PHYSICAL REVIEW D VOLUME 9, NUMBER 3 1 FEBRUARY 1974

Self-consistent model for diffractive scattering *

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(Received 20 August 1973)

Crossed-channel unitarity is imposed on a Regge eikonal model. An integral equation is thereby found for the elastic π-π scattering amplitude. The simplest solution is the formation of a fixed singularity at $J=1$, which yields behavior identical to the droplet model of Chou, Wu, and Yang. Integral equations for the residue function of the fixed pole are presented. The possibility of other solutions is discussed.

I. INTRODUCTION

The question of the asymptotic energy dependence of the elastic scattering amplitudes in strong-interaction physics is still a puzzling and very open one. Experimentally, the total p-p cross section seems constant up to logarithmic factors.[1] The differential cross sections seem to be shrinking slowly with increasing energy. For t greater than 0.25 GeV^2 they are quite independent of energy. Numerous models have been proposed to explain this asymptotic behavior, but none of these has won over a majority of particle physicists. We wish to present here another such model even though the market is already flooded.

Direct (s-) channel unitarity certainly plays a dominant role in diffractive scattering. This constraint is conveniently handled in the eikonal formalism, where the Froissart bound is easily satisfied. In most eikonal models no attempt is made to appease the branch point at $t=4\mu^2$. Crossed (t-) channel unitarity must also be satisfied by the elastic amplitude, but it is not clear how important this is in determining the asymptotic behavior. Simple multiperipheral models satisfy elastic t-channel unitarity exactly: however, they do not satisfy s-channel unitarity. In this paper we will try to formulate a model which incorporates both s- and t-channel unitarity constraints.

The model we propose is essentially a marriage between the t-channel Bethe-Salpeter equation and the s-channel eikonal approximation. In many models the Born term of the eikonal expansion is taken to be a set of ladder or ladderlike Feynman graphs. This is the case in the massive quantum electrodynamics calculation of Cheng and Wu[2] and in the simpler ϕ^3 calculation of Chang and Yan.[3] It is not known which Feynman graphs are the correct ones for describing the Born term of the eikonal series. We postulate here, as in the above models, that the Born term is some set of crossed-channel two-particle reducible graphs. Since pions are the lightest hadrons, we include only two-pion reducible graphs in the Born term. It follows from the nature of the eikonal iteration that all the other terms of the amplitude do not contribute to two-pion cuts, and thus they are two-pion irreducible. We also assume that this Born term is produced multiperipherally.

We will presume in what follows that the Froissart bound is satisfied off the mass shell and that elastic pion amplitudes are strongly damped off the pion mass shell (Gribov finite-mass hypothesis).

Our model then consists of the following propositions:

1. The full elastic amplitude asymptotically is of the eikonal form. The Born term of this series we assume to be some set of t-channel, two-pion reducible Feynman diagrams.
2. Because of the nature of the eikonal iteration, all other Feynman diagrams in our model besides the Born term must be two-pion irreducible in the t channel.
3. Therefore the Born term in our model must be the full amplitude, which is two-pion reducible in the crossed channel asymptotically.
4. We assume that this Born term is produced multiperipherally. The precise meaning of this will be made clear later in the paper.
5. We demand that the full amplitude satisfy the Bethe-Salpeter equation. The meaning of this will also be made clear.
6. We assume that the full asymptotic amplitude is strongly damped off the mass shell, so that in the Bethe-Salpeter equation, one need not worry about the masses of the virtual pions becoming very large.
7. We assume that the Froissart bound is satisfied by the full off-shell amplitude, with two particles on shell and two with negative (mass).[2]

We have not been able to find a set of Feynman graphs which satisfies these conditions, but if they are true we can derive a self-consistent integral equation involving only the full elastic π-π am-

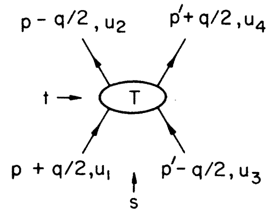

FIG. 1. Definition of variables. Here $s=(p+p')^2$, $t=q^2$, $u_1=(p+\frac{1}{2}q)^2$, $u_2=(p-\frac{1}{2}q)^2$, $u_3=(p'-\frac{1}{2}q)^2$, $u_4=(p'+\frac{1}{2}q)^2$.

plitude. This equation is the main point of this paper. We have found only one nontrivial solution, and this is a fixed pole in the angular momentum plane at $J=1$. The possibility of other solutions will be discussed. Although we consider a field-theory framework with only neutral pseudoscalar mesons, we expect to find the same fixed-pole solution in cases involving charge, spin, and coupled channels.

