Coherent states and particle production*

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Hadron production at high energies is discussed in terms of coherent states. We suggest that the scattering operator and the pion field density operator take on simple forms in the coherent-state representation. In particular, we show that for a wide variety of configurations of the pion field the density operator can be written in diagonal form in the coherent-state representation. When this is possible, one obtains the statistical theory of pion production which we have discussed recently. We also construct a solvable unitary model in which the total, elastic, and inclusive cross sections go to constants at high energy, while the exclusive cross sections go to zero like a power of energy. In order to take into account the isotopic spin of the pions, we generalize the concept of coherent states and construct states with definite charge and isotopic spin.

I. INTRODUCTION

In most discussions of particle production one starts by considering matrix elements of the scattering operator between states containing definite numbers of particles. This is the natural approach when only a few particles are produced. However, at present accelerator energies the average number of particles produced in hadron-hadron collisions is already large, and it is expected to increase indefinitely with energy. Under these circumstances it may be advantageous to express the scattering operator and other operators of interest in terms of states which are not eigenfunctions of the number operator. A well-known example is the use of coherent states of photons to study the statics of the radiation field in systems such as lasers, where the average number of photons is large.[1] In this paper we shall use this approach to study the production of pions at high energies.[2]

It should be emphasized that we do not envision the pions produced in a high-energy collision as coming off in a single coherent state. Pions produced in such a state would have no correlations in their rapidity or transverse momenta aside from those imposed by energy and momentum conservation. What we do wish to suggest is that the scattering operator and the pion field density operator take on simple forms in the coherent-state representation. We show that this is the case for a wide variety of final configurations for the pion field, ranging from ones in which the pion distribution is chaotic to ones in which it is highly coherent. Although pions produced in a single coherent state do not have any correlations in momentum space, there is no problem in introducing both short- and long-range correlations in our formalism.

In order to present our ideas in the simplest possible framework, we start by considering the production of spinless, isoscalar "pions." In Sec. II the properties of the coherent states of such particles are briefly reviewed. In Sec. III the density operator for the isoscalar field is introduced, and expressions for the inclusive and semi-inclusive cross sections are given in terms of it. The density operator is then expressed in terms of coherent states and some simple examples are presented. The connection between the present approach and the statistical theory recently presented by two of us is discussed.[3]

In Sec. IV our formalism is used to construct generalizations of the models of Auerbach *et al.*,[4] which satisfy exact s-channel unitarity. We present a solvable unitary model in which the total and elastic cross sections go to constants at high energy, while the exclusive cross sections for the production of pions in the central region go to zero like a power of the energy. This model contains many of the features of the two-component models recently discussed in the literature[5]; however, in the present case the diffractive component arises naturally; it does not have to be put in by hand.

In Sec. V we confront the problems arising from the fact that the pion has isospin one. An ordinary coherent state of charged pions will not have definite charge or definite isospin. It is therefore necessary to generalize the concept of coherent states in order to construct ones with definite charge and isospin. The techniques developed here can be applied equally well to particles with larger isospin than the pion or to higher-symmetry groups.

Finally in Sec. VI we briefly discuss our results and consider possible generalizations of our work.

II. COHERENT STATES

We begin by briefly reviewing the properties of the coherent states of a spinless, isoscalar particle.[6] The four-momenta of these particles will be expressed in terms of their rapidity y and their transverse momenta, $\vec{q}_\perp$. When no confusion can arise, we shall denote the three variables $(y, \vec{q}_\perp)$ by the single symbol q. The creation and annihilation operators are normalized so that their commutation relation takes on the Lorentz-covariant form

$$[a(y,\vec{q}_\perp), a^\dagger(y',\vec{q}_\perp')] = \delta(y-y')\delta^2(\vec{q}_\perp - \vec{q}_\perp')\,. \tag{1}$$

The coherent states are, by definition, the eigenstates of the destruction operator. Thus,

$$a(q)|\Pi\rangle = \Pi(q)|\Pi\rangle\,, \tag{2}$$

where $\Pi(q)$ is an arbitrary complex function of q. $|\Pi\rangle$ can be written in the form

$$|\Pi\rangle = \exp\left[-\tfrac{1}{2}\int dq\,|\Pi(q)|^2\right] \times \exp\left[\int dq\,\Pi(q)a^\dagger(q)\right]|0\rangle\,. \tag{3}$$

$|0\rangle$ is the vacuum state defined by $a(q)|0\rangle = 0$ and $dq \equiv dyd^2q_\perp$ is the invariant phase-space volume element. It is sometimes useful to introduce the coherent-state displacement operator

$$D(\Pi) = \exp\left\{\int dq\,[\Pi(q)a^\dagger(q) - \Pi^*(q)a(q)]\right\}\,. \tag{4}$$

Notice that

$$|\Pi\rangle = D(\Pi)|0\rangle\,. \tag{5}$$

$D(\Pi)$ is a unitary operator, $D^\dagger(\Pi) = D(-\Pi) = D^{-1}(\Pi)$.

The states $|\Pi\rangle$ are normalized to unity, but they are not mutually orthogonal. In fact

$$\langle\Pi'|\Pi\rangle = \exp\left\{-\tfrac{1}{2}\int dq\,[|\Pi'(q)|^2 + |\Pi(q)|^2 - 2\Pi'^*(q)\,\Pi(q)]\right\}\,. \tag{6}$$

The average number of particles in the state $|\Pi\rangle$ is

$$\bar{n} = \left\langle\Pi\left|\int dq\,a^\dagger(q)a(q)\right|\Pi\right\rangle = \int dq\,|\Pi(q)|^2\,. \tag{7}$$

