Lagrangian formulation for arbitrary spin. II. The fermion case*†

L. P. S. Singh and C. R. Hagen
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
(Received 10 October 1973)

The Rarita-Schwinger formalism for fermion fields is brought to a Lagrangian form in the case of arbitrary spin. The requirement that all differential equations of the field should follow from the variation of an action integral necessitates the introduction of additional fields in the theory. By demanding that these auxiliary variables vanish in the case of no interaction, an explicit form is obtained for the Lagrangian. The resulting theory is found to reproduce the usual formalism in the case of spin $\frac{3}{2}$, and turns out to be in agreement with results obtained by Chang for spin values $\frac{5}{2}$ and $\frac{7}{2}$. The Galilean limit of the minimally coupled equations yields the minimal Galilean-invariant theory of Hagen and Hurley. The g factor turns out to be $1/s$, in accordance with a long-standing conjecture.

I. INTRODUCTION

In the preceding paper[1] a Poincaré-invariant Lagrangian theory was presented for a massive boson field of arbitrary spin. The spin-s boson field was taken to be a symmetric traceless tensor of rank s, which, along with symmetric traceless tensors of ranks $s-2, s-3, \ldots, 0$, allowed the construction of the second-order Lagrangian. This Lagrangian was then brought to first-order form and some simple aspects of the resulting theory were studied. In this paper[2] the corresponding program is carried out for a massive arbitrary-spin fermion field.

Following the Rarita-Schwinger scheme[3] we select the spin-s fermion field to be a symmetric tensor-spinor $\psi^{(n)}$ of rank n $(\equiv s-\frac{1}{2})$ which satisfies the spinor trace condition[4]

$$\gamma^\mu \psi^{(n)}_{\mu\cdots} = 0 . \tag{1}$$

Thus $\psi^{(n)}$ transforms according to the Lorentz-group representation $D(\frac{1}{2}(n+1), \frac{1}{2}n) \oplus D(\frac{1}{2}n, \frac{1}{2}(n+1))$ and satisfies

$$(-i\gamma\cdot\partial + m)\psi^{(n)}_{\mu_1\cdots\mu_n} = 0 \tag{2a}$$

and

$$\partial^\mu \psi^{(n)}_{\mu\cdots} = 0 . \tag{2b}$$

Our aim is to write down a set of Lorentz-invariant first-order differential equations which yield (2) and are obtainable from a Lagrangian.[5] Note that since Eq. (2b) follows from (2a) and (1), it would appear that one has only to obtain Eq. (2a). The latter, however, is not a Lagrange equation since it does not consist of traceless terms, whereas an equation obtained from a Lagrangian by varying it with respect to a symmetric traceless field such as $\psi^{(n)}$ should consist only of symmetric traceless terms.[6] The correct Lagrange equation corresponding to (2a) is

$$(-i\gamma\cdot\partial + m)\psi^{(n)}_{\mu_1\cdots\mu_n} + \frac{i}{(n+1)}\sum_{j=1}^{n}\gamma_{\mu_j}(\partial\psi^{(n)})_{\mu_1\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_n} = 0, \tag{3}$$

a result which reduces to (2a) provided (2b) is satisfied. However, the latter result is not obtainable from (3), and more equations and consequently more fields are needed. The resulting equations should be such as to imply that all the auxiliary fields vanish, and in addition (2b) should be obtained.

In Sec. II we consider the question as to what types of auxiliary fields are to be used. The arguments and procedure are similar to those used in the preceding paper. Since the number of auxiliary-field components should be kept to a minimum, we restrict ourselves to tensor-spinors of rank less than n. In view of the fact that these can couple only to $(\partial\psi^{(n)})$, one can restrict consideration to symmetric, traceless objects. We start by analyzing the situation for some lower spin values. A pattern emerges which is then used to construct the Lagrangian for the general case. In Sec. III minimal electromagnetic interactions are introduced and the Galilean limit of the resulting equations is obtained. The result is found to coincide with the Galilean-invariant theory of Hagen and Hurley.[7]

II. THE FREE FIELD

As noted in the previous section, the Lagrange equation (3) does not reduce to (2a) unless (2b) is satisfied. The problem is thus seen to be that of eliminating $(\partial\psi^{(n)})$—a tensor-spinor of rank $n-1$. It is natural to try an equation involving tensor-spinors of that rank. We therefore introduce a symmetric traceless tensor-spinor $\psi^{(n-1)}$ of rank $n-1$. In the spin-$\frac{3}{2}$ case, then, the most general Lagrange equations involving $\psi^{(1)}$ and $\psi^{(0)}$ are[8]

$$(-i\gamma\cdot\partial+m)\psi^{(1)}_\mu+\tfrac{1}{2}i\gamma_\mu(\partial\psi^{(1)})=c[\partial_\mu+\tfrac{1}{4}\gamma_\mu(\gamma\cdot\partial)]\psi^{(0)} \quad (4)$$

and

$$(\partial\psi^{(1)})=-(i\gamma\cdot\partial+am)\psi^{(0)}\ . \quad (5)$$

The real coefficients c and a are to be determined from the requirement that (4) and (5) yield $\psi^{(0)}=0=(\partial\psi^{(1)})$. Contracting (4) with ∂^μ and subsequently eliminating $(\partial\psi^{(1)})$ with the help of (5), one obtains

$$[(1-\tfrac{3}{2}c)\partial^2+(2-a)(-im\gamma\cdot\partial)-2am^2]\psi^{(0)}=0\ .$$

This will imply $\psi^{(0)}=0$ [hence $(\partial\psi^{(1)})=0$ from (5)] if and only if the coefficients of the derivative terms vanish and that of the last term is nonzero. The coefficients c and a are thus uniquely determined to be $\frac{2}{3}$ and 2, respectively, thereby leading to the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}\psi^{(1)\mu}\beta(i\gamma\cdot\partial-m)\psi^{(1)}_\mu-\tfrac{2}{3}\psi^{(0)}\beta(\partial\psi^{(1)})$$
$$-\tfrac{1}{3}\psi^{(0)}\beta(i\gamma\cdot\partial+2m)\psi^{(0)}\ .$$

Introducing a new vector-spinor ψ_μ (which is not traceless) by

$$\psi_\mu=\psi^{(1)}_\mu-\tfrac{1}{6}i(\omega+\tfrac{1}{2})^{-1}\psi^{(0)},\quad \omega\neq-\tfrac{1}{2}$$

this is seen to reduce to the one-parameter family of Lagrangians used in the usual formulation of the spin-$\frac{3}{2}$ field.[9]

In the general case, the analogs of Eqs. (4) and (5) are[10]

$$(-i\gamma\cdot\partial+m)\psi^{(n)}-\mathrm{Tr}=c\{\partial\psi^{(n-1)}\}_{\mathrm{S.T.}} \quad (6)$$

and

$$(\partial\psi^{(n)})=-(i\gamma\cdot\partial+a_1m)\psi^{(n-1)}-\mathrm{Tr}\ . \quad (7)$$

These imply $\psi^{(n-1)}=0=(\partial\psi^{(n)})$ only when $c=2n^2/(2n+1)$, $a_1=(n+1)/n$, and

$$(\partial\psi^{(n-1)})=(\partial\partial\psi^{(n)})=0\ . \quad (8)$$

The latter result cannot be obtained from Eqs. (6) and (7) if the above values for c and a_1 are used, so more equations are needed in order to obtain (8). Again the obvious choice is the introduction of a symmetric traceless tensor-spinor $\chi^{(n-2)}$ of rank $n-2$.

