Some general theorems in Brans-Dicke and Hoyle-Narlikar cosmologies

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(Received 12 April 1973)

Raychaudhuri-type equations are written for cosmological models filled with a perfect fluid and obeying the Brans-Dicke and Hoyle-Narlikar field equations. In addition, the following three theorems are proved: (1) The only possible spatially-homogeneous stationary models of the universe of perfect fluid obeying the Brans-Dicke theory are the radiation-filled and the perfectly empty universe. (2) If the creation is absent in the C-field theory of Hoyle and Narlikar, the existence of the slightest pressure will prevent an expansion or a contraction. (3) A spatially homogeneous rotating universe filled with incoherent dust and having a shear-free expansion is ruled out by C-field cosmology (of Hoyle and Narlikar) without creation.

I. INTRODUCTION

We know that the dominant force controlling the large-scale behavior of our universe is gravitation. Einstein was the first to give a theory of gravitation in a generally covariant form. Various other theories have since been proposed. The theories which have received some amount of attention are those proposed by Brans and Dicke[1] and the steady-state theory proposed first by Bondi and Gold.[2] Hoyle and Narlikar later introduced a metric and field equations in the latter theory.[3,4,5] They allowed for the continuous creation of matter by bringing in a scalar field.

Raychaudhuri[6] discovered an important equation for arbitrary cosmological models with incoherent matter. This equation is very useful for making general predictions about the evolution of cosmological models obeying Einstein's law of gravitation. The equations were generalized to the case of matter exerting pressure by Ehlers.[7]

Raychaudhuri's equation was generalized to apply to the steady-state theory of Hoyle and Narlikar[4] by Raychaudhuri and Banerji.[8] The author[9] pointed out that the geodesic postulate implies a vanishing of rotation, if matter is created to ensure a steady state. Nariai,[10] however, considered Pryce's equations (as reported by Hoyle and Narlikar[4]) in the more general case when a scalar field exists, but matter is not necessarily created. Hoyle and Narlikar[11] later showed that the divergence-free C field (i.e., no creation) can prevent the collapse of a massive star. Faulkes[12] later studied the world lines of dust in the general C-field cosmology including rotation.

We propose to write Raychaudhuri-type equations for the gravitation theories of Brans and Dicke and of Hoyle and Narlikar in the general case of a universe filled with a perfect fluid. We shall also prove some general theorems for the two cosmologies. In Sec. II we shall discuss the Brans-Dicke theory, and in Sec. III the Hoyle-Narlikar theory.

II. THE BRANS-DICKE COSMOLOGY

The field equations in the Brans-Dicke theory[1] are obtained from the variational principle

$$\delta \int \left[\Phi R + \left(\frac{16\pi}{c^4}\right) L - \Omega(\Phi_\mu \Phi^\mu/\Phi)\right](-g)^{1/2} d^4x = 0, \tag{1}$$

where $\Phi_\mu \equiv \Phi_{,\mu} \equiv \partial\Phi/\partial x^\mu$, Φ being the strength of a scalar field. R is the scalar curvature and L is the Lagrangian density of matter.

The field equations are

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -\frac{8\pi}{c^4 \Phi} T_{\mu\nu} + \frac{\Omega}{\Phi^2}(\Phi_\mu \Phi_\nu - \tfrac{1}{2} g_{\mu\nu} \Phi_\alpha \Phi^\alpha) + \frac{1}{\Phi}(\Phi_{\mu;\nu} - g_{\mu\nu} \Phi^\alpha{}_{;\alpha}), \tag{2}$$

and Φ satisfies the equation

$$\Phi^\alpha{}_{;\alpha} = (-g)^{-1/2}[(-g)^{1/2}\Phi^\alpha]_{,\alpha} = -\frac{8\pi}{(3+2\Omega)c^4} T, \tag{3}$$

where semicolons represent covariant derivatives. $T \equiv T_\mu{}^\mu$, the contracted form of the energy-momentum tensor $T_{\mu\nu}$, and Ω is a positive number which is related to the ratio of the tensor coupling to the scalar coupling. In the limit $\Omega \to \infty$, $\Phi^{-1} \to G$, the gravitation constant and the theory reduces to the conventional Einstein theory.

