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Perturbation analysis for gravitational and electromagnetic radiation in a Reissner-Nordström geometry

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We consider the gravitational and electromagnetic fields produced by a charged (or uncharged) test particle moving in a Reissner-Nordström geometry as perturbations on the background Reissner-Nordström geometry and its associated electric field, respectively. The gravitational perturbations are expanded in tensor harmonics in the manner of Regge and Wheeler, while the electromagnetic field is expanded in vector harmonics. Following a previously proposed convention, we find that in the Einstein-Maxwell system of equations, electric gravitational multipoles couple only to electric (TM) electromagnetic multipoles and similarly for magnetic multipoles. It is possible to reduce the entire Einstein-Maxwell system for each type of multipole to two second-order Schrödinger-type equations.

I. INTRODUCTION

The problem of gravitational radiation emitted by moving bodies has had significant attention in recent years (due in no small part to Weber's pioneering work in gravitational radiation detectors). In 1957 Regge and Wheeler[1] outlined a harmonic analysis for perturbations on a Schwarzschild background geometry. This was developed by Thorne and colleagues,[2] Vishveshwara,[3] and others. Also, a suitable scheme to determine the gravitational radiation emitted by a body moving in a Schwarzschild field was outlined and a simple self-adjoint (Schrödinger-type) differential equation was found to describe "electric" multipole gravitational radiation.[4] Regge and Wheeler had previously found the self-adjoint equation for "magnetic multipoles." These equations have since been studied analytically[5] and integrated numerically[6] to yield results of astrophysical interest.

Concurrently, the problem of gravitational radiation in a flat-space background has received significant attention and the combined problem of electromagnetic and gravitational radiation has been analyzed.[7] However, much of the peculiarly general-relativistic effects are slighted in flat-space treatments. Thus we wish to look at the following specific, consistent, and fundamental problem: Consider a charged test particle moving according to the Lorentz force law in a Reissner-Nordström background geometry and find the gravitational and electromagnetic fields produced by this test particle as perturbations on the background electromagnetic field and geometry.

We decompose the gravitational and electromagnetic field perturbations and their matter and current sources into tensor and vector harmonics. Just as there are electric and magnetic multipoles for the electromagnetic field, there are corresponding "electric" and "magnetic" gravitational multipoles, and, with the proper choice of names, only electric gravitational multipoles couple to electric electromagnetic multipoles in the Einstein-Maxwell equations and likewise for magnetic multipoles. Then, for each type of multipole, denoted by the superscript e or m, we derive a "superpotential" $R_{LM}^{(e,m)}$ for the gravitational field and a "superpotential" $f_{LM}^{(e,m)}$ for the electromagnetic field which satisfy equations of the form

$$\frac{d^2R_{LM}^{(e,m)}}{dr^{*2}}+(\omega^2-V_L^{(\mathrm{grav})})R_{LM}^{(e,m)} = a_L^{(e,m)}f_{LM}^{(e,m)}+\text{grav. source}\ , \quad (1)$$

$$\frac{d^2 f_{LM}^{(e,m)}}{dr^{*2}} + (\omega^2 - V_L^{(\mathrm{em})}) f_{LM}^{(e,m)}$$
$$= b_L^{(e,m)} R_{LM}^{(e,m)} + c_L^{(e,m)} \frac{dR_{LM}^{(e,m)}}{dr^*} \quad \text{source.} \qquad (2)$$

These equations have been numeric[illegible]y integrated for certain specific sources: an uncharged test particle falling radially into a Reissner-Nordström black hole—this produces significant electromagnetic radiation; a charged test particle falling radially in, and a charged test particle in a circular orbit. The results are presented elsewhere.[8]

II. PERTURBATIONS ON A REISSNER-NORDSTRÖM GEOMETRY

The combined Einstein-Maxwell equations describe the gravitational and electromagnetic fields[9]:

$$\tilde{G}_{\mu\nu} = 8\pi(T_{\mu\nu} + \tilde{E}_{\mu\nu}), \qquad (3)$$

$$((-\tilde{g})^{1/2}\tilde{F}^{\mu\nu})_{,\nu} = 4\pi(-\tilde{g})^{1/2}J^\mu, \qquad (4)$$

where the tilde denotes quantities associated with the total electromagnetic and gravitational fields. The corresponding quantities without the tilde refer to the background Reissner-Nordström geometry

$$ds^2_{\mathrm{RN}} = -e^\nu dt^2 + e^{-\nu} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (5)$$

and electromagnetic field

$$F_{\mathrm{RN}} = -\frac{e}{r^2} dt \wedge dr, \qquad (6)$$

where

$$e^\nu = 1 - \frac{2m}{r} + \frac{e^2}{r^2}$$

and

$$\tilde{E}_{\mu\nu} = \frac{1}{4\pi}(\tilde{g}^{\rho\sigma}\tilde{F}_{\rho\mu}\tilde{F}_{\sigma\nu} - \tfrac{1}{4}\tilde{g}_{\mu\nu}\tilde{F}_{\rho\sigma}\tilde{F}^{\rho\sigma}).$$

$T_{\mu\nu}$ is the matter energy-momentum tensor and J^μ is the electromagnetic current. For a point charge of mass m_0 and charge q,

$$T^{\mu\nu} = m_0 \int_{-\infty}^{\infty} \delta^{(4)}(x - z(s)) \frac{dz^\mu}{ds}\frac{dz^\nu}{ds}\, ds, \qquad (7)$$

where $z^\mu(s)$ is the world line of the particle as a function of arc length, and

$$J^\mu = q\int_{-\infty}^{\infty} \delta^{(4)}(x - z(s))\frac{dz^\mu}{ds}\, ds, \qquad (8)$$

where

$$\iiiint \delta^{(4)}(x)(-g)^{1/2} d^4x = 1.$$

We wish to consider first-order perturbations, $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ and $\tilde{F}_{\mu\nu} = F_{\mu\nu} + f_{\mu\nu}$. Keeping terms to first order we obtain

$$\delta G_{\mu\nu} = 8\pi(T_{\mu\nu} + \delta E_{\mu\nu}), \qquad (9)$$

$$\delta((-\tilde{g})^{1/2}\tilde{F}^{\mu\nu})_{,\nu} = 4\pi(-g)^{1/2}J^\mu. \qquad (10)$$

