Note that the transmission-coefficient formula is the same for zeroth- and first-order approximations. However, the parameter E in Eq. (14) satisfies the first-order approximation given by Eq. (7) rather than the zeroth-order approximation given by Eq. (6).

The calculated transmissions coefficients to zeroth and first orders for the Eckart potential are shown in Table I along with the exact results. We see a remarkable improvement in the results by including the first-order terms $\hbar^2$.

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Lagrangian formulation for arbitrary spin. I. The boson case*†

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An explicit form is obtained for the Lagrangian of an arbitrary-spin boson field. This is achieved by introducing auxiliary field variables which are required to vanish in the free-field limit. For $s \leq 4$ the results are found to be in agreement with those obtained by Chang. Canonical commutation rules are derived and the equations of motion are brought to first-order form, thereby facilitating the introduction of minimal electromagnetic coupling. It is found that, upon taking the Galilean limit, the $(6s+1)$-component Galilean-invariant theory of Hagen and Hurley results. The g factor is found to be $1/s$, thereby confirming a long-standing conjecture.

I. INTRODUCTION

The long-standing problem of how to construct a theory of higher-spin fields was first undertaken by Dirac[1] as a generalization of his celebrated spin-$\frac{1}{2}$ equation. In that paper he wrote that "the underlying theory is of considerable mathematical interest." And so it has turned out to be. After more than three decades of intensive investigations the problem is still only partially solved, and has turned out to be among the most intriguing and challenging in theoretical physics. It touches upon some of the most basic ingredients of present-day physical theory—causality and the positive-definiteness of the Hilbert-space metric.

Various approaches have been tried—equations describing many masses and spins, non-Lagrangian theories, and theories with indefinite metric.[2] In this paper[3] we consider the "simplest" formulation, namely a Lagrangian formalism for fields of unique mass and spin. At present Lagrangian field theory is the only formalism which provides a unified framework for the study of all aspects of the operator formalism of a given theory (e.g., equations of motion, canonical commutators, Green's functions, and the energy-momentum tensor).

All relativistic field theories are based on invariance under the full Poincaré group (including reflections).[4] Thus an "elementary" free field is taken to transform according to an irreducible representation of this group.[5] The two group invariants

$$P^2 = P_\mu P^\mu$$

and

$$P^2 S^2 = \tfrac{1}{2} J_{\mu\nu} J^{\mu\nu} P^2 - J^{\mu\nu} J_{\mu\lambda} P_\nu P^\lambda$$

define the two basic quantum numbers, mass and spin, respectively, of the field through

(i) $P^2 = -m^2$

and

(ii) $S^2 = s(s+1)$.

In this paper we restrict ourselves to massive boson fields, i.e., to fields with $m^2 > 0$ and integer s.

It is important to note that even the specification of (i) and (ii) does not determine uniquely the transformation properties of the field under the Lorentz group, inasmuch as one can choose a set of functions transforming according to any of the representations $D(s-\frac{1}{2}n, \frac{1}{2}n) \oplus D(\frac{1}{2}n, s-\frac{1}{2}n)$, $0 \leq n \leq s$, of the (proper orthochronous) Lorentz group. However, we make the usual choice by selecting the representation $D(\frac{1}{2}s, \frac{1}{2}s)$, for which case the field $\phi^{(s)}$ is a symmetric traceless tensor of rank s, which by condition (i) is seen to satisfy the Klein-Gordon equation with mass m, i.e.,

$$(-\partial^2 + m^2)\phi^{(s)}_{\mu_1 \cdots \mu_s} = 0\,. \tag{1a}$$

Under the subgroup O(3) of spatial rotations, the representation $D(\frac{1}{2}s, \frac{1}{2}s)$ is reducible:

$$D(\tfrac{1}{2}s, \tfrac{1}{2}s) = \sum_{j=0}^{s} D(j)\,,$$

that is, all spin values from 0 to s are present. However, (ii) implies that all lower spin values should be eliminated, a result which is well known to be accomplished by imposing the "Lorentz condition"

$$\partial^{\mu_1}\phi^{(s)}_{\mu_1 \cdots \mu_s} = 0\,. \tag{1b}$$

Since we shall be using a Lagrangian formulation here, it is appropriate to recall that the necessity of such an approach was first pointed out by Fierz and Pauli.[6] They noted that the introduction of minimal electromagnetic interactions in Eqs. (1) leads to inconsistencies which can be avoided by requiring that all equations involving derivatives be obtainable from a Lagrangian. They further noted that it is impossible to construct a Lagrangian that will yield (1) by using only $\phi^{(s)}$, and that additional fields (the so-called auxiliary fields) have to be introduced. A procedure for introducing these fields and constructing Lagrangians was later suggested by Fronsdal[7] and by Chang.[8] Although the latter author obtained the Lagrangians for $s=2$, 3, and 4, the method does not yield a closed form for the Lagrangian of a general-spin field.

In this paper we obtain the Lagrange function for the general case and study some of its simple properties. In Sec. II we review in part the work of Fierz and Pauli in an attempt to motivate the introduction of auxiliary fields. The explicit form is found for the Lagrangian of an arbitrary-spin field. The basic requirement on the equations is that the auxiliary fields vanish in the free-field case. Consequently the free-field commutation rules, etc. can be obtained without detailed knowledge of that part of the Lagrangian which depends on the auxiliary fields. These have been derived by Chang[8] and will be summarized in Sec. III. In Sec. IV minimal electromagnetic interactions are introduced and the Galilean limit obtained. The latter is found to coincide with the Galilean-invariant theory of Hagen and Hurley.[9]

II. FREE-FIELD EQUATIONS

Our aim is to write down Lorentz-invariant, linear, second-order differential equations which reduce to (1) for $\phi^{(s)}$, and are obtainable from a Lagrangian. The latter requirement demands that the equation obtained by the variation of the Lagrangian with respect to a certain field should carry the symmetry of that field, i.e., an equation obtained by varying the Lagrangian with respect to a symmetric, traceless tensor should itself consist of symmetric, traceless terms.

Let us start with spin 1, for which case one has a vector field ϕ_μ. It is well known that Eq. (1) can be obtained from

$$-\partial^\mu(\partial_\mu\phi_\nu - \partial_\nu\phi_\mu) + m^2\phi_\nu = 0\,. \tag{2}$$

Contraction with ∂^ν yields $\partial^\nu\phi_\nu = 0$ ($m^2 \neq 0$ is crucial), and (2) trivially reduces to the Klein-Gordon equation.

In the general case, then, one might be tempted to try

$$-\partial^\mu(\partial_\mu\phi^{(s)}_{\mu_1 \cdots \mu_s} - \partial_{\mu_1}\phi^{(s)}_{\mu\mu_2 \cdots \mu_s}) + m^2\phi^{(s)}_{\mu_1 \cdots \mu_s} = 0 \tag{3}$$

for the symmetric, traceless field $\phi^{(s)}_{\mu_1 \cdots \mu_s}$. These, however, are not Lagrange equations, as they do not carry the symmetry of the field; the second term is neither symmetric nor traceless in the μ_j. Thus for the spin-2 field the correct form is

$$(-\partial^2 + m^2)\phi^{(2)}_{\mu\nu} + 2\{\partial_\mu(\partial\phi^{(2)})_\nu\}_{\text{S.T.}} = 0\,, \tag{4}$$

where the subscript S.T. is used to indicate that the symmetric, traceless part of the enclosed tensor is taken—a notation used throughout this work. Contraction of (4) with ∂^ν yields $(\partial\phi^{(2)})_\mu = 0$ only if the coefficient of the second term is 2, and $(\partial\partial\phi^{(2)}) = 0$. However, $(\partial\partial\phi^{(2)}) = 0$ cannot be obtained from Eq. (4). Thus one needs more equations and consequently more fields, i.e., auxiliary fields are required. The resulting equations should imply $(\partial\partial\phi^{(2)}) = 0$ and in addition the auxiliary fields clearly should vanish for the case of no interaction.

The question now arises as to what types of auxiliary fields should be introduced; we bear in mind that one would like to keep their number at a minimum. Thus it is natural to look for tensors of ranks lower than s. As these can couple only to $(\partial\phi^{(s)})$, one needs to consider only symmetric, traceless objects.