We now take the energy-momentum tensor of a perfect fluid, viz.,

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu}, \tag{4}$$

with the restriction $\rho > 0$ and $p > 0$. Since $T_\mu{}^\nu{}_{;\nu} = 0$ here as in Einstein's theory, we must have (see, e.g., Ref. 13)

$$\dot{u}^\mu = \frac{p_{,\nu} h^{\mu\nu}}{p + \rho} \tag{5}$$

and

$$\theta = -\frac{\rho}{\rho + p}, \tag{6}$$

where $\dot{u}^\mu = u^\mu{}_{;\nu} u^\nu$ and in the following, the dot will indicate the covariant derivative along the world line. $h^{\mu\nu}$ and θ are the projection tensor and the scalar of expansion, respectively, defined as follows:

$$h^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu, \qquad \theta = u^\mu{}_{;\mu}. \tag{7}$$

We may introduce l by the equation

$$\tfrac{1}{3}\theta = \frac{\dot{l}}{l}; \tag{8}$$

then

$$\frac{\dot{\rho}}{\rho + p} = -3\frac{\dot{l}}{l}. \tag{6a}$$

Substituting from (4) into (3) we have

$$\Phi^\alpha{}_{;\alpha} = \frac{8\pi}{(3 + 2\Omega)c^4}(3p - \rho). \tag{9}$$

We may write Eq. (9) in the form

$$[\Phi^\alpha(-g)^{1/2}]_{,\alpha} = \frac{8\pi}{(3 + 2\Omega)c^4}(3p - \rho)(-g)^{1/2}. \tag{9a}$$

We now assume that our universe is spatially homogeneous. We then choose the time lines along the world lines of matter (co-moving coordinates) and define the homogeneous varieties at the t- constant spaces. In view of this choice of the coordinate system and spatial homogeneity, g_{44} is at most a function of t alone and can be reduced to unity by a suitable transformation of t. Therefore the line element is given by

$$ds^2 = dt^2 + 2g_{4i}\,dt\,dx^i + g_{ik}\,dx^i\,dx^k. \tag{10}$$

In this paper Greek indices stand for the numbers 1 to 4, and Latin indices for 1 to 3.

On account of spatial homogeneity we shall have $\Phi_i = 0$, $\Phi_4 = \dot{\Phi}$ as in this coordinate system, $d/dt = d/ds$, and $p_{,i} = \rho_{,i} = 0$. Then (9a) reduces to the form

$$[g^{\alpha 4}\dot{\Phi}(-g)^{1/2}]_{,\alpha} = \frac{8\pi}{(3 + 2\Omega)c^4}(3p - \rho)(-g)^{1/2}. \tag{9b}$$

The left-hand side will obviously be zero for a stationary universe. But the right-hand side can be zero only if $p = \frac{1}{3}\rho$, which holds for diffuse radiation or $p = \rho = 0$. Hence we have the theorem:

Theorem I: The only possible spatially homogeneous stationary models of the universe of perfect fluid obeying Brans-Dicke theory are the radiation-filled and the perfectly empty universe.

This result does not hold for Einstein's theory, where Gödel's solution[14] is an example of a spatially homogeneous stationary cosmological model filled with perfect fluid having the equation of state $p = \rho$ (if the cosmological constant $\Lambda = 0$).

From Eqs. (7) and (8) we have in this coordinate system

$$\frac{d}{dt}\ln(-g)^{1/2} = \frac{d}{dt}\ln l^3. \tag{8a}$$

Putting the constant of integration zero we may write

$$(-g)^{1/2} = l^3. \tag{8b}$$

For incoherent dust, $p = 0$ and we have from (6a)

$$\rho l^3 = f(x^i). \tag{6b}$$

But if we have a spatially homogeneous universe with both ρ and p constant on the homogeneous varieties, then

$$l = S(x^4)W(x^i). \tag{11}$$

This result was obtained by the author[13] for a perfect fluid and for an incoherent dust by Schücking.[15] The former paper will be referred to as I in the following.

