gamma_0^{\alpha\beta}\}_n$ is finite.

(c) The first term on the left-hand side is made finite by $\tilde{Z}_\alpha$ and X_α; the second term does not contribute to the left-hand side of (4.6) because both $L^{\alpha a}$ and Γ^{abc} vanish in the tree approximation; the last term does not contribute because $L_\mu^{\alpha\beta a}$ and $T_0^{\alpha a}$ vanish in the tree approximation. Thus we have

$$t_{ac}^\alpha \{\Gamma_0^{cb}(p^2)\}_n^{\text{div}} - \{\Gamma_0^{ac}(p^2)\}_n^{\text{div}} t_{cb}^\alpha = 0 \tag{5.10}$$

because $\{L^{\alpha ab}\}_n$ is made finite by $x_{\alpha n}$. Writing

$$\{\Gamma_0^{ab}(p^2)\}_n^{\rm div}=p^2\delta^{ab}H^a(\epsilon)+F^{ab}(\epsilon)\,,$$

we see that $H^a(\epsilon)$ is removed by z_{an}, and $F^{ab}(\epsilon)$ is removed by $\{\delta M^2{}_{ab}(\zeta^r, c^r;\epsilon)\}_n$ satisfying (5.5).

(d) Repeating an analysis similar to (c) above, we find that

$$(t^{\alpha}_{aa'}\delta_{bb'}\delta_{cc'}+t^{\alpha}_{bb'}\delta_{aa'}\delta_{cc'}+t^{\alpha}_{cc'}\delta_{aa'}\delta_{bb'})\{\Gamma^{a'b'c'}(pqr)\}_n^{\rm div}=0\,. \quad (5.11)$$

This means that $\{\Gamma^{abc}\}_n^{\rm div}$ must be an invariant under G. But this is impossible unless

$$\{\Gamma^{abc}\}_n^{\rm div}=0 \quad (5.12)$$

because the group theoretic structure of Γ^{abc} must be of the form

$$\{\Gamma^{abc}\}_n^{\rm div}=A_d t^{abcd}\times \text{const}, \quad (5.13)$$

where A_d is the constant vector defined in (5.2) and t^{abcd} is a G-invariant tensor. [Note that the terms which break the $\psi\to-\psi$ invariance in (5.2) are all proportional to c^{α}_a.]

This concludes the proof that $Z_i(\zeta^r,0)$, $\tilde{Z}_\alpha(\zeta^r,0)$, $X_\alpha(\zeta^r,0)$, and $\delta M^2(\zeta^r,c^r)$ remove divergences from the perturbation series for proper vertices in the linear gauge specified by the two sets of parameters ζ^r and c^r after the vacuum expectation values of ψ_a are duly taken into account. The question as to whether $\delta M^2(\zeta^r,c^r)$ is also independent of c^r cannot be discussed meaningfully in the context of an unbroken theory because of the impossibility of defining the S matrix. We shall return to this question after we discuss the renormalization of spontaneously broken theories.

VI. RENORMALIZATION OF SBGT

This section is devoted to augmenting the discussion of LZII on the renormalizability of spontaneously broken gauge theories (SBGT), so as to make it applicable to arbitrary linear gauges. The Higgs mechanism[20] (for a historical review of the subject, see Ref. 21) takes place in general when the condition $M^2>0$ is violated. For the following discussion, it is convenient to keep in mind a comparison theory given by the same g^r and λ^r, but with a positive definite M_0^2.

Consider the effective action

$$L_{\rm eff}(\zeta^r,c^r,M_0^2)+\sum_a \gamma_a\psi_a^r\,, \quad (6.1)$$

where γ_a's are finite constants. The vacuum expectation values of ψ_a^r of this theory, $u_a(\gamma)$, are given by the solution of

$$\left.\frac{\delta\Gamma^r}{\delta\Phi_a^r}\right|_{\Phi^r=u^r,B_\mu^r=0}=-\gamma_a\,, \quad (6.2)$$

satisfying the positivity condition that Γ^r be convex at $\Phi^r=u$, $B_\mu^r=0$. Here Γ^r is the generating functional of proper vertices of the theory given by $L_{\rm eff}(M_0^2)$. In perturbation theory, we may define $\psi'^{(r)}$ by

$$\psi_a^r=\psi_a'^{(r)}+u_a\,, \quad (6.3)$$

$$u_a=u_{a0}+u_{a1}+u_{a2}+\cdots \quad (6.4)$$

and determine u_{a0} by the condition that

$$\left.\frac{\delta L^0_{\rm eff}(M_0^2)}{\delta\psi_a^r}\right|_{\psi_a^r=u_{a0},b_\mu=0}=-\gamma_a \quad (6.5)$$

subject to the positivity condition, where $L^0_{\rm eff}$ is the effective action with all counterterms set equal to zero, and u_{an} by the condition that the ψ' tadpoles vanish in the n-loop approximation.

