Demonstration of noncausality for the Rarita-Schwinger equation*

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Using the Rarita-Schwinger equation for a spin-3/2 particle in a a constant magnetic field, we explicitly calculate the propagator in 2 + 1 dimensions. From the behavior of the propagator, it is seen that the propagation is noncausal.

I. INTRODUCTION

Recently, there has been some interest on higher-spin equations. Velo and Zwanziger showed in Ref. 1 that the solutions of the Rarita-Schwinger (RS) equation are noncausal, using the method of characteristics. There are two main criticisms of the Velo and Zwanziger result. First, they are applying the method of characteristics outside the domain of the existing proofs.[2] Secondly, since the metric used in the RS equation is not positive definite, even the existence of the solutions is not certain. The positivity of the metric is used in an essential manner for the existence proofs of symmetric equations.[3]

There has been further work on this subject, and the existence of solutions for the RS equation has been shown quite recently.[4] However, we think it is still worthwhile to communicate this work, which was done some time ago, which might be complementary to Ref. 1 in some respect. Here, without treating in pure mathematical rigor the problems of domains of definition, regularity of the solutions, etc., we explicitly construct the propagator for the RS equation in $2+1$ dimensions.

We use Velo and Zwanziger's techniques[1] to reduce the RS equation into the modified form where we have a genuine equation of motion, which is equivalent to the original equation if we impose certain conditions at time t_0. For any electromagnetic potential, we do not know any solutions of this equation in $3+1$ dimensions. In $2+1$ dimensions, however, this equation can be easily solved for a constant magnetic field. From the solutions, we construct the propagator and explicitly see that the propagation of the RS field in $2+1$ dimensions is noncausal.

II. THE CALCULATION AND DISCUSSION

In $2+1$ dimensions, the Rarita-Schwinger equation with minimal electromagnetic interaction is

$$L^\nu = (\not{D} + m)\psi^\nu - (\gamma^\nu D_\mu + D^\nu \gamma_\mu)\psi^\mu - \gamma^\nu(\not{D} - m)\gamma_\mu \psi^\mu = 0\,, \qquad \nu = 0, 1, 2\,, \quad (1)$$

$$D^\mu = \partial^\mu - ieA^\mu\,.$$

Using the subsidiary conditions

$$\gamma^\mu \psi_\mu = \frac{ie}{2m}\left(-F_{\nu\mu}\gamma^\mu\psi^\nu + \tfrac{1}{2}F_{\nu\lambda}\gamma^\nu\gamma^\lambda\gamma_\mu\psi^\mu\right) \quad (2)$$

one gets the modified RS equation

$$(\not{D} + m)\psi^\kappa - (\gamma^\kappa m + D^\kappa)ge(F_{\nu\mu}\gamma^\mu\psi^\nu - \tfrac{1}{2}F_{\nu\lambda}\gamma^\nu\gamma^\lambda\gamma_\mu\psi^\mu) = 0\,,$$

$$g = -\frac{i}{2m^2}\,. \quad (3)$$

Equation (1) has two subsidiary conditions: Eq. (2) and the zeroth component of L^ν. Take

$$\chi \equiv L^0 = (\gamma^l D_l - m)\gamma_k \psi^k + D_l \psi^l = 0\,, \qquad l, k = 1, 2 \text{ for } t = t_0 \quad (4)$$

$$\phi = \gamma_\mu\psi^\mu - ge(F_{\nu\mu}\gamma^\mu\psi^\nu - \tfrac{1}{2}F_{\nu\lambda}\gamma^\nu\gamma^\lambda\gamma_\mu\psi^\mu) = 0\,, \qquad \text{for } t = t_0\,. \quad (5)$$

By straightforward computation[5] one can show that χ and ϕ satisfy the equations of motion

$$(-\not{D} + m)\phi + 2(D_k\gamma^k - m)\phi - 2\chi = 0\,, \quad (6)$$

$$(\gamma_0 D_0 - \gamma^l D_l - m)\chi = \tfrac{1}{2}eF_{lk}\gamma^l\gamma^k\phi\,, \quad (7)$$

which gives the wave equation for ϕ:

$$(D_\mu D^\mu + m^2)\phi + eF_{\mu\nu}\gamma^\mu\gamma^\nu\phi = 0\,. \quad (8)$$

So, if we impose $\phi = 0$, $\chi = 0$, $\partial_k\phi = 0$ at $t = t_0$, Eqs. (6) and (7) give $\chi = 0 = \phi$ for all time. This shows that Eqs. (2)–(4) are equivalent to Eq. (1), i.e., if we impose the subsidiary conditions $L^0 = 0$ and Eq. (2) on the solutions of Eq. (3) at $t = t_0$, we get the solutions of Eq. (1). One can also get to Eq. (3) using the methods described in Ref. 6.