The probability density of finding n particles with momenta $q_1, \ldots, q_n$ in this state is

$$|\langle 0|a(q_1)\cdots a(q_n)|\Pi\rangle|^2 = |\Pi(q_1)|^2\cdots|\Pi(q_n)|^2 e^{-\bar{n}}\,, \tag{8}$$

so the probability of finding n particles of any momenta is

$$P_n = \frac{(\bar{n})^n}{n!}e^{-\bar{n}}\,. \tag{9}$$

It is sometimes convenient to introduce a complete set of states, $f_k(q)$, normalized so that

$$\int dq\,f_k^*(q)f_{k'}(q) = \delta_{k,k'}\,. \tag{10}$$

For most purposes it will not be necessary to specify the functional form of the $f_k(q)$. The creation and annihilation operators for a particular normal mode are defined by

$$a_k = \int dq\,f_k^*(q)a(q)\,, \tag{11}$$

so

$$[a_k, a_{k'}^\dagger] = \delta_{k,k'}\,. \tag{12}$$

Writing

$$\Pi(q) = \sum_k \alpha_k f_k(q)\,, \tag{13}$$

we see that the coherent state defined in Eq. (3) can be written in the form

$$|\Pi\rangle = \exp\left(-\tfrac{1}{2}\sum_k|\alpha_k|^2\right)\exp\left(\sum_k\alpha_k a_k^\dagger\right)|0\rangle \equiv |\{\alpha_k\}\rangle\,. \tag{14}$$

Although the coherent states are not linearly independent, they do form a complete set. Denoting by $\int\delta\Pi$ a functional integration over all forms of the complex field $\Pi(q)$ and by

$$\int d^2\alpha_k = \int d(\mathrm{Re}\,\alpha_k)d(\mathrm{Im}\,\alpha_k)$$

an integration over all complex values of α_k, we see that

$$\begin{aligned}\int\delta\Pi\,|\Pi\rangle\langle\Pi| &= \int\prod_k\frac{d^2\alpha_k}{\pi}|\{\alpha_k\}\rangle\langle\{\alpha_k\}| \\ &= \sum_{n_1,n_2,\ldots}\frac{(a_1^\dagger)^{n_1}}{(n_1!)^{1/2}}\frac{(a_2^\dagger)^{n_2}}{(n_2!)^{1/2}}\cdots|0\rangle \\ &\qquad\times\langle 0|\frac{(a_1)^{n_1}}{(n_1!)^{1/2}}\frac{(a_2)^{n_2}}{(n_2!)^{1/2}}\cdots \\ &= 1\,.\end{aligned} \tag{15}$$

The measure of the functional integration is defined by the first line of Eq. (15). In going from line 1 to line 2 of Eq. (15) we have made use of the fact that

$$\int\frac{d^2\alpha_k}{\pi}e^{-|\alpha_k|^2}(\alpha_k)^n(\alpha_k^*)^{n'} = \delta_{n,n'}n!\,. \tag{16}$$

Using Eq. (15), one can expand any vector or operator in terms of coherent states. For example,

$$|f\rangle=\int \delta\Pi\ |\Pi\rangle\langle\Pi|f\rangle\ . \tag{17}$$

However, since the coherent states are not linearly independent, the expansion is not unique.[1]

III. THE PION FIELD DENSITY OPERATOR

Most of the properties of the pions produced in a high-energy hadron-hadron collision can be expressed simply in terms of a pion field density operator. As in Sec. II we shall neglect the complications of spin and isospin. Proceeding formally we write

$$\tilde{\rho}=T|P_1P_2\rangle\langle P_1P_2|T^\dagger\tilde{Z}^{-1}, \tag{18}$$

with

$$\begin{aligned}\tilde{Z}&=\mathrm{tr}[T|P_1P_2\rangle\langle P_1P_2|T^\dagger]\\&=\langle P_1P_2|T^\dagger T|P_1P_2\rangle\ .\end{aligned} \tag{19}$$

$|P_1P_2\rangle$ is the incident state of two hadrons with momenta P_1 and P_2, and T is related to the scattering operator by $T=2i(1-S)$. Clearly $\tilde{Z}$ is proportional to the total cross section σ.

The inclusive cross sections for the production of a pion with momentum q is given in terms of $\tilde{\rho}$ by

$$\begin{aligned}\frac{1}{\sigma}\frac{d\sigma}{dq}&=\tilde{Z}^{-1}\sum_n\langle n|a(q)T|P_1P_2\rangle\langle P_1P_2|T^\dagger a^\dagger(q)|n\rangle\\&=\mathrm{tr}[a^\dagger(q)a(a)\tilde{\rho}]\ .\end{aligned} \tag{20}$$

Again $dq=dyd^2q_\perp$. Notice that we can use closure in Eq. (20) because the total energy-momentum conservation δ function is included in the operator T. Similarly expressions for all inclusive and exclusive cross sections can be written in the form $\mathrm{tr}[O\tilde{\rho}]$.

If we restrict our interest to one type of particle, say pions, then we can perform the trace over the coordinates of all other particles at the beginning. We therefore define the pion field density operator ρ by

$$\rho=\mathrm{tr}'\tilde{\rho}\,, \tag{21}$$

where tr′ indicates a trace over the coordinates of all particles except pions. Clearly ρ satisfies the normalization condition

$$\mathrm{tr}\rho=1\,. \tag{22}$$

From now on tr will denote a trace over pion coordinates only. The inclusive cross section for the production of n pions is given in terms of ρ by

$$\begin{aligned}\frac{1}{\sigma}\frac{d\sigma}{dq_1\cdots dq_n}&=\mathrm{tr}[a^\dagger(q_1)\cdots a^\dagger(q_n)a(q_1)\cdots a(q_n)\rho]\\&\equiv I(q_1,\ldots,q_n)\,,\end{aligned} \tag{23}$$

and the semi-inclusive cross sections for the production of exactly n pions plus any number of other types of particles by

$$\begin{aligned}\frac{1}{\sigma}\sigma_n(q_1,\ldots,q_n)&=\mathrm{tr}[|q_1,\ldots,q_n\rangle\langle q_1,\ldots,q_n|\rho]\\&\equiv S(q_1,\ldots,q_n)\,.\end{aligned} \tag{24}$$

Here $|q_1,\ldots,q_n\rangle$ is a state of n pions with momenta $q_1,\ldots,q_n$.

In performing the trace in Eq. (21) one integrates out the energy-momentum conservation δ functions in T, thereby introducing constraints into ρ. Since the transverse momenta of the pions are limited by the dynamics, the most important constraint is on the rapidities. In the laboratory system y must lie in the approximate range $0\leq y\leq Y$, where Y is the rapidity of the incident projectile. There are, of course, more complicated constraints which are most important for pions produced near the edges of the rapidity plot, i.e., near 0 or Y. Since we expect most of the pions to come off in the central region, the only constraint that we shall explicitly build in is that their rapidities be restricted to the range $0\leq y\leq Y$.