There are two parameters of smallness of the perturbation: the mass m_0 of the test particle and its charge q. However, we can express the charge as $q = \epsilon m_0$, where ϵ is the charge to mass ratio of the test particle. Given a charge to mass ratio which need not be small, the perturbation, nevertheless, will be small if m_0 is sufficiently small (so that $q = m_0\epsilon$ is also small). To first order, $\tilde{g}^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu}$, where $h^{\mu\nu} = g^{\mu\alpha}g^{\mu\beta}h_{\alpha\beta}$ and $(-\tilde{g})^{1/2} = (-g)^{1/2}(1 + \frac{1}{2}g^{\mu\nu}h_{\mu\nu})$. Then, the Einstein-Maxwell equations become

$$h_{\mu\nu;\alpha}{}^{;\alpha} - (k_{\mu;\nu} + k_{\nu;\mu}) + 2R^\rho{}_\mu{}^\alpha{}_\nu h_{\rho\alpha} + h^\alpha{}_{\alpha;\mu;\nu}$$
$$- (R^\rho{}_\nu h_{\mu\rho} + R^\rho{}_\mu h_{\nu\rho}) + Rh_{\mu\nu}$$
$$+ g_{\mu\nu}(k_\lambda{}^{;\lambda} - h^\alpha{}_{\alpha;\lambda}{}^{;\lambda} - R^{\alpha\beta}h_{\alpha\beta})$$
$$= -16\pi(T_{\mu\nu} + \delta E_{\mu\nu}), \qquad (11)$$

$$((-g)^{1/2}f^{\mu\nu})_{,\nu} = 4\pi(-g)^{1/2}(J^\mu + j^\mu), \qquad (12)$$

where $k_\mu = h_{\mu\alpha}{}^{;\alpha}$. The quantities on the right-hand side of Eqs. (11) and (12) are

$$\delta E_{\mu\nu} = \delta E^{(h)}_{\mu\nu} + \delta E^{(f)}_{\mu\nu},$$

where

$$\delta E^{(h)}_{\mu\nu} = \frac{-1}{4\pi}[g^{\rho\alpha}g^{\sigma\beta}(F_{\rho\mu}F_{\sigma\nu} - \tfrac{1}{2}g_{\mu\nu}g^{k\lambda}F_{\rho k}F_{\sigma\lambda})h_{\alpha\beta}$$
$$+ \tfrac{1}{4}F_{\rho\sigma}F^{\rho\sigma}h_{\mu\nu}],$$

$$\delta E^{(f)}_{\mu\nu} = \frac{1}{4\pi}[g^{\rho\sigma}(F_{\sigma\nu}f_{\rho\mu} + F_{\sigma\mu}f_{\rho\nu})$$
$$- \tfrac{1}{2}g_{\mu\nu}g^{\alpha\rho}g^{\beta\sigma}F_{\alpha\beta}f_{\rho\sigma}],$$

and

$$4\pi(-g)^{1/2}j^\mu = [(-g)^{1/2}(g^{\nu\beta}g^{\mu\rho}g^{\alpha\sigma} + g^{\mu\alpha}g^{\nu\rho}g^{\beta\sigma})F_{\alpha\beta}h_{\rho\sigma}$$
$$- \tfrac{1}{2}(-g)^{1/2}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}h]_{,\nu}.$$

We expand $h_{\mu\nu}$ and $f_{\mu\nu}$ in tensor harmonics. The geometrical perturbations $h_{\mu\nu}$ are given in Table I. The freedom to make infinitesimal coordinate transformations allows us to set $h_2 \equiv 0$ in the magnetic multipoles and $h_0^{(e)} \equiv h_1^{(e)} \equiv G \equiv 0$ in the electric multipoles (Regge-Wheeler gauge). The electromagnetic field harmonics are given in Table II. Let $f_{\mu\nu}$ be derived from the potential a_μ, i.e.,

$$f_{\mu\nu} = a_{\nu,\mu} - a_{\mu,\nu}.$$

Then, for magnetic multipoles,

TABLE I. Tensor harmonic expansion of geometric perturbations $h_{\mu\nu}$. The asterisks denote elements obtained by symmetry. Y_{LM} are normalized spherical harmonics.

magnetic multipoles:

$$\|h_{\mu\nu}\| = \begin{bmatrix} 0 & 0 & -h_0 \dfrac{1}{\sin\theta}\dfrac{\partial Y_{LM}}{\partial\phi} & h_0 \sin\theta \dfrac{\partial Y_{LM}}{\partial\theta} \\ 0 & 0 & -h_1 \dfrac{1}{\sin\theta}\dfrac{\partial Y_{LM}}{\partial\phi} & h_1 \sin\theta \dfrac{\partial Y_{LM}}{\partial\theta} \\ * & * & h_2 \dfrac{1}{2\sin\theta} X_{LM} & -h_2 \tfrac{1}{2}\sin\theta\, W_{LM} \\ * & * & * & -h_2 \tfrac{1}{2}\sin\theta\, X_{LM} \end{bmatrix}$$

electric multipoles:

$$\|h_{\mu\nu}\| = \begin{bmatrix} e^{\nu} H_0 Y_{LM} & H_1 Y_{LM} & h_0^{(e)} \dfrac{\partial Y_{LM}}{\partial\theta} & h_0^{(e)} \dfrac{\partial Y_{LM}}{\partial\phi} \\ * & e^{-\nu} H_2 Y_{LM} & h_1^{(e)} \dfrac{\partial Y_{LM}}{\partial\theta} & h_1^{(e)} \dfrac{\partial Y_{LM}}{\partial\phi} \\ * & * & r^2\left(K Y_{LM} + G \dfrac{\partial Y_{LM}}{\partial\theta}\right) & \tfrac{1}{2} r^2 G X_{LM} \\ * & * & * & r^2 \sin^2\theta \left[K Y_{LM} + G\left(\dfrac{\partial^2 Y_{LM}}{\partial\theta^2} - W_{LM}\right)\right] \end{bmatrix}$$

where

$$X_{LM} = 2\frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial\phi} - \cot\theta\right) Y_{LM},$$

$$W_{LM} = \left(\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) Y_{LM}.$$