The most general equations for the spin-$\frac{5}{2}$ case, in view of the above discussion, are

$$(-i\gamma\cdot\partial+m)\psi^{(2)}-\mathrm{Tr}=\tfrac{8}{5}\{\partial\psi^{(1)}\}_{\mathrm{S.T.}}\ , \quad (9)$$

$$(\partial\psi^{(2)})=-\{(i\gamma\cdot\partial+\tfrac{3}{2}m)\psi^{(1)}-\mathrm{Tr}\}+c_1\{\partial\chi^{(0)}\}_{\mathrm{S.T.}}\ , \quad (10)$$

and

$$(\partial\psi^{(1)})=(-i\gamma\cdot\partial+d_2m)\chi^{(0)}, \quad (11)$$

where the values of c and a_1 obtained above have been substituted. A simple calculation convinces one of the fact that these equations cannot be adjusted to yield $\chi^{(0)}=0$, and the introduction of only $\chi^{(0)}$ is consequently not sufficient. The simplest choice is to add another scalar-spinor $\psi^{(0)}$, yielding

$$\mathcal{L}'=(c_1\chi^{(0)}+c_1'\psi^{(0)})\beta(\partial\psi^{(1)})$$
$$-\tfrac{1}{2}c_1\chi^{(0)}\beta(-id_2'\gamma\cdot\partial+d_2m)\chi^{(0)}$$
$$-\tfrac{1}{2}c_1b_2\psi^{(0)}\beta(ia_2'\gamma\cdot\partial+a_2m)\psi^{(0)}$$
$$-c_1\chi^{(0)}\beta(-ie_2\gamma\cdot\partial+b_2m)\psi^{(0)}$$

as the most general form (up to an over-all constant) for that part of the spin-$\frac{5}{2}$ Lagrangian which involves $\chi^{(0)}$ and $\psi^{(0)}$. Since the auxilary fields are required to vanish, the results should not be altered by the replacement

$$\begin{pmatrix}\chi^{(0)}\\ \psi^{(0)}\end{pmatrix}\to R\begin{pmatrix}\chi^{(0)}\\ \psi^{(0)}\end{pmatrix}, \quad (12)$$

where R is an arbitrary real, nonsingular 2×2 matrix. Consequently one can fix any four of the coefficients in $\mathcal{L}'$ at will without any loss of generality. We set $c_1'=e_2=0$, $d_2'=a_2'=1$. The resulting Lagrange equations are (9), (10),

$$(\partial\psi^{(1)})=(-i\gamma\cdot\partial+d_2m)\chi^{(0)}+b_2m\psi^{(0)}, \quad (13)$$

and

$$m\chi^{(0)}=-(i\gamma\cdot\partial+a_2m)\psi^{(0)}\ . \quad (14)$$

Now it is straightforward to verify that the necessary and sufficient conditions for these equations to imply $\psi^{(0)}=0$ [hence $\chi^{(0)}=0=(\partial\psi^{(1)})=(\partial\partial\psi^{(2)})$]

are

$$d_2 = a_2 = 3,$$
$$c_1 = \tfrac{18}{5},$$

and

$$b_2 = -\tfrac{5}{3}.$$

The resulting Lagrangian is

$$\mathfrak{L} = \tfrac{1}{2}\psi^{(2)\mu\nu}\beta(i\gamma\cdot\partial - m)\psi^{(2)}_{\mu\nu} + \tfrac{8}{5}\psi^{(1)\mu}\beta(\partial\psi^{(2)})_\mu + \tfrac{4}{5}\psi^{(1)\mu}\beta(i\gamma\cdot\partial + \tfrac{3}{2}m)\psi^{(1)}_\mu$$
$$+ (\tfrac{12}{5})^2[\chi^{(0)}\beta(\partial\psi^{(1)}) - \tfrac{1}{2}\chi^{(0)}\beta(-i\gamma\cdot\partial + 3m)\chi^{(0)} + \tfrac{5}{6}m(\chi^{(0)}\beta\psi^{(0)} + \psi^{(0)}\beta\chi^{(0)}) + \tfrac{5}{6}\psi^{(0)}\beta(i\gamma\cdot\partial + 3m)\psi^{(0)}];$$

the formalism is invariant under the transformations

$$\psi^{(1)} \to \text{const}\,\psi^{(1)}$$

and (12).

The same procedure can be carried out for spin $\frac{7}{2}$. It turns out that in addition to $\psi^{(3)}$, $\psi^{(2)}$, $\psi^{(1)}$, and $\chi^{(1)}$, one must introduce two "scalar-spinors" $\psi^{(0)}$ and $\chi^{(0)}$. We omit the details here and pass on to the general case.

One can discern the following pattern: The result $(\partial\psi^{(n)}) = 0$ can be obtained by introducing a symmetric traceless traceless tensor-spinor $\psi^{(n-1)}$ of rank $n-1$, provided that Eq. (8) is satisfied. The latter situation can be achieved by successively obtaining $\psi^{(n,\lambda)} = 0 = \psi^{(n-1,\lambda)}$ for $\lambda = n, n-1, n-2, \ldots, 2$, where

$$\psi^{(n,\lambda)}_{\mu_{\lambda+1}\cdots\mu_n} \equiv \partial^{\mu_1}\cdots\partial^{\mu_\lambda}\psi^{(n)}_{\mu_1\cdots\mu_n}$$

and

$$\psi^{(n-1,\lambda)}_{\mu_{\lambda+1}\cdots\mu_n} \equiv \partial^{\mu_2}\cdots\partial^{\mu_\lambda}\psi^{(n-1)}_{\mu_2\cdots\mu_n}$$

are symmetric traceless tensor-spinors of rank $n-\lambda$. At each stage one needs to introduce two symmetric traceless tensor-spinors each of the same rank as $\psi^{(n,\lambda)}$, i.e., two symmetric traceless tensor-spinors of ranks $0, 1, 2, \ldots, n-2$ have to be introduced in addition to $\psi^{(n-1)}$. Thus one has a $\frac{4}{3}(n+1)(n^2+2n+3)$-component theory involving the representations

$$D(\tfrac{1}{2}(n+1), \tfrac{1}{2}n) \oplus D(\tfrac{1}{2}n, \tfrac{1}{2}(n+1))$$
$$\oplus D(\tfrac{1}{2}n, \tfrac{1}{2}(n-1)) \oplus D(\tfrac{1}{2}(n-1), \tfrac{1}{2}n)$$
$$\oplus\, 2\sum_{j=0}^{n-2}[D(\tfrac{1}{2}(j+1), \tfrac{1}{2}j) \oplus D(\tfrac{1}{2}j, \tfrac{1}{2}(j+1))]$$

of the Lorentz group.