If one had a fundamental theory of strong interaction dynamics, one could of course calculate ρ. In the absence of such a theory it seems useful to attempt to find a simple phenomenological parameterization of ρ in the hopes of learning something about the underlying dynamics. One knows from the study of quantum optics that because of the overcompleteness of the coherent states, a wide class of density operators can be written in diagonal form in the coherent-state representation.[1,7] Since calculations are particularly simple for such density operators, let us start by considering them.

We write ρ in the form

$$\rho=Z^{-1}\int\delta\Pi\,|\Pi\rangle\,e^{-F[\Pi]}\langle\Pi|\,, \tag{25}$$

with

$$Z=\int\delta\Pi\ e^{-F[\Pi]}\,. \tag{26}$$

$F[\Pi]$ is an arbitrarily functional of Π. Because of the energy-momentum conservation constraint, the functional integration is to run only over those functions of $\Pi(y,\vec{q}_\perp)$ which vanish for y outside the range $0\leq y\leq Y$. Recalling that $a(y,\vec{q}_\perp)|\Pi\rangle=\Pi(y,\vec{q}_\perp)|\Pi\rangle$, we see that the inclusive and semi-inclusive cross sections take on the particularly simple forms

$$I(q_1, \ldots, q_n) = Z^{-1}\int \delta\Pi\, |\Pi(q_1)|^2 \cdots |\Pi(q_n)|^2 \times e^{-F[\Pi]}, \tag{27}$$

and

$$S(q_1, \ldots, q_n) = Z^{-1}\int \delta\Pi\, |\Pi(q_1)|^2 \cdots |\Pi(q_n)|^2 \times e^{-F[\Pi]} \exp\left[-\int dq\, |\Pi(q)|^2\right]. \tag{28}$$

Mueller's generating function[8] $\Omega(z)$ is given by

$$\Omega(z) = 1 + \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!} \int dq_1 \cdots dq_n I(q_1, \ldots, q_n) = Z^{-1} \int \delta\Pi\, e^{-F[\Pi]} e^{-(1-z)\int dq |\Pi(q)|^2}. \tag{29}$$

Equations (27) and (29) were the starting points for the statistical theory of particle production recently proposed by two of us.[3]

A particularly simple form for the density matrix is obtained by taking $e^{-F[\Pi]}$ to be a functional δ function. Then the density matrix is given by

$$\rho = |\Pi_0\rangle\langle\Pi_0|. \tag{30}$$

Clearly,

$$I(q_1, \ldots, q_n) = |\Pi_0(q_1)|^2 \cdots |\Pi_0(q_n)|^2, \tag{31}$$

$$S(q_1, \ldots, q_n) = |\Pi_0(q_1)|^2 \cdots |\Pi_0(q_n)|^2 e^{-\bar{n}}, \tag{32}$$

and

$$\Omega(z) = e^{(z-1)\bar{n}}, \tag{33}$$

with

$$\bar{n} = \int_0^Y dy \int d^2q_\perp |\Pi_0(y, q_\perp)|^2. \tag{34}$$

If $\Pi_0(y, \vec{q}_\perp)$ is independent of y for $0 \le y \le Y$, or a slowly varying function of y in this region, then $\bar{n}$ grows like Y and the $S(q_1, \ldots, q_n)$ fall like a power of s. The inclusive cross sections are precisely what one would obtain in a multi-Regge model in which there was a single Regge trajectory with intercept one. This model is of course tremendously oversimplified as there are no correlations in either the rapidity or the transverse momentum.

In I we considered a simple form for $F[\Pi]$ based upon an analogy with the generalized Ginsburg-Landau theory of superconductivity. We wrote

$$F[\Pi] = \int_0^Y dy \left[a|\Pi(y)|^2 + b|\Pi(y)|^4 + c\left|\frac{\partial\Pi}{\partial y}\right|^2\right]. \tag{35}$$

For simplicity the dependence on the transverse momenta has been suppressed. Although Eq. (35) could be obtained by retaining the first few terms in the power-series expansion of a general $F[\Pi]$, it should be emphasized that it is applicable even when the fluctuations of the pion field are not small. The main justification for this form of $F[\Pi]$ is that by suitably adjusting the phenomenological parameters a, b, and c, it can describe a wide range of possible final configurations of the pion field.

To see just how wide a class of phenomena can be described by the Ginzburg-Landau functional let us consider two extreme examples. First take $a, c > 0$ and $b = 0$. In this case it is convenient to expand $\Pi(y)$ in terms of a complete set of normal-mode functions. Since we do not wish to restrict the values $\Pi(y)$ can take on at the boundaries,[3] the appropriate choice is

$$f_0(y) = Y^{-1/2}, \quad f_k(y) = (\tfrac{1}{2}Y)^{-1/2}\cos\left[\frac{\pi k y}{Y}\right], \quad k = 1, 2, \ldots . \tag{36}$$

From Eq. (13) and the orthogonality of the $f_k(y)$ we see that

$$F[\Pi] = \sum_{k=0}^{\infty} |\alpha_k|^2 [a + c(\pi k/Y)^2]. \tag{37}$$

Then from Eqs. (14) and (15) we find

$$\rho = \prod_{k=0}^{\infty} \rho_k, \tag{38}$$

with

$$\rho_k = (\bar{n}_k)^{-1} \int \frac{d^2\alpha_k}{\pi} |\alpha_k\rangle\langle\alpha_k| e^{-|\alpha_k|^2/\bar{n}_k} \tag{39}$$

and

$$\bar{n}_k = [a + c(\pi k/Y)^2]^{-1}. \tag{40}$$

In the occupation-number basis we have

$$\rho_k = (1+\bar{n}_k)^{-1} \sum_{m_k} |m_k\rangle\langle m_k| [\bar{n}_k/(1+\bar{n}_k)]^{m_k}, \tag{41}$$

where

$$|m_k\rangle = (m_k!)^{-1/2} (a_k^\dagger)^{m_k} |0\rangle. \tag{42}$$

Thus, each normal mode of the pion field is described by a Gaussian density operator which corresponds to random excitation characteristic of noncoherent sources.[1]

The opposite extreme, a highly coherent pion field, can be obtained by taking $a<0$, $b>0$, and $c\gg 1$. From Eqs. (26), (29), and (35) one sees that the generating function is given by

$$\Omega(z)=Z(a+1-z)/Z(a)\,. \tag{43}$$

The functional integral needed to calculate $Z(a)$ will be strongly peaked near the mean-field value Π_0 defined by

$$\frac{\delta F[\Pi_0]}{\delta\Pi_0^*}=0\,, \tag{44}$$

which gives

$$|\Pi_0|^2=-a/2b\,. \tag{45}$$

One then finds[3]

$$\Omega(z)\simeq[1+(1-z)/a]^{-1/4}\exp\{Y[(z-1)(-a/2b)+(z-1)^2/4b]\}\,. \tag{46}$$

Clearly by making $|a|$ and b large, one can come arbitrarily close to the Poisson distribution associated with a pure coherent state [Eq. (33)].