TABLE II. Tensor harmonic expansion of electromagnetic perturbations $f_{\mu\nu}$. The asterisk denotes components obtainable by (anti)symmetry. Y_{LM} are normalized spherical harmonics. The tilde $(\tilde{f}_{\mu\nu})$ denotes the angle-independent parts of $f_{\mu\nu}$.

magnetic multipoles:

$$\|f_{\mu\nu}\| = \begin{bmatrix} 0 & 0 & \tilde{f}_{02} \dfrac{1}{\sin\theta}\dfrac{\partial Y_{LM}}{\partial\phi} & -\tilde{f}_{02} \sin\theta \dfrac{\partial Y_{LM}}{\partial\theta} \\ 0 & 0 & \tilde{f}_{12} \dfrac{1}{\sin\theta}\dfrac{\partial Y_{LM}}{\partial\phi} & -\tilde{f}_{12} \sin\theta \dfrac{\partial Y_{LM}}{\partial\theta} \\ * & * & 0 & \tilde{f}_{23} \sin\theta\, Y_{LM} \\ * & * & * & 0 \end{bmatrix}$$

electric multipoles:

$$\|f_{\mu\nu}\| = \begin{bmatrix} 0 & \tilde{f}_{01} Y_{LM} & \tilde{f}_{02} \dfrac{\partial Y_{LM}}{\partial\theta} & \tilde{f}_{02} \dfrac{\partial Y_{LM}}{\partial\phi} \\ * & 0 & \tilde{f}_{12} \dfrac{\partial Y_{LM}}{\partial\theta} & \tilde{f}_{12} \dfrac{\partial Y_{LM}}{\partial\phi} \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{bmatrix}$$

$$a_0=0,$$

$$a_1=0,$$

$$a_2=\frac{\tilde f_{23}\left(\frac{\partial Y_{LM}}{\partial\phi}\right)}{L(L+1)\sin\theta},$$

$$a_3=-\tilde f_{23}\left(\frac{\partial Y_{LM}}{\partial\theta}\right)\sin\theta\, L(L+1).$$

We denote by $\tilde f_{\mu\nu}$ the angle-independent parts of $f_{\mu\nu}$. The field equations $f_{\mu\nu,\lambda}+f_{\lambda\mu,\nu}+f_{\nu\lambda,\mu}=0$ (or equivalently, the fact that $f_{\mu\nu}$ is derived from a potential) tell us that

$$\tilde f_{12}=\frac{1}{L(L+1)}\frac{\partial f_{23}}{\partial r}, \tag{13}$$

$$\tilde f_{02}=\frac{1}{L(L+1)}\frac{\partial f_{23}}{\partial t}. \tag{14}$$

For electric multipoles we have $a_0=-\tilde f_{02}Y_{LM}$, $a_1=-\tilde f_{12}Y_{LM}$, $a_2=a_3=0$. Thus

$$\tilde f_{01}=\frac{\partial\tilde f_{02}}{\partial r}-\frac{\partial\tilde f_{12}}{\partial t}. \tag{15}$$

If we now substitute in Eqs. (11) and (12) the Reissner-Nordström values for $F_{\mu\nu}$ and $g_{\mu\nu}$, we obtain the perturbation equations. We write the equations for the Fourier transforms of the field functions ($\partial/\partial t\to -i\omega$). For magnetic multipoles (note that only magnetic electromagnetic multipoles couple to magnetic gravitational multipoles) we obtain

$$-\omega^2e^{-\nu}h_1+i\omega e^{-\nu}\frac{dh_0}{dr}-\frac{2i\omega}{r}e^{-\nu}h_0+e^{\nu}\left(\nu''+\nu'^2+\frac{2\nu'}{r}\right)h_1+2\lambda r^{-2}h_1=2e^2r^{-4}h_1-4er^{-2}e^{-\nu}\tilde f_{02}+A_{12}, \tag{16}$$

$$-e^{\nu}\frac{d^2h_0}{dr^2}-i\omega e^{\nu}\frac{dh_1}{dr}-\frac{2i\omega}{r}h_1+e^{\nu}\left(\nu''+\nu'^2+\frac{2\nu'}{r}+\frac{2}{r^2}\right)h_0+2\lambda r^{-2}h_0=2e^2r^{-4}h_0-4er^{-2}e^{\nu}\tilde f_{12}+A_{02}, \tag{17}$$

$$2i\omega e^{-\nu}h_0+2e^{\nu}\frac{dh_1}{dr}+2e^{\nu}\nu'h_1=A_{22}, \tag{18}$$

$$e^{\nu}\frac{d}{dr}e^{\nu}\frac{d\tilde f_{23}}{dr}+\omega^2\tilde f_{23}-\frac{1}{r^2}L(L+1)e^{\nu}\tilde f_{23}=\frac{e}{r^2}L(L+1)e^{\nu}\left(i\omega h_1+r^2\frac{d}{dr}\frac{h_0}{r^2}\right)-4\pi L(L+1)e^{\nu}y, \tag{19}$$

where $\lambda=\frac{1}{2}(L-1)(L+2)$ and primes denote differentiation with respect to r. The quantities A_{12}, A_{02}, A_{22}, and y are coefficients in the harmonic expansion of the matter and current sources (see Table III).