Returning momentarily to the spin-2 case one sees that since a scalar, namely $(\partial\partial\phi^{(2)})$, is to be eliminated, the obvious choice is a scalar equation. One consequently introduces a scalar field $\phi^{(0)}$. This can couple to $\phi^{(2)}$ only through a term $(\partial\phi^{(2)})_\mu(\partial^\mu\phi^{(0)})$, thereby yielding the most general Lagrange equations,[10]

$$(-\partial^2+m^2)\phi^{(2)}_{\mu\nu}+2\{\partial_\mu(\partial\phi^{(2)})_\nu\}_{\text{S.T.}}=c\{\partial_\mu\partial_\nu\phi^{(0)}\}_{\text{S.T.}} \tag{5}$$

and

$$(\partial\partial\phi^{(2)})=(\partial^2-am^2)\phi^{(0)}\,. \tag{6}$$

Contracting (5) with $\partial^\mu\partial^\nu$ and eliminating $(\partial\partial\phi^{(2)})$ with the help of (6), one gets

$$[(1-\tfrac{3}{2}c)\partial^4+(2-a)m^2\partial^2-2am^4]\,\phi^{(0)}=0\,. \tag{7}$$

Thus the desired result $\phi^{(0)}=0=(\partial\partial\phi^{(2)})$ is obtained if and only if $c=\frac{2}{3}$, $a=2$. All the coefficients are thus uniquely determined.

Next consider spin 3. The tensor $\phi^{(3)}_{\mu\nu\lambda}$ satisfies

$$(-\partial^2+m^2)\phi^{(3)}_{\mu\nu\lambda}+3\{\partial_\mu(\partial\phi^{(3)})_{\nu\lambda}\}_{\text{S.T.}} = \text{terms involving auxiliary fields}\,.$$

Since one needs $(\partial\partial\phi^{(3)})_\mu=0$ in order to obtain (1b), a vector field $\phi^{(1)}_\mu$ must be introduced. The analogs of Eqs. (5) and (6) are thus

$$(-\partial^2+m^2)\phi^{(3)}_{\mu\nu\lambda}+3\{\partial_\mu(\partial\phi^{(3)})_{\nu\lambda}\}_{\text{S.T.}}=c\{\partial_\mu\partial_\nu\phi^{(1)}_\lambda\}_{\text{S.T.}} \tag{8}$$

and

$$(\partial\partial\phi^{(3)})_\mu=(\partial^2-a_2m^2)\phi^{(1)}_\mu+b_2\partial_\mu(\partial\phi^{(1)})\,. \tag{9}$$

In this case, however, $\phi^{(1)}_\mu$ and $(\partial\partial\phi^{(3)})_\mu$ cannot be eliminated unless

$$(\partial\phi^{(1)})=0=(\partial\partial\partial\phi^{(3)})\,. \tag{10}$$

Once the latter result is obtained, contraction of (8) with $\partial^\nu\partial^\lambda$ and elimination of $(\partial\partial\phi^{(3)})_\mu$ with the help of (9) yields the necessary and sufficient conditions for $\phi^{(1)}_\mu$ and $(\partial\partial\phi^{(3)})_\mu$ to vanish as $c=\frac{12}{5}$, $a_2=\frac{3}{2}$. With these values for c and a_2, it is impossible to obtain (10) from Eqs. (8) and (9). More equations and hence more auxiliary fields are needed. Again scalar variables are to be eliminated so that the most economical choice is a scalar field $\phi^{(0)}$. The restriction to second-order equations dictates that it couple only to $\phi^{(1)}$. Equation (8) is thus unchanged, while (9) is modified to

$$(\partial\partial\phi^{(3)})_\mu=(\partial^2-a_2m^2)\phi^{(1)}_\mu+b_2\partial_\mu(\partial\phi^{(1)}) + c_2m\,\partial_\mu\phi^{(0)}\,; \tag{11}$$

the additional equation is

$$m(\partial\phi^{(1)})=(\partial^2-a_3m^2)\phi^{(0)}\,. \tag{12}$$

Equations (8), (11), and (12) are the most general ones involving these fields. A straightforward calculation now shows that these yield $\phi^{(0)}=0$ [consequently (10) from (12) and (11)] if and only if $b_2=\frac{1}{5}$, $c_2=\frac{3}{5}$, and $a_3=2$. [One contracts (8) with $\partial^\mu\partial^\nu\partial^\lambda$ and (11) with ∂^μ, eliminates $(\partial\partial\partial\phi^{(3)})$ and $(\partial\phi^{(1)})$ from the resulting equations and (12), and equates to zero the coefficients of various powers of ∂^2 in the equation so obtained.]

The following pattern emerges: One must successively obtain $\phi^{(s,\lambda)}=0$ for $\lambda=s, s-1, s-2, \ldots, 2$, where

$$\phi^{(s,\lambda)}_{\mu_{\lambda+1}\cdots\mu_s}\equiv\partial^{\mu_1}\cdots\partial^{\mu_\lambda}\phi^{(s)}_{\mu_1\cdots\mu_s}$$

is a symmetric, traceless tensor of rank $s-\lambda$. At each stage an auxiliary field—a symmetric, traceless tensor of the same rank as $\phi^{(s,\lambda)}$—is needed. Thus one introduces symmetric, traceless tensors of ranks $0, 1, 2, \ldots, s-2$. These will be labeled $\phi^{(0)}, \phi^{(1)}, \ldots, \phi^{(s-2)}$, respectively, and correspond to the representations $D(\frac{1}{2}j, \frac{1}{2}j)$, $j=0, 1, 2, \ldots, s-2$, of the Lorentz group. Thus the second-order theory requires $(s+1)^2+\frac{1}{6}s(s-1)\times(2s-1)$ field components.

The most general quadratic second-order Lagrangian involving these fields is[10,11]

$$\begin{aligned}\mathcal{L}&=\tfrac{1}{2}\phi^{(s)}(\partial^2-m^2)\phi^{(s)}+\tfrac{1}{2}s(\partial\phi^{(s)})^2\\&\quad+c\Big[\phi^{(s-2)}(\partial\partial\phi^{(s)})-\tfrac{1}{2}\phi^{(s-2)}(\partial^2-a_2m^2)\phi^{(s-2)}+\tfrac{1}{2}b_2(\partial\phi^{(s-2)})^2-d_2\phi^{(s-4)}(\partial\partial\phi^{(s-2)})\\&\quad-\sum_{q=3}^{s}\Big(\prod_{k=2}^{q-1}c_k\Big)[\tfrac{1}{2}\phi^{(s-q)}(\partial^2-a_qm^2)\phi^{(s-q)}-\tfrac{1}{2}b_q(\partial\phi^{(s-q)})^2-m\phi^{(s-q)}(\partial\phi^{(s-q+1)})+d_q\phi^{(s-q+2)}(\partial\partial\phi^{(s-q)})]\Big]\,,\end{aligned}$$

yielding the equations

$$(-\partial^2+m^2)\phi^{(s)}+s\{\partial(\partial\phi^{(s)})\}_{\text{S.T.}}=c\{\partial\partial\phi^{(s-2)}\}_{\text{S.T.}}\,, \tag{13, 0}$$

$$(\partial\partial\phi^{(s)})=(\partial^2-a_2m^2)\phi^{(s-2)}+b_2\{\partial(\partial\phi^{(s-2)})\}_{\text{S.T.}}+mc_2\{\partial\phi^{(s-3)}\}_{\text{S.T.}}+d_2\{\partial\partial\phi^{(s-4)}\}_{\text{S.T.}}\,, \tag{13, 2}$$

$$m(\partial\phi^{(s-2)})=(\partial^2-a_3m^2)\phi^{(s-3)}+b_3\{\partial(\partial\phi^{(s-3)})\}_{\text{S.T.}}+mc_3\{\partial\phi^{(s-3)}\}_{\text{S.T.}}+d_3\{\partial\partial\phi^{(s-5)}\}_{\text{S.T.}}\,, \tag{13, 3}$$

$$mc_{q-2}c_{q-1}(\partial\phi^{(s-q+1)})-d_{q-2}(\partial\partial\phi^{(s-q+2)})=c_{q-2}c_{q-1}[(\partial^2-a_qm^2)\phi^{(s-q)}+b_q\{\partial(\partial\phi^{(s-q)})\}_{\text{S.T.}}$$
$$+mc_q\{\partial\phi^{(s-q-1)}\}_{\text{S.T.}}+d_q\{\partial\partial\phi^{(s-q-2)}\}_{\text{S.T.}}]\,,$$
$$q=4,5,\ldots,s-2 \tag{13, q}$$

$$mc_{s-3}c_{s-2}(\partial\phi^{(2)})-d_{s-3}(\partial\partial\phi^{(3)})=c_{s-3}\,c_{s-2}[(\partial^2-a_{s-1}m^2)\phi^{(1)}+b_{s-1}\{\partial(\partial\phi^{(1)})\}+mc_{s-1}\{\partial\phi^{(0)}\}]\,, \tag{13, s-1}$$

and

$$mc_{s-2}\,c_{s-1}(\partial\phi^{(1)})-d_{s-2}(\partial\partial\phi^{(2)})=c_{s-2}\,c_{s-1}(\partial^2-a_s\,m^2)\phi^{(0)}\,. \tag{13, s}$$