The most general Lagrangian involving these fields is

$$\mathfrak{L} = \tfrac{1}{2}\psi^{(n)}\beta(i\gamma\cdot\partial - m)\psi^{(n)}$$
$$+ c\{\psi^{(n-1)}\beta(\partial\psi^{(n)}) + \tfrac{1}{2}\psi^{(n-1)}\beta(i\gamma\cdot\partial + a_1 m)\psi^{(n-1)} + c_1\chi^{(n-2)}\beta(\partial\psi^{(n-1)}) + c_1'\psi^{(n-2)}\beta(\partial\psi^{(n-1)})$$
$$-c_1[\tfrac{1}{2}\chi^{(n-2)}\beta(-id_2'\gamma\cdot\partial + d_2 m)\chi^{(n-2)} - \chi^{(n-2)}\beta(ie_2\gamma\cdot\partial - b_2 m)\psi^{(n-2)}$$
$$+\tfrac{1}{2}b_2\psi^{(n-2)}\beta(ia_2'\gamma\cdot\partial + a_2 m)\psi^{(n-2)} - (c_2'\psi^{(n-3)} + c_2''\chi^{(n-3)})\beta(\partial\chi^{(n-2)}) + c_2'''\psi^{(n-3)}\beta(\partial\psi^{(n-2)})]\}$$
$$+ cc_1\Bigg\{\sum_{q=3}^{n}(-)^q\left(\prod_{j=2}^{q-1}c_j b_j\right)[\chi^{(n-q)}\beta(\partial\psi^{(n-q+1)}) + (c_q'\psi^{(n-q-1)} + c_q''\chi^{(n-q-1)})\beta(\partial\chi^{(n-q)})$$
$$+c_q'''\psi^{(n-q-1)}\beta(\partial\psi^{(n-q)}) + \tfrac{1}{2}\chi^{(n-q)}\beta(id_q'\gamma\cdot\partial + d_q m)\chi^{(n-q)}$$
$$+\chi^{(n-q)}\beta(ie_q\gamma\cdot\partial - b_q m)\psi^{(n-q)} - \tfrac{1}{2}b_q\psi^{(n-q)}\beta(ia_q'\gamma\cdot\partial + a_q m)\psi^{(n-q)}]\Bigg\}.$$

The coefficients are to be so adjusted that the equations imply Eq. (8) and $\psi^{(p)} = \chi^{(p)} = 0$, $0 \le p \le n-2$.

Before embarking on the program the following fact should be noted: Since all the auxiliary fields are required to vanish as the end result, the transformations

$$\psi^{(n-1)} \to \rho\psi^{(n-1)},$$
$$\begin{pmatrix}\chi^{(p)}\\ \psi^{(p)}\end{pmatrix} \to R^{(p)}\begin{pmatrix}\chi^{(p)}\\ \psi^{(p)}\end{pmatrix}, \quad 0 \le p \le n-2 \tag{15}$$

should not affect the procedure; ρ is an arbitrary nonzero real number and $R^{(p)}$ is any real nonsingular 2×2 matrix. Using this freedom one can set[11]

$$a_1'=1\,,\quad e_2=0\,,$$

and (16)

$$a_q'=d_q'=1,\quad c_q''=c_q'''=0 \text{ for } 2\leq q\leq n$$

without any loss of generality. With these substitutions the equations are

$$(-i\gamma\cdot\partial+m)\psi^{(n)}-\mathrm{Tr}=c\{\partial\psi^{(n-1)}\}_{\mathrm{S.T.}}\,, \qquad (17,1)$$

$$(\partial\psi^{(n)})=-[(i\gamma\cdot\partial+a_1 m)\psi^{(n-1)}-\mathrm{Tr}] + c_1\{\partial\chi^{(n-2)}\}_{\mathrm{S.T.}}\,, \qquad (17,2)$$

$$(\partial\psi^{(n-1)}=[(-i\gamma\cdot\partial+d_2 m)\chi^{(n-2)}-\mathrm{Tr}] + b_2 m\psi^{(n-2)}+c_2'\{\partial\psi^{(n-3)}\}_{\mathrm{S.T.}}\,, \qquad (17,3\text{a})$$

$$m\chi^{(n-2)}=-[(i\gamma\cdot\partial+a_2 m)\psi^{(n-2)}-\mathrm{Tr}] + c_2 m\{\partial\chi^{(n-3)}\}_{\mathrm{S.T.}}\,, \qquad (17,3\text{b})$$

$$\cdots$$

$$(\partial\psi^{(n-q+1)})=[(-i\gamma\cdot\partial+d_q m)\chi^{(n-q)} + (-i\gamma\cdot\partial e_q+b_q m)\psi^{(n-q)}-\mathrm{Tr}] + c_q'\{\partial\psi^{(n-q-1)}\}_{\mathrm{S.T.}}\,, \qquad (17,q+1\text{a})$$

$$\left(\frac{c_{q-1}'}{b_{q-1}c_{q-1}}\right)(\partial\chi^{(n-q+1)})+[(-ie_q\gamma\cdot\partial+b_q m)\chi^{(n-q)}-\mathrm{Tr}] = -b_q[(i\gamma\cdot\partial+a_q m)\psi^{(n-q)}-\mathrm{Tr}]+b_q c_q\{\partial\chi^{(n-q-1)}\}_{\mathrm{S.T.}}\,,$$

$$q=3,4,5,\ldots,n-1 \qquad (17,q+1\text{b})$$

$$(\partial\psi^{(1)})=(-\gamma\cdot\partial+d_n m)\chi^{(0)}+(-ie_n\gamma\cdot\partial+b_n m)\psi^{(0)}\,, \qquad (17,n+1\text{a})$$

and

$$\left(\frac{c_{n-1}'}{b_{n-1}c_{n-1}}\right)(\partial\chi^{(1)})+(-ie_n\gamma\cdot\partial+b_n m)\chi^{(0)} = -b_n(i\gamma\cdot\partial+a_n m)\psi^{(0)}\,. \qquad (17,n+1\text{b})$$

As seen above, one proceeds step by step in the following way: Given

$$\psi^{(p)}=\chi^{(p)}=0,\quad p<n-\lambda$$

and (18)

$$\psi^{(p,\lambda+1)}=\chi^{(p,\lambda+1)}=0,\quad p\geq n-\lambda$$

for any integer λ, $2\leq\lambda\leq n$, where[12]

$$\psi^{(n-q,\lambda)}_{\mu_{\lambda-q+1}\cdots\mu_{n-q}}\equiv\partial^{\mu_1}\cdots\partial^{\mu_{\lambda-q}}\psi^{(n-q)}_{\mu_1\cdots\mu_{n-q}}$$

($\chi^{(p,\lambda)}$ is defined in a similar fashion), one obtains the conditions for Eqs. (17, 1), ..., (17, $\lambda+1$) to imply $\psi^{(n-\lambda)}=0$. Contracting Eqs. (17, q) with $\partial^{\mu_1}\cdots\partial^{\mu_{\lambda-q+1}}$, $q=1,2,\ldots,\lambda+1$, and using (18) one obtains

$$\left(-i\,\frac{n-\lambda+1}{n+1}\,\gamma\cdot\partial+m\right)\psi^{(n,\lambda)}=cg(0,\lambda)\partial^2\psi^{(n-1,\lambda)}\,, \qquad (19,1)$$

$$\psi^{(n,\lambda)}=-\left(i\,\frac{n-\lambda+1}{n}\,\gamma\cdot\partial+a_1 m\right)\psi^{(n-1,\lambda)} + c_1 g(1,\lambda)\partial^2\chi^{(n-2,\lambda)}\,, \qquad (19,2)$$

$$\psi^{(n-1,\lambda)}=\left(-i\,\frac{n-\lambda+1}{n-1}\,\gamma\cdot\partial+d_2 m\right)\chi^{(n-2,\lambda)} + mb_2\psi^{(n-2,\lambda)}+c_2' g(2,\lambda)\partial^2\psi^{(n-3,\lambda)}\,, \qquad (19,3\text{a})$$

$$m\chi^{(n-2,\lambda)}=-\left(i\,\frac{n-\lambda+1}{n-1}\,\gamma\cdot\partial+a_2 m\right)\psi^{(n-2,\lambda)} + c_2 g(2,\lambda)\partial^2\chi^{(n-3,\lambda)}\,, \qquad (19,3\text{b})$$

$$\psi^{(n-q+1,\lambda)}=\left(-i\,\frac{n-\lambda+1}{n-q+1}\,\gamma\cdot\partial+d_q m\right)\chi^{(n-q,\lambda)} + \left(-i\,\frac{n-\lambda+1}{n-q+1}\,e_q\gamma\cdot\partial+b_q m\right)\psi^{(n-q,\lambda)} + c_q' g(q,\lambda)\partial^2\psi^{(n-q-1,\lambda)}\,, \qquad (19,q+1\text{a})$$

$$\left(\frac{c_{q-1}'}{b_{q-1}c_{q-1}}\right)\chi^{(n-q+1,\lambda)} + \left(-ie_q\,\frac{n-\lambda+1}{n-q+1}\,\gamma\cdot\partial+b_q m\right)\chi^{(n-q,\lambda)} = -b_q\left(i\,\frac{n-\lambda+1}{n-q+1}\,\gamma\cdot\partial+a_q m\right)\psi^{(n-q,\lambda)} + b_q c_q g(q,\lambda)\partial^2\chi^{(n-q-1,\lambda)}\,,$$

$$q=3,4,5,\ldots,\lambda, \qquad (19,q+1\text{b})$$

where

$$g(q,\lambda)\equiv\tfrac{1}{2}(\lambda-q)(2n-\lambda-q+2)(n-q)^{-1}(n+1-q)^{-1}.$$