TABLE III. Coefficients of the harmonic expansion of matter and current sources.

magnetic multipoles:

$$A_{12}\frac{1}{\sin\theta}\frac{\partial Y_{LM}}{\partial\phi}=-16\pi T_{12},$$

$$A_{02}\frac{1}{\sin\theta}\frac{\partial Y_{LM}}{\partial\phi}=-16\pi T_{02},$$

$$A_{22}\frac{1}{\sin\theta}\tfrac{1}{2}X_{LM}=-16\pi T_{22},$$

$$y\frac{1}{\sin\theta}\frac{\partial Y_{LM}}{\partial\phi}=J_2.$$

electric multipoles:

$$A_{00}Y_{LM}=16\pi T_{00},\quad A_{01}Y_{LM}=-16\pi T_{01},\quad A_{11}Y_{LM}=-16\pi T_{11},$$

$$A_{02}\frac{\partial Y_{LM}}{\partial\theta}=-16\pi T_{02},\quad A_{12}\frac{\partial Y_{LM}}{\partial\theta}=-16\pi T_{12},$$

$$A_{23}\tfrac{1}{2}X_{LM}=-16\pi T_{23},$$

$$A_{22}Y_{LM}+A_{23}\tfrac{1}{2}W_{LM}=-16\pi T_{22},$$

$$vY_{LM}=J_0,$$

$$uY_{LM}=J_1$$

$$w\frac{\partial Y_{LM}}{\partial\theta}=J_2.$$

We also have the relations

$$\tilde f_{12}=\frac{\frac{d\tilde f_{23}}{dr}}{L(L+1)}$$

and

$$\tilde f_{02}=-\frac{i\omega\tilde f_{23}}{L(L+1)}.$$

Thus the electromagnetic field is determined by one function which we call $f_{LM}^{(m)}$ and choose to be

$$f_{LM}^{(m)}=\frac{1}{L(L+1)}\tilde f_{23}.$$

Now, solve Eq. (18) for h_0 and substitute it into Eq. (16). We obtain a second-order equation for h_1 in terms of the sources and $f_{LM}^{(m)}$. Equation (17) is satisfied identically if the energy-momentum tensor of matter plus the electromagnetic field has zero divergence. In the case of a point mass-charge this is equivalent to the requirement that the motion of the particle satisfy the Lorentz force law

$$m_0 Du^\mu = qF^\mu{}_\nu u^\nu ,$$

where D is the covariant derivative and $u^\mu = dz^\mu/ds$. Define the function $R_{LM}^{(m)}(r) = (1/r)e^\nu h_1$ and use the variable r^* where $dr^*/dr = e^{-\nu}$, then

$$\frac{d^2R_{LM}^{(m)}}{dr^{*2}} + \left\{\omega^2 - e^\nu\left[\frac{L(L+1)}{r^2} - \frac{6m}{r^3} + \frac{4e^2}{r^4}\right]\right\}R_{LM}^{(m)} = -\frac{4i\omega e}{r^3}e^\nu f_{LM}^{(m)} - \frac{e^{2\nu}}{r}A_{12} + \tfrac{1}{2}re^\nu\frac{d}{dr}\left(\frac{e^\nu}{r^2}A_{22}\right), \tag{20}$$

while Eq. (19) becomes

$$\frac{d^2f_{LM}^{(m)}}{dr^{*2}} + \left\{\omega^2 - e^\nu\left[\frac{L(L+1)}{r^2} + \frac{4e^2}{r^4}\right]\right\}f_{LM}^{(m)} = -\frac{e(L-1)(L+2)}{i\omega r^3}e^\nu R_{LM}^{(m)} + \frac{e}{i\omega r^2}e^{2\nu}A_{12} - 4\pi e^\nu y . \tag{21}$$

Thus the problem is reduced to the solution of two coupled second-order Schrödinger-type equations. The matter source terms for a point test particle are given in Table IV. Given a solution $R_{LM}^{(m)}$, $f_{LM}^{(m)}$, the metric and field functions are

$$h_{1LM} = re^{-\nu}R_{LM}^{(m)},$$

$$h_{0LM} = -\frac{1}{i\omega}\frac{d}{dr^*}(rR_{LM}^{(m)}) + \left(\frac{e^\nu}{2i\omega}\right)A_{22},$$

$$\tilde{f}_{01} = 0,$$

$$\tilde{f}_{02} = -i\omega f_{LM}^{(m)},$$

$$\tilde{f}_{12} = \frac{df_{LM}^{(m)}}{dr},$$

$$\tilde{f}_{23} = L(L+1)f_{LM}^{(m)} .$$

The electric multipole equations are slightly more complicated. They are as follows:

$$e^{2\nu}\left[2\frac{d^2K}{dr^2} - \frac{2}{r}\frac{dH_2}{dr} + \left(\nu' + \frac{6}{r}\right)\frac{dK}{dr} - 2\left(\frac{1}{r^2} + \frac{\nu'}{r}\right)(H_0 + H_2)\right] + e^\nu\left(\frac{2}{r^2}H_0 - \frac{2\lambda}{r^2}K\right) = \frac{2e^2}{r^4}e^\nu H_2 + \frac{4e}{r^2}e^\nu\tilde{f}_{01} + A_{00}, \tag{22}$$

$$-2e^{-2\nu}\omega^2K + \frac{4i\omega}{r}e^{-\nu}H_1 + \frac{2}{r}\frac{dH_0}{dr} - \left(\nu' + \frac{2}{r}\right)\frac{dK}{dr} + \frac{2}{r^2}e^{-\nu}H_2 - \frac{1}{r^2}L(L+1)e^{-\nu}H_0 + \frac{2\lambda}{r^2}e^\nu K = \frac{2e^2}{r^4}e^{-\nu}H_0 - \frac{4e}{r^2}e^{-\nu}\tilde{f}_{01} + A_{11}, \tag{23}$$