If, however, the rotation vanishes, i.e., the homogeneous varieties are orthogonal to the t lines, then we can always reduce the g_{i4} to zero. Then Eqs. (9b) and (11) give

$$\dot{\Phi}S^3 = \frac{8\pi}{(3 + 2\Omega)c^4}\int(3p - \rho)S^3 dt. \tag{12}$$

In the case of the spatially homogeneous dust-filled nonrotating universe we have

$$\dot{\Phi} = -\frac{8\pi}{(3 + 2\Omega)c^4}\rho(t - t_0), \tag{13}$$

on account of (6b) and (11). t_0 is the constant of integration. This has been derived by Brans and Dicke[1] for Friedmann models.

So far we have used only the divergence relations following from Bianchi identities. Now we shall write Raychaudhuri-type equations using the field equations. From Eqs. (2) and (4) we have

$$R_{\mu\nu} = -\frac{8\pi}{c^4\Phi}\left[\rho\left(u_\mu u_\nu - \frac{1+\Omega}{3 + 2\Omega}g_{\mu\nu}\right) + p\left(u_\mu u_\nu + \frac{\Omega}{3 + 2\Omega}g_{\mu\nu}\right)\right] + \frac{\Omega}{\Phi^2}\Phi_\mu\Phi_\nu + \frac{\Phi_{\mu;\nu}}{\Phi}. \tag{14}$$

From the notation of Ehlers, we have

$$R_{\mu\nu}u^\mu u^\nu = \dot{\theta} + \tfrac{1}{3}\theta^2 - \dot{u}^\mu{}_{;\mu} + 2(\sigma^2 - \omega^2), \tag{15}$$

and

$$h^{\alpha\mu}R_{\mu\nu}u^\nu = h^\alpha{}_\mu(\omega^{\mu\nu}{}_{;\nu} - \sigma^{\mu\nu}{}_{;\nu} + \tfrac{2}{3}\theta^{,\mu}) - (\omega^\alpha{}_\mu + \sigma^\alpha{}_\mu)\dot{u}^\mu, \tag{16}$$

where θ and $h_{\alpha\beta}$ are given by Eq. (7). $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ are shear and rotation tensors, respectively, given by

$$\sigma_{\mu\nu} = u_{(\mu;\nu)} - \dot{u}_{(\mu}u_{\nu)} - \tfrac{1}{3}h_{\mu\nu}\theta,$$
$$\omega_{\mu\nu} = u_{[\mu;\nu]} - \dot{u}_{[\mu}u_{\nu]}, \tag{17}$$
$$\sigma^2 \equiv \tfrac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu},$$

and

$$\omega^2 \equiv \tfrac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu}.$$

Instead of the rotation tensor, we introduce the rotation vector defined by

$$\omega^\alpha \equiv \tfrac{1}{2}\eta^{\alpha\beta\mu\nu}u_\beta\omega_{\mu\nu} = \tfrac{1}{2}\eta^{\alpha\beta\mu\nu}u_\beta u_{\mu,\nu}, \tag{18}$$

then $\omega^2 = -\omega_\alpha\omega^\alpha$.

As shown in I (see Ref. 13) we can write

$$R^\alpha{}_\mu u^\mu = \theta_{,\beta}(\tfrac{2}{3}g^{\alpha\beta} + \tfrac{1}{3}u^\alpha u^\beta) - \sigma^{\alpha\beta}{}_{;\beta} + \tfrac{1}{3}\theta^2 u^\alpha - \dot{u}^\mu{}_{;\mu}u^\alpha - \sigma^\alpha{}_\beta\dot{u}^\beta - 2\omega^2 u^\alpha + \eta^{\mu\nu\beta\alpha}(\omega_{\mu,\beta}u_\nu - 2\omega_\mu u_\nu\dot{u}_\beta). \tag{19}$$