Next the fields $\psi^{(p,\lambda)}$ and $\chi^{(p,\lambda)}$ (with $p\geq n-\lambda+1$) and $\chi^{(n-\lambda)}$ are eliminated from Eqs. (19). The necessary and sufficient conditions for the resulting equation to yield $\psi^{(n-\lambda)}=0$ are obtained by requiring that the coefficient of each of the derivative terms in that equation vanish, and the coefficient of the term with no derivatives be nonzero.

As seen above, this procedure for $\lambda=1$ yields $c=2n^2/(2n+1)$, $a_1=(n+1)/n$. Using these values, one obtains for $\lambda=2$

$$c_1=\frac{2(n+1)^2}{2n+1},$$

$$a_2=d_2=\frac{n+1}{n-1},$$

and

$$b_2=-\frac{2n+1}{2n-1}.$$

Next consider $\lambda=3$. The necessary and sufficient conditions to obtain $\psi^{(n-3)}=0$, after some simplification, are found to be

$$b_3c_2(b_3-a_3d_3)\neq 0, \tag{20}$$

$$c_2'^2\left[c_2+\frac{2(n-2)^2}{2n-3}\right]=0, \tag{21}$$

$$c_2'\left[c_2e_3-\frac{(n-2)(2n-1)}{2(n-1)(2n+1)}\left(d_3-\frac{n+1}{n-2}\right)c_2'\right]=0, \tag{22}$$

$$\left[b_3c_2+\frac{(n+1)(2n-1)}{2(n-1)(2n+1)}c_2'd_3\right]\frac{c_2'}{(n-2)(2n+1)}+\frac{b_3c_2^2}{8(n-1)^2}-\frac{n^2}{4(n-1)^2(2n-1)}c_2(e_3^2+b_3)=0, \tag{23}$$

$$a_3-\frac{n+1}{n-2}=0, \tag{24}$$

$$\left[b_3c_2+\frac{(n+1)(2n-1)}{2(n-1)(2n+1)}c_2'd_3\right]\frac{c_2'}{4(n-1)(n-2)}-\frac{n^2c_2}{4(n-1)^2(2n-1)}\left[(b_3-a_3d_3)b_3+\frac{4n}{(2n-3)}(b_3+e_3^2)\right]-\frac{(n+1)^2b_3c_2^2}{8(n-1)^2(n-2)^2}=0, \tag{25}$$

and

$$2e_3-a_3+d_3=0. \tag{26}$$

As shown in Appendix A, these conditions uniquely determine c_2' to be zero, i.e., $\psi^{(n-3)}$ does not couple to any field of rank higher than itself. Therefore a replacement

$$\begin{pmatrix}\chi^{(n-3)}\\ \psi^{(n-3)}\end{pmatrix}\to R^{(n-3)}\begin{pmatrix}\chi^{(n-3)}\\ \psi^{(n-3)}\end{pmatrix}$$

with $R_{12}^{(n-3)}=0$ does not affect any part of the Lagrangian determined so far. Thus one can, using the above transformation, set $e_3=0$ without any loss of generality; the remaining coefficients are uniquely determined as

$$a_3=d_3=\frac{n+1}{n-2},$$

$$c_2=\frac{2n^2}{2n-1},$$

$$b_3=-\frac{4n}{2n-3}.$$

We can now proceed to a general value of λ. We prove the following: Let

$$c=\frac{2n^2}{2n+1}, \tag{27, 1}$$

$$a_q=d_q=\frac{n+1}{n+1-q}, \tag{27, 2}$$

$$b_q=-\frac{(q-1)(2n-q+3)}{2n-2q+3}, \tag{27, 3}$$

$$c_q'=e_q=0, \tag{27, 4}$$

and

$$c_{q-1}=2(n-q+3)^2(2n-2q+5)^{-1} \tag{27, 5}$$

for $q\leq\lambda-1$. Then the requirement that Eqs. (19) imply $\chi^{(p,\lambda)}=\psi^{(p,\lambda)}=0$ for $p\geq n-\lambda$ determines the coefficients $c_{\lambda-1}$, $c_{\lambda-1}'$, a_λ, b_λ, d_λ, and e_λ uniquely [up to the arbitrariness implied by transformations (15)] to be as given by Eqs. (27, 2)–(27, 5) with $q=\lambda$. From the values of the coefficients determined so far it can be seen that Eqs. (27) are satisfied for $q\leq 3$. It follows by induction, then, that all the coefficients are given by (27).

Proof. One can easily verify that the elimination of $\psi^{(n-q+1,\lambda)}$ and $\chi^{(n-q,\lambda)}$ from Eqs. (19, $q+1$a) and (19, $q+1$b), and

$$\left[\partial^2 + \frac{(q-1)(2n-q+3)}{(\lambda-q+1)(2n-\lambda-q+3}\, m^2\right] \psi^{(n-q+1,\lambda)} = \frac{n+1}{n+1-q}\left[-i\,\frac{n-\lambda+1}{n+1}\,\gamma\cdot\partial + m\right]\chi^{(n-q)}, \qquad (28, q)$$

with the use of (27), results in $(28, q+1)$, i.e., the equation obtained from $(28, q)$ by the replacement $q \to q+1$. Now, eliminating $\psi^{(n,\lambda)}$ from Eqs. (19, 1) and (19, 2), one obtains Eq. (28, 2), i.e., Eq. $(28, q)$ with $q=2$. It follows by induction that the elimination of $\psi^{(p+1,\lambda)}$ and $\chi^{(p,\lambda)}$ $(p \geq n-\lambda+2)$ from Eqs. (19, 1)–(19, $\lambda-2$b) with the use of Eqs. (27) results in (28, $\lambda-1$). Further elimination of $\psi^{(n-\lambda+2,\lambda)}$, $\psi^{(n-\lambda+1,\lambda)}$, $\chi^{(n-\lambda+1,\lambda)}$, and $\chi^{(n-\lambda)}$ with the help of Eqs. (19, $\lambda-1$a)–(19, $\lambda+1$b) yields

$$\sum_{j=0}^{6} A_j\,(-i\gamma\cdot\partial)^{6-j}\, m^j\, \psi^{(n-\lambda)} = 0\,.$$