$$r^2\left[-\omega^2e^{-\nu}K - e^\nu\frac{d^2K}{dr^2} + e^\nu\left(\nu' + \frac{2}{r}\right)\frac{dK}{dr} + i\omega\left(\nu' + \frac{2}{r}\right)H_1 - \omega^2e^{-\nu}H_2 + 2i\omega\frac{dH_1}{dr} + e^\nu\frac{d^2H_0}{dr^2} + e^\nu\left(\tfrac{1}{2}\nu' + \frac{1}{r}\right)\frac{dH_2}{dr} + e^\nu\left(\tfrac{3}{2}\nu' + \frac{1}{r}\right)\frac{dH_0}{dr} + \frac{1}{2r^2}L(L+1)(H_2 - H_0) + e^\nu\left(\nu'' + \nu'^2 + \frac{2\nu'}{r}\right)(H_2 - K)\right] = 4e\tilde{f}_{01} + A_{22}, \tag{24}$$

TABLE IV. Harmonic coefficients for matter and current sources for a point particle: magnetic multipoles. $T(s)$, $R(s)$, $\Theta(s)$, $\Phi(s)$ is the trajectory in Schwarzschild-type coordinates vs. arc length (proper distance) s. The mass of the test particle is m_0 and its charge is q. The quantity $\gamma = dT/ds$.

matter:

$$A_{12} = 16\pi m_0\gamma[L(L+1)]^{-1}e^{-\nu}(dR/dt)\delta(r - R(t))\left[(1/\sin\Theta)(\partial Y_{LM}^*/\partial\Phi)d\Theta/dt - \sin\Theta(\partial Y_{LM}^*/\partial\Theta)d\Phi/dt\right],$$

$$A_{22} = 16\pi m_0[L(L+1)(L-1)(L+2)]^{-1}\delta(r - R(t)) \times\left\{(1/\sin\Theta)X_{LM}^*(\Theta,\Phi)[(d\Theta/dt)^2 - \sin^2\Theta(d\Phi/dt)^2] - 2\sin\Theta W_{LM}^*(\Theta,\Phi)(d\Theta/dt)(d\Phi/dt)\right\} .$$

current:

$$y = q[L(L+1)]^{-1}\delta(r - R(t))\left\{(1/\sin\Theta)(\partial Y_{LM}^*/\partial\Phi)(d\Theta/dt) - \sin\Theta(\partial Y_{LM}^*/\partial\Theta)(d\Phi/dt)\right\}$$

$$-2i\omega\frac{dK}{dr}+\frac{2i\omega}{r}H_2+i\omega\left(\nu'-\frac{2}{r}\right)K-e^{\nu}\left(\frac{2\nu'}{r}+\frac{2}{r^2}\right)H_1-\frac{2\lambda}{r^2}H_1=\frac{2e^2}{r^4}H_1+A_{01}\ , \tag{25}$$

$$-e^{\nu}\frac{dH_1}{dr}-i\omega K-i\omega H_2-e^{\nu}\nu' H_1=-\frac{4e}{r^2}e^{\nu}\tilde{f}_{12}+A_{02}\ , \tag{26}$$

$$-i\omega e^{-\nu}H_1-\frac{dH_0}{dr}+\frac{dK}{dr}-\left(\tfrac{1}{2}\nu'+\frac{1}{r}\right)H_2-\left(\tfrac{1}{2}\nu'-\frac{1}{r}\right)H_0=-\frac{4e}{r^2}e^{-\nu}f_{02}+A_{12}\ , \tag{27}$$

$$H_2-H_0=A_{23}\ , \tag{28}$$

$$\frac{1}{r^2}\frac{d}{dr}(r^2\tilde{f}_{01})-\frac{1}{r^2}L(L+1)e^{-\nu}\tilde{f}_{02}=-\frac{e}{2r^2}\frac{d}{dr}(H_2-H_0-2K)+4\pi e^{-\nu}v\ , \tag{29}$$

$$-i\omega\tilde{f}_{01}-\frac{1}{r^2}L(L+1)e^{\nu}\tilde{f}_{12}=\frac{i\omega e}{2r^2}(H_2-H_0-2K)+4\pi e^{\nu}u\ , \tag{30}$$

$$-i\omega e^{-\nu}\tilde{f}_{02}-\frac{d}{dr}(e^{\nu}\tilde{f}_{12})=4\pi w\ . \tag{31}$$

We also have the homogeneous Maxwell equation

$$\frac{d\tilde{f}_{02}}{dr}+i\omega\tilde{f}_{12}-\tilde{f}_{01}=0\ . \tag{32}$$

Again, A_{00}, A_{11}, A_{22}, A_{01}, A_{02}, A_{12}, A_{23}, v, u, y are coefficients in the harmonic expansion of the matter and current sources (see Table III).

This set of equations may be reduced to two coupled second-order equations just as in the case of magnetic multipoles. We outline the method. Substitute (28) into Eqs. (25), (26), and (27), eliminating the function H_2. This gives us three first-order equations for H_0, H_1, and K. If we substitute dH_1/dr, dK/dr, and dH_0/dr as given by these equations into Eq. (23) we obtain an algebraic relation involving H_0, H_1, K, and the electromagnetic $\tilde{f}_{01}$, $\tilde{f}_{02}$, $\tilde{f}_{12}$. We solve this equation for H_0 in terms of the other quantities and substitute this result into Eqs. (25) and (26). The remaining Einstein equations are satisfied identically (assuming zero divergence of the total energy-momentum tensor) by a solution of the resulting pair of equations which we may write as

$$\frac{dK}{dr}=\alpha_\omega(r)K+\omega^{-1}\beta_\omega(r)H_1+S_1\ , \tag{33}$$

$$\omega^{-1}\frac{dH_1}{dr}=\gamma_\omega(r)K+\omega^{-1}\delta_\omega(r)H_1+S_2\ . \tag{34}$$