From (14), (15), and (8) we can write

$$3\frac{\ddot{l}}{l} - \dot{u}^\mu{}_{;\mu} + 2(\sigma^2 - \omega^2) = -\frac{8\pi}{c^4\Phi}\left[\frac{2+\Omega}{3+2\Omega}\rho + \frac{3(1+\Omega)}{3+2\Omega}p\right] + \frac{\Omega}{\Phi^2}\dot{\Phi}^2 + \frac{\ddot{\Phi}}{\Phi} - \frac{\Phi_\mu\dot{u}^\mu}{\Phi}. \tag{20}$$

This is analogous to Raychaudhuri's equation for a Friedmann universe in general relativity written in the notation used by Ehlers.[7,16] $\dot{u}^\mu$ in the above equation is given by Eq. (5) and is zero for $p=0$.

For incoherent matter (20) reduces to

$$3\frac{\ddot{l}}{l} = 2(\omega^2 - \sigma^2) - \frac{8\pi}{c^4\Phi}\cdot\frac{2+\Omega}{3+2\Omega}\rho + \frac{\Omega}{\Phi^2}\dot{\Phi}^2 + \frac{\ddot{\Phi}}{\Phi}. \tag{20a}$$

Unlike Einstein's theory, the sign of $\ddot{l}/l$ depends on the sign and magnitude of the derivatives of Φ. So we cannot conclude as in Einstein's theory that $\ddot{l}/l$ is always negative when $\omega = 0$. So a nonrotating cosmological model may, in general, have a minimum volume.

However, if we assume spatial homogeneity, then for a nonrotating dust-filled cosmological model we obtain from (13) in the coordinate system introduced earlier

$$\ddot{\Phi} + 3\dot{\Phi}\frac{\dot{S}}{S} = -\frac{8\pi}{(3+2\Omega)c^4}\rho.$$

Hence from (20a) we obtain in this case

$$3\frac{\ddot{S}}{S} = -2\sigma^2 - \frac{8\pi(3+\Omega)}{c^4\Phi(3+2\Omega)}\rho + \frac{\Omega}{\Phi^2}\dot{\Phi}^2 - \frac{3\dot{\Phi}}{\Phi}\frac{\dot{S}}{S}. \tag{20b}$$

The term $(\Omega/\Phi^2)\dot{\Phi}^2$ is positive and hence the sign of $\ddot{S}/S$ will be determined by the magnitude of this term.

Using Eqs. (14), (19), and (20) we obtain, in general,

$$\tfrac{2}{3}h^{\alpha\beta}\theta_{,\beta} - \sigma^{\alpha\beta}{}_{;\beta} - \sigma^{\alpha\beta}\dot{u}_\beta - 2\sigma^2 u^\alpha + \eta^{\mu\nu\beta\alpha}(\omega_{\mu,\beta}u_\nu - 2\omega_\mu u_\nu\dot{u}_\beta) = h^{\alpha\beta}\left(\frac{\Omega}{\Phi^2}\dot{\Phi}\Phi_\beta + \frac{\dot{\Phi}_\beta}{\Phi}\right). \tag{21}$$

A similar equation was obtained in I (see Ref. 13) for Einstein's theory. We can also use Ehlers' equation (16) and write instead the following equivalent equation:

$$h^\alpha{}_\mu(\omega^{\mu\nu}{}_{;\nu} - \sigma^{\mu\nu}{}_{;\nu} + \tfrac{2}{3}\theta^{,\mu}) - (\omega^\alpha{}_\mu + \sigma^\alpha{}_\mu)\dot{u}^\mu = h^{\alpha\beta}\left(\frac{\Omega}{\Phi^2}\dot{\Phi}\Phi_\beta + \frac{\dot{\Phi}_\beta}{\Phi}\right). \tag{22}$$

III. THE HOYLE-NARLIKAR COSMOLOGY

The field equations may be derived from the variational principle given by Pryce:

$$\delta A = 0,$$

where

$$A = \frac{1}{16\pi}\int R(-g)^{1/2}d^4x - \sum m\int ds + \tfrac{1}{2}f\int C_\alpha C^\alpha(-g)^{1/2}d^4x - \sum m\int C_\alpha\frac{dx^\alpha}{ds}ds, \tag{23}$$

where f is a coupling constant and $C_\alpha \equiv C_{,\alpha}$, C being the strength of a scalar field. The field equations are