The coefficients are to be determined so as to yield $\phi^{(p)}=0$, $p=0,1,2,\ldots,s-2$, and $(\partial\partial\phi^{(s)})=0$. This, as seen above, happens in steps. One first obtains $\phi^{(p,s)}=0$, $p=s,s-1,s-2,s-3,\ldots,0$, where[12]

$$\phi^{(s-q,\lambda)}_{\mu_{\lambda-q+1}\cdots\mu_{s-q}}\equiv\partial^{\mu_1}\cdots\partial^{\mu_{\lambda-q}}\phi^{(s-q)}_{\mu_1\cdots\mu_{s-q}}\,.$$

Given this, one next obtains $\phi^{(p,s-1)}=0\;\forall\,p$, and so on. In short, for $2\le\lambda\le s$, one assumes

$$\phi^{(p)}=0\,,\quad p\le s-\lambda-1$$
$$\phi^{(p,\lambda+1)}=0\,,\quad p\ge s-\lambda \tag{14}$$

and arranges the equations $(13,0)\cdots(13,\lambda)$ so as to obtain $\phi^{(p,\lambda)}=0$, $p\ge s-\lambda$. To this end one contracts Eqs. $(13,q)$ with $\partial^{\mu_1}\cdots\partial^{\mu_{\lambda-q}}$. Using (14), one obtains

$$\left[\partial^2+\frac{s}{(s-1)^2f_1(\lambda)}\,m^2\right]\phi^{(s,\lambda)}=\frac{s}{(s-1)^2}\,f_0(\lambda)c\,\partial^4\phi^{(s-2,\lambda)}\,, \tag{15, 0}$$

$$\phi^{(s,\lambda)}=\{[1+b_2f_2(\lambda)]\,\partial^2-a_2m^2\}\,\phi^{(s-2,\lambda)}+mc_2f_2(\lambda)\partial^2\phi^{(s-3,\lambda)}+d_2f_2(\lambda)f_3(\lambda)\partial^4\phi^{(s-4,\lambda)}\,, \tag{15, 2}$$

$$m\phi^{(s-2,\lambda)}=\{[1+b_3f_3(\lambda)]\,\partial^2-a_3m^2\}\,\phi^{(s-3,\lambda)}+mc_3f_3(\lambda)\partial^2\phi^{(s-4,\lambda)}+d_3f_3(\lambda)f_4(\lambda)\partial^4\phi^{(s-5,\lambda)}\,, \tag{15, 3}$$

and

$$mc_{q-1}\,c_{q-2}\phi^{(s-q+1,\lambda)}-d_{q-2}\phi^{(s-q+2,\lambda)}=c_{q-1}c_{q-2}(\{[1+b_qf_q(\lambda)]\,\partial^2-a_q\,m^2\}\,\phi^{(s-q,\lambda)}+mc_qf_q(\lambda)\partial^2\phi^{(s-q-1,\lambda)}$$
$$+d_qf_q(\lambda)f_{q+1}(\lambda)\partial^4\phi^{(s-q-2,\lambda)})\,,\qquad q=4,5,\ldots,\lambda \tag{15, q}$$

where

$$f_q(\lambda)\equiv\tfrac{1}{2}(s-q)^{-2}(\lambda-q)(2s-\lambda-q+1)\,.$$

One then eliminates $\phi^{(p,\lambda)}$, $p=(s-\lambda+1)$, $(s-\lambda+2),\ldots,s$, from Eqs. (15) and equates to zero the coefficients of $\partial^{2n}\phi^{(s-\lambda)}$ in the resulting equation, thereby obtaining the necessary and sufficient conditions for the equations to imply $\phi^{(s-\lambda)}=0$.

Let us start with $\lambda=2$. Equations (15) are then

$$\left(\partial^2+\frac{s}{s-1}m^2\right)\phi^{(s,2)}=\frac{(2s-1)}{s(s-1)^2}\,c\,\partial^4\phi^{(s-2)}, \tag{15, 0$'$}$$

and

$$\phi^{(s,2)}=(\partial^2-a_2m^2)\phi^{(s-2)}\,. \tag{15, 2$'$}$$

Eliminating $\phi^{(s,2)}$, there follows

$$\left\{\left[1-\frac{(2s-1)c}{s(s-1)^2}\right]\partial^4+\left(\frac{s}{s-1}-a_2\right)\partial^2m^2-\frac{s}{s-1}\,a_2m^4\right\}\phi^{(s-2)}=0\,. \tag{16}$$

The necessary and sufficient condition for this to imply $\phi^{(s-2)}=0$, which in turn yields $\phi^{(s,2)}=0$ from (15, 2)$'$, is that the coefficients of terms with ∂^{2n} vanish. This uniquely fixes c and a_2 to be $s(s-1)^2/(2s-1)$ and $s/(s-1)$, respectively.

Next consider $\lambda=3$. A straightforward though cumbersome calculation, upon substituting above values for c and a_2, leads to

$$b_2=\frac{(s-1)^2}{2s-1}\,,$$

$$c_2=\frac{s^2(s-2)^2}{(s-1)(2s-3)(2s-1)}\,,$$

and

$$a_3=\frac{3(s-1)^2}{(s-2)(2s-3)}$$

as the necessary and sufficient conditions for the equations to yield $\phi^{(s-3)}=0$, which in turn implies $\phi^{(s,3)}=\phi^{(s-2,3)}=0$. Similarly, using the values of the coefficients determined so far, and carrying out the same procedure for $\lambda=4$, one obtains

$$d_2=0\,,$$

$$b_3=\frac{(s-3)^2}{2s-3}\,,$$

$$c_3=\frac{3(s-3)^2(s-1)(2s-1)}{2(2s-3)(2s-5)(s-2)}\,,$$

and

$$a_4=\frac{2(2s-3)(s-2)}{(2s-5)(s-3)}\,.$$

Now we can consider the general case. We prove the following: Let

$$a_q=\frac{q(2s-q+1)(s-q+2)}{2(2s-2q+3)(s-q+1)}\,,$$

$$b_{q-1}=\frac{(s-q+1)^2}{2s-2q+5}\,, \qquad (17)$$

$$c_{q-1}=\frac{(q-1)(s-q+1)^2(s-q+3)(2s-q+3)}{2(s-q+2)(2s-2q+3)(2s-2q+5)}\,,$$

and

$$d_{q-2}=0$$

for $q=2,3,\ldots,\lambda-1$.

Then (17) also holds for $q=\lambda$ in order that the equations imply $\phi^{(p,\lambda)}=0$, $p\geq s-\lambda$. Looking at the coefficients determined so far, one can see that (17) is true for $q=2,3,4$. It follows by induction, then, that (17) holds for all q.

Proof. Setting $d_q=0$, $2\leq q\leq\lambda-3$, Eqs. (15) become

$$\left[\partial^2+\frac{s}{(s-1)f_1(\lambda)}m^2\right]\phi^{(s,\lambda)}=\frac{sf_0(\lambda)}{(s-1)^2}c\,\partial^4\phi^{(s-2,\lambda)}\,, \qquad (15,0)''$$

$$\phi^{(s,\lambda)}=\{[1+b_2f_2(\lambda)]\partial^2-a_2m^2\}\phi^{(s-2,\lambda)}+mc_2f_2(\lambda)\partial^2\phi^{(s-3,\lambda)}\,, \qquad (15,2)''$$

$$m\phi^{(s-q+1,\lambda)}=\{[1+b_qf_q(\lambda)]\partial^2-a_qm^2\}\phi^{(s-q,\lambda)}+mc_qf_q(\lambda)\partial^2\phi^{(s-q-1,\lambda)}\,,\quad 3\leq q\leq\lambda-3 \qquad (15,q)''$$

$$m\phi^{(s-\lambda+3,\lambda)}=\{[1+b_{\lambda-2}f_{\lambda-2}(\lambda)]\partial^2-a_{\lambda-2}m^2\}\phi^{(s-\lambda+2,\lambda)}+mc_{\lambda-2}f_{\lambda-2}(\lambda)\partial^2\phi^{(s-\lambda+1,\lambda)}$$
$$+d_{\lambda-2}f_{\lambda-2}(\lambda)f_{\lambda-1}(\lambda)\partial^4\phi^{(s-\lambda)}\,, \qquad (15,\lambda-2)''$$

$$m\phi^{(s-\lambda+2,\lambda)}=\{[1+b_{\lambda-1}f_{\lambda-1}(\lambda)]\partial^2-a_{\lambda-1}m^2\}\phi^{(s-\lambda+1,\lambda)}+mc_{\lambda-1}f_{\lambda-1}(\lambda)\partial^2\phi^{(s-\lambda)}\,, \qquad (15,\lambda-1)''$$

and

$$mc_{\lambda-2}c_{\lambda-1}\phi^{(s-\lambda+1,\lambda)}-d_{\lambda-2}\phi^{(s-\lambda+2,\lambda)}=c_{\lambda-2}c_{\lambda-1}(\partial^2-a_\lambda m^2)\phi^{(s-\lambda)}\,. \qquad (15,\lambda)$$

Substituting for c, a_2, b_2, and c_2, and eliminating $\phi^{(s,\lambda)}$ from $(15,0)''$ and $(15,2)''$, one gets

$$\left[\partial^2-\frac{f_2(0)}{f_2(\lambda)}m^2\right]m\phi^{(s-2,\lambda)}=\frac{\partial^2}{f_1(3)}[f_1(\lambda)\partial^2-f_1(0)m^2]\phi^{(s-3,\lambda)}\,. \qquad (18,3)$$

A tedious but straightforward calculation shows that, using a_q, b_q, and c_q from (17), the elimination of $\phi^{(s-q+1,\lambda)}$ from $(15,q)''$ and

$$m\left[\partial^2-\frac{f_{q-1}(0)}{f_{q-1}(\lambda)}m^2\right]\phi^{(s-q+1,\lambda)}=\frac{\partial^2}{f_{q-2}(q)}[f_{q-2}(\lambda)\partial^2-f_{q-2}(0)m^2]\phi^{(s-q,\lambda)} \qquad (18,q)$$

results in $(18,q+1)$, which is the equation obtained from $(18,q)$ by the replacement $q\to q+1$. Consequently it follows that the elimination of $\phi^{(p,\lambda)}$, $p=s,s-2,s-3,\ldots,s-\lambda+4$, from Eqs. $(15,0),\ldots,(15,\lambda-3)$ results in $(18,\lambda-2)$.