A detailed study of the models arising from the Ginzburg-Landau form for $F[\Pi]$ is given in I. The problem of performing the functional integrations reduces to the problem of solving the Schrödinger equation for the anharmonic oscillator. The point we wish to emphasize here is that by making appropriate choices for the parameters a, b, and c, this simple form for $F[\Pi]$ can describe pion field configurations ranging from chaotic to highly coherent. The present experimental data favor the range $a<0$ and $c\gg 1$. (See Ref. 3.) The $|\partial\Pi/\partial y|^2$ term in $F[\Pi]$ gives rise to nontrivial short-range correlations in rapidity. The Ginzburg-Landau form for $F[\Pi]$ can also be used to generate three-dimensional models in which the dependence on $\vec{q}_\perp$ is retained. In this case, terms which contain gradients, with respect to the transverse momentum or transverse impact parameter, will give rise to short-range correlations in $\vec{q}_\perp$.

Although a wide class of density operators can be written in diagonal form in the coherent-state representation, this is not always convenient or even possible. An example of a simple density operator that is most conveniently written in nondiagonal form is given in Sec. IV.

IV. A UNITARY MODEL

In the models that we have discussed so far the incident hadrons can be pictured as exciting the pion field via a single interaction. In fact for the diagonal density operators discussed in Sec. III, the inclusive cross sections have the same structure as a function of rapidity as one finds in the multi-Regge model.[3] Recently a rather different class of models [AABS models[4]] was presented for which the scattering operator satisfies exact s-channel unitarity. In these models the incident hadrons are pictured as propagating through the interaction region, without making significant fractional changes in their rapidities. However, in order to satisfy s-channel unitarity they must be allowed to interact an arbitrary number of times with pions being emitted and absorbed at each interaction. In the original version of these models each interaction corresponded to the exchange of a multiperipheral-like chain. In this section we shall use the coherent-state representation to discuss generalization of the AABS models.

In the AABS models the scattering operator is diagonal in the rapidity difference Y, and the relative impact parameter, $\vec{B}$, of the two incident hadrons. We write it in the form

$$S(Y,\vec{B})=\exp[i\chi(Y,\vec{B})]\,. \tag{47}$$

The Hermitian operator $\chi(Y,\vec{B})$ determines the amplitude for emitting or absorbing a given number of pions with each interaction. In general it is a functional of the pion creation and annihilation operators. If $\chi(Y,\vec{B})$ has a finite Hilbert-Schmidt norm[9] then it can always be written in the form[10]

$$\chi(Y,\vec{B})=\int\delta\Pi\,D(\Pi)[\chi(\Pi;Y,\vec{B})+\chi^*(-\Pi;Y,\vec{B})]\,, \tag{48}$$

where $D(\Pi)$ is the coherent-state displacement operator defined in Eq. (4). The Hermiticity of the operator $\chi(Y,\vec{B})$ follows from the fact that $D^\dagger(\Pi)=D(-\Pi)$.

The weight function $\chi(\Pi;Y,\vec{B})$ can be parameterized using techniques similar to those discussed in Sec. III. In particular it is not difficult to introduce short-range correlations in rapidity and transverse momentum among the pions produced in a particular interaction. However, in order to simplify our presentation we shall neglect such correlations and take $\chi(\Pi;Y,\vec{B})$ to be proportional to a functional δ function. We write

$$\chi(Y,\vec{B})=\tfrac{1}{2}\Lambda(\vec{B})[D(\Pi_0)+D(-\Pi_0)]\,. \tag{49}$$

The only requirements that we shall make on the function $\Pi_0(y_1\vec{q}_\perp)$ are that it be nonzero only in the interval $0\leq y\leq Y$ and that inside this interval it be slowly varying enough so that

$$\int_0^Y dy\int d^2q_\perp|\Pi_0(y,\vec{q}_\perp)|^2=\lambda Y\,, \tag{50}$$

where λ goes to a constant at high energies. In principle Λ could also depend on Y. With our choice, the Born approximation to the elastic scattering amplitude is

$$2e^{Y}\int d^2B\langle 0|\chi(Y,B)|0\rangle = 2e^{Y(1-\lambda/2)}\int d^2B\,\Lambda(\vec{B}), \tag{51}$$

which corresponds to a fixed pole in the angular momentum plane at $l=1-\lambda$. Our choice seems to be the most natural since we are neglecting the correlation in the transverse momenta necessary to generate a moving pole.[4] One could generalize the model by allowing the incident hadrons to be diffractively excited.[11] The only change would be that Λ would become a matrix.

For any two functions Π_1 and Π_2, one easily sees that

$$D(\Pi_1)D(\Pi_2)=D(\Pi_1+\Pi_2)e^{(1/2)\int dq[\Pi_2^*\Pi_1-\Pi_1^*\Pi_2]}. \tag{52}$$

As a result,

$$[D(\Pi_0), D(-\Pi_0)]=0 \tag{53}$$

and

$$D(\Pi_0)^n D(-\Pi_0)^m = D((n-m)\Pi_0). \tag{54}$$

The scattering operator therefore has the simple series expansion

$$S(Y,\vec{B})=\sum_{n,m=0}^{\infty}\frac{(i\Lambda/2)^{n+m}}{n!\,m!}D((n-m)\Pi_0). \tag{55}$$

Since the two incident hadrons are assumed not to be pions, in order to construct the elastic scattering amplitude we need the matrix element of S in the state with no pions:

$$\langle 0|S(Y,\vec{b})|0\rangle=\sum_{n,m}\frac{(i\Lambda/2)^{n+m}}{n!\,m!}e^{-(1/2)(n-m)^2\lambda Y}$$
$$\simeq J_0(\Lambda). \tag{56}$$

In the last step in Eq. (56) we have retained only the terms of leading power in $s\sim e^{Y}$. The total cross section is given by

$$\sigma=\int d^2B\,\mathrm{Im}\{2i[1-\langle 0|S(Y,\vec{B})|0\rangle]\}$$
$$\simeq 2\int d^2B[1-J_0(\Lambda(\vec{B}))], \tag{57}$$

and the elastic cross section by

$$\sigma_{el}=\int d^2B|1-\langle 0|S(Y,\vec{B})|0\rangle|^2$$
$$\simeq\int d^2B|1-J_0(\Lambda(\vec{B}))|^2. \tag{58}$$

Clearly the total and elastic cross section go to constants at high energy for any value of the parameters λ and Λ.