The coefficients α, β, γ, δ, are functions of r and ω, where

$$\alpha_\omega(r)=\alpha_0(r)+\omega^2\alpha_2(r)\ ,$$

$$\beta_\omega(r)=\beta_0(r)+\omega^2\beta_2(r)\ ,$$

$$\gamma_\omega(r)=\gamma_0(r)+\omega^2\gamma_2(r)\ ,$$

$$\delta_\omega(r)=\delta_0(r)+\omega^2\delta_2(r)\ .$$

The quantities S_1 and S_2 contain $\tilde{f}_{01}$, $\tilde{f}_{02}$, $\tilde{f}_{12}$ plus the matter and current sources. It is possible to reduce this pair of equations to a single second-order equation. For the moment, let us turn to the Maxwell equations (29)–(32). Equation (30) gives $\tilde{f}_{01}$ in terms of $e^{\nu}\tilde{f}_{12}$, while (31) gives $\tilde{f}_{02}$ in terms of $e^{\nu}\tilde{f}_{12}$. Equation (29) is satisfied identically if the current satisfies the divergence condition. If we substitute these values of $\tilde{f}_{01}$ and $\tilde{f}_{02}$ into Eq. (32) we obtain a second-order equation for $e^{\nu}\tilde{f}_{12}$ (denote $e^{\nu}\tilde{f}_{12}$ by $f_{LM}^{(e)}$):

$$\frac{d^2 f_{LM}^{(e)}}{dr^{*2}}+[\omega^2-e^{\nu}r^{-2}L(L+1)]f_{LM}^{(e)}=\frac{i\omega e}{2r^2}e^{\nu}(A_{23}-2K_{LM})+4\pi\left[e^{2\nu}u-\frac{d}{dr^*}(e^{\nu}w)\right]\ . \tag{35}$$

Our task now is to reduce (33) and (34) to a single second-order equation for a "superpotential" $R_{LM}^{(e)}$ and to express K_{LM} in Eq. (35) in terms of $R_{LM}^{(e)}$ and the matter sources.

A. Transformation of electric equations to a single second-order equation

The idea is to find a transformation from the functions H_1 and K to a new pair of functions $\hat{K}$ and $\hat{L}$, where $\hat{K}$ and $\hat{L}$ satisfy the following equations:

$$\frac{d\hat{K}}{d\hat{r}}=\hat{L}+\hat{S}_1\ , \tag{36}$$

$$\frac{d\hat{L}}{d\hat{r}}=-[\omega^2-V(\hat{r})]\hat{K}+\hat{S}_2\ . \tag{37}$$

The new variable $\hat{r}$ is given in terms of r by $d\hat{r}/dr=1/n(r)$. If we can find such a transformation, the problem is solved since, letting $R_{LM}^{(e)}=\hat{K}_{LM}$, we have

$$\frac{d^2R_{LM}^{(e)}}{d\hat{r}^2}+[\omega^2-V^{(e)}(r)]R_{LM}^{(e)}=S_{LM}\ , \tag{38}$$

where $S_{LM}=\hat{S}_2+d\hat{S}_1/d\hat{r}$. Thus we wish to find functions of r only, $f(r)$, $g(r)$, $h(r)$, and $k(r)$ such that if

$$K(r)=f(r)\hat{K}(\hat{r})+g(r)\hat{L}(\hat{r}),$$

$$\omega^{-1}H_1(r)=h(r)\hat{K}(\hat{r})+k(r)\hat{L}(r)\ ,$$

$$\frac{d\hat{r}}{dr}=\frac{1}{n(r)}\ ,$$

then $\hat{K}$ and $\hat{L}$ satisfy (36) and (37). Fortunately, the problem has a unique solution. Let us use a matrix notation for compactness. If we make the definitions

$$\psi=\begin{pmatrix}K\\ \omega^{-1}H_1\end{pmatrix},\quad A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},\quad S=\begin{pmatrix}S_1\\ S_2\end{pmatrix},$$

then Eqs. (34) and (35) can be written

$$\frac{d\psi}{dr}=A\psi+S\ . \tag{39}$$

Let

$$\hat{\psi}=\begin{pmatrix}\hat{K}\\ \hat{L}\end{pmatrix},\quad F=\begin{pmatrix}f & g\\ h & k\end{pmatrix},$$

then

$$\psi=F\hat{\psi}\ . \tag{40}$$

Thus

$$\frac{d\psi}{dr}=F\frac{d\hat{\psi}}{dr}+\frac{dF}{dr}\hat{\psi}=A\psi+S=AF\hat{\psi}+S$$

so that

$$\frac{d\hat{\psi}}{dr}=-n(r)F^{-1}\left(\frac{dF}{dr}-AF\right)\hat{\psi}+n(r)F^{-1}S\ .$$

Thus, to satisfy our requirements, we want

$$n(r)F^{-1}\left(AF-\frac{dF}{dr}\right)=\begin{pmatrix}0 & 1\\ -\omega^2+V & 0\end{pmatrix}. \tag{41}$$