It is easy to see that the proper vertices of the theory (6.1) are rendered finite by the counterterms $[Z_i(M_0^2)-1]$, $[Z_\alpha(M_0^2)-1]$, $[X_\alpha(M_0^2)-1]$, $\delta\lambda(M_0^2)$, and $\delta M^2(M_0^2)$ of the comparison theory $L_{\rm eff}(M_0^2)$. The argument involved here is completely analogous to that given for the σ model,[22] (see also Ref. 23) and relies on the so-called spurion analysis.[24]

Let us now consider the theory

$$L_{\rm eff}(\zeta^r,c^r,M^2)+\sum_a\gamma_a\psi_a^r\,, \quad (6.6)$$

where M^2 is no longer positive definite. The scalar propagators of the theory are of the form

$$\begin{aligned}\left\{\left[k^2\underline{1}-M^2-P^{cd}u_{c0}u_{d0}-\sum_\alpha\frac{1}{\zeta_\alpha}Q^\alpha\right]^{-1}\right\}_{ab} &=\left\{\left[k^2\underline{1}-M_0^2-P^{cd}u_{c0}u_{d0}-\sum_\alpha\frac{1}{\zeta_\alpha}Q^\alpha\right]^{-1}\right\}_{ab}+\left\{\left[k^2\underline{1}-M_0^2-P^{cd}u_{c0}u_{d0}-\sum_\alpha\frac{1}{\zeta_\alpha}Q^\alpha\right]^{-1}\right\}_{ae}\\ &\quad\times(M^2-M_0^2)_{ef}\left\{\left[k^2\underline{1}-M^2-P^{cd}u_{c0}u_{d0}-\sum_\alpha\frac{1}{\zeta_\alpha}Q^\alpha\right]^{-1}\right\}_{fb}\,,\end{aligned} \quad (6.7)$$

where $u_{a0}=u_{a0}(\gamma, M^2)$ are obtained from (6.5) with $L_{\rm eff}=L_{\rm eff}(M^2)$, subject to the positivity condition [which guarantees that $M^2+Puu+(1/\zeta)Q$ is positive semidefinite], and where P^{cd} and Q^{α} are matrices acting on the scalar-field indices. Now consider a renormalization part of the theory (6.16). If we substitute the right-hand side of (6.7) for every scalar propagator in the diagram, there will result a number of terms. The first term, in which all scalar propagators are replaced by the first term on the right-hand side of (6.7), is the corresponding renormalization part of a theory of the form (6.2) with a different set of γ's [because u_{c0} appearing here is $u_{c0}(M^2,\gamma)$ and not $u_{c0}(M_0{}^2,\gamma)$. But we can choose γ's such that $u_{c0}(M^2,\gamma) = u_{c0}(M_0{}^2,\gamma')$], and this term is made finite by a counterterm of the comparison theory $L_{\rm eff}(M_0{}^2)$. If the superficial degree of divergence D of the renormalization part in question is zero, the rest of the terms are superficially convergent, and we require no more over-all subtractions. If D is 2 (scalar self-energy), the terms in which only one scalar propagator is replaced by the second term on the right-hand side of (6.7) are still logarithmically divergent, but this divergence is removed by a suitable choice of the mass counterterm

$$\delta M^2(M^2, M_0{}^2, \gamma):$$
$$[t^a, \delta M^2(M^2, M_0{}^2, \gamma)]_{ab}=0\,. \qquad (6.8)$$

When we let $\gamma=0$, we have

$$u_a(\gamma=0)=v_\alpha\,, \qquad v_{\alpha 0}\neq 0\,,$$

and we have an (intermediately) renormalized SBGT.

To recapitulate: A SBGT is renormalizable in any linear gauge (ζ^r, c^r) by the vertex and field renormalization transformations of a comparison unbroken theory $L_{\rm eff}(\zeta^r, c^r, M_0{}^2>0)$ [also of $L_{\rm eff}(\zeta^r, 0, M_0{}^2)$; see Sec. V] and a suitably chosen G-invariant mass counterterm $\delta M^2(M^2)=\delta M^2(M_0{}^2)+\delta M^2(M^2, M_0{}^2,\gamma=0)$. It is to be noted[25] that in a linear gauge, the vacuum expectation values of $\psi'^{(r)}$ are in general infinite and gauge-dependent—they are not observables. This in no way implies the nonfiniteness of renormalized vertices, since the role of the infinite vacuum expectation values is precisely to cancel another infinity.