In our present normalization the density operator is given by

$$\rho(Y,\vec{B})=\frac{1}{\sigma}[1-e^{i\chi(Y,\vec{B})}]|0\rangle\langle 0|[1-e^{-i\chi(Y,\vec{B})}]. \tag{59}$$

Notice that in this class of models the density operator is always separable in the impact-parameter representation. The inclusive cross section for the production of l pions is given by

$$I(q_1,\ldots,q_l)=\frac{1}{\sigma}\int d^2B\langle 0||e^{-i\chi(Y,\vec{B})}a^\dagger(q_1)\cdots a^\dagger(q_l)a(q_1)\cdots a(q_l)\,e^{i\chi(Y,\vec{B})}|0\rangle$$
$$=\frac{1}{\sigma}\prod_{i=1}^{l}|\Pi_0(q_i)|^2\int d^2B\sum_{\substack{n,m\\n',m'}}\frac{(i\Lambda/2)^{n+m}}{n!\,m!}\,\frac{(-i\Lambda/2)^{n'+m'}}{n'!\,m'!}\,[(n-m)(n'-m')]^l e^{-(1/2)[(n-m)-(n'-m')]^2\lambda Y}. \tag{60}$$

Retaining the terms of leading power in s gives

$$I(q_1,\ldots,q_n)\simeq\prod_{i=1}^{l}|\Pi_0(q_i)|^2\,2C(l)/\sigma, \tag{61}$$

where

$$C(l)=\sum_{p=1}^{\infty}p^{2l}\int d^2B\,J_p^{\,2}(\Lambda(\vec{B})). \tag{62}$$

Thus, the inclusive cross sections approach energy-independent limits at high energies. Although we have omitted the short-range correlations among the pions associated with a single χ, the correlation functions do not vanish. For example,

$$C_2(q_1,q_2)=\left[\frac{I(q_1,q_2)}{I(q_1)I(q_2)}-1\right]$$
$$=[\sigma C(2)/C(1)^2-1], \tag{63}$$

which is independent of the q_i. This long-range correlation arises from interference effects among pions associated with different χ's. This type of long-range correlation has been discussed previously using the more familiar S-matrix Regge-pole language.[12]

Since the only particles in this model besides the pions are the two incident hadrons which are neither created nor destroyed the semi-inclusive cross sections are identical to the exclusive cross sections. We have

$$S(q_1,\ldots,q_l)=\frac{1}{\sigma}\int d^2B\,|\langle q_1,\ldots,q_l|(1-e^{i\chi(Y,\vec{B})})|0\rangle|^2$$
$$=\frac{1}{\sigma}\prod_{i=1}^{l}|\Pi_0(q_i)|^2\sum_{n,m}\int d^2B\frac{(i\Lambda/2)^{n+m}}{n!\,m!}(n-m)^l e^{-(1/2)(n-m)^2\lambda Y^2}\,. \tag{64}$$

Notice that all the exclusive cross sections except the elastic go to zero at high energy like a power of the energy.

The generating function takes on an interesting form in this model. From Eqs. (29), (50), and (61) we see that

$$\Omega(z)=1+\sum_{n=1}^{\infty}\frac{(z-1)^n}{n!}(\lambda Y)^n\,2C(n)/\sigma\,. \tag{65}$$

Using the identity

$$1=J_0^2(\Lambda)+2\sum_{p=1}^{\infty}J_p^2(\Lambda)\,, \tag{66}$$

we see that $\Omega(z)$ can be written in the form

$$\Omega(z)=\frac{\sigma_{el}}{\sigma}+\frac{2}{\sigma}\sum_{p=1}^{\infty}e^{(z-1)\lambda p^2 Y}\int d^2B\,J_p^2(\Lambda(B))\,. \tag{67}$$

This generating function has a form recently suggested by several authors[5]; namely, it contains a diffractive component σ_{el}/σ plus a sum of multiperipheral-like components. However, it should be emphasized that in this model the diffractive component in $\Omega(z)$ has not been put in by hand. It arises naturally from interference effects among pions produced in different independent interactions between the incident hadrons. (Multiperipheralists may wish to think of these pions as being produced on different multiperipheral chains.) Thus the constant elastic cross section really does arise from the shadow cast by the inelastic channels. Notice that if Λ is small, all of the pions are produced in a single interaction. However, the important terms in the elastic amplitude are those proportional to χ^2. This is natural since it is only through terms quadratic or greater in χ that the inelastic channels can affect the elastic amplitude.

To summarize, we have constructed a simple unitary model in which the total, elastic, and inclusive cross sections approach constants at high energy, while the exclusive cross sections fall like a power of the energy. These results should be contrasted with those obtained in the solvable model of AABS,[4] which is very similar in spirit to the present model. In the AABS model χ was chosen to have the general features of the multiperipheral model. In the present notation χ_{AABS} can be written

$$\chi_{AABS}=e^{-(\alpha-1-\lambda/2)Y}f(\vec{B})\,e^{\int dq[\Pi_0(q)a^\dagger(q)+\Pi_0^*(q)a(q)]}\,. \tag{68}$$