Further elimination of $\phi^{(s-\lambda+3,\lambda)}$, $\phi^{(s-\lambda+2,\lambda)}$, and $\phi^{(s-\lambda+1,\lambda)}$ from Eqs. $(18,\lambda-2)$, $(15,\lambda-2)''$, $(15,\lambda-1)''$, and $(15,\lambda)$ results in

$$\sum_{j=0}^{4}A_jm^{2j}\partial^{2(4-j)}\phi^{(s-\lambda)}=0\,,$$

where A_j depend on $d_{\lambda-2}$, $c_{\lambda-1}$, $b_{\lambda-1}$, and a_λ. The necessary and sufficient conditions for this to imply $\phi^{(s-\lambda)}=0$ are

$$A_0=A_1=A_2=A_3=0\,,$$

$$A_4 = a_\lambda a_{\lambda-1}\frac{f_{\lambda-2}(0)}{f_{\lambda-2}(\lambda)} \neq 0\,,$$

which, after a lengthy calculation, are found to yield

$$d_{\lambda-2}=0\,,$$

$$B_1 \equiv \frac{f_{\lambda-3}(\lambda)}{f_{\lambda-3}(\lambda-1)} - 1 - b_{\lambda-1} f_{\lambda-1}(\lambda) = 0\,,$$

$$B_2 \equiv a_{\lambda-1} + [1 + b_{\lambda-1} f_{\lambda-1}(\lambda)]\frac{f_{\lambda-2}(0)}{f_{\lambda-2}(\lambda)} - \frac{f_{\lambda-3}(0)}{f_{\lambda-3}(\lambda-1)} - c_{\lambda-1} f_{\lambda-1}(\lambda) - a_\lambda B_1 = 0\,,$$

and

$$[c_{\lambda-1} f_{\lambda-1}(\lambda) - a_{\lambda-1}]\frac{f_{\lambda-2}(0)}{f_{\lambda-2}(\lambda)} - a_\lambda [B_2 + a_\lambda B_1 + c_{\lambda-1} f_{\lambda-1}(\lambda)] = 0\,.$$

These uniquely determine $b_{\lambda-1}$, $c_{\lambda-1}$, and a_λ to be as given by (17).

Thus we have demonstrated that in order that Eqs. (13) reduce to (1) along with $\phi^{(p)}=0$, $0 \leq p \leq s-2$, the coefficients must be as given by (17). The "minimal"[13] second-order Lagrangian is, thus,

$$\mathcal{L} = \tfrac{1}{2}\phi^{(s)}(\partial^2 - m^2)\phi^{(s)} + \tfrac{1}{2}s(\partial\phi^{(s)})^2 + c\Big\{\phi^{(s-2)}(\partial\partial\phi^{(s)}) - \tfrac{1}{2}\phi^{(s-2)}(\partial^2 - a_2 m^2)\phi^{(s-2)} + \tfrac{1}{2}b_2(\partial\phi^{(s-2)})^2 - \sum_{q=3}^{s}\Big(\prod_{k=2}^{q-1} c_k\Big) \times [\tfrac{1}{2}\phi^{(s-q)}(\partial^2 - a_q m^2)\phi^{(s-q)} - \tfrac{1}{2}b_q(\partial\phi^{(s-q)})^2 - m\phi^{(s-q)}(\partial\phi^{(s-q-1)})]\Big\}. \tag{19}$$

As noted earlier,[10] the formalism is invariant under the transformation

$$\phi^{(s-q)} \to \alpha_q \phi^{(s-q)}\,, \quad 2 \leq q \leq s\,, \quad \alpha_q \neq 0$$

so different choices of the α's will yield equivalent though different Lagrangians. Suitable choices of the α's are easily seen to yield the Lagrangians obtained by Chang[8] for the spin values 2, 3, and 4.

The next task which must be undertaken is to obtain a first-order formalism, i.e., a formulation involving only first-order equations for the fields. This is necessary in order that electromagnetic interactions can be introduced in an unambiguous fashion. For this purpose more fields have to be introduced. There are many ways of doing this, and analysis of the spin-2 case shows that the theory of the interacting field is extremely sensitive to the particular first-order formalism which is chosen. Furthermore, in the presence of interactions all auxiliary fields do not necessarily vanish. Consistency demands that only $2(2s+1)$ variables satisfy equations of motion (i.e., equations with time derivatives), and that all other variables should be determined in terms of these through constraints (i.e., equations with no time derivatives).[14] This is not the case with all the formalisms, and in some cases the number of constraints actually decreases with the introduction of interaction.[15] Therefore we follow the obvious generalization of Chang's formalism for spin 2 and introduce fields $H^{(p)}_{\mu_1\cdots\mu_{p-1},\mu\alpha}$ ($p=s, s-2, s-3, \ldots, 2$), $H^{(1)}_{\mu\alpha}$, $H^{(0)}_{\alpha}$, and H, which are tensors of ranks $(s+1)$, $(s-1)$, $(s-2), \ldots, 0$, respectively, through[16]

$$H^{(s)}_{\mu_1\cdots\mu_{s-1},\mu\alpha} = 2\{\partial_\alpha\phi^{(s)}_{\mu\mu_1\cdots\mu_{s-1}} + (s-1)a g_{\alpha\mu_1}(\partial\phi^{(s)})_{\mu\mu_2\cdots\mu_{s-1}} + (s-1)^2 b g_{\mu\mu_1}\partial_\alpha\phi^{(s-2)}{}_{\mu_2\cdots\mu_{s-1}}\}_A\,, \tag{20,0}$$

$$H^{(s-2)}_{\mu_1\cdots\mu_{s-3},\mu\alpha} = 2\{\partial_\alpha\phi^{(s-2)}_{\mu\mu_1\cdots\mu_{s-3}}\}_A\,, \tag{20,2}$$

$$H^{(s-q)}_{\mu_1\cdots\mu_{s-q-1},\mu\alpha} = 2\{\partial_\alpha\phi^{(s-q)}_{\mu\mu_1\cdots\mu_{s-q-1}} + (s-q-1)\lambda_q g_{\alpha\mu_1}(\partial\phi^{(s-q)})_{\mu\mu_2\cdots\mu_{s-q-1}}\}_A\,, \quad 3 \leq q \leq s-2 \tag{20,q}$$

$$H^{(1)}_{\mu\alpha} = \partial_\alpha\phi^{(1)}_\mu - \partial_\mu\phi^{(1)}_\alpha\,, \qquad H = (\partial\phi^{(1)})\,, \tag{20,s-1}$$

and

$$H^{(0)}_\alpha = \partial_\alpha\phi^{(0)}\,, \tag{20,s}$$

where a and b are given in terms of another constant λ by

$$a=\frac{s-1-\lambda}{\lambda(s+1)-(s-1)},$$

$$b=\frac{s(s-1)}{(2s-1)[\lambda(s+1)-(s-1)]},$$

λ and λ_q being arbitrary constants subject to the restrictions

$$\lambda \neq \frac{s-1}{s+1},$$

$$\lambda_q \neq -(s-q+1)^{-1}.$$

These tensors have the following properties:

(i) They are antisymmetric with respect to μ and α, and are symmetric and traceless in the remaining indices,

(ii) $\epsilon^{\mu_1\mu\alpha\beta}H^{(p)}_{\mu_1\cdots\mu_{p-1},\mu\alpha}=0$,

and

(iii) $g^{\mu\mu_1}H^{(s-2)}_{\mu_1\cdots\mu_{s-3},\mu\alpha}=0$.