II. THE BETHE-SALPETER EQUATION

The normalization of our amplitude is defined by the relation

$$\operatorname{Im} T(s,t,\mu^2,\mu^2,\mu^2,\mu^2)=2|\vec{k}|\sqrt{s}\,\sigma_{\text{tot}}\,, \tag{2.1}$$

where $T(s,t,u_1,u_2,u_3,u_4)=T(q,p,p')$ is the scattering amplitude for two off-shell pseudoscalar mesons (see Fig. 1) and

$$s=(p+p')^2,\quad t=q^2,\quad u_1=(p+\tfrac{1}{2}q)^2,$$
$$u_2=(p-\tfrac{1}{2}q)^2\,,\quad u_3=(p'-\tfrac{1}{2}q)^2,\quad u_4=(p'+\tfrac{1}{2}q)^2, \tag{2.2}$$

μ = pion mass , $\vec{k}$ = c.m. momentum of one pion.

We define $I(s,t,u_1,u_2,u_3,u_4)=I(q,p,p')$ to be the full amplitude that is two-pion irreducible in the t channel. This means that $I(q,p,p')$ is the sum of all Feynman graphs, which give no contribution to the discontinuity of that cut along the real t axis which begins at the normal threshold branch point

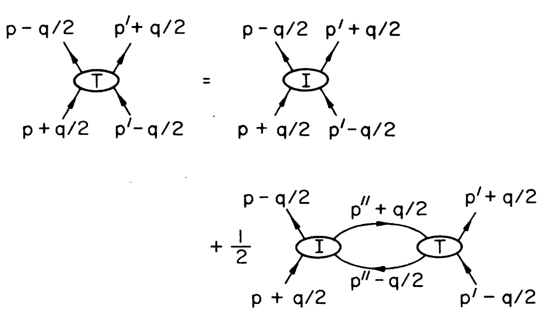

FIG. 2. The Bethe-Salpeter equation.

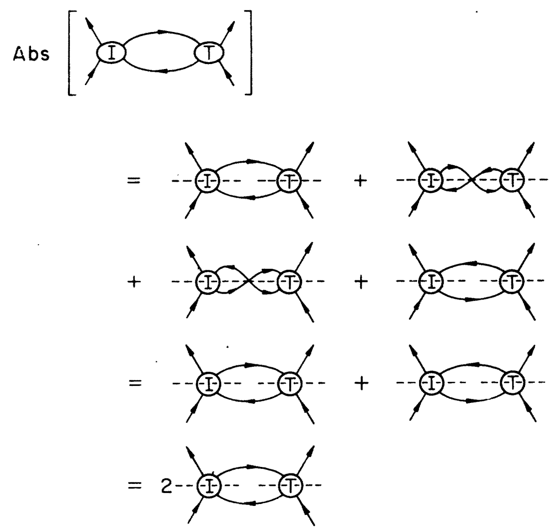

FIG. 3. Cutkosky rules for taking the s-channel absorptive part of a t-channel iteration of crossing-invariant objects.

at $t=4\mu^2$. Likewise we define $T_2(s,t,u_1,u_2,u_3,u_4)=T_2(q,p,p')$ to be the full amplitude that is two-particle reducible in that channel (i.e., it is the sum of all graphs which do contribute to this cut). Both T_2 and I are crossing symmetric in the s-u sense.