Here α is the position of the input pole exchanged along the multiperipheral chain and λ is defined in Eq. (50). The principal difference between χ_{AABS} and the χ defined in Eq. (49) is the plus sign before $\Pi_0^*(q)a(q)$, which makes χ_{AABS} an unbounded operator. Notice that when χ_{AABS} is applied to the pion vacuum it gives rise to a coherent state proportional to $|\Pi_0\rangle$. The contributions to the elastic scattering amplitude proportional to χ_{AABS} and $\chi_{AABS}{}^2$ have the energy dependence $e^{\alpha Y}$ and $e^{(2\alpha-1+\lambda)Y}$, respectively,[4] which is of course what one would expect from the multiperipheral model in the weak coupling limit. In the present model, the corresponding terms have the energy dependence $e^{(1-\lambda/2)Y}$ and e^Y. So, as was mentioned before, the total and elastic cross sections go to constants at high energy independent of the value of λ, which is the effective pion coupling constant. On the other hand, the input pole, which controls the energy dependence of the exclusive cross sections [see Eq. (64)], is located at $l=1-\frac{1}{2}\lambda$. This is an example of the dependence of cross sections on coupling constants recently conjectured by Harari.[13]

In the solvable AABS model the total cross section always goes to zero at high energies for $\alpha\leq 1$ because any dynamical pole that approaches $l=1$ is washed out by rapid oscillations of the S matrix. These oscillations are associated with large eigenvalues of χ_{AABS}, which is an unbounded operator. In the present model χ is a bounded operator and no such oscillations occur.

The model that we have been discussing can be made to look even more like the AABS model by replacing $\Lambda(\vec{B})$ by the quantity $\Lambda(\vec{B})e^{(\alpha-1+\lambda/2)Y}$. Then expanding the elastic amplitude in powers of χ, one finds that the first terms have energy dependence $e^{\alpha Y}$ and the Nth term $e^{Y[1+N(\alpha-1+\lambda/2)]}$ for $N\geq 2$. For $\alpha<1-\frac{1}{2}\lambda$ the total cross section goes to zero at high energies, for $\alpha>1-\frac{1}{2}\lambda$ it goes like Y^2 and the Froissart bound is saturated, and for $\alpha=1-\frac{1}{2}\lambda$ we regain the model we have discussed in detail. This generalized model provides an example of the saturation of multichain forces discussed in the second and third papers of Ref. 4.

V. ISOTOPIC SPIN

In order to present our ideas in the simplest possible setting, we have ignored internal quantum numbers in the preceding sections. We now turn to the problem of finding a set of states suitable for describing physical pions.

The creation and annihilation operators for the pions are vectors in isotopic-spin space and will be written in the form

$$\vec{a}(q)=(a_1(q), a_2(q), a_3(q)). \tag{69}$$

The creation operators for the physical pions are

$$a_+^\dagger(q)=\frac{-1}{\sqrt{2}}[a_1^\dagger(q)+i\,a_2^\dagger(q)],$$
$$a_-^\dagger(q)=\frac{1}{\sqrt{2}}[a_1^\dagger(q)-i\,a_2^\dagger(q)], \tag{70}$$
$$a_0^\dagger(q)=a_3^\dagger(q).$$

The subscripts on the creation operators on the left-hand side of Eq. (70) stand for the charges of the pions.

The eigenfunctions of the annihilation operators can be written in the form

$$\vec{a}(q)|\vec{\Pi}\rangle=\vec{\Pi}(q)|\vec{\Pi}\rangle, \tag{71}$$

where

$$|\vec{\Pi}\rangle=e^{-(1/2)\int dq|\vec{\Pi}(q)|^2}e^{\int dq\,\vec{\Pi}(q)\cdot\vec{a}^\dagger(q)}|0\rangle, \tag{72}$$

and

$$\vec{\Pi}(q)=(\Pi_1(q), \Pi_2(q), \Pi_3(q)) \tag{73}$$

is an isospin vector whose components $\Pi_i(q)$ are, in general, complex. The states $|\vec{\Pi}\rangle$ are not likely to be a useful set of basis states for describing pion production since they are not eigenstates of the total charge or the total isotopic spin.

Let us begin by constructing a set of states of definite charge and isospin in which all pions have the momentum-space wave function $\Pi(q)$. We write the vector $\vec{\Pi}$ in the form

$$\vec{\Pi}(q)=\Pi(q)\hat{n}, \tag{74}$$

where

$$\hat{n}=(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta) \tag{75}$$

gives the direction of $\vec{\Pi}$ in isotopic-spin space. The required state with total isotopic spin I and z component of isotopic spin I_z is

$$|\Pi, I, I_z\rangle=\int d\Omega_{\hat{n}}\, Y^*_{I,I_z}(\hat{n})e^{-(1/2)\int dq|\vec{\Pi}|^2}e^{\int dq\,\vec{\Pi}\cdot\vec{a}^\dagger}|0\rangle. \tag{76}$$

$Y_{I,I_z}(\hat{n})=Y_{I,I_z}(\theta,\varphi)$ is the familiar spherical harmonic of angular momentum I and z component I_z. Notice that $|\Pi, I, I_z\rangle$ is not an eigenstate of the destruction operator $\vec{a}(q)$. That $|\Pi, I, I_z\rangle$ is indeed a state of isotopic spin I and z component I_z can be seen by recalling that under a rotation R in isospin space,

$$O_R a_i^\dagger O_R^{-1}=\sum_j R_{ij}a_j^\dagger, \tag{77}$$

so

$$\begin{aligned} O_R|\Pi, I, I_z\rangle &= \int d\Omega_{\hat{n}}\, Y^*_{I,I_z}(\hat{n}) \times e^{-(1/2)\int dq|\vec{\Pi}|^2}e^{\int dq\,\vec{\Pi}\cdot R\vec{a}^\dagger}|0\rangle \\ &= \int d\Omega_{\hat{n}}\, Y^*_{I,I_z}(R\hat{n}') \times e^{-(1/2)\int dq|\vec{\Pi}|^2}e^{\int dq\,\vec{\Pi}'\cdot\vec{a}^\dagger}|0\rangle \\ &= \sum_{I_z'} D^I_{I_z',I_z}(R)|\Pi, I, I_z'\rangle, \end{aligned} \tag{78}$$

where $\hat{n}'=R^{-1}\hat{n}$ and $D^I(R)$ is the Wigner D matrix.