In the general case, i.e., $s\geq 3$, the representations of the Lorentz group involved are

$$D(\tfrac{1}{2}(s+1),\tfrac{1}{2}(s-1))\oplus D(\tfrac{1}{2}(s-1),\tfrac{1}{2}(s+1))\oplus D(\tfrac{1}{2}(s-1),\tfrac{1}{2}(s-1)),$$

$$D(\tfrac{1}{2}(s-1),\tfrac{1}{2}(s-3))\oplus D(\tfrac{1}{2}(s-3),\tfrac{1}{2}(s-1)),$$

$$D(1+\tfrac{1}{2}j,\tfrac{1}{2}j)\oplus D(\tfrac{1}{2}j,1+\tfrac{1}{2}j)\oplus D(\tfrac{1}{2}j,\tfrac{1}{2}j),\qquad j=0,1,2,\ldots,s-4$$

and $D(\frac{1}{2},\frac{1}{2})$. Thus, in addition to the $\phi^{(p)}$, $5s^2+\frac{1}{2}(2s-1)(s-2)(s-3)+4$ components have to be introduced in order to bring the theory to first-order form. The spin values 0, 1, and 2 are exceptions, however. As is well known, one has to introduce a vector field in the spin-0 case, whereas the spin-1 theory can be brought to first-order form by introducing a second-rank antisymmetric tensor corresponding to the representation $D(1,0)\oplus D(0,1)$. As for the spin-2 case, one can see from Eq. (21, 2) below that $H^{(0)}$ does not appear in the equations, so only the 20-component field $H^{(2)}$ corresponding to the representations $D(\frac{3}{2},\frac{1}{2})\oplus D(\frac{1}{2},\frac{3}{2})\oplus D(\frac{1}{2},\frac{1}{2})$ is needed.

We omit the result of substituting Eqs. (20) into (19) to obtain the first-order Lagrangian. It is sufficient to remark that the resulting equations, after some simple algebraic manipulations, reduce to (20) and

$$\{\partial^\alpha H^{(s)}_{\mu_1\cdots\mu_{s-1},\mu_s\alpha}+\lambda\partial_{\mu_1}\overline{H}^{(s)}_{\mu_2\cdots\mu_s}\}_{\text{S.T.}}-m^2\phi^{(s)}_{\mu_1\cdots\mu_s}=0, \tag{21, 0}$$

$$\frac{1}{1+a(s+1)}\partial^\alpha\overline{H}^{(s)}_{\mu_1\cdots\mu_{s-2}\alpha}-\frac{s-2}{2s-1}\{\partial^\alpha H^{(s-2)}_{\mu_1\cdots\mu_{s-3},\mu_{s-2}\alpha}\}_{\text{S.T.}}-a_2m^2\phi^{(s-2)}_{\mu_1\cdots\mu_{s-2}}+mc_2\{\partial_{\mu_1}\phi^{(s-3)}_{\mu_2\cdots\mu_{s-2}}\}_{\text{S.T.}}=0, \tag{21, 2}$$

$$\{\partial^\alpha H^{(p)}_{\mu_1\cdots\mu_{p-1},\mu_p\alpha}+A_p\partial_{\mu_1}\overline{H}^{(p)}_{\mu_2\cdots\mu_p}\}_{\text{S.T.}}-m(\partial\phi^{(p+1)})_{\mu_1\cdots\mu_p}+mc_{s-p}\{\partial_{\mu_1}\phi^{(p-1)}_{\mu_2\cdots\mu_p}\}_{\text{S.T.}}-a_{s-p}m^2\phi^{(p)}_{\mu_1\cdots\mu_p}=0,$$

$$p=s-3,s-4,\ldots,2 \tag{21, s-p}$$

$$\partial^\alpha\{H^{(1)}_{\mu\alpha}+\tfrac{6}{5}g_{\mu\alpha}H\}-m(\partial\phi^{(2)})_\mu+mc_{s-1}\partial_\mu\phi^{(0)}-a_{s-1}m^2\phi^{(1)}_\mu=0, \tag{21, s-1}$$

and

$$\partial^\alpha H^{(0)}_\alpha-a_sm^2\phi^{(0)}-m(\partial\phi^{(1)})=0, \tag{21, s}$$

where

$$A_p=\frac{\lambda_{s-p}(p+1)-1-b_{s-p}}{1+\lambda_{s-p}(p+1)}$$

and

$$\overline{H}^{(p)}_{\mu_1\cdots\mu_{p-1}}\equiv g^{\mu\nu}H^{(p)}_{\mu_1\cdots\mu_{p-2}\mu,\nu\mu_{p-1}}.$$

It follows from the tensor properties (i)–(iii) that $H^{(p)}$ is completely symmetric and traceless.

III. QUANTIZATION OF THE FREE FIELD

The commutation relations of the spin-s system can now readily be obtained from the action principle.[17] According to this approach the generator for an infinitesimal variation of the fields is obtained from the surface terms resulting from the variation of the action integral, i.e.,

$$\delta S=\int_{\sigma_1}^{\sigma_2}\delta\mathcal{L}dx=G(\sigma_2)-G(\sigma_1)$$

$$i[G,\chi(x)]=\tfrac{1}{2}\delta\chi(x).$$

The Lagrangian for a free spin-s boson field reduces to[18]

$$\begin{aligned}\mathcal{L}=&\tfrac{1}{2}(\phi_{\mu_1\cdots\mu_s}\partial_\alpha H^{\mu_1\cdots\mu_{s-1},\mu_s\alpha}\\&-H^{\mu_1\cdots\mu_{s-1}\mu_s\,\alpha}\partial_\alpha\phi_{\mu_1\cdots\mu_s})\\&+\tfrac{1}{4}H_{\mu_1\cdots\mu_{s-1},\mu\alpha}H^{\mu_1\cdots\mu_{s-1},\mu\alpha}\\&-\tfrac{1}{2}m^2\phi_{\mu_1\cdots\mu_s}\phi^{\mu_1\cdots\mu_s},\end{aligned}$$

the equations of motion being

$$H_{\mu_1\cdots\mu_{s-1},\mu\alpha}=\partial_\alpha\phi_{\mu\mu_1\cdots\mu_{s-1}}-\partial_\mu\phi_{\alpha\mu_1\cdots\mu_{s-1}}, \tag{22}$$

$$\{\partial_\alpha H^{\mu_1\cdots\mu_{s-1},\mu_s\alpha}\}_{\text{S.T.}}-m^2\phi^{\mu_1\cdots\mu_s}=0 \tag{23}$$

and

$$\partial^\mu\phi_{\mu\cdots}=0=\tilde{H}_{\cdots}\,. \tag{24}$$

The generator for the variations at time t is consequently given by

$$G=\tfrac{1}{2}\int d^3x(\phi_{\mu_1\cdots\mu_s}\delta H^{\mu_1\cdots\mu_{s-1},\mu_s0}-H^{\mu_1\cdots\mu_{s-1},\mu_s0}\delta\phi_{\mu_1\cdots\mu_s})$$

$$=\tfrac{1}{2}\int d^3x\sum_{q=1}^{s}{}^sC_q[\phi(q)\delta\tilde{H}(q)-\tilde{H}(q)\delta\phi(q)]\,,$$

where

$$\phi^{(q)}_{k_1\cdots k_q}\equiv\phi_{k_1\cdots k_q00\cdots0}=\phi_{ll\,k_1\cdots k_q}(q+2)$$

and

$$\tilde{H}^{k_1\cdots k_q}(q)\equiv\tilde{H}^{k_1\cdots k_q00\cdots0}=H^{llk_1\cdots k_q}(q+2)\,,$$

with

$$\tilde{H}^{\mu_1\cdots\mu_s}\equiv(1/s)\sum_{j=1}^{s}H^{\mu_1\cdots\mu_{j-1}\mu_{j+1}\cdots\mu_s,\mu_j0}\,.$$

Thus the fields appearing in the generator are $\phi_{k_1\cdots k_s}$, $\phi_{k_1\cdots k_{s-1}}(1)=\phi_{0k_1\cdots k_{s-1}}$, $\tilde{H}^{k_1\cdots k_s}$, and $\tilde{H}^{0k_1\cdots k_{s-1}}(1)$. Not all of these are independent nor are their variations. They are related by certain constraints resulting from the field equations, namely

$$\Lambda_{kl}\phi_{klk_3\cdots k_s}\equiv\delta_{kl}-\frac{\partial_k\partial_l}{\nabla^2-m^2}\quad\phi_{klk_3\cdots k_s}=0\,, \tag{25}$$

$$\left\{m^2-\frac{s-1}{s}\nabla^2\right\}\phi_{0k_1\cdots k_{s-1}}+\frac{s-1}{s}\{\partial_{k_1}\partial_l\phi_{0lk_2\cdots k_{s-1}}\}_S=\partial_k\tilde{H}^{kk_1\cdots k_{s-1}}, \tag{26}$$

$$\tilde{H}^{0k_1\cdots k_{s-1}}=\frac{s-1}{s}\{\partial_k\phi_{kk_1\cdots k_{s-1}}-\partial_{k_1}\phi_{llk_2\cdots k_{s-1}}\}_S\,, \tag{27}$$

and

$$\tilde{H}^{kkk_1\cdots k_{s-2}}=\frac{s-2}{s}\{\partial_{k_1}\phi_{0llk_2\cdots k_{s-2}}-\partial_l\phi_{0lk_1\cdots k_{s-2}}\}_S\,. \tag{28}$$