VII. GAUGE INDEPENDENCE OF THE S MATRIX

Left so far unresolved is the question whether δM^2 is gauge-dependent. To answer this question, we must consider the S matrix. Fortunately for SBGT, at least in the presently contemplated applications to weak and electromagnetic interactions, it is possible to do so, because the infrared singularities of these models are no worse than that of quantum electrodynamics.

Thus we adopt here the conventional (and perhaps unsatisfactory) tactics of assigning the photon a small mass μ, and keeping it finite until physical quantities—cross sections, etc.—are computed, and then taking the limit $\mu\to 0$ accounting at the same time for the particular experimental setups in measurements. There is a problem, here, though and it has to do with the introduction of the photon mass in a way not destroying the underlying non-Abelian gauge symmetry. This one can do easily if the gauge group in question has an Abelian factor group as in the Weinberg-Salam theory[26] simply by giving the Abelian gauge boson a mass. In a theory such as the Georgi-Glashow theory,[27] the above option is not available, and one must invent some other ways—for example, by including more (fictitious) scalar mesons as was done by Hagiwara.[28]

Therefore, it suffices to consider the case in which all physical particles are massive. We shall call a pole in the propagator of a regularized theory physical, if the location of the pole does not depend on the gauge-fixing parameters ζ and c. What we have described in previous sections is an intermediate renormalization procedure, after which renormalized Green's functions are finite. It is therefore possible to normalize asymptotic physical particle states to unity by final, *finite* multiplicative renormalizations. Henceforth Z_i will refer to the complete renormalization constant when i refers to a physical particle.

Let us choose a particular gauge (ζ_0^r, c_0^r) and write the effective Lagrangian always in terms of renormalized fields and constants appropriate to this guage. We reall the important conclusion[5] which follows from (2.4); for the *same* Lagrangian, a change in the gauge-fixing term has the same effect on Green's functions as a change in the source term. In particular, this means that

$$\Delta_i(k;\zeta^r, c^r;\epsilon)=z_i(k;^2\zeta^r, c^r;\zeta_0^r, c_0^r;\epsilon)\Delta_i(k;\zeta_0^r, c_0^r;\epsilon) + \text{terms not having poles at } k^2=m_i^2, \qquad (7.1)$$

where $\Delta_i(k;\zeta^r, c^r;\epsilon)$ is the full regularized propagator in the gauge (ζ^r, c^r) for the physical field ϕ_i^r as renormalized in the fiducial gauge (ζ_0^r, c_0^r). Since for physical particles the two propagators appearing on both sides of (7.1) have the pole at the same value of $k^2=m_i{}^2$, the mass counterterm δM^2 is the same in all gauges provided that the renormalization conditions for the scalar masses are expressed in terms of observables, i.e., "the physical mass of the stable particle i shall be m_i." One must give precisely as many conditions of this type as there are independent parameters in M^2.

The value of z_i at $k^2=m_i^2$ is the relative field renormalization constant:

$$z_i(m_i^2;\zeta^r,c^r;\zeta_0^r,c_0^r)=Z_i(\zeta^r,c^r)/Z_i(\zeta_0^r,c_0^r)\,, \quad (7.2)$$

where the Z_i's are the complete renormalization constants in the respective gauges. The relative renormalization constant z_i is in general infinite, except where $\zeta^r=\zeta_0^r$, because in the latter case both Z_i's are relatively finite with respect to $Z_i(\zeta^r,0;M_0^2)$ of Sec. V.

The renormalized (with respect to external lines) physical T-matrix elements T are the same in all gauges:

$$T(k_1,k_2,\ldots,k_n;\zeta^r,c^r)=T(k_1,k_2,\ldots,k_n;\zeta_0^r,c_0^r), \quad (7.3)$$

where

$$T(k_1,k_2,\ldots,k_n;\zeta^r,c^r)=\lim_{\substack{k_i^2\to m_i^2\\ \epsilon\to 0}}\left\{\prod_{i=1}^{n}\frac{1}{[z_i(\zeta^r,c^r;\zeta_0^r,c_0^r)]^{1/2}}(k_i^2-m_i^2)\right\}G(k_1,k_2,\ldots,k_n;\zeta^r,c^r;\zeta_0^r,c_0^r),$$
$$k_1+k_2+\cdots+k_n=0 \quad (7.4)$$