In order to understand the properties of these states let us consider the state with isospin zero in some detail. The states defined in Eq. (75) are not normalized to unity. In particular

$$\begin{aligned} \langle\Pi, 0, 0|\Pi, 0, 0\rangle &= e^{-c}\int d\Omega_{\hat{n}}d\Omega_{\hat{n}'}e^{c\hat{n}\cdot\hat{n}'} \\ &= (4\pi)^2 j_0(-ic)e^{-c}\equiv N, \end{aligned} \tag{79}$$

where

$$c=\int dq|\Pi(q)|^2, \tag{80}$$

and j_0 is the spherical Bessel function of order zero. The probability amplitude for finding n pions with momenta $q_1, \ldots, q_n$ and isospin indices $i_1, \ldots, i_n$ in the state $|\Pi, 0, 0\rangle$ is

$$\begin{aligned} &\langle 0|a_{i_1}(q_1)\cdots a_{i_n}(q_n)|\Pi, 0, 0\rangle N^{-1/2} \\ &\quad = N^{-1/2}e^{-(1/2)c}\Pi(q_1)\cdots\Pi(q_n)\int d\Omega_{\hat{n}}\hat{n}_{i_1}\cdots\hat{n}_{i_n}. \end{aligned} \tag{81}$$

This amplitude clearly vanishes unless the pions are in a state of total isotopic spin zero. The probability of finding n_+ π_+'s, n_- π_-'s, and n_0 π_0's is

$$P(n_+,n_-,n_0) = \frac{N^{-1}e^{-c}c^{n_++n_-+n_0}}{n_+!\,n_-!\,n_0!\,2^{n_++n_-}} \times \left| \int d\Omega_{\hat n}\, e^{i(n_+-n_-)\varphi}(\sin\theta)^{n_++n_-}(\cos\theta)^{n_0}\right|^2 . \tag{82}$$

P obviously vanishes unless $n_+=n_-\equiv\frac{1}{2}n_c$, and n_0 is even. Under these circumstances

$$P(\tfrac{1}{2}n_c,\tfrac{1}{2}n_c,n_0)=\frac{(2\pi)^2N^{-1}c^{n_c+n_0}e^{-c}}{n_0!\,2^{n_c}} \times\left[\frac{\Gamma(\frac{1}{2}n_0+\frac{1}{2})}{\Gamma(\frac{1}{2}n_c+\frac{1}{2}n_0+\frac{3}{2})}\right]^2 . \tag{83}$$

Finally, the average number of particles is given by

$$\begin{aligned}\bar n &= N^{-1}\langle \Pi,0,0|\sum_i\int dq\, a_i^\dagger(q)a_i(q)|\Pi,0,0\rangle \\ &= N^{-1}e^{-c}c\int d\Omega_{\hat n}\,d\Omega_{\hat n'}\,\hat n\cdot\hat n'\,e^{c\hat n\cdot\hat n'} \\ &= icj_1(-ic)/j_0(-ic)=c\coth(c)-1\end{aligned} \tag{84}$$

and

$$\bar n_+=\bar n_-=\bar n_0=\tfrac{1}{3}\bar n . \tag{85}$$

The states $|\Pi,I,I_z\rangle$ do not form a complete set. The difficulty is that in general the real and imaginary parts of the vector $\vec\Pi$ need not point in the same direction in isotopic-spin space. Consider a particular normal-mode function $f_k(q)$. The most general $\vec\Pi$ that can be constructed from it can be written in the form

$$\vec\Pi_k(q)=f_k(q)[\mathrm{Re}\alpha_k\,\hat n_R+i\,\mathrm{Im}\alpha_k\,\hat n_I] , \tag{86}$$

where $\hat n_R$ and $\hat n_I$ are two independent unit vectors and α_k is any complex number. A complete set of states for the kth normal mode is

$$\begin{aligned}|f_k;I,I_z;I_R,I_I\rangle &= \sum_m \langle I_R,m;I_I,I_z-m|I,I_z;I_R,I_I\rangle \\ &\times\int d\Omega_{\hat n_R}\,d\Omega_{\hat n_I} \\ &\times Y_{I_R,m}(\hat n_R)^*\,Y_{I_I,I_z-m}(\hat n_I)^* \\ &\times e^{-\int dq|\vec\Pi_k|^2}e^{\int dq\,\vec\Pi_k\cdot\vec a^\dagger}|0\rangle ;\end{aligned} \tag{87}$$

$\langle I_R,m;I_I,I_z-m|I,I_z;I_R,I_I\rangle$ is the standard Clebsch-Gordan coefficient. One can check, using the techniques of Eq. (15), that the states obtained by taking direct products over all normal modes are complete. Since the problem of coupling the isospins of the various normal modes to form a definite total isotopic spin is formidable, this approach is likely to be useful only if the dynamics is such that only a few modes enter or that the isospin in each mode is zero.

In the states that we have just described, the isotopic spin is a global property. Pions which are well separated in rapidity and transverse momentum are correlated to the extent that their individual isotopic spins are coupled to form a definite total isotopic spin. In some models, such as the multi-Regge model, one assumes that there are significant correlations only among particles with small relative rapidities. It is not difficult to construct states of definite isotopic spin with this property. Let us consider an example. We write the vector $\vec\Pi$ in the form

$$\vec\Pi(y)=\Pi(y)\hat n(y) . \tag{88}$$

The dependence on the transverse momentum has been suppressed for simplicity. As usual we restrict y to the range $0\le y\le Y$.

Consider the state

$$|\Pi;I_0,I_{0z};I_Y,I_{Yz}\rangle = \int\delta\Omega\left\{\exp\int_0^Y dy\left[-c\left(\frac{d\hat n}{dy}\right)^2+\vec\Pi(y)\cdot\vec a^\dagger(y)\right]\right\}|0\rangle . \tag{89}$$

$\int\delta\Omega$ indicates a functional integration over all possible forms for the unit vector $\hat n(y)$. (I_0,I_{0z}) is the isospin transferred to the pion field at rapidity 0 and (I_Y,I_{Yz}) is the isospin transferred from the pion field at rapidity Y. These quantities determine the boundary conditions on $\hat n(y)$. In order to understand the content of this state let us consider the probability amplitude for finding two pions with rapidities y_1 and y_2 and isospin indices i and j:

$$\begin{aligned}A_{ij}(y_1,y_2) &= N^{-1/2}\langle 0|a_i(y_1)a_j(y_2)|\Pi;I_0,I_{0z};I_Y,I_{Yz}\rangle \\ &= N^{-1/2}\Pi(y_1)\Pi(y_2)\int\delta\Omega\,\hat n_i(y_1)\hat n_j(y_2) \\ &\qquad\times\exp\left[-c\int_0^Y dy\left(\frac{d\hat n}{dy}\right)^2\right], \\ &\equiv N^{-1/2}\Pi(y_1)\Pi(y_2)J_{ij} ,\end{aligned} \tag{90}$$

with

$$N=\langle \Pi; I_0, I_{0z}; I_Y, I_{Yz} | \Pi; I_0, I_{0z}; I_Y, I_{Yz}\rangle . \quad (91)$$