Now we study Eq. (3) with a constant magnetic field,

$$A_1 = Bx_2, \quad A_2 = A_3 = 0\,.$$

Then Eq. (3) reads

$$(\not{D} + m)\psi^\nu - (\gamma^\nu m + D^\nu)egB\gamma^1\gamma^2\gamma_0\psi^0 = 0\,. \quad (9)$$

For $\nu = 0$, ψ^0 totally decouples from ψ_1 and ψ_2.

$$(\not\partial+m)\psi^0-(\gamma^0 m+\partial^0)egB\gamma^1\gamma^2\gamma^0\psi^0-ie\gamma_1 A^1\psi_0=0\ . \tag{10}$$

One can solve the $\nu=0$ equation for ψ^0, then use ψ^0 as a source term in the equations for ψ^1 and ψ^2. This simplification of the problem is our main reason to go to $2+1$ dimensions.

We have three wave equations for ψ^0, ψ^1, and ψ^2, and two subsidiary conditions, Eqs. (2) and (4). We can solve Eq. (2) for ψ^2 in terms of ψ^1 and ψ^0. Then Eq. (4) gives a differential equation for ψ^1, where ψ^0 is treated as a source term, and where only space derivatives appear. We do not have any restrictions on ψ^0 and $\partial_0\psi^0$ at $t=t_0$; fixing them determines ψ^1 and ψ^2. So, we have a well-defined Cauchy problem for ψ^0.

In $2+1$ dimensions γ^μ has the representation of the Pauli τ matrices

$$\gamma_0=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix},\quad \gamma_1=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\quad \gamma_2=\begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix}.$$

After eliminating the first-order time derivative by taking $\psi_0=e^{i\alpha t}\phi$, where

$$\alpha=\frac{eB}{m(1-e^2B^2/4m^4)}\ , \tag{11}$$

Eq. (10) goes into

$$\left\{-1\left(1-\frac{e^2B^2}{4m^4}\right)(\partial^0)^2+(\partial_1-iex_2B)^2+\partial_2{}^2+\left[i\sigma_0 eB-\frac{e^2B^2}{m(1-e^2B^2/4m^4)}-m^2\left(1-\frac{e^2B^2}{4m^4}\right)\right]\right\}\phi=0\ , \tag{12}$$

where

$$-i\sigma_0=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}.$$

Note that if $e^2B^2>4m^4$, the equation above is not hyperbolic. A similar situation occurs in $3+1$ dimensions as shown in Ref. 1.

For the sake of simplicity we change to variables

$$x=(eB)^{1/2}x_1,\quad y=(eB)^{1/2}x_2,\quad t=(eB)^{1/2}x_0\ .$$

Equation (12) has a stationary solution,

$$\phi_n=\frac{e^{-iE_nt}}{2\pi(n!)^{1/2}}\,\frac{H_n((y+p_1)/\sqrt{2})}{\pi^{1/4}}\,e^{-(y+p_1)^2/2}e^{-ip_1x}\ , \tag{13}$$

where

$$E_n=\left(\frac{2n+1+M^2}{1-e^2B^2/4m^4}\right)^{1/2},$$
$$M^2=\frac{eB}{m(1-e^2B^2/4m^4)}+\frac{m^2}{eB}\left(1-\frac{e^2B^2}{4m^4}\right)-i\sigma_0\ . \tag{14}$$

H_n are the Hermite polynomials of degree n.

Here we want to study the propagation properties of

$$\psi^0(x)=\int D(x,x')\overleftrightarrow{\partial}_0\psi_0(x')d^3x'\ . \tag{15}$$

Imposing Eqs. (2) and (4) at time $t=t_0$, we get a solution for Eq. (1). The propagator $D(x,x')$ propagates that solution to time t. Therefore, it is sufficient to establish the noncausality for $D(x,x')$ to show the noncausality for the solutions of the RS equation in $2+1$ dimensions.

Using Ref. 7, we write down the homogeneous propagator D for ψ^0,

$$D=\sum_n\int dp_1 e^{ip_1(x-x')}e^{-[(y+p_1)^2+(y'+p_1)^2]/2}\times e^{i\alpha(t-t')}\frac{\sin E_n(t-t')}{E_n}\times\frac{H_n((y+p_1)/\sqrt{2})H_n((y'+p_1)/\sqrt{2})}{\sqrt{\pi}\,n!}\ . \tag{16}$$

D satisfies

$$LD=0\ , \tag{17}$$

where L is the differential operator of Eq. (10),

$$\left.\frac{\partial D}{\partial x_0}\right|_{x_0=0}=\delta(\vec{x}-\vec{x}'),\qquad D=D^{\rm adv}-D^{\rm ret}\ . \tag{18}$$