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -8\pi[T_{\mu\nu} - f(C_\mu C_\nu - \tfrac{1}{2}g_{\mu\nu}C_\alpha C^\alpha)]. \tag{24}$$

C satisfies the equation

$$C^{\alpha}{}_{;\alpha}=\frac{1}{f}j^{\alpha}{}_{;\alpha}\,, \tag{25}$$

where $j^{\alpha}\equiv\rho u^{\alpha}$ is the mass current. Hoyle and Narlikar have always considered incoherent matter, but have indicated that $T_{\mu\nu}$ can be taken for a perfect fluid in the form of Eq. (4).[4,5] They later applied the above equation to the local phenomenon of the gravitational collapse of a star[11] and supposed that $C^{\alpha}{}_{;\alpha}=(\rho u^{\alpha})_{;\alpha}=0$ in this case because the cosmological phenomenon of creation of matter has practically no influence on local behavior. They constructed a solution to demonstrate that the C field can arrest the gravitational collapse. However they neglected the pressure in their treatment, which is unrealistic.

Taking the divergence of (24) we obtain

$$T^{\alpha\beta}{}_{;\beta}=fC^{\alpha}C^{\beta}{}_{;\beta}=C^{\alpha}(\rho u^{\beta})_{;\beta}\,. \tag{26}$$

Let us introduce λ by the equation:

$$\rho\lambda\equiv(\rho u^{\beta})_{;\beta}=\dot{\rho}+\rho\theta\,. \tag{27}$$

From (4) and (26) we get

$$\dot{u}^{\alpha}=\frac{\rho\lambda}{\rho+p}(C^{\alpha}-u^{\alpha})+\frac{h^{\alpha\beta}p_{,\beta}}{p+\rho}-\frac{p\theta u^{\alpha}}{\rho+p}\,. \tag{28}$$

Using the relation $u_{\alpha}\dot{u}^{\alpha}=0$ we have

$$\rho\lambda(\dot{C}-1)=p\theta\,. \tag{29}$$

If $\rho\lambda\neq 0$ we may write

$$\dot{C}=\frac{p\theta}{\rho\lambda}+1\,. \tag{29a}$$

Substituting from (29) in (28) we obtain

$$\dot{u}^{\alpha}=\frac{h^{\alpha\beta}}{p+\rho}(\rho\lambda C_{\beta}+p_{,\beta})\,. \tag{30}$$

In the local problem considered by Hoyle and Narlikar the creation may be taken to be zero, i.e., $\rho\lambda=0$ and we get from (29) the interesting result that $p\theta=0$. Hence we have the theorem:

Theorem II: If the creation is absent in the C-field theory, the existence of the slightest pressure will prevent an expansion or a contraction.

This is an amazing result, as it is generally believed that gravitational collapse has occurred in certain stars like white dwarfs and pulsars. In Einstein's theory, the conservation of baryon number does not necessarily imply that $(\rho u^{\beta})_{;\beta}=0$, which condition is analogous to the conservation of mass in Newtonian theory. However, if the baryons do not interact with each other then the above condition holds [cf. Eq. (6)]. But when there is interaction, a part of the total mass-energy is present in the form of the energy of pressure and other forms of binding energy. We cannot in this case write $\rho=m_0N$ where m_0 is the proper mass, and N the number density of baryons[17] as was done by Hoyle and Narlikar.[18]

Faulkes has divided the case of pressureless dust into four classes. We shall discuss these cases for a perfect fluid with nonvanishing pressure.

A (i) $\lambda=0$ and $C_{\alpha}=u_{\alpha}$. In this case $\theta=0$. However the world lines are not geodesics unless the pressure gradient is tangential to the world lines. So, in general, $\omega\neq 0$.