It is convenient to break up the region $0\leq y\leq Y$ into N equal intervals of length Δy so that $\Delta y N = Y$. Then

$$J_{ij}=\lim_{\Delta y\to 0}(c/\Delta y \pi)^N \int \prod_{l=0}^{N} d\Omega_l\, \hat{n}_i(l_1)\hat{n}_j(l_2) \times Y^*_{I_0,I_{0z}}(\hat{n}(0))\, Y^*_{I_Y,I_{Yz}}(\hat{n}(N)) \times \exp\left[-c\sum_{l=0}^{N-1}[\hat{n}(l+1)-\hat{n}(l)]^2/\Delta y\right], \quad (92)$$

where

$$\hat{n}(l)\equiv \hat{n}(l\,\Delta y), \quad l=0,1,2,\ldots,N$$

and

$$\hat{n}(l_1)=\hat{n}(y_1), \quad \hat{n}(l_2)=\hat{n}(y_2). \quad (93)$$

The factor $(c/\Delta y \pi)^N$ defines the measure of the functional integration. The spherical harmonics $Y_{I_0,I_{0z}}$ and $Y_{I_Y,I_{Yz}}$ appear in Eq. (92) because we have insisted that a definite isospin be transferred at the boundaries $y=0, Y$. More generally one could replace them by arbitrary functions of $\hat{n}(0)$ and $\hat{n}(N)$. The forms of these functions would then determine the probability of a given isospin being transferred at the boundary.

The general integral which must be performed to evaluate J_{ij} is

$$J=(c/\Delta y \pi)\int d\Omega_l e^{-c[\hat{n}(l+1)-\hat{n}(l)]^2/\Delta y}\psi(\hat{n}(l)) = (4c/\Delta y)\sum_{I,I_z} e^{-2c/\Delta y} j_I(-2ic/\Delta y) i^I\, Y_{I,I_z}(\hat{n}(l+1)) \times \int d\Omega_l\, Y^*_{I,I_z}(\hat{n}(l))\psi(\hat{n}(l)). \quad (94)$$

Using the asymptotic behavior of the spherical Bessel functions we find that to first order in Δy,

$$J\simeq \sum_{I,I_z}[1-I(I+1)\Delta y/4c]\, Y_{I,I_z}(\hat{n}(l+1)) \times \int d\Omega_l Y^*_{I,I_z}(\hat{n}(l))\psi(\hat{n}(l)) = [1-\vec{I}^2\Delta y/4c]\,\psi(\hat{n}(l+1)) \simeq e^{-\vec{I}^2\Delta y/4c}\psi(\hat{n}(l+1)), \quad (95)$$

where $\vec{I}$ is the isotopic-spin operator. Using this result in Eq. (92) one finds for $0<y_1<y_2<Y$

$$J_{ij}=\sum_{I,I_z} e^{-y_1 I_0(I_0+1)/4c} g^i_{I_0,I_{0z};I,I_z} \times e^{-(y_2-y_1)I(I+1)/4c} g^j_{I,I_z;I_Y,I_{Yz}} \times e^{-(Y-y_2)I_Y(I_Y+1)/4c}, \quad (96)$$

where

$$g^i_{I,I_z;I',I'_z}=\int d\Omega\, Y_{I,I_z}(\hat{n})\hat{n}_i\, Y_{I',I'_z}(\hat{n}). \quad (97)$$

In particular for $I_0=I_Y=0$,

$$J_{ij}=\delta_{ij}(4\pi/3)e^{-|y_2-y_1|/2c}. \quad (98)$$

Notice that for $I_0=I_Y=0$ the state $|\Pi; 0, 0; 0, 0\rangle$ has total isotopic spin zero.

The probability amplitude for finding N pions can be obtained in an analogous manner. The isospin structure is the same as in the multi-Regge model when there is only one Regge trajectory of each isospin. Secondary trajectories for each isospin will arise naturally when we introduce functional integrations over the $\Pi(y)$ as discussed in Sec. III.

VI. DISCUSSION

In this work we have constructed a general framework for describing high-energy pion production. We have seen that all of the pion cross sections can be obtained from a pion field density operator. For a wide range of configurations of the pion field, this operator can be simply expressed in the coherent-state representation.

We have considered a variety of correlations among the pions. In Sec. III we showed how to introduce short-range correlations in the rapidity and transverse momentum. In Sec. IV we presented a simple model that satisfied exact s-channel unitarity, and so how the constraints of unitarity led to long-range correlations. Finally in Sec. V we studied the correlations associated with isospin conservation. We believe that all of these correlations can and should be taken into account in a realistic theory of pion production.

In Sec. III we pointed out that the density operator could often be written in diagonal form in the coherent-state representation, and that when it could, the various pion cross section took on a particularly simple form. If one then uses the Ginsburg-Landau parameterization for $F[\Pi]$ or a more general form that leads only to short-range correlations, then the pion field can be thought of as being excited by a single interaction of the incident hadrons. As we saw in Sec. IV, multiple independent interactions lead to long-range correlations. When only short-range correlations are present, it can be shown from Eq. (29) that $\Omega(z)$ falls like a power of the energy for z near zero. Crudely speaking this follows from the fact that $\int dq|\Pi(q)|^2$ is typically of order Y. As a result, the elastic and total cross sections in such a model will always go to zero at high energies. In Sec. IV we considered a model in which the elastic and total cross sections do go to constants. In this model if one retains only the terms that contribute to leading power in s, then the density operator can be written in diagonal form. The present problem is avoided because the weight functional $e^{-F[\Pi]}$ has a singularity at $\Pi = 0$ which has the form of a functional δ function. In more general unitary models in which the incident hadrons can be diffractively excited to states which subsequently decay into pions,[11] we would expect this functional δ function and its derivatives to be present in $e^{-F[\Pi]}$.

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