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$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2g^{\alpha\beta},$$

and

$$\gamma^0 \gamma^{\alpha\dagger} \gamma^0 = \gamma^\alpha.$$

We put $\hbar = c = 1$.

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Higher-order calculation of transmission above the potential barrier

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By extending the Miller-Good modified WKB approximation to include the higher-order terms, as derived by Lu and Measure, we are able to calculate the transmission coefficients for energies above the potential barrier. We show that the higher-order terms are essential to the approximation, especially for energies near the barrier top.

I. INTRODUCTION

For potentials other than the square potential barrier, the ordinary WKB approximation fails to give the correct transmission coefficient for energies near the top of the barrier. This limitation of the ordinary WKB method was pointed out by Ford *et al.*[1]

Miller and Good[2] approached the problem by formulating a model potential qualitatively similar to the actual potential and whose Schrödinger equation could be solved exactly. Using the exact solutions as the bases of the approximation, they were able to approximate the wave function of the actual potential. In order to demonstrate this method, they chose the Eckart potential and used their approximation of the actual potential wave function to calculate the transmission coefficients. Miller and Good give a detailed analysis, and the reader is referred to their paper. However, owing to a divergence in the higher-order terms, they could only use the approximation to zeroth order in $\hbar^2$, so their results are limited in accuracy. The method developed by Lu and Measure[3] removes the apparent divergence and allows for the inclusion of higher-order terms in the approximation. But since the divergence is removed, the formula obtained is only valid for energies above the top of the potential barrier. This will be discussed in Sec. II.

The Eckart potential is selected to demonstrate the method since we have the exact transmission coefficients[4] for comparison. The calculated transmission coefficients, for energies near the potential barrier, agree only qualitatively with the exact results if we use the zeroth-order approximation but are accurate to at least four significant figures if we use the first-order approximation. This demonstrates the need to include the higher-order terms in the approximation.

II. METHOD OF APPROXIMATION

In general, we wish to solve the Schrödinger equation

$$\left(\frac{d^2}{dx^2} + \frac{P_1^2(x)}{\hbar^2}\right)\psi(x) = 0 \tag{1}$$

for a given potential $V(x)$, where

$$P_1^2(x) = t_1(x) = 2m[W - V(x)]. \tag{2}$$

The classical turning points correspond to the condition $P_1(x_i)=0$, where x_i is the ith turning point.

We now construct a model potential $U(s)$ qualitatively similar to $V(x)$ and whose Schrödinger equation can be solved exactly. Thus we have

$$\left(\frac{d^2}{ds^2}+\frac{P_2^2(s)}{\hbar^2}\right)\phi(s)=0, \tag{3}$$

where

$$P_2^2(s)=t_2(s)=E-U(s). \tag{4}$$

The turning points of the model potential correspond to the condition $P_2(s_i)=0$, where s_i is the ith turning point of the model problem. Both the actual potential and the model potential must have the same number of turning points.

The solution to the Schrödinger equation of the actual potential is given by

$$\psi(x)=[s'(x)]^{-1/2}\phi(s(x)). \tag{5}$$

Substituting Eqs. (3) and (5) into Eq. (1), we obtain to zeroth order in $\hbar^2$ (see Ref. 2)

$$\int_{s_1}^{s_2}P_2(s)ds=\int_{x_1}^{x_2}P_1(x)dx \tag{6}$$

and to first order in $\hbar^2$ (see Ref. 3)

$$\int_{s_1}^{s_2}P_2ds+\frac{\hbar^2}{24}\int_{s_1}^{s_2}\left(\frac{t_2''^2}{t_2^{1/2}t_2'^2}-\frac{t_2'''}{t_2^{1/2}t_2'}\right)ds=\int_{x_1}^{x_2}P_1dx+\frac{\hbar^2}{24}\int_{x_1}^{x_2}\left(\frac{t_1''^2}{t_1^{1/2}t_1'^2}-\frac{t_1'''}{t_1^{1/2}t_1'}\right)dx, \tag{7}$$

where x_1, x_2 and s_1, s_2 are the respective turning points. In obtaining Eq. (7), we have introduced a divergence when $t'=0$, so Eq. (7) is valid only along the path $t'\neq 0$. We can always, in principle, achieve this condition when the integration limits are complex, but then Eq. (7) is limited to energies above the potential barrier.

The transmission coefficient is then given by

$$T=\frac{J_{\text{trans}}}{J_{\text{inc}}}, \tag{8}$$

where

$$J=\frac{\hbar}{2mi}\left[\psi^*\frac{d}{dx}\psi-\left(\frac{d}{dx}\psi^*\right)\psi\right]$$

and

$$\psi_{\text{trans}}\underset{\substack{x\to+\infty\\ s\to+\infty}}{=}[s'(x)]^{1/2}\phi(s(x)),$$

$$\psi_{\text{inc}}\underset{\substack{x\to-\infty\\ s\to-\infty}}{=}[s'(x)]^{-1/2}\phi(s(x)).$$

We have omitted the normalization constants for the wave functions, since Eq. (8) only involves the ratio of the wave functions.