The subscript S indicates that the symmetric part of the bracketed tensor is to be taken. Using these, the equal-time commutators can be obtained. Thus, after a tedious calculation, one finds

$$[\phi_{k_1\cdots k_s}(x),\phi_{l_1\cdots l_s}(x')]=0\,, \tag{29}$$

$$-i[\tilde{H}^{k_1\cdots k_s}(x),\phi^{l_1\cdots l_s}(x')]=\left\{[\Lambda^{-1}(s)]_{k_1\cdots k_s,l_1\cdots l_s}+\frac{1}{s}\sum_{\mu=1}^{s}\frac{\partial_{k_\mu}\partial_k}{m^2-\nabla^2}[\Lambda^{-1}(s)]_{kk_1\cdots k_{\mu-1}k_{\mu+1}\cdots k_s,l_1\cdots l_s}\right\}\delta^{(3)}(\vec{x},\vec{x}')\,, \tag{30}$$

and

$$-i[\phi_{0k_1\cdots k_{s-1}}(x),\phi_{l_1\cdots l_s}(x')]=\frac{\partial_k}{m^2-\nabla^2}[\Lambda^{-1}(s)]_{kk_1\cdots k_{s-1},l_1\cdots l_s}\delta^{(3)}(\vec{x},\vec{x}')\,, \tag{31}$$

where

$$[\Lambda^{-1}(s)]_{k_1\cdots k_s,l_1\cdots l_s}=\left\{\sum_{p=0}^{[s/2]}\frac{(-)^p s!(2s-2p-1)!!}{2^p p!(s-p)!(2s-1)!!}\Lambda^{-1}{}_{k_1k_2}\Lambda^{-1}{}_{l_1l_2}\cdots\Lambda^{-1}{}_{k_{2p-1}k_{2p}}\Lambda^{-1}{}_{l_{2p-1}l_{2p}}\times\Lambda^{-1}{}_{k_{2p+1}l_{2p+1}}\cdots\Lambda^{-1}{}_{k_sl_s}\right\}_{\text{sym}(k),\ \text{sym}(l)}\,, \tag{32}$$

where $\Lambda^{-1}{}_{kl}=\delta_{kl}-\partial_k\partial_l/m^2$, and $[s/2]$ is the largest integer in $s/2$. The subscripts on the right-hand side of (32) mean that the bracketed expression is to be symmetrized with respect to each set of indices. Equations (30) and (31) together with Eq. (22) lead to the covariant commutator

$$i[\phi_{\mu_1\cdots\mu_s}(x),\phi_{\nu_1\cdots\nu_s}(x')]=\Theta^{(s)}_{\mu_1\cdots\mu_s,\nu_1\cdots\nu_s}|_{\partial^2=m^2}\Delta(x-x',m^2)\,, \tag{33}$$

where

$$\Delta(x-x',m^2)=i\int\frac{dp}{(2\pi)^3}e^{ip\cdot(x-x')}\epsilon(p)\delta(p^2+m^2)\,,$$

and $\Theta_{(\mu),(\nu)}(s)$ is the projection operator introduced by Fronsdal[7] and subsequently used by Chang,[8] and is obtained by replacing $\Lambda^{-1}{}_{kl}$ in (33) by $\Theta_{\mu\nu}=g_{\mu\nu}-\partial_\mu\partial_\nu/\partial^2$.

One can readily verify the positive-definiteness of the energy. The energy density up to divergence terms is

$$\mathcal{H}(x)\equiv\sum_\alpha \dot{\chi}_\alpha\frac{\partial\mathcal{L}}{\partial\dot{\chi}_\alpha}-\mathcal{L}$$

$$=\tfrac{1}{2}m^2\phi_{\mu_1\cdots\mu_s}\phi^{\mu_1\cdots\mu_s}$$

$$-\tfrac{1}{4}H_{\mu_1\cdots\mu_{s-1},\mu\alpha}H^{\mu_1\cdots\mu_{s-1},\mu\alpha}$$

$$-\tfrac{1}{2}\{\phi_{\mu_1\cdots\mu_s}\partial_k H^{\mu_1\cdots\mu_{s-1},\mu_s k}$$

$$-H^{\mu_1\cdots\mu_{s-1},\mu_s k}\partial_k\phi_{\mu_1\cdots\mu_s}\},$$

which can be easily reduced to

$$\tfrac{1}{2}\{\phi_{\mu_1\cdots\mu_s}(m^2-\nabla^2)\phi^{\mu_1\cdots\mu_s}+\dot{\phi}_{\mu_1\cdots\mu_s}\dot{\phi}^{\mu_1\cdots\mu_s}\}$$
$$+\text{divergence terms}.$$

Thus

$$P^0=\tfrac{1}{2}\int d^3x\{\psi^{k_1\cdots k_s}(m^2-\nabla^2)\Lambda^{-1}{}_{k_1l_1}\cdots\Lambda^{-1}{}_{k_sl_s}\psi^{l_1\cdots l_s}$$
$$+\dot{\psi}^{k_1\cdots k_s}\Lambda^{-1}{}_{k_1l_1}\cdots\Lambda^{-1}{}_{k_sl_s}\dot{\psi}^{l_1\cdots l_s}\},$$

where

$$\psi_{k_1\cdots k_s}\equiv\Lambda_{k_1l_1}\Lambda_{k_2l_2}\cdots\Lambda_{k_sl_s}\phi_{l_1\cdots l_s}.$$

Now

$$\int d^3x\{\psi_{(k)}\Lambda^{-1}{}_{k_1l_1}\cdots\Lambda^{-1}{}_{k_sl_s}\psi_{(l)}\}$$
$$=\int d^3x\sum_{p=0}^{s}{}^sC_p m^{-2p}\{\partial_{k_1}\cdots\partial_{k_p}\psi^{k_1\cdots k_s}\}^2>0,$$

from which the positive-definiteness of P^0 immediately follows.

IV. ELECTROMAGNETIC INTERACTIONS

Minimal electromagnetic coupling is most easily introduced into the equations for a spin-s particle by doubling the field components (we are using Hermitian fields), introducing the charge matrix

$$q=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

in the two dimensional charge space, and making the replacement

$$\partial_\mu\to D_\mu=\partial_\mu-ieqA_\mu$$

in Eqs. (20) and (21). As noted earlier, the auxiliary variables no longer vanish, so consistency requires a proof that only $2(2s+1)$ components satisfy equations of motion, with all others being determined in terms of these through equations of constraint. Unfortunately, the equations become extremely complicated, and a general proof has consequently not yet been obtained.[19]

Because of this we restrict consideration here to a discussion of the Galilean limit, where things become considerably more manageable. To this end we introduce the following conventions: The factors of c are written explicitly, e.g., $m\to mc$, $x^0\to ct$. We separate the rest energy by writing

$$D_0=-i\left(\frac{E}{c}+mc+\frac{e}{c}qA_0\right),$$

where $E=i\partial/\partial t$ corresponds to the "nonrelativistic" energy. Also, "trace" in the following refers to the three-dimensional trace unless otherwise specified. We also define $\phi^{(p,\beta)}_{i_1\cdots i_\beta}$ to be the component of $\phi^{(p)}$ with $p-\beta$ indices equal to zero, the remaining ones being $i_1,\ldots,i_\beta$,

$$\psi_{i_1\cdots i_s}=\phi^{(s)}_{i_1\cdots i_s}-\text{Tr},$$

$$\chi_{i_1\cdots i_{s-1}}=c\phi^{(s,s-1)}_{i_1\cdots i_{s-1}}-\text{Tr},$$

and define $G_{i_1\cdots i_{s-1},jk}$ to be the component of $H^{(s)}_{i_1\cdots i_{s-1},jk}$ which satisfies

$$G_{\cdots i_\mu\cdots i_\nu\cdots,jk}=G_{\cdots i_\nu\cdots i_\mu\cdots,jk},\tag{34a}$$

$$G_{\cdots,jk}=-G_{\cdots,kj},\tag{34b}$$

$$G_{ii\cdots,jk}=0=G_{\cdots j,jk},\tag{34c}$$

and

$$\epsilon_{jkl}G_{\cdots j,kl}=0,\tag{34d}$$

where ϵ_{jkl} is the Levi-Civita tensor in three dimensions.