and $G(\zeta^r,c^r;\zeta_0^r,c_0^r)$ is the momentum-space Green's function in the gauge (ζ^r,c^r), wherein the fields are renormalized with respect to the fiducial gauge. This was the main conclusion of LZIV. Now if we adopt the renormalization conditions that g_α is the value of the T-matrix element for a particular trilinear coupling of three vector bosons, then it follows that

$$\frac{Y_\alpha(\zeta^r,c^r)}{[Z_\alpha(\zeta^r,c^r)Z_\beta(\zeta^r,c^r)Z_\gamma(\zeta^r,c^r)]^{1/2}} \quad (7.5)$$

is independent of (ζ^r,c^r), where $Y_\alpha(\zeta^r,c^r)$ is the vertex renormalization constant which will meet the on-mass-shell renormalization condition, and the indices α, β, and γ refer to the same factor group. Note that the ratio in (7.5) and

$$\frac{X_\alpha(\zeta^r,0)}{Z_\alpha^{1/2}(\zeta^r,0)\tilde{Z}_\alpha(\zeta^r,0)}$$

which appears in (3.2) is relatively finite. A similar statement can be made also for the quartic scalar couplings: With on-mass-shell renormalization conditions on these vertices, we find that

$$[1+\lambda_{abcd}^{(r)\,-1}\delta\lambda_{abcd}(\zeta^r,c^r)]$$
$$\times[Z_a(\zeta^r,c^r)Z_b(\zeta^r,c^r)Z_c(\zeta^r,c^r)Z_d(\zeta^r,c^r)]^{1/2}.$$

is gauge-independent.

In conclusion, the renormalized S matrix, starting from the same Lagrangian, is the same in all linear gauges.

ACKNOWLEDGMENTS

Much of this work was done while the author enjoyed the hospitality of Theory Group of CERN. The author benefited greatly from the wisdoms of S. Coleman, G. 't Hooft, M. Veltman, and J. Zinn-Justin.

*On leave of absence from the Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11790.

† Operated by Universities Research Association, Inc., under contract with the United States Atomic Energy Commission.

[1]B. W. Lee, Phys. Lett. 46B, 214 (1973).
[2]B. W. Lee and J. Zinn-Justin, Phys. Rev. D 5, 3121 (1972).
[3]B. W. Lee and J. Zinn-Justin, Phys. Rev. D 5, 3137 (1972).
[4]G. 't Hooft and M. Veltman, Nucl. Phys. B50, 318 (1972).
[5]B. W. Lee and J. Zinn-Justin, Phys. Rev. D 7, 1049 (1973).
[6]A. Slavnov, Teor. Mat. Fiz. 10, 152 (1972); [Theor. and Math. Phys. 10, 99 (1972)].
[7]J. C. Taylor, Nucl. Phys. B33, 436 (1971).
[8]K. Fujikawa, B. W. Lee, and A. I. Sanda, Phys. Rev. D 6, 2923 (1972).
[9]D. J. Gross and R. Jackiw, Phys. Rev. D 6, 477 (1972).
[10]H. Georgi and S. L. Glashow, Phys. Rev. D 6, 429 (1972).
[11]W. A. Bardeen, in *Proceedings of the XVI International Conference on High Energy Physics, Chicago-Batavia, Ill., 1972,* edited by J. D. Jackson and A. Roberts (NAL, Batavia, Ill., 1973), Vol. 2, p. 295.
[12]G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972).
[13]E. S. Abers and B. W. Lee, Phys. Rep. (to be published).
[14]G. 't Hooft, Nucl. Phys. B33, 173 (1971).
[15]R. P. Feynman, Acta Phys. Pol. 24, 697 (1963).
[16]B. S. DeWitt, Phys. Rev. Lett. 12, 742 (1964); Phys. Rev. 162, 1195 (1967); 162, 1239 (1967).
[17]L. D. Faddeev and V. N. Popov, Phys. Lett. 25B, 29 (1967).
[18]G. Jona-Lasinio, Nuovo Cimento 34, 1790 (1964).
[19]N. N. Bogoliubov and O. Parasiuk, Acta Math. 97,