III. APPLICATION TO THE ECKART POTENTIAL

The Eckart potential demonstrated in Ref. 2 is given as

$$V(x)=1.922\,e^x(1+e^x)^{-1}+11.2\,e^x(1+e^x)^{-2}. \tag{9}$$

The turning points are obtained from the condition $P_1(x_i)=0$ so that

$$y_1=e^{x_1}=\frac{A}{C}+i\frac{B}{C} \tag{10a}$$

and

$$y_2=e^{x_2}=\frac{A}{C}-i\frac{B}{C}, \tag{10b}$$

where

$$A=13.122-2W,$$

$$B=[44.8\,W+(13.122)^2]^{1/2},$$

$$C=2(W-1.922),$$

and

$$P_1^2(x)=t_1(x)=2[W-V(x)].$$

The model potential is given by $U(s)=-s^2$ such that $P_2^2(s)=t_2(s)=E+s^2$ and the Schrödinger equation for the model potential is

$$\left(\frac{d^2}{ds^2}+\frac{E+s^2}{\hbar^2}\right)\phi(s)=0. \tag{11}$$

The exact solution $\phi(s)=D_n(z)$, where $n=\frac{1}{2}(\pm iE-1)$ and $z=2^{1/2}s\,e^{i\pi/4}$, is given in Whittaker and Watson.[5] The asymptotic representation for $D_n(z)$ can be found in the Appendix of Ref. 2. The turning points of the model potential corresponds to the condition $P_2(s_i)=0$ and are given by

$$s_1=+iE^{1/2} \tag{12a}$$

and

$$s_2=-iE^{1/2}. \tag{12b}$$

Following the procedure outlined in Sec. II, we obtain to zeroth order in $\hbar^2$

$$\int_{x_1}^{x_2}P_1(\xi)d\xi=\int_{s_1}^{s_2}P_2(\sigma)d\sigma$$

$$=\int_{s_1}^{s_2}(E+\sigma^2)^{1/2}d\sigma$$

$$=-\tfrac{1}{2}iE\pi,$$

and using Eq. (8) we obtain to zeroth order

$$T=(1+e^{-E\pi})^{-1}.$$

Thus E is a parameter that satisfies Eq. (6). However, to first order in $\hbar^2$, E is a parameter that now satisfies

$$\int_{s_1}^{s_2} P_2\,d\sigma+\frac{\hbar^2}{24}\int_{s_1}^{s_2}\left(\frac{t_2''^2}{t_2^{1/2}t_2'^2}-\frac{t_2'''}{t_2^{1/2}t_2'}\right)d\sigma=\int_{x_1}^{x_2} P_1\,d\xi+\frac{\hbar^2}{24}\int_{x_1}^{x_2}\left(\frac{t_1''^2}{t_1^{1/2}t_1'^2}-\frac{t_1'''}{t_1^{1/2}t_1'}\right)d\xi, \qquad (7')$$

where

$$P_2^2=t_2=E+\sigma^2, \quad P_1^2=t_1=2[W-V(\xi)].$$

and x_1, x_2 and s_1, s_2 are the respective turning points. We see that the $\hbar^2$ terms on the left-hand side of Eq. (7) cannot contribute because of parity. Therefore these two terms reduce to zero as

$$\frac{\hbar^2}{24}\int_{-iE^{1/2}}^{iE^{1/2}}\frac{d\sigma}{(E+\sigma^2)^{1/2}\sigma^2}=\frac{\hbar^2}{24}\lim_{\epsilon\to 0}\int_{-iE^{1/2}}^{\epsilon}\frac{d\sigma}{(E+\sigma^2)^{1/2}\sigma^2}+\frac{\hbar^2}{24}\lim_{\epsilon\to 0}\int_{\epsilon}^{iE^{1/2}}\frac{d\sigma}{(E+\sigma^2)^{1/2}\sigma^2}$$

$$=\frac{\hbar^2}{24}\lim_{\epsilon\to 0}\int_{iE^{1/2}}^{\epsilon}\frac{d(-\sigma)}{[E+(-\sigma)^2]^{1/2}(-\sigma)^2}+\frac{\hbar^2}{24}\lim_{\epsilon\to 0}\int_{\epsilon}^{iE^{1/2}}\frac{d\sigma}{(E+\sigma^2)^{1/2}\sigma^2}=0.$$

Thus we can substitute $P_2(s)=t_2^{1/2}(s)=(E+s^2)^{1/2}$ into the left-hand side of Eq. (7) and obtain

$$\int_{s_1}^{s_2} P_2\,d\sigma+\frac{\hbar^2}{24}\int_{s_1}^{s_2}\left(\frac{t_2''^2}{t_2^{1/2}t_2'^2}-\frac{t_2'''}{t_2^{1/2}t_2'}\right)d\sigma=-\tfrac{1}{2}iE\pi.$$