The necessary and sufficient conditions for this to imply $\psi^{(n-\lambda)}=0$ are

$$A_6 \neq 0;\; A_j = 0 \text{ for } j \leq 5.$$

In terms of the coefficients a_λ, $b_\lambda, c_{\lambda-1}, c'_{\lambda-1}, d_\lambda$, and e_λ these conditions are

$$b_\lambda c_{\lambda-1}(b_\lambda - a_\lambda d_\lambda) \neq 0\,, \qquad (29)$$

$$c'^{2}_{\lambda-1}\left[c_{\lambda-1} + \frac{2(n-\lambda+1)^2}{(2n-2\lambda+3)}\right] = 0\,, \qquad (30)$$

$$c'_{\lambda-1}\left[c_{\lambda-1}e_\lambda - \frac{(n-\lambda+1)(2n-2\lambda+5)c'_{\lambda-1}}{2(\lambda-2)(n-\lambda+2)(2n-\lambda+4)}\left(d_\lambda - \frac{n+1}{n-\lambda+1}\right)\right] = 0\,, \qquad (31)$$

$$\frac{c'_{\lambda-1}}{(\lambda-2)(n-\lambda+1)(2n-\lambda+4)}\left[b_\lambda c_{\lambda-1} + \frac{(n+1)(2n-2\lambda+5)c'_{\lambda-1}d_\lambda}{2(\lambda-2)(n-\lambda+2)(2n-\lambda+4)}\right] - \frac{(n-\lambda+3)^2 c_{\lambda-1}(b_\lambda + e_\lambda{}^2)}{4(n-\lambda+2)^2(2n-2\lambda+5)} + \frac{b_\lambda c_{\lambda-1}{}^2}{8(n-\lambda+2)^2} = 0\,, \qquad (32)$$

$$a_\lambda - \frac{n+1}{n-\lambda+1} = 0\,, \qquad (33)$$

$$\frac{c'_{\lambda-1}}{4(n-\lambda+1)(n-\lambda+2)}\left[b_\lambda c_{\lambda-1} + \frac{(n+1)(2n-2\lambda+5)c'_{\lambda-1}d_\lambda}{2(\lambda-2)(n-\lambda+2)(2n-\lambda+4)}\right] - \frac{(n-\lambda+3)^2 c_{\lambda-1}}{4(n-\lambda+2)^2(2n-2\lambda+5)}\left[(b_\lambda - a_\lambda d_\lambda)b_\lambda + \frac{(\lambda-1)(2n-\lambda+3)}{(2n-2\lambda+3)}(b_\lambda + e_\lambda{}^2)\right] - \frac{(n+1)^2 b_\lambda c_{\lambda-1}{}^2}{8(n-\lambda+1)^2(n-\lambda+2)^2} = 0\,, \qquad (34)$$

and

$$2e_\lambda + d_\lambda - a_\lambda = 0\,. \qquad (35)$$

As shown in Appendix A, these conditions yield $c'_{\lambda-1}=0$. Thus $\psi^{(n-\lambda)}$ does not couple to fields of rank higher than its own. Consequently one can make a transformation

$$\begin{pmatrix}\chi^{(n-\lambda)}\\ \psi^{(n-\lambda)}\end{pmatrix} \to R^{(n-\lambda)}\begin{pmatrix}\chi^{(n-\lambda)}\\ \psi^{(n-\lambda)}\end{pmatrix}$$

with $R_{12}^{(n-\lambda)}=0$ without altering any part of the Lagrangian determined so far, i.e., without changing Eqs. (16) or (27) for $q \leq \lambda-1$. Thus one can set $e_\lambda = 0$ without any loss of generality. The remaining coefficients are then uniquely obtained to be as given by Eqs. (27) with $q=\lambda$.

Thus it has been demonstrated that, in order that Eqs. (17) reduce to (2) along with $\psi^{(p)} = \chi^{(p)} = 0$ for $0 \leq p \leq n-1$, the coefficients appearing in the Lagrangian are as given by (27). The minimal[13] Lagrangian for a spin-$(n+\frac{1}{2})$ field is thus

$$
\begin{aligned}
\mathcal{L} = & \tfrac{1}{2}\psi^{(n)}\beta(i\gamma\cdot\partial - m)\psi^{(n)} + c\psi^{(n-1)}\beta(\partial\psi^{(n)}) + \tfrac{1}{2}c\psi^{(n-1)}\beta(i\gamma\cdot\partial + a_1 m)\psi^{(n-1)} \\
& + cc_1\Big\{\chi^{(n-2)}\beta(\partial\psi^{(n-1)}) - \tfrac{1}{2}\chi^{(n-2)}(-i\gamma\cdot\partial + d_2 m)\chi^{(n-2)} - \tfrac{1}{2}mb_2(\chi^{(n-2)}\beta\psi^{(n-2)} + \psi^{(n-2)}\beta\chi^{(n-2)}) \\
& \quad - \tfrac{1}{2}b_2\psi^{(n-2)}\beta(i\gamma\cdot\partial + a_2 m)\psi^{(n-2)} \\
& \quad + \sum_{q=3}^{n}(-)^q\Big(\prod_{j=2}^{q-1}c_j b_j\Big)\,[\chi^{(n-q)}\beta(\partial\psi^{(n-q+1)}) - \tfrac{1}{2}\chi^{(n-q)}\beta(-i\gamma\cdot\partial + d_q m)\chi^{(n-q)} \\
& \qquad - \tfrac{1}{2}b_q m(\chi^{(n-q)}\beta\psi^{(n-q)} + \psi^{(n-q)}\beta\chi^{(n-q)}) - \tfrac{1}{2}b_q\psi^{(n-q)}\beta(i\gamma\cdot\partial + a_q m)\psi^{(n-q)}]\Big\};
\end{aligned}
$$

the coefficients are given by (27).

As noted earlier, the formalism is invariant under the transformations (15), so different choices of ρ and $R^{(p)}$ yield different Lagrangians which are completely equivalent.[14] Appropriate values of ρ and $R^{(p)}$ are seen to yield the Lagrangians obtained by Chang[15] for the spin values $\frac{3}{2}$, $\frac{5}{2}$, and $\frac{7}{2}$.

In view of the fact that all auxiliary fields vanish in the absence of interactions, many of the free-field properties can be obtained without the knowledge of the full Lagrangian. The free-field Lagrangian becomes

$$\mathcal{L} = \tfrac{1}{2}\psi^{(n)}\beta(i\gamma\cdot\partial - m)\psi^{(n)},$$

the field equations being (1) and (2). Using these and the action principle[16] the free-field anticommutators have been found by Chang[15] to be

$$
\begin{aligned}
&\{\psi_{\mu_1\cdots\mu_n}(x),\ \psi_{\nu_1\cdots\nu_n}(x')\beta\} \\
&\qquad = -i(i\gamma\cdot\partial + m)\Theta_{(\mu),(\nu)}(s)\,|_{\partial^2=m^2}\,\Delta(x-x', m^2),
\end{aligned}
$$

where $\Theta(s)$ is a spin projection operator introduced by Fronsdal[17] and subsequently used by Chang,[15] and $\Delta(x-x', m^2)$ is the invariant function

$$\Delta(x-x', m^2) = i\int\frac{dp}{(2\pi)^3}\, e^{ip\cdot(x-x')}\,\epsilon(p)\delta(p^2+m^2).$$

The positive-definiteness of the anticommutator is readily verified; the reader is referred to Chang's paper (Ref. 15) for details.