Condition (41) gives us four equations involving f, g, h, k, n, V in terms of α, β, γ, δ. If we equate separately the coefficients of ω^0 and ω^2 we obtain eight equations plus the condition $kf-gh\neq 0$ for six functions f, g, h, k, n, V. The system is overdetermined, but the functions α, β, γ, δ are such that the system is consistent and has a unique solution:

$$f(r)=\frac{L(L+1)/2r-e^{\nu}(3mr-4e^2)}{r(\lambda r^2+3mr-2e^2)}\ , \tag{42}$$

$$g(r)=1\ , \tag{43}$$

$$k(r)=-ire^{-\nu}\ , \tag{44}$$

$$h(r)=-i\left[\frac{1-e^{-\nu}r^{-2}(mr-e^2)-(3mr-4e^2)}{\lambda r^2+3mr-2e^2}\right]\ , \tag{45}$$

$$n(r)=e^{\nu}\ . \tag{46}$$

Thus our new variable $\hat{r}$ is just the variable r^* that we used previously for the magnetic multipoles. The effective potential for gravitational perturbations is

$$V_L^{(e)}(r)=e^{\nu}\left[\frac{L(L+1)}{r^2}-\frac{6m}{r^3}+\frac{8e^2}{r^4}+\frac{4}{r^4}\frac{(3mr-4e^2)(mr-e^2)+e^{\nu}r^2(3mr-5e^2)}{\lambda r^2+3mr-2e^2}+\frac{2}{r^2}\frac{e^{\nu}(3mr-4e^2)^2}{(\lambda r^2+3mr-2e^2)^2}\right]\ . \tag{47}$$

We then have

$$K_{LM}=f(r)R_{LM}^{(e)}+\frac{dR_{LM}^{(e)}}{dr^*}-\hat{S}_1\ ,$$

TABLE V. Harmonic coefficients for matter and current source for a point particle: electric multipoles. The Fourier transforms of these quantities must be used in Eqs. (16)–(50).

matter:

$$A_{00}=-16\pi m_0\gamma e^{2\nu}r^{-2}\delta(r-R(t))Y_{LM}^*(\Omega(t))\ ,$$

$$A_{01}=16\pi m_0\gamma(dR/dt)r^{-2}\delta(r-R(t))Y_{LM}^*(\Omega(t))\ ,$$

$$A_{11}=-16\pi m_0\gamma e^{-2\nu}(dR/dt)^2r^{-2}\delta(r-R(t))Y_{LM}^*(\Omega(t))\ ,$$

$$A_{02}=-16\pi m_0\gamma[L(L+1)]^{-1}e^{\nu}\delta(r-R(t))(d/dt)Y_{LM}^*(\Omega(t))\ ,$$

$$A_{12}=-16\pi m_0\gamma[L(L+1)]^{-1}e^{-\nu}\times(dR/dt)\delta(r-R(t))(d/dt)Y_{LM}^*(\Omega(t))\ ,$$

$$A_{22}=-8\pi r^2m_0\gamma\delta(r-R(t))\times\{(d\Theta/dt)^2+\sin^2\Theta(d\Phi/dt)^2\}Y_{LM}^*(\Omega(t))\ ,$$

$$A_{23}=-16\pi m_0\gamma r^2[L(L+1)(L-1)(L+2)]^{-1}\delta(r-R(t))\times\{2X_{LM}^*(\Omega(t))(d\Theta/dt)(d\Phi/dt)+[(d\Theta/dt)^2-\sin^2\Theta(d\Phi/dt)^2]W_{LM}^*(\Omega(t))\}\ .$$

current:

$$v=-qe^{\nu}r^{-2}\delta(r-R(t))Y_{LM}^*(\Omega(t))\ ,$$

$$u=qe^{-\nu}(dR/dt)r^{-2}\delta(r-R(t))Y_{LM}^*(\Omega(t))\ ,$$

$$w=q[L(L+1)]^{-1}\delta(r-R(t))(d/dt)Y_{LM}^*(\Omega(t))\ .$$

where

$$\hat{S}_1 = e^{\nu}(\lambda r^2 + 3mr - 2e^2)^{-1} f_{LM}^{(e)} + \hat{S}_1^{(\text{matter})},$$

$$\hat{S}_1^{(\text{matter})} = -(1/i\omega) e^{\nu} r^3 (\lambda r^2 + 3mr - 2e^2)^{-1} (\tfrac{1}{2}A_{01} + r^{-1}A_{02}) .$$

We may substitute this value for K_{LM} in Eq. (35) to obtain the second-order electromagnetic equation. Thus we obtain the following system:

$$\frac{d^2 R_{LM}^{(e)}}{dr^{*2}} + (\omega^2 - V_L^{(e)}) R_{LM}^{(e)} = -e^{2\nu}(\lambda r^2 + 3mr - 2e^2)^{-1}[e^{-\nu} r^{-2}(\lambda r + m) + (2\lambda r + 3m)(\lambda r^2 + 3mr - 2e^2)^{-1}](8e/i\omega) f_{LM}^{(e)} + S_{LM}^{(\text{matter})}, \tag{48}$$

$$\frac{d^2 f_{LM}^{(e)}}{dr^{*2}} + \left\{\omega^2 - e^{\nu}\left[\frac{L(L+1)}{r^2} + e^{\nu}\frac{4e^2}{r^2(\lambda r^2 + 3mr - 2e^2)}\right]\right\} f_{LM}^{(e)} = -\frac{i\omega e}{r^2} e^{\nu}\left[f(r) R_{LM}^{(e)} + \frac{dR_{LM}^{(e)}}{dr^*}\right] + \frac{i\omega e}{2r^2} e^{\nu} A_{23} + \frac{i\omega e}{r^2}\hat{S}_1 + 4\pi e^{\nu}\left[e^{\nu} u - \frac{d}{dr^*}(e^{\nu} w)\right], \tag{49}$$

where the matter source term $S_{LM}^{(\text{matter})}$ is

$$S_{LM}^{(\text{matter})} = -\frac{1}{i\omega}\frac{d}{dr^*}[e^{\nu} r^3 (\lambda r^2 + 3mr - 2e^2)^{-1}(\tfrac{1}{2}A_{01} + r^{-1}A_{02})] + e^{\nu} r^{-1} A_{23}$$
$$- e^{2\nu} r^3 (\lambda r^2 + 3mr - 2e^2)^{-1}\{(1/2i\omega)[r^{-1} - (2\lambda r + 3m)(\lambda r^2 + 3mr - 2e^2)^{-1}] A_{01}$$
$$+ (1/i\omega r)[2/r - (2\lambda r + 3m)(\lambda r^2 + 3mr - 2e^2)^{-1} - e^{-\nu} r^{-1}(\lambda + 1)] A_{02}\}$$
$$- e^{\nu} r^{-1} \tilde{B}_{LM}, \tag{50}$$

where

$$\tilde{B}_{LM} = \tfrac{1}{2} e^{\nu} r^4 (\lambda r^2 + 3mr - 2e^2)^{-1}[A_{11} + (2/r)A_{12} + (8e/i\omega r^3)4\pi w + (4e/i\omega r^2)4\pi u] .$$