In the Appendix we prove that

$$\phi^{(p,\beta)}=O\left(\frac{1}{c^{s-\beta}}\{\psi,\chi\}\right),\qquad \beta\leq s-2.$$

Thus all auxiliary fields are at least $O(1/c^2)$ compared to ψ and χ. Also, in view of the fact that $\phi^{(s)}$ is traceless in four dimensions, one has

$$\phi^{(s)}_{iii_3\cdots i_s}=\phi^{(s)}_{00i_3\cdots}=O(c^{-2})$$

and so forth. Consequently one can write

$$\phi^{(s)}_{i_1\cdots i_s}=\psi_{i_1\cdots i_s}+O(c^{-2})$$

and

$$\phi^{(s)}_{0i_1\cdots i_{s-1}}=c^{-1}\chi_{i_1\cdots i_{s-1}}+O(c^{-3}).$$

From Eq. (21, 0) one has

$$m^2c^2\phi^{(s)}_{0i_1\cdots i_{s-1}}=\frac{1}{s}\Big[D_l H^{(s)}_{i_1\cdots i_{s-1},0l}+\lambda D_0\overline{H}^{(s)}_{i_1\cdots i_{s-1}}+\sum_{\mu=1}^{s-1}(D^\alpha H^{(s)}_{0i_1\cdots i_{\mu-1}i_{\mu+1}\cdots i_{s-1},i_\mu\alpha}+\lambda D_{i_\mu}\overline{H}^{(s)}_{0i_1\cdots i_{\mu-1}i_{\mu+1}\cdots i_{s-1}})-\frac{1+\lambda}{s}\sum_{\substack{\mu=1\\ \mu<\nu}}^{s-1}\sum_{\nu=1}^{s-1}\delta_{i_\mu i_\nu}D^\alpha\overline{H}^{(s)}_{0\alpha i_1\cdots i_{\mu-1}i_{\mu+1}\cdots i_{\nu-1}i_{\nu+1}\cdots i_{s-1}}\Big]. \quad (35)$$

As the left-hand side is $O(c)$, one looks for terms of the same order on the right-hand side. From Eq. (20, 0) one has

$$H^{(s)}_{i_1\cdots i_{s-1},0l}=imc\psi_{li_1\cdots i_{s-1}}+O(c^{-1}),$$

$$\overline{H}^{(s)}_{i_1\cdots i_{s-1}}=-\{1+a(s-1)\}\times\{D_l\psi_{li_1\cdots i_{s-1}}+im\chi_{i_1\cdots i_{s-1}}\}+O(c^{-2}),$$

$$H^{(s)}_{0i_1\cdots i_{s-2},i_{s-1}k}=O(c^{-1}),$$

and

$$H^{(s)}_{0i_1\cdots i_{s-2},i_{s-1}0}=-(1+a)im\chi_{i_1\cdots i_{s-1}}-aD_l\psi_{li_1\cdots i_{s-1}}+O(c^{-2}).$$

Substituting these in Eq. (35) and separating the traceless part, one obtains

$$m^2c\chi_{i_1\cdots i_{s-1}}=i\frac{mc}{s}\{(1+\lambda)+a[\lambda(s+1)-(s-1)]\}D_l\psi_{li_1\cdots i_{s-1}}+\frac{m^2c}{s}\{(s-1)-\lambda-a[\lambda(s+1)-(s-1)]\}\chi_{i_1\cdots i_{s-1}}+O(c^{-1}),$$

or, to order c^{-2},

$$m\chi_{i_1\cdots i_{s-1}}=iD_l\psi_{li_1\cdots i_{s-1}}\,. \quad (36)$$

Also, one has without approximation

$$G_{i_1\cdots i_{s-1},jk}=\Big\{D_k\psi_{ji_1\cdots i_{s-1}}-\frac{1}{s}\sum_{\mu=1}^{s-1}\delta_{ki_\mu}D_l\psi_{lji_1\cdots i_{\mu-1}i_{\mu+1}\cdots i_{s-1}}\Big\}-\{j\leftrightarrow k\}\,. \quad (37)$$

Equation (21, 0), for the case when none of the indices is zero, reads

$$m^2c^2\phi^{(s)}_{i_1\cdots i_s}=\Big\{D_lH^{(s)}_{i_1\cdots i_{s-1},i_sl}-D_0H^{(s)}_{i_1\cdots i_{s-1},i_s0}+\lambda D_{i_1}\overline{H}^{(s)}_{i_2\cdots i_s}-\frac{(1+\lambda)(s-1)}{2s}\delta_{i_1i_2}D^\alpha\overline{H}^{(s)}_{\alpha i_3\cdots i_s}\Big\}_S$$

which, upon separation of the traceless part, yields

$$m^2c^2\psi_{i_1\cdots i_s}=\{D_lG_{i_1\cdots i_{s-1},i_sl}\}_{\text{S.T.}}+\{D_l(H^{(s)}-G)_{i_1\cdots i_{s-1},i_sl}-D_0H^{(s)}_{i_1\cdots i_{s-1},i_s0}+\lambda D_{i_1}\overline{H}^{(s)}_{i_2\cdots i_s}\}_{\text{S.T.}}\,. \quad (38)$$

From Eqs. (20, 0), (21, 0), (36), and (37) one has, to $O(c^{-2})$,

$$\overline{H}^{(s)}_{i_2\cdots i_s}=0\,,$$

$$\{D_0H^{(s)}_{i_1\cdots i_{s-1},i_s0}\}_{\text{S.T.}}=D_0^2\psi_{i_1\cdots i_s}+im\{D_{i_1}\chi_{i_2\cdots i_s}\}_{\text{S.T.}}\,,$$

and

$$\{D_l(H^{(s)}-G)_{i_1\cdots i_{s-1},i_sl}\}_{\text{S.T.}}=-im(s-1)\{D_{i_1}\chi_{i_2\cdots i_s}\}_{\text{S.T.}}$$

These, upon substituting in Eq. (38), yield

$$\{D_lG_{i_1\cdots i_{s-1},i_sl}-ims^{-1}(2s-1)D_{i_1}\chi_{i_2\cdots i_s}\}_{\text{S.T.}}+2m(E+eqA_0)\psi_{i_1\cdots i_s}=0\,. \quad (39)$$

Equations (36), (37), and (39) constitute the desired set. The $(2s+1)$ components $\psi_{i_1\cdots i_s}$ satisfy equations of motion, and transform like a pure spin-s object under spatial rotations. The remaining fields G and χ are dependent components determined by the constraints (36) and (37), and transform like spin-s and spin-$(s-1)$ objects, respectively. In fact we have obtained the minimal $(6s+1)$-component Galilean theory of Hagen and Hurley.[9]

The elimination of the dependent components from Eq. (39) results in

$$\left(E+\frac{\vec{D}^2}{2m}+eqA_0\right)\psi_{i_1\cdots i_s}=\frac{ie}{2smc}B_k\left\{\sum_{\mu=1}^{s}\epsilon_{ki_\mu i'_\mu}\delta_{i_1i'_1}\delta_{i_2i'_2}\cdots\delta_{i_{\mu-1}i'_{\mu-1}}\delta_{i_{\mu+1}i'_{\mu+1}}\cdots\delta_{i_si'_s}\right\}\psi_{i'_1\cdots i'_s}$$

$$=\frac{e}{2mc}\frac{1}{s}(\vec{B}\cdot\vec{S})_{i_1\cdots i_s,i'_1\cdots i'_s}\psi_{i'_1\cdots i'_s}, \qquad (40)$$

which, not surprisingly, yields the value $1/s$ for the g factor, as first conjectured by Belinfante[20] and subsequently obtained by Hagen and Hurley.[9]

APPENDIX

We show that

$$\phi^{(p,\beta)}=O(c^{-1}\phi^{(r,\beta+1)},c^{-2}\phi^{(r,\beta+2)}), \qquad (A1)$$

and consequently

$$\phi^{(p,\beta)}=O\left(\frac{1}{c^{s-\beta}}\{\psi,\chi\}\right),\quad \beta\leqslant s-2.$$

It is convenient to work with second-order equations. Substituting for $H^{(p)}$ from (20) into (21) (with minimal coupling), one gets

$$(-D^2+m^2c^2)\phi^{(s)}+s\{D(D\phi^{(s)})\}_{\text{S.T.}}+i\frac{e}{c}q\{F\phi^{(s)}\}_{\text{S.T.}}=\frac{s(s-1)^2}{2s-1}\{DD\phi^{(s-2)}\}_{\text{S.T.}}, \qquad (A2,0)$$

$$(DD\phi^{(s)})=(D^2-a_2m^2c^2)\phi^{(s-2)}+b_2\{D(D\phi^{(s-2)})\}_{\text{S.T.}}+\frac{s(s-2)}{2s-1}i\frac{e}{c}q\{F\phi^{(s-2)}\}_{\text{S.T.}}+mcc_2\{D\phi^{(s-3)}\}_{\text{S.T.}}, \qquad (A2,2)$$

$$mc(D\phi^{(s-q+1)})=(D^2-a_qm^2c^2)\phi^{(s-q)}+b_q\{D(D\phi^{(s-q)})\}_{\text{S.T.}}-i\frac{e}{c}q\{F\phi^{(s-q)}\}_{\text{S.T.}}+mcc_q\{D\phi^{(s-q-1)}\}_{\text{S.T.}}$$

$$q=3,4,\ldots,s-1 \qquad (A2,q)$$

and

$$mc(D\phi^{(1)})=(D^2-a_sm^2c^2)\phi^{(0)}, \qquad (A2,s)$$

where the same notation has been used as was employed for Eqs. (13) (with the replacement $\partial\to D$), and

$$(F\phi^{(p)})_{\mu_1\cdots\mu_p}\equiv F_{\mu_1}{}^{\mu}\phi^{(p)}_{\mu\mu_2\cdots\mu_p},$$

$F_{\mu\nu}$ being the electromagnetic field tensor.