III. ELECTROMAGNETIC INTERACTIONS

One readily introduces minimal electromagnetic interactions by doubling the number of field components and making the replacement

$$\partial_\mu \to D_\mu = \partial_\mu - ieqA_\mu$$

in the field equations, q being the two-dimensional matrix

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

In the presence of interactions the auxiliary variables do not necessarily vanish, so in order for the formalism to be consistent the resulting equations should be such that $2(2s+1)$ field components satisfy equations of motion (i.e., equations involving time derivatives) while all remaining components are to be determined in terms of the aforementioned ones through equations of constraints (equations involving no time derivatives). Because of the enormous complexity of the equations in the presence of interactions a consistency proof has not been worked out for the arbitrary-spin case, and the discussion here is consequently limited to a consideration of the Galilean limit. For this purpose it is convenient to use complex fields (i.e., a representation of the charge space in which q is diagonal) and the standard representation of the Dirac matrices. We introduce the following conventions: The factors of c are to be written explicitly, e.g., $m\to mc$, $x^0\to ct$. The rest energy is separated by writing

$$D_0 \to -i\left(\frac{E}{c} + mc + \frac{e}{c}A_0\right),$$

$E \equiv i\partial/\partial t$ being the "nonrelativistic" energy. Unless otherwise specified, trace in the following means the three-dimensional spinor trace (i.e., contraction with γ^k for four-component spinors and with σ^k for two-component ones). We also define

$$\psi^{(n)}_{k_1\cdots k_n} \equiv \begin{pmatrix} \zeta_{k_1\cdots k_n} \\ \frac{1}{c}\eta_{k_1\cdots k_n} \end{pmatrix}$$

and

$$\psi^{(n)}_{k_1\cdots k_n} - \mathrm{Tr} \equiv \psi_{k_1\cdots k_n} \equiv \begin{pmatrix} \phi_{k_1\cdots k_n} \\ \frac{1}{c}\chi_{k_1\cdots k_n} \end{pmatrix},$$

where ζ, η, ϕ, and χ are two-component objects in the spinor space. In view of $\gamma^\mu\psi^{(n)}_{\mu\cdots} = 0$, one has, then,

$$\psi^{(n)}_{0k_1\cdots k_{n-1}}=\begin{pmatrix}\frac{1}{c}\sigma^k\eta_{kk_1\cdots k_{n-1}}\\ -\sigma^k\zeta_{kk_1\cdots k_{n-1}}\end{pmatrix}\equiv\begin{pmatrix}\frac{1}{c}\xi_{k_1\cdots k_{n-1}}\\ \xi'_{k_1\cdots k_{n-1}}\end{pmatrix}.$$

It is shown in Appendix B that for $\lambda\le n-2$

$$\psi^{(p,\lambda)},\ \chi^{(p,\lambda)}=O\left(\frac{1}{c^{n-\lambda}}\psi\right), \tag{36}$$

where $\psi^{(p,\lambda)}$ ($\chi^{(p,\lambda)}$) is the component of $\psi^{(p)}$ ($\chi^{(p)}$), with λ nonzero tensor indices. It is also shown that[18]

$$\psi^{(n-1,n-1)}=O(c^{-2}\psi) \tag{37}$$

and

$$\xi'=O(c^{-1}\phi). \tag{38}$$

It follows from (36) and (38) that, to order c^{-1}, $\phi=\zeta$ and

$$\sigma^k\xi_k\ldots=0. \tag{39}$$

Thus the tensor-spinors ϕ, χ, and ξ are all symmetric and traceless in the Galilean limit.

Using Eqs. (36)–(39), the traceless part of Eq. (17, 1) with all tensor indices nonzero yields, to order c^{-1},

$$i(E+eA_0)\phi_{k_1\cdots k_n}=\left\{(\vec{\sigma}\cdot\vec{\mathrm{D}})\chi_{k_1\cdots k_n}+\left(\frac{2n}{2n+1}\right)D_{k_1}\xi_{k_2\cdots k_n}\right\}_{\text{S.T.}}, \tag{40}$$

and

$$2m\chi_{k_1\cdots k_n}=-i\{(\vec{\sigma}\cdot\vec{\mathrm{D}})\phi_{k_1\cdots k_n}-\mathrm{Tr}\}. \tag{41}$$

Similarly the traceless part of Eq. (17, 1) with one tensor index zero yields

$$m\xi_{k_1\cdots k_{n-1}}=-iD_k\phi_{kk_1\cdots k_{n-1}}. \tag{42}$$

Equations (39)–(41) are the desired set. One can immediately see that the $(6s+1)$-component Galilean-invariant theory of Hagen and Hurley[7] has been reproduced. Substituting for the dependent fields χ and ξ into (40), one obtains

$$\left(E+\frac{\vec{\mathrm{D}}^2}{2m}+eA_0\right)\phi_{k_1\cdots k_n}=\frac{ie}{(2n+1)mc}B_k\left\{\sum_{\mu=1}^{n}\epsilon_{kk_\mu l_\mu}\delta_{k_1l_1}\delta_{k_2l_2}\cdots\delta_{k_{\mu-1}l_{\mu-1}}\delta_{k_{\mu+1}l_{\mu+1}}\cdots\delta_{k_nl_n}\right\}\phi_{l_1\cdots l_n}+\frac{e}{2(2n+1)mc}(\vec{\mathrm{B}}\cdot\vec{\sigma})\phi_{k_1\cdots k_n}$$

or

$$\left(E+\frac{\vec{\mathrm{D}}^2}{2m}+eA_0\right)\phi_{k_1\cdots k_n}=\frac{1}{s}\frac{e}{2mc}(\vec{\mathrm{B}}\cdot\vec{\mathrm{S}})_{k_1\cdots k_n,l_1\cdots l_n}\phi_{l_1\cdots l_n},$$

which yields the value $1/s$ for the g factor.

IV. SUMMARY

The Fierz-Pauli program of constructing the Lagrange functions for higher-spin fields by using auxiliary variables has been brought to a conclusion in this and the preceding paper. Although a prescription for this process was suggested by Fronsdal and was later used by Chang to obtain the Lagrangians for spin values less than or equal to 4, the method failed to yield a closed form for the Lagrangian in the general case. In the present papers the auxiliary fields needed in the general case have in fact been determined. It turns out that in the case of a spin-s boson field represented by a symmetric traceless tensor of rank s, one has to introduce symmetric traceless tensors of ranks $s-2, s-3, \ldots, 0$ in order to obtain a second-order formalism; still more variables are required to bring the theory to a first-order form.[1] In the case of a spin-s fermion field, a symmetric traceless tensor-spinor of rank $s-\frac{3}{2}$ and two each of ranks $s-\frac{5}{2}$, $s-\frac{7}{2}, \ldots, 0$ are needed. Using these the explicit form for the Lagrangian of an arbitrary-spin field has been obtained, thereby enabling one to discuss higher-spin fields in a unified fashion. Although the complicated nature of the minimally coupled equations has prevented, for the time being, a complete proof of their consistency,[19] it has been possible to demonstrate that the theory has a consistent Galilean limit. In particular the g factor has been determined to be $1/s$ (as has been anticipated for some time), and it has now been made possible, using the formalism presented here, to obtain all higher electromagnetic moments of an arbitrary-spin particle.

APPENDIX A

Here we prove the assertions made in Sec. II to the effect that conditions (29)–(35) imply $c'_{\lambda-1}=0$. Since conditions (20)–(26) are same as conditions (29)–(35) with $\lambda=3$, it follows that the former set implies $c'_2=0$.