The other metric functions are

$$H_{1LM} = \omega h(r) R_{LM}^{(e)} - i\omega r e^{-\nu}\left(\frac{dR_{LM}^{(e)}}{dr^*} - \hat{S}_1\right),$$

$$H_{0LM} = \frac{\lambda r^3 e^{\nu} + mr(r - 3m) + 2me^2 - \omega^2 r^5}{r(\lambda r^2 + 3mr - 2e^2)e^{\nu}} K_{LM} + \frac{(\lambda + 1)(mr - e^2) - \omega^2 r^3}{i\omega r(\lambda r^2 + 3mr - 2e^2)} H_{1LM}$$
$$- \frac{4e}{i\omega} r (\lambda r^2 + 3mr - 2e^2)^{-1}\left(\frac{df_{LM}^{(e)}}{dr^*} + \frac{1}{2r} f_{LM}^{(e)}\right) - \tilde{B}_{LM} + (e^{\nu}\nu' r^4/4i\omega)(\lambda r^2 + 3mr - 2e^2)^{-1} A_{01}.$$

The electromagnetic field functions are

$$\tilde{f}_{12} = e^{-\nu} f_{LM}^{(e)},$$

$$\tilde{f}_{02} = -\frac{1}{i\omega}\frac{df_{LM}^{(e)}}{dr^*} - \frac{4\pi}{i\omega} e^{\nu} w,$$

$$\tilde{f}_{01} = -(1/i\omega r^2) L(L+1) f_{LM}^{(e)} + (e/r^2) K_{LM} - (e/2r^2) A_{23} - (4\pi/i\omega) e^{\nu} u .$$

The matter source coefficients for a point mass-charge are given in Table V.

III. CONCLUSION

For each type of multipole, electric and magnetic, it is possible to reduce the original Einstein-Maxwell system of fourteen functions and fourteen equations to a pair of second-order Schrödinger-type equations for two functions—one which determines the gravitational field and one which determines the electromagnetic field. We have found the effective potentials for each case. Numerical integrations of the equations for certain cases of interest—charged and uncharged test particles in various orbits are being carried out.

Note added in proof: Vincent Moncrief points out that the magnetic multipole equations may be decoupled by a simple linear transformation.

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[9]We use units in which $G=c=1$. We follow the convention proposed by C. W. Misner, K. S. Thorne, and J. A. Wheeler, in "An Open Letter to Relativity Theorists," 1968 (unpublished): Latin indices run from 1 to 3 and indicate spacelike coordinates; Greek indices run from 0 to 4, 0 representing the timelike coordinate. The metric has signature +2 (spacelike convention). The connection coefficients and Riemann tensor are

$$\Gamma^{\lambda}{}_{\mu\nu} = \tfrac{1}{2} g^{\lambda\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})$$

and

$$R^{\alpha}{}_{\lambda\mu\nu} = \Gamma^{\alpha}{}_{\lambda\nu,\mu} - \Gamma^{\alpha}{}_{\lambda\mu,\nu} + \Gamma^{\alpha}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\lambda\nu} - \Gamma^{\alpha}{}_{\sigma\mu}\Gamma^{\alpha}{}_{\lambda\mu}.$$

The contracted Riemann tensor is $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ and the Einstein equations are $G_{\mu\nu}(g_{\alpha\beta}) \equiv R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$.

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Transitions to the ground state in synchrotron radiation

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This paper will deal with the high-energy tip of the synchrotron spectrum. Specifically, we consider exact relativistic wave functions of an electron in a transverse magnetic field as the basis for the calculation of the transition rate to the ground state.

I. INTRODUCTION

Several articles[1-5] and at least one text[6] have been written dealing with the quantum-mechanical description of an electron undergoing synchrotron radiation. The general picture[7] is that an electron moving in a transverse magnetic field occupies a definite set of energy levels with energy eigenvalues

$$E_n = mc^2[1 + 2n(H/H_0) + \gamma_3{}^2]^{1/2}, \qquad (1.1)$$

where

$$n = n_p + S + \tfrac{1}{2}, \qquad (1.2a)$$

n_p = principal quantum number $(0, 1, 2, \ldots)$, (1.2b)

S = spin quantum number $(\pm\tfrac{1}{2})$, (1.2c)

H = magnetic field strength, (1.2d)

$$H_0 = \frac{m^2c^3}{e\hbar} = 4.414\times 10^{13}\ \text{gauss}, \qquad (1.2e)$$

$$\gamma_3 = p_3/mc, \qquad (1.2f)$$

p_3 = component of electron's linear momentum along $\vec{H}$, (1.2g)

and m, c, e, and $\hbar$ have their usual significance.

Spontaneous radiative transitions are allowed between these energy levels. These rates have been computed in first[6] and second[4] order by time-dependent perturbation theory. The matrix elements which enter in the calculation of the transition rates between initial state n and final state n' are essentially Laguerre functions, $I_{nn'}(x)$, where the argument

$$x = A \sin^2\theta \qquad (1.3)$$

contains the following parameters:

θ = polar angle in the system of spherical coordinates where $\vec{H}$ is parallel to the z axis and the electrons initially move in the x-y plane (the component of the emitted photon momentum along $\vec{H}$ is proportional to $\cos\theta$), (1.4a)

$$A = (H_0/2H)(\gamma_n - \gamma_{n'})^2, \qquad (1.4b)$$