T is related to I by the Bethe-Salpeter equation, which has been derived in an axiomatic framework[4]:

$$T(q,p,p')=I(q,p,p') - \tfrac{1}{2}i\int\frac{d^4p''}{(2\pi)^4}\,I(q,p,p'')T(q,-p'',p') \times\Delta(p''-\tfrac{1}{2}q)\Delta(p''+\tfrac{1}{2}q)\,, \tag{2.3}$$

where $\Delta(p)$ is the full renormalized pion propagator. This equation is illustrated graphically in Fig. 2. We will use Eq. (2.3), which is a statement of two-particle t-channel unitarity in the s-channel physical region. We are interested in the kinematic region

$$t<0,\ |t|\ll s,\ s\gg 1\,. \tag{2.4}$$

T, T_2, and I are related simply by

$$T(q,p,p')=I(q,p,p')+T_2(q,p,p')\,. \tag{2.5}$$

Let us now use the Cutkosky rules to take the absorptive part in s of Eq. (2.3). We assume that the external particles and the vacuum are stable and therefore neither of the exchanged pions in the integral of (2.3) can ever be put on shell. All particle lines which are put on shell in this process must be inside either I or T. This clearly causes the integrand to become a product of the absorptive parts of I and T in the subenergies. Both I and T are invariant under crossing, therefore each has a left-hand cut in s which contributes to the absorptive part of the integral.

If we let dashed lines through a graph represent only the right-hand discontinuity in the subenergy which is cut by the dashed line, then we can include the contribution from the left-hand cut by simply reversing the momenta of the two virtual pions. This is illustrated in Fig. 3. The two surviving terms are identical since the exchanged pions are integrated over all momenta and the two amplitudes are invariant under crossing. This scheme simply yields a factor of 2 times an integral over only the right-hand absorptive parts. This factor of 2 nicely cancels the factor of $\frac{1}{2}$ outside the integral in (2.3), yielding (see Fig. 4)

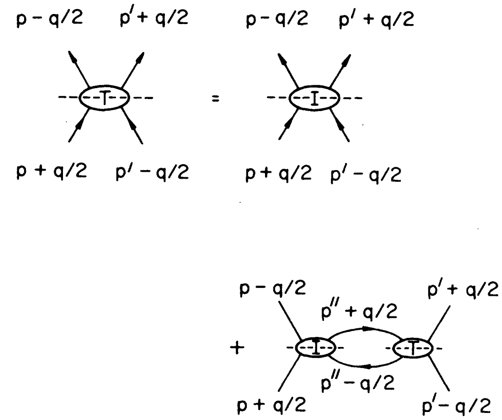

FIG. 4. s-channel absorptive part of Fig. 2, the ABFST equation.

$$\text{Abs}\,T(q,p,p')=\text{Abs}I(q,p,p')+\int\frac{d^4p''}{(2\pi)^4}\text{Abs}I(q,p,p'')\text{Abs}\,T(q,-p'',p')\,\Delta(p''-\tfrac{1}{2}q)\Delta(p''+\tfrac{1}{2}q)\,. \tag{2.6}$$

Here $\text{Abs}\,T=2\,\text{Im}\,T$ on shell in the s-channel physical region and the integration is only over positive subenergies $(p+p'')^2$ and $(p'-p'')^2$.

Equation (2.6) is the famous ABFST (Amati-Bertocchi-Fubini-Stanghellini-Tonin) equation[5] studied by them in the ladder approximation. Since we used a stability condition in deriving (2.6), we must restrict its use to values of the external masses below the first inelastic threshold [all external (masses)^2 less than $4\mu^2$ for scalar mesons and $9\mu^2$ for pseudoscalar mesons]. When the full irreducible kernel I is used, (2.6) is an exact equation, independent of the details of the interaction (I will of course be determined by this interaction).