A (ii) $\lambda\neq 0$ but $C_{\alpha}=u_{\alpha}$. In this case $\theta=0$ [from (29)]. The world lines are not geodesics as before, and in general $\omega\neq 0$.

B (i) $C_{\alpha}\neq u_{\alpha}$, $\lambda=0$. Again $\theta=0$, the world lines are not geodesics and in general $\omega\neq 0$.

B (ii) $C_{\alpha}\neq u_{\alpha}$, $\lambda\neq 0$. In general $\theta\neq 0$, the world lines are not geodesics, and $\omega\neq 0$. However if $\theta=0$ then $u_{\alpha}(C^{\alpha}-u^{\alpha})=0$. Hence $A^{\alpha}\equiv C^{\alpha}-u^{\alpha}$ is a spacelike vector.

It should, however, be noted that rotation is not necessarily absent if u_{α} is equal to the gradient of a scalar and hence $u_{[\alpha;\beta]}=0$, because the expression for ω in Eq. (17) contains $\dot{u}_{\alpha}$. In the case B (ii) considered by Faulkes ($p=0$), the absence of rotation does not mean $u_{[\alpha;\beta]}=0$, and hence the absence of rotation in this case is not ruled out.

We may now write Raychaudhuri-type equations as before. Using Eqs. (24), (15), and (4) we have

$$3\frac{\ddot{l}}{l}-\dot{u}^{\mu}{}_{;\mu}+2(\sigma^2-\omega^2)=-4\pi(\rho+3p)+8\pi f\dot{C}^2\,. \tag{31}$$

$\dot{C}$ in the above equation is given by (29), and $\dot{u}^{\mu}$ by (30). We know that in Hoyle-Narlikar theory nonrotating incoherent matter can remain in equilibrium.

Similarly using Eqs. (24), (19), and (31) we obtain

$$\tfrac{2}{3}h^{\alpha\beta}\theta_{,\beta}-\sigma^{\alpha\beta}{}_{;\beta}-\sigma^{\alpha\beta}\dot{u}_{\beta}-2\sigma^2u^{\alpha}-\eta^{\mu\nu\beta\alpha}\{\omega_{\mu,\beta}u_{\nu}-2\omega_{\mu}u_{\nu}\dot{u}_{\beta}\}=8\pi f\dot{C}h^{\alpha\beta}C_{\beta}\,. \tag{32}$$

We can also write instead, using Eqs. (24), (16), and (4)

$$h^{\alpha}{}_{\beta}(\omega^{\beta\gamma}{}_{;\gamma}-\sigma^{\beta\gamma}{}_{;\gamma}+\tfrac{2}{3}\theta^{,\beta})-(\omega^{\alpha}{}_{\beta}+\sigma^{\alpha}{}_{\beta})\dot{u}^{\beta}=8\pi f\dot{C}h^{\alpha\beta}C_{\beta}\,. \tag{33}$$

We now introduce spatial homogeneity and a coordinate system as before. We shall consider the case called B (i) by Faulkes with $p=0$ and $\lambda=0$. In this case we deduce Eqs. (6b) and (11) as before Further we have $C^{\beta}{}_{;\beta}=0$. Hence we can write the following equation analogous to (9b):

$$[g^{\alpha 4}\dot{C}(-g)^{1/2}]_{,\alpha}=0. \tag{34}$$

For a nonrotating universe we thus obtain

$$\dot{C}S^3=\text{const}. \tag{35}$$

We shall now try to extend Schücking's theorem in Einstein's cosmology: that a spatially homogeneous rotating and expanding universe filled with incoherent dust must necessarily have shear. This was extended in I (see Ref. 13) to the case of a perfect fluid with the equation of state $p=\alpha\rho$, where α is a constant other than $\frac{1}{9}$ and lying within the range 0 to +1. In the Hoyle-Narlikar cosmology we cannot have an expanding universe if $\lambda=0$ and $p\neq 0$. Hence we take the case $p=0$, $\sigma=0$, and $\lambda=0$. By using the conditions $\sigma_{\mu\nu}=0$ and Eq. (11) we can write all the components $g_{\mu\nu}$ as products of time-dependent and space-dependent functions as in I (Ref. 13):