Using the completeness property of e^{-ip_1x} and $H_n((p_1+y)/\sqrt{2})$, we perform the integration[8] and get

$$\frac{1}{\sqrt{\pi}}\frac{1}{n!}\int_{-\infty}^{\infty}dp_1 e^{ip_1(x-x')}e^{-[(y+p_1)^2+(y'+p_1)^2]/2}H_n\left(\frac{y+p_1}{\sqrt{2}}\right)H_n\left(\frac{y'+p_1}{\sqrt{2}}\right)=e^{-(x-x')(y+y')/2}L_n(\sigma')e^{-\sigma'/2}\ , \tag{19}$$

where L_n is the Laguerre function and

$$\sigma'=\tfrac{1}{2}[(x-x')^2+(y-y')^2]\ . \tag{20}$$

Using the integral representation for Bessel functions.[9]

$$\frac{\sin E_n(t-t')}{E_n}=(\tfrac{1}{2}\pi)^{1/2}\frac{1}{2\pi i}(t-t')\int_{c-i\infty}^{c+i\infty}d\beta\,\frac{\exp\{\frac{1}{2}[\beta-E_n(t-t')/\beta]^2\}}{\beta^{3/2}}\ ,\quad c>0 \tag{21}$$

and the generating function for the Laguerre polynomials,

$$\sum_n u^n L_n(\sigma')e^{-\sigma'/2} = \frac{\exp[\frac{1}{2}\sigma'(1+u)/(1-u)]}{1-u}\ , \quad u<1 \tag{22}$$

after changing variables, $\phi = \beta/(t-t')^2$, and restoring to the original variables x_0, x_1, x_2,[10] we end up with

$$D = \frac{1}{2\pi i}(\tfrac{1}{2}\pi)^{1/2}\exp\left[\frac{ieB(x_0-x_0')}{m(1-e^2B^2/4m^4)}\right]\ \exp[\tfrac{1}{2}ieB(x_2+x_2')(x_1-x_1')]$$

$$\times\int_{c-i\infty}^{c+i\infty}\frac{d\phi}{2\phi^{3/2}}\exp\left(\frac{1}{2}\left\{-\lambda\phi-\frac{1}{\phi}\left[\frac{e^2B^2}{m^2(1-e^2B^2/4m^4)^2}+m'^2\right]\right\}\right)\frac{\exp\left\{\frac{1}{2}\sigma eB\left[\frac{2\phi}{eB}-\frac{2eB\phi}{4m^4}-\coth\frac{eB}{2\phi(1-e^2B^2/4m^4)}\right]\right\}}{\sinh\left[\frac{eB}{2\phi(1-e^2B^2/4m^4)}\right]}\ , \tag{23}$$

where

$$-\lambda = 2\sigma\left(1-\frac{e^2B^2}{4m^4}\right)+(x_0-x_0')^2\ , \tag{24}$$

$$\sigma = \tfrac{1}{2}[(x_1-x_1')^2+(x_2-x_2')^2]\ , \tag{25}$$

$$m'^2 = m^2 - \frac{i\sigma_0 eB}{(1-e^2B^2/4m^4)}\ , \tag{26}$$

$$c>0\ .$$

For $\lambda>0$, we can close the contour on the right-hand side without encountering any singularities, since all the singularities lie on the imaginary axis. Therefore, for $\lambda>0$, $D=0$. This corresponds to the region where

$$(x_0-x_0')^2-\left(1-\frac{e^2B^2}{4m^4}\right)[(x_1-x_1')^2+(x_2-x_2')^2]<0\ .$$

If $B=0$, we retain the causal behavior, since the propagator is zero for spacelike separations.

When $B\neq 0$, λ does not equal the spacelike distance. We see that the light-cone singularity occurs at $\lambda=0$, which is shifted to the spacelike region.[11] This establishes the noncausal behavior of the ψ^0 component of the RS field in 2+1 dimensions. Once we find a solution which is nonzero in a spacelike region; as with the singular solution we found for $\lambda=0$, we have established the noncausuality of the propagator. From the other components of Eq. (9), one easily sees that the other two components propagate causally. So, when all three components are taken together, the RS equation exhibits noncausality in 2+1 dimensions.

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[1] G. Velo and D. Zwanziger, Phys. Rev. 186, 1337 (1969).

[2] R. Seiler, in *Broken Scale Invariance and the Light Cone*, 1971 Coral Gables Conference on Fundamental Interactions at High Energy, edited by M. Dal Cin, G. J. Iverson, and A. Perlmutter (Gordon and Breach, New York, 1971); J. Bellissard and R. Seiler, University of Provence report, 1972 (unpublished); M. Hortaçsu, University of Pittsburgh dissertation, 1971 (unpublished).

[3] Fritz John, in *Partial Differential Equations*, Proceedings of the Summer Seminar, Boulder, Colorado, 1957 (Interscience, New York, 1964), Vol. III.

[4] R. Seiler, private communication.

[5] M. Hortaçsu, in Ref. 2.

[6] J. Bellissard and R. Seiler, in Ref. 2.

[7] G. Källén, in *Handbuch der Physik*, edited by S. Flügge (Springer, Berlin, 1958), Vol. V (Part I).

[8] J. Geheniau, Physica 16, 822 (1950).

[9] *Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 2, p. 189.

[10] Reference 5. Similar manipulations are described in Refs. 7 and 8.

[11] For $\lambda=0$, the propagator is reduced to a finite part coming from integrating in a finite region, plus terms like

$$\int_c^\infty \frac{d\phi}{\phi^{1/2}}+\int_c^\infty d\phi\, O(\phi^{-3/2}),\quad c\gg 1$$

which diverges.