Setting $y=e^x$ and substituting

$$P_1=t_1^{1/2}=2^{1/2}[W-1.922y(1+y)^{-1}-11.2y(1+y)^{-2}]^{1/2}$$

into the right-hand side of Eq. (7), Eq. (7) becomes

$$\tfrac{1}{2}E\pi=2^{1/2}\left(\int_{y_1}^{y_2}\frac{1}{y}(-Z)^{1/2}dy-\int_{y_1}^{y_2}\frac{1}{1+y}(-Z)^{1/2}dy+\tfrac{3}{2}\int_{y_1}^{y_2}\frac{1}{1+y}(-Z)^{-1/2}dy\right.$$

$$\left.+2(13.122)\int_{y_1}^{y_2}\frac{1}{9.278y-13.122}(-Z)^{-1/2}dy+2(293.938)\int_{y_1}^{y_2}\frac{1}{(9.278y-13.122)^2}(-Z)^{-1/2}dy\right), \qquad (13)$$

where

$$Z=a+by+cy^2,$$

$$a=W,$$

$$b=2W-13.122,$$

$$c=W-1.922.$$

In obtaining Eq. (13), we multiplied both sides of Eq. (7) by i and brought i inside the square root of the integrand on the right-hand side. Since the maximum of the potential barrier is at $W=3.8$, $-a<0$ and $-c<0$, so that Eq. (13) can be integrated immediately and we obtain E.

The transmission coefficient, using Eq. (8), becomes

$$T=(1+e^{-E\pi})^{-1}. \qquad (14)$$

TABLE I. Transmission coefficients T for various energies W. (a) Modified WKB approximation to zeroth order in $\hbar^2$; (b) modified WKB approximation to first order in $\hbar^2$; (c) exact results. As mentioned in the text, the transmission coefficients are for $W>W_C$, where $W_C=3.8$ is the maximum of the potential barrier.

	T		
W	(a)	(b)	(c)
4.00	0.699 126	0.732 924	0.732 933
4.25	0.894 610	0.909 285	0.909 302
4.50	0.967 049	0.971 951	0.971 958
4.75	0.989 763	0.991 317	0.991 319
5.00	0.996 735	0.997 233	0.997 234
5.25	0.998 923	0.999 088	0.999 088
5.50	0.999 633	0.999 689	0.999 689
5.75	0.999 871	0.999 891	0.999 891
6.00	0.999 953	0.999 960	0.999 960

Note that the transmission-coefficient formula is the same for zeroth- and first-order approximations. However, the parameter E in Eq. (14) satisfies the first-order approximation given by Eq. (7) rather than the zeroth-order approximation given by Eq. (6).

The calculated transmissions coefficients to zeroth and first orders for the Eckart potential are shown in Table I along with the exact results. We see a remarkable improvement in the results by including the first-order terms $\hbar^2$.

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Lagrangian formulation for arbitrary spin. I. The boson case*†

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An explicit form is obtained for the Lagrangian of an arbitrary-spin boson field. This is achieved by introducing auxiliary field variables which are required to vanish in the free-field limit. For $s \leq 4$ the results are found to be in agreement with those obtained by Chang. Canonical commutation rules are derived and the equations of motion are brought to first-order form, thereby facilitating the introduction of minimal electromagnetic coupling. It is found that, upon taking the Galilean limit, the $(6s+1)$-component Galilean-invariant theory of Hagen and Hurley results. The g factor is found to be $1/s$, thereby confirming a long-standing conjecture.

I. INTRODUCTION

The long-standing problem of how to construct a theory of higher-spin fields was first undertaken by Dirac[1] as a generalization of his celebrated spin-$\frac{1}{2}$ equation. In that paper he wrote that "the underlying theory is of considerable mathematical interest." And so it has turned out to be. After more than three decades of intensive investigations the problem is still only partially solved, and has turned out to be among the most intriguing and challenging in theoretical physics. It touches upon some of the most basic ingredients of present-day physical theory—causality and the positive-definiteness of the Hilbert-space metric.

Various approaches have been tried—equations describing many masses and spins, non-Lagrangian theories, and theories with indefinite metric.[2] In this paper[3] we consider the "simplest" formulation, namely a Lagrangian formalism for fields of unique mass and spin. At present Lagrangian field theory is the only formalism which provides a unified framework for the study of all aspects of the operator formalism of a given theory (e.g., equations of motion, canonical commutators, Green's functions, and the energy-momentum tensor).

All relativistic field theories are based on invariance under the full Poincaré group (including reflections).[4] Thus an "elementary" free field is taken to transform according to an irreducible representation of this group.[5] The two group invariants

$$P^2 = P_\mu P^\mu$$

and

$$P^2 S^2 = \tfrac{1}{2} J_{\mu\nu} J^{\mu\nu} P^2 - J^{\mu\nu} J_{\mu\lambda} P_\nu P^\lambda$$

define the two basic quantum numbers, mass and spin, respectively, of the field through

(i) $P^2 = -m^2$