$$g_{ik}=S^2(x^4)\psi_{ik}(x^j)+\psi_{i4}(x^j)\psi_{k4}(x^j),$$
$$g_{i4}=\psi_{i4},\quad g^{ik}=S^{-2}\psi^{ik}, \tag{36a}$$

where

$$\psi_{ik}\psi^{kl}=\delta_i{}^l,\quad g^{i4}=-S^{-2}\psi^{ik}\psi_{k4}, \tag{36b}$$

$$g^{44}=1+S^{-2}\psi^{ik}\psi_{i4}\psi_{k4}. \tag{36c}$$

From Eq. (31) we obtain

$$3\frac{\ddot{S}}{S}-\frac{2A}{S^4}+\frac{4\pi B}{S^3}=8\pi f\dot{C}^2, \tag{37}$$

where $\omega^2=A/S^4$, A being a positive constant. ω^2 is a scalar in a spatially-homogeneous cosmology and therefore, can depend on t alone. Further $\rho S^3=B$, and B is a second positive constant. This reduces to Eq. (35) of I if $\dot{C}=0$ and $p=0$. In this coordinate system the dot denotes a derivative with respect to t.

From the fourth equation of (32) ($\alpha=4$) we can similarly write

$$\frac{\ddot{S}}{S}-\frac{\dot{S}^2}{S^2}=4\pi f\dot{C}^2+\frac{E}{S^2}, \tag{38}$$

where

$$E=\epsilon^{ijk4}\epsilon^{m4np}\psi_{4j}\{W^{-3}\psi_{im}\psi_{4n,p}\}_{,k}/4W^3(\psi^{ik}\psi_{i4}\psi_{k4}).$$

This is analogous to Eq. (36) of I.

Now from Eq. (34) we obtain

$$\ddot{C}\left(1+\frac{n}{S^2}\right)+\dot{C}\frac{\dot{S}}{S}\left(3+\frac{n}{S^2}\right)-m\frac{\dot{C}}{S^2}+0, \tag{39}$$

where $n=\psi^{ik}\psi_{i4}\psi_{k4}$ and $m=(\psi^{ik}\psi_{k4}W^3)_{,i}/W^3$. From (37) and (38) we obtain

$$\frac{\ddot{S}}{S}+2\frac{\dot{S}^2}{S^2}=-\frac{2E}{S^2}-\frac{4\pi B}{S^3}+\frac{2A}{S^4}. \tag{40}$$

Integration yields

$$\frac{\dot{S}^2}{S^2}=-\frac{E}{S^2}-\frac{8\pi B}{3S^3}+\frac{F}{S^6}+\frac{2A}{S^4}, \tag{41}$$

where F is a constant. Substituting this result into (38) we obtain

$$4\pi f\dot{C}^2=\frac{4\pi B}{S^3}-\frac{3F}{S^6}-\frac{4A}{S^4}. \tag{42}$$

On substituting from (42) in (39) we obtain

$$\dot{S}\left(\frac{6\pi B}{S^4}-\frac{4A}{S^5}-\frac{2\pi Bn}{S^6}+\frac{4An}{S^7}+\frac{6Fn}{S^9}\right)=\frac{4\pi Bm}{S^5}-\frac{4Am}{S^6}-\frac{3Fm}{S^8}. \tag{43}$$

By squaring this equation and substituting from (41) we find, on equating the coefficients of like powers of S on both sides, that $B=0$, i.e., $\rho=0$. Hence we have the theorem:

Theorem III: A spatially-homogeneous rotating universe filled with incoherent dust and having a shear-free expansion is ruled out by C-field cosmology (of Hoyle-Narlikar) without creation.

So the presence of the C field does not modify Schücking's theorem. If, however, $\omega=0$ Eq. (32) reduces to an identity for $\sigma=0$. So, for shear-free expansion, we must have either $\omega=0$ or $\theta=0$, i.e., $\omega\theta=0$.

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