Equation (30) has two solutions: $c'_{\lambda-1}=0$ or

$$c_{\lambda-1}=-\frac{2(n-\lambda+1)^2}{2n-2\lambda+3}\quad . \tag{A1}$$

We show that the assumption $c'_{\lambda-1}\neq 0$ with conditions (29)–(35) leads to inconsistencies. Equations (31), (33), and (35) yield

$$2e_\lambda\left\{c_{\lambda-1}+\frac{(n-\lambda+1)(2n-2\lambda+5)}{(\lambda-2)(n-\lambda+2)(2n-\lambda+4)}\,c'_{\lambda-1}\right\}=0\,, \tag{A2}$$

implying

$$e_\lambda=0 \tag{A2a}$$

or

$$c'_{\lambda-1}=-\frac{(\lambda-2)(n-\lambda+2)(2n-\lambda+4)}{(n-\lambda+1)(2n-2\lambda+5)}\,c_{\lambda-1}\,. \tag{A2b}$$

Let us consider (A2a) first. Equations (33) and (35) yield

$$a_\lambda=d_\lambda=\frac{n+1}{n-\lambda+1}\,. \tag{A3}$$

Using (A2a), (A1), (A3), (32), and (34), one readily obtains

$$b_\lambda=\left(\frac{n+1}{n-\lambda+1}\right)^2,$$

in clear violation of (29), which with (A3) reads

$$b_\lambda-\left(\frac{n+1}{n-\lambda+1}\right)^2\neq 0\,.$$

Thus (A2a) cannot be true. Let us try (A2b). Elimination of $(b_\lambda+e_\lambda{}^2)$ from Eqs. (32) and (34) and subsequent use of (29) results in

$$b_\lambda=-\frac{\lambda(n-\lambda+2)^2(2n-\lambda+2)(2n-2\lambda+5)}{(n-\lambda+3)^2(2n-2\lambda+3)^2}\,. \tag{A4}$$

Using (A1), (A2b), (A4), (33), and (35), Eq. (32) becomes after some manipulations,

$$\left[\left(\frac{n-\lambda+3}{n-\lambda+2}\right)e_\lambda-\frac{4(n+1)(n-\lambda+2)^2}{(n-\lambda+1)(n-\lambda+3)(2n-2\lambda+3)}\right]^2+\frac{4(\lambda-1)(n-\lambda+2)(2n-2\lambda+5)}{(n-\lambda+3)^2(2n-2\lambda+3)^2}\left[2(n-\lambda+2)(n-\lambda+3)^2+(\lambda-1)(n-\lambda+1)^2\right]=0\,.$$

This clearly is absurd, since the left-hand side is a positive-definite quantity. Thus it has been shown that $c'_{\lambda-1}=0$ is the only value that can satisfy conditions (29)–(35).

APPENDIX B

Here the relations (36)–(38) are obtained. We begin by proving (36), i.e., for $\lambda\leq n-2$, $\psi^{(p,\lambda)}$ and $\chi^{(p,\lambda)}$ are at most of order $c^{-n+\lambda}\psi$.

Consider the field equations (17) with minimal coupling, i.e., with

$$\partial_\mu\to D_\mu=\partial_\mu-i\,\frac{e}{c}\,A_\mu\,.$$

In view of the fact that

$$D_0=-i\left(mc+\frac{E}{c}+\frac{e}{c}A_0\right)$$

and

$$-i\gamma^\mu D_\mu=-\beta\left(mc+\frac{E}{c}+\frac{e}{c}A_0\right)+\vec{\gamma}\cdot\vec{D}\,,$$

the highest-order terms (in powers of c) involving a certain field component are the ones with either D_0 or $\gamma^\mu D_\mu$ acting on the component. Separating these in the minimally coupled equations (17) with λ nonzero tensor indices and using (27), one obtains after some simplification

$$P_-\psi^{(n,\lambda)}=-i\,\frac{n(n-\lambda)(n+\lambda+2)}{(2n+1)}\,\psi^{(n-1,\lambda)}+\text{O.T.}\,, \tag{A5, 1}$$

$$\psi^{(n,\lambda)}=\frac{i}{n}\,P_+\psi^{(n-1,\lambda)}-\frac{(n+1)^2(n-\lambda+1)(n+\lambda+1)}{n(n-1)(2n+1)}\,\chi^{(n-2,\lambda)}+\text{O.T.}\,, \tag{A5, 2}$$

$$\psi^{(p+1,\lambda)}=-\frac{i}{p+1}\,P_-\chi^{(p,\lambda)}+\frac{i(n-p-1)(n+p+3)}{(2p+3)}\,\psi^{(p,\lambda)}+\text{O.T.} \tag{A5, $n-p+1$,a}$$

and

$$\chi^{(p,\lambda)} = -\frac{1}{p+1} P_+ \psi^{(p,\lambda)} - \frac{i(p+2)^2(p-\lambda)(p+\lambda+2)}{p(p+1)(2p+3)} \chi^{(p-1,\lambda)} + \text{O.T.}, \qquad (\text{A5}, n-p+1, \text{b})$$

where

$$p = n-2, n-3, \ldots, \lambda,$$

$$P_\pm \equiv \{(n+1) \pm (\lambda+1)\beta\},$$

and O.T. stands for "other terms" which are of order $\psi^{(r,\lambda-1)}$, $\chi^{(r,\lambda-1)}$, $\psi^{(r,\lambda-2)}$, $\chi^{(r,\lambda-2)}$, and $c^{-1}(\psi^{(r,\lambda+1)}, \chi^{(r,\lambda+1)}, \psi^{(r,\lambda)}, \chi^{(r,\lambda)})$.

We proceed to show that Eqs. (A5) yield

$$\psi^{(p,\lambda)}, \chi^{(p,\lambda)} = O\big(c^{-1}(\psi^{(r,\lambda+1)}, \chi^{(r,\lambda+1)}), \psi^{(r,\lambda-2)}, \chi^{(r,\lambda-2)}, \psi^{(r,\lambda-1)}, \chi^{(r,\lambda-1)}\big) \qquad (\text{A6})$$

The following assertion is easily verified: The elimination of $\psi^{(p+1,\lambda)}$ and $\chi^{(p,\lambda)}$ from Eqs. (A5, $n-p+1$) and

$$\psi^{(p+1,\lambda)} = -\frac{i(p-\lambda+1)(p+\lambda+3)}{(p+1)(n-\lambda)(n+\lambda+2)} P_- \chi^{(p,\lambda)} + \text{O.T.} \qquad (\text{A7}, p)$$

results in (A7, $p-1$), which is the equation obtained from (A7, p) by the substitution $p \to p-1$. Since the elimination of $\psi^{(n,\lambda)}$ from Eqs. (A5, 1) and (A5, 2) yields (A7, $n-2$), i.e., Eq. (A7, p) with $p=n-2$, it follows by induction that subsequent elimination of $\psi^{(p+1,\lambda)}$ and $\chi^{(p,\lambda)}$ ($p=n-2, n-3, \ldots, \lambda$) results in (A7, $\lambda-1$), that is,

$$\psi^{(\lambda,\lambda)} = O\big(c^{-1}(\psi^{(r,\lambda+1)}, \chi^{(r,\lambda+1)}), \psi^{(r,\lambda-2)}, \chi^{(r,\lambda-2)}, \psi^{(r,\lambda-1)}, \chi^{(r,\lambda-1)}\big),$$

whence Eqs. (A5) immediately yield (A6). Now for $\lambda=0$, this reads

$$\psi^{(p,0)}, \chi^{(p,0)} = O\big(c^{-1}(\psi^{(r,1)}, \chi^{(r,1)})\big),$$

that is

$$\psi^{(p,\lambda)}, \chi^{(p,\lambda)} = O\big(c^{-1}(\psi^{(r,\lambda+1)}, \chi^{(r,\lambda+1)})\big) \qquad (\text{A8})$$

is true for $\lambda=0$. Using this, (A6) reduces to (A8) for $\lambda=1$. Also, if (A8) is true for $\lambda=\lambda_0$ and $\lambda=\lambda_0+1$, (A6) becomes (A8) for $\lambda=\lambda_0+2$. Consequently, it follows by induction that (A8) is true for all λ, whence (36) is immediately obtained.