In view of the fact that

$$D_0=-i\left(mc+\frac{E}{c}+\frac{e}{c}qA_0\right),$$

the highest-order terms (in powers of c) involving a certain field component are the ones with the maximum number of factors of D_0 acting on that component. Separating these in Eqs. (A2) with β nonzero indices, one obtains, after some simplification,

$$\phi^{(s,\beta)}=-\frac{(s+\beta)(s-\beta-1)}{2(2s-1)}\phi^{(s-2,\beta)}+\text{O.T.}, \qquad (A3,0)$$

$$\phi^{(s,\beta)}=\left[a_2-\frac{(s-\beta)(s+\beta+1)}{2(2s-1)}\right]\phi^{(s-2,\beta)}-ic_2\frac{(s-\beta-2)(s+\beta-1)}{2(s-2)}\phi^{(s-3,\beta)}+\text{O.T.}, \qquad (A3,2)$$

$$\phi^{(p+1,\beta)}=-i\left[\frac{(p+\beta+3)(p-\beta+2)}{2(2p+3)}-a_{s-p}\right]\phi^{(p,\beta)}-c_{s-p}\frac{(p-\beta)(p+\beta+1)}{2p}\phi^{(p-1,\beta)}+\text{O.T.},$$

$$p=s-3,s-4,\ldots,\beta \qquad (A3,s-p)$$

where the indices have been omitted for convenience. O.T. stands for "other terms" which involve objects of order $c^{-2}\phi^{(r,\beta+2)}$, $c^{-2}\phi^{(r,\beta)}$, $c^{-1}\phi^{(r,\beta\pm1)}$, and $\phi^{(r,\beta-2)}$.

The following statement is easily verified: The elimination of $\phi^{(p+1,\beta)}$ from $(A3,s-p)$ and

$$\phi^{(p+1,\beta)}=-i\frac{(p-\beta+1)(p+\beta+2)}{2(2p+3)}\phi^{(p,\beta)}+\text{O.T.} \qquad (A4,s-p)$$

results in $(A4,s-p+1)$, i.e., the equation obtained from $(A4,s-p)$ by the replacement $p\to p-1$. Since the elimination of $\phi^{(s,\beta)}$ from Eqs. (A3, 0) and (A3, 2) is readily seen to yield (A4, 3), it follows from induction that the elimination of $\phi^{(p,\beta)}$ $(p=s,s-2,s-3,\ldots,\beta+1)$ from Eqs. (A3) results in

$$\phi^{(\beta,\beta)}=\text{O.T.}=O(c^{-2}\phi^{(r,\beta+2)},c^{-2}\phi^{(r,\beta)},c^{-1}\phi^{(r,\beta\pm1)},\phi^{(r,\beta-2)}),$$

which with the help of Eqs. (A4) leads to

$$\phi^{(p,\beta)}=O(c^{-1}\phi^{(r,\beta\pm1)},c^{-2}\phi^{(r,\beta+2)},\phi^{(r,\beta-2)}). \qquad (A5,\beta)$$

Equation (A5, 0) [which is the equation obtained from (A5, β) by the substitution $\beta = 0$] implies (A1) for $\beta = 0$, which with the help of (A5, 1) yields (A1) for $\beta = 1$. Further, if (A1) is satisfied for $\beta = \beta_0$ and $\beta = \beta_0 + 1$, it follows from (A5, $\beta_0 + 2$) that (A1) is satisfied for $\beta = \beta_0 + 2$ also. It therefore follows that (A1) is true for all β.

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[1]P. A. M. Dirac, Proc. R. Soc. A155, 447 (1936).

[2]To mention just a few, H. J. Bhabha, Rev. Mod. Phys. 17, 200 (1945); 21, 451 (1949); S. Weinberg, Phys. Rev. 133, B1318 (1964); 134, B882 (1964); D. L. Pursey, Ann. Phys. (N.Y.) 32, 157 (1965); W.-K. Tung, Phys. Rev. Lett. 16, 763 (1966); Phys. Rev. 156, 1385 (1967); W. J. Hurley, Phys. Rev. Lett. 29, 1475 (1972).

[3]We use natural units (except in Sec. IV), and $g_{\mu\nu} \equiv \mathrm{diag}(-1,1,1,1)$. Greek tensor indices run from 0 to 3, the Latin ones from 1 to 3.

[4]The term "relativistic" is used here in the conventional sense; strictly speaking, the invariance group need not be the Poincaré group. For instance one can have Galilei-invariant field theories (which are considerably easier to construct).

[5]E. P. Wigner, Ann. Math. 40, 149 (1939).

[6]M. Fierz and W. Pauli, Proc. R. Soc. A173, 211 (1939).

[7]C. Fronsdal, Nuovo Cimento Suppl. 9, 416 (1958).

[8]S.-J. Chang, Phys. Rev. 161, 1308 (1967).

[9]C. R. Hagen and W. J. Hurley, Phys. Rev. Lett. 24, 1381 (1970).

[10]Actually the most general form for Eq. (6) will be $(\partial\partial\phi^{(2)}) = (A\,\partial^2 - am^2)\phi^{(0)}$, involving one more coefficient: A. However, since the result is to be $\phi^{(0)} = 0$, a transformation $\phi^{(0)} \to \text{constant} \times \phi^{(0)}$ should not alter the result. Thus, since one of the coefficients can be arbitrarily chosen, we take $A = 1$. In the general case, i.e., in Eqs. (13, 0)...(13, s), the same argument has been used to set one coefficient in each equation equal to unity.

[11]Throughout this work we have omitted tensor indices where they are obvious. Thus $\phi^{(p)}$ carries indices $\mu_1\mu_2\cdots\mu_p$. $(\partial\phi^{(p)})$ carries indices $\mu_2\cdots\mu_p$ and stands for $\partial^\mu\phi^{(p)}_{\mu\mu_2\cdots\mu_p}$. On the other hand $\{\partial\phi^{(p)}\}$ stands for $\partial_{\mu_{p+1}}\phi^{(p)}_{\mu_1\cdots\mu_p}$. Similarly, $\{\partial\partial\phi^{(p)}\} \equiv \partial_{\mu_{p+1}}\partial_{\mu_{p+2}}\phi^{(p)}_{\mu_1\cdots\mu_p}$, whereas $(\partial\partial\phi^{(p)}) \equiv \partial^\mu\partial^\nu\phi^{(p)}_{\mu\nu\mu_3\cdots\mu_p}$.

[12]Thus $\phi^{(s-\lambda,\lambda)} = \phi^{(s-\lambda)}$, $\phi^{(s-\lambda+1,\lambda)} = (\partial\phi^{(s-\lambda+1)})$, and so on.

[13]Although we have not "proved" that the formalism is minimal, i.e., no formalism with fewer auxiliary variables can be obtained, the discussion in the earlier part of this section strongly suggests this conclusion.

[14]In fact it was an increase in the number of constraints that led Fierz and Pauli to use the Lagrangian approach.

[15]Fierz and Pauli give an example of this phenomenon (Ref. 6, Appendix 1). More relevant is the case of a charged spin-2 field. Two alternative first-order formalisms turn out to be completely different in the presence of electromagnetic interactions. One of them suffers a loss of constraints [P. Federbush, Nuovo Cimento 19, 572 (1961); G. Velo and D. Zwanziger, Phys. Rev. 188, 2218 (1969)], whereas the other one with fewer dependent components [S.-J. Chang, Phys. Rev. 148, 1259 (1966)] remains entirely consistent as far as the number of constraints is concerned [C. R. Hagen, Phys. Rev. D 6, 984 (1972)].

[16]For a tensor $T_{\mu_1\cdots\mu_{p+1}}$, $\{T\}_A$ denotes the component which has properties (i)–(iii) below.

[17]J. Schwinger, Phys. Rev. 91, 713 (1953).

[18]We have omitted the superscript (s) since $\phi^{(s)}$ and $H^{(s)}$ are the only fields involved.

[19]Consistency in the case of the charged spin-2 field has been demonstrated by Hagen in the paper cited in Ref. 15.

[20]F. J. Belinfante, Phys. Rev. 92, 997 (1953).