227 (1957); K. Hepp, Commun. Math. Phys. 2, 301 (1966); W. Zimmermann, in *Lectures on Elementary Particles and Quantum Field Theory,* edited by S. Deser *et al.*, (MIT Press, Cambridge, Mass., 1970).
[20]P. Higgs, Phys. Lett. 12, 132 (1964); Phys. Rev. Lett. 13, 508 (1964); Phys. Rev. 145, 1156 (1966); G. Guralnik, C. Hagen, and T. Kibble, Phys. Rev. Lett. 13, 585 (1964); T. Kibble, Phys. Rev. 155, 1554 (1967); F. Englert and R. Brout, Phys. Rev. Lett. 13, 321(1964).
[21]M. Veltman, invited talk at the International Symposium on Electrons and Photon Interactions, Bonn, 1973 (unpublished).
[22]B. W. Lee, Nucl. Phys. B9, 646 (1969); J. L. Gervais and B. W. Lee *ibid.* B12, 627 (1969), B. W. Lee, *Chiral Dynamics* (Gordon and Breach, New York, 1972).
[23]K. Symanzik, Commun. Math. Phys. 16, 48 (1970); Nuovo Cimento Lett. 16, 705 (1970).
[24]K. Symanzik, in *Fundamental Interactions at High Energy,* edited by A. Perlmutter *et al.* (Gordon and Breach, New York, 1970), p. 263.
[25]T. Appelquist, J. Carazzone, T. Goldman, and H. R. Quinn, Phys. Rev. D 8, 1747 (1973).
[26]S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); A. Salam, in *Elementary Particle Theory,* edited by N. Svartholm (Almquist and Wiksell, Stockholm, 1968), p. 367
[27]H. Georgi and S. L. Glashow, Phys. Rev. Lett. 28, 1497 (1972).
[28]T. Hagiwara, thesis, State University of New York at Stony Brook, 1973 (unpublished).

PHYSICAL REVIEW D VOLUME 9, NUMBER 4 15 FEBRUARY 1974

Application of conformal symmetry to quantum electrodynamics*

N. Christ
Columbia University, New York, New York 10027
(Received 5 October 1973)

Conformal symmetry is used to determine the position-space vertex function $\Gamma_\lambda(x,y;z)$ of massless quantum electrodynamics in terms of the current-current correlation function $\pi_{\mu\nu}(x,y)$ and coefficients E_1 and E_2 appearing in the Wilson expansion of the product $\bar\psi(y)\psi(x)$ of Fermi fields. This relationship holds order by order in perturbation theory for graphs containing a single electron line, evaluated in a special position-dependent gauge. The relationship provides an explicit and perhaps simplified method for computing the vacuum-polarization amplitude $F_1(\alpha)$and is used to analyze the behavior of single-electron-loop amplitudes in quantum electrodynamics for values of the coupling constant α near a zero of $F_1(\alpha)$.

I. INTRODUCTION

We study the short-distance behavior of the connected, improper vertex function $\Gamma_\lambda(x, y; z)$ computed in quantum electrodynamics from graphs containing a single electron line. Examples of such graphs, with the position coordinates x, y, and z labled, are shown in Fig. 1. Throughout this paper we will work with amplitudes in position space[1] and achieve the short-distance limit by setting the electron mass equal to zero.

It is demonstrated in Sec. II that such zero-mass Feynman amplitudes, corresponding to graphs containing a single electron line, are conformally covariant[2] provided they are computed in a special gauge.[3] In this gauge the free photon propagator is given by

$$D^B_{\mu\nu}(x, y)=-\frac{1}{2}\,\frac{(x-y)^2}{(x-B)^2(y-B)^2}\,\frac{\partial}{\partial x_\mu}\,\frac{\partial}{\partial y_\nu}\,\frac{(x-B)^2(y-B)^2}{(x-y)^2}+(\tfrac{1}{4}\kappa-1)\frac{\partial}{\partial x_\mu}\,\frac{\partial}{\partial y_\nu}\ln(x-y)^2\,. \qquad (1)$$

The new gauge parameters B_μ, $1\le\mu\le4$, must be transformed as the coordinates of a position four-vector in order to realize conformal covariance. In the limit $\kappa=0$, $B_\mu\to\infty$ the above propagator reduces to the usual Feynman gauge propagator. The parameter κ is chosen as that power series in α,

$$\kappa=1-\frac{3\alpha}{8\pi}+\cdots\,, \qquad (2)$$

which makes the electron wave-function renormalization constant Z_2 finite.[4]

In Sec. III the vertex function $\Gamma_\lambda(x, y; z)$ is first studied in the generalized Landau gauge with photon propagator

$$D_{\mu\nu}(x-y)=\frac{\delta^{\mu\nu}}{(x-y)^2}+\tfrac{1}{4}\kappa\,\frac{\partial}{\partial x_\mu}\,\frac{\partial}{\partial y_\nu}\ln(x-y)^2\,. \qquad (3)$$

Conformal symmetry is then used to determine the dependence of the improper vertex function $\Gamma^B_\lambda(x, y; z)$, evaluated in the gauge (1), on the gauge parameters B_μ. It is found that the limit