In order to get (37) and (38), one contracts (17, 1) with D^{μ_n}. In view of (36) the resulting equation with all tensor indices nonzero is

$$[-in\gamma\cdot D + (n+1)mc](D\psi^{(n)})_{k_1\cdots k_{n-1}} = nD^2\psi^{(n-1)}_{k_1\cdots k_{n-1}} + O(c\psi^{(n-1)}, \psi^{(n)}).$$

Also, Eq. (17, 2) can be written, using (36), as

$$(D\psi^{(n)})_{k_1\cdots k_{n-1}} = -\left(i\gamma\cdot D + \frac{n}{n+1} mc\right)\psi^{(n-1)}_{k_1\cdots k_{n-1}} + O(\psi^{(n-1)}).$$

The elimination of $(D\psi^{(n)})_{k_1\cdots k_{n-1}}$ from these two equations results in

$$m^2c^2\psi^{(n-1)}_{k_1\cdots k_{n-1}} = O(\psi^{(n)}),$$

whence $\psi^{(n-1)}_{k_1\cdots k_{n-1}} = O(c^{-2}\psi^{(n)})$, which is (37). Using this, Eq. (A5, 1) with $\lambda=n-1$ immediately yields

$$\psi^{(n)}_{0k_1\cdots k_{n-1}} = O(c^{-1}\psi),$$

whence Eq. (38), i.e.,

$$\sigma^k\zeta_{kk_1\cdots k_{n-1}} = O(c^{-1}\zeta)$$

follows.

---

*Research supported in part by the U. S. Atomic Energy Commission.

†Based in part on the Ph.D. dissertation to be submitted by one of us (L.P.S.S.) to the Physics Department of the University of Rochester.

[1]L. P. S. Singh and C. R. Hagen, preceding paper, Phys. Rev. D 9, 898 (1974).

[2]Throughout this paper (except in Sec. III) we use natural units and $g_{\mu\nu} \equiv \text{diag}(-1,1,1,1)$. All Greek tensor indices run from 0 to 3, the Latin ones from 1 to 3. The spinor index is suppressed throughout. The Dirac matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = -2g_{\mu\nu}$. A Majorana representation for these matrices is employed and all fields are taken to be real (again, except in Sec. III). Thus all the coefficients in the Lagrangian are real.

[3]W. Rarita and J. Schwinger, Phys. Rev. 60, 61 (1941).

[4]In the following, trace is taken to refer to the spinor trace of a tensor spinor. We note that, in view of the symmetry of the fields with respect to the tensor indices, vanishing of the spinor trace implies tracelessness with respect to the Lorentz indices as well, i.e.,

$$g^{\mu\nu}\psi^{(n)}_{\mu\nu\ldots} = 0.$$

[5]The necessity of a Lagrangian approach was first pointed out by M. Fierz and W. Pauli [Proc. R. Soc. A173, 211 (1939)], who showed that Eqs. (2) become inconsistent when minimal electromagnetic interactions

are introduced.

[6]This vital requirement has sometimes been ignored in the past. A case in point is the treatment of P. A. Moldauer and K. M. Case [Phys. Rev. 102, 279 (1956)], who arrive at the strange result that the quadrupole moment of a spin-$\frac{5}{2}$ particle is dependent upon a free parameter in the theory. As will be seen later (see Ref. 14 below), this is certainly not the case—no physical property is found to depend on the arbitrary parameters in the theory.

[7]C. R. Hagen and W. J. Hurley, Phys. Rev. Lett. 24, 1381 (1970).

[8]Using the freedom to transform $\psi^{(0)}$ by multiplying it with an arbitrary real constant [see discussion preceding (15) below] the coefficient of $-i\gamma\cdot\partial\psi^{(0)}$ has been set equal to unity.

[9]See, for instance, K. Johnson and E. C. G. Sudarshan, Ann. Phys. (N.Y.) 13, 126 (1961).

[10]For the sake of convenience we omit tensor indices whenever it is possible. The notation introduced in Ref. 1 is used (see Ref. 11 of that paper) except that trace in the present case refers to the spinor trace. As in Ref. 1, the subscript S.T. is used to denote the symmetric traceless part of a tensor spinor.

[11]It is obvious that one can set $a'_q = d'_q = 1$, $c_q = 0$. As for c'_q and c''_q, there are two possibilities; one can have either $c'_q = 0$ or $c''_q = 0$. The latter case is discussed below. Similar treatment for $c'_q = 0$ leads in a much more straightforward manner to the same final result, i.e., c''_q must also vanish.

[12]Thus $\psi^{(n-\lambda,\lambda)} = \psi^{(n-\lambda)}$, $\psi^{(n-\lambda+1,\lambda)} = (\partial\psi^{(n-\lambda+1)})$, and so on.

[13]Although we have not rigorously demonstrated that the formalism is minimal, the discussion in the earlier part of this section strongly suggests this conclusion.

[14]It is obvious that two Lagrangians related by transformations (15) remain equivalent even in the presence of electromagnetic interactions. Thus by performing a suitable transformation of the auxiliary fields, one can bring any Lagrangian in the equivalence class to the form displayed above, with no arbitrary constants left in it. Consequently no physical result depends on the arbitrary parameters in the theory.

[15]S.-J. Chang, Phys. Rev. 161, 1308 (1967).

[16]J. Schwinger, Phys. Rev. 91, 713 (1953).

[17]C. Fronsdal, Nuovo Cimento Suppl. 9, 416 (1958).

[18]Thus all auxiliary fields vanish in the Galilean limit. It is not suprising, therefore, that even the treatments using "incorrect" auxiliary variables have yielded the right value for the g factor.

[19]We refer here to the consistency problems related to the number of degrees of freedom. Inconsistencies of another type—the existence of noncausal modes of propagation and the related problem of non-positive-definiteness of fermion anticommutators—have been shown to exist for spin values $\frac{3}{2}$ and 2 [G. Velo and D. Zwanziger, Phys. Rev. 186, 1337 (1969); G. Velo, Nucl. Phys. B43, 389 (1972); Johnson and Sudarshan, Ref. 9]. These problems are more basic in nature and will require a much deeper understanding of the very foundations of relativistic quantum field theory for their clarification.

PHYSICAL REVIEW D VOLUME 9, NUMBER 4 15 FEBRUARY 1974

Example of Schwinger's modified propagation function*

Alain J. Phares†

Department of Physics, Harvard University, Cambridge, Massachusetts 02138

(Received 29 December 1972; revised manuscript received 9 October 1973)

The aim of this paper is to investigate a specific example of the two-point–function expression, as proposed in source theory by Schwinger, which represents the inverse modified propagation function as a spectral form. Since this expression for the propagation function most naturally (but not exclusively) refers to spin-1 particles, our program is explicitly carried out in the context of spin-1 electrodynamics, with the progagation function referring to the charged particle. The two-point transverse and longitudinal spectral weight functions are calculated in lowest order. We find that the inclusion of those contributions due to source radiation leads to spectral forms which are infrared-convergent and have non-negative spectral weight functions. Furthermore, the spectral integrals are explicitly evaluated and we see the expected rate of falloff, faster than $1/p^2$, of the propagation function at large p^2. The complex p^2 poles of the propagation function are large and shown to be physically acceptable within the framework of source theory. We also demonstrate that these results remain unaffected when magnetic and quadrupole couplings are present.

I. INTRODUCTION

By general source-theoretical arguments, Schwinger[1] has shown that the spin-1 propagation function has an asymptotic decrease at least as rapid as $(1/p^2)^2$ and can approach $(1/p^2)^3$. This is the case if the spectral weight function $A(M^2)$ associated with the $G^{\mu\nu}G_{\mu\nu}$ coupling, where $G^{\mu\nu}$ is the usual antisymmetric spin-1 tensor field, increases less rapidly than M^2 as M^2 approaches infinity.