For convenience we define the following operation for the various four-point functions that we will be working with:

$$a\times_t b=\int\frac{d^4p''}{(2\pi)^4}\,a(q,p,p'')b(q,-p'',p')\times\Delta(p''+\tfrac{1}{2}q)\Delta(p''-\tfrac{1}{2}q)\,, \tag{2.7}$$

where a and b are understood to have only right-hand support in the variables $(p+p'')^2$ and $(p'-p'')^2$, respectively, when they represent absorptive parts, and also

$$a\times_t a=a^2,\qquad a\times_t a^{n-1}=a^{n-1}\times_t a=a^n\,. \tag{2.8}$$

We will also use this notation for raising regular numbers to powers since no confusion is likely to arise. Then (2.3) and (2.6) become

$$T=I-\tfrac{1}{2}iI\times_t T\,, \tag{2.9}$$

$$\text{Abs}\,T=\text{Abs}I+\text{Abs}I\times_t\text{Abs}\,T\,. \tag{2.10}$$

These equations are solved by iteration yielding

$$T=2i\sum_{n=1}^{\infty}(-\tfrac{1}{2}i)^n(I)^n\,, \tag{2.11}$$

$$\text{Abs}\,T=\sum_{n=1}^{\infty}(\text{Abs}I)^n\,, \tag{2.12}$$

$$T_2=2i\sum_{n=2}^{\infty}(-\tfrac{1}{2}i)^n(I)^n\,, \tag{2.13}$$

$$\text{Abs}\,T_2=\sum_{n=2}^{\infty}(\text{Abs}I)^n\,. \tag{2.14}$$

It is our aim to avoid plugging in a particular model for I, although we will end up with a statement about I. Our assumptions enable us to eliminate I from the problem. To see how this comes about, let us introduce the following generating functions which prove to be very useful:

$$T(\lambda)=2i\sum_{n=1}^{\infty}\lambda^n(-\tfrac{1}{2}i)^n(I)^n\,, \tag{2.15}$$

$$T_2(\lambda)=2i\sum_{n=2}^{\infty}\lambda^n(-\tfrac{1}{2}i)^n(I)^n\,, \tag{2.16}$$

$$\text{Abs}\,T(\lambda)=\sum_{n=1}^{\infty}\lambda^n(\text{Abs}I)^n\,, \tag{2.17}$$

$$\text{Abs}\,T_2(\lambda)=\sum_{n=2}^{\infty}\lambda^n(\text{Abs}I)^n\,. \tag{2.18}$$

Varying λ, as we see it, corresponds to varying the strength of the basic t-channel force. It plays the same role here as the coupling constant in the

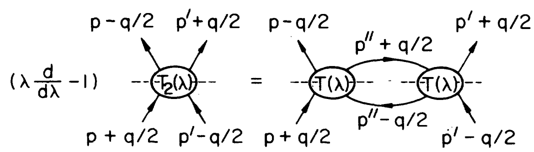

FIG. 5. The bilinear equation associated with Fig. 4.

ladder approximation to the Bethe-Salpeter equation. It might seem pointless at present to introduce these unphysical functions; however, they enable us to do some rather complex manipulations quite easily. The actual physical amplitudes are gotten by taking $\lambda = 1$ at the end of the calculation.

The functions (2.15)–(2.18) enable us to derive and work with the following relations,[6] which can be checked by substitution (see Fig. 5):

$$\left(\lambda \frac{\partial}{\partial\lambda} - 1\right) \frac{T_2(\lambda)}{2} = \left(\lambda \frac{\partial}{\partial\lambda} - 1\right) \frac{T(\lambda)}{2} = -\tfrac{1}{4} i\, T(\lambda) \times_t T(\lambda)\,, \qquad (2.19)$$

$$\left(\lambda \frac{\partial}{\partial\lambda} - 1\right) \mathrm{Abs}\, T_2(\lambda) = \left(\lambda \frac{\partial}{\partial\lambda} - 1\right) \mathrm{Abs}\, T(\lambda) = \mathrm{Abs}\, T(\lambda) \times_t \mathrm{Abs}\, T(\lambda)\,, \qquad (2.20)$$

$$\tfrac{1}{2} T(\lambda) = \tfrac{1}{2} T(\lambda = 1) + i \sum_{n=1}^{\infty} (\lambda - 1)^n [(-\tfrac{1}{2} i T(\lambda = 1))^n + (-\tfrac{1}{2} i T(\lambda = 1))^{n+1}]\,, \qquad (2.21)$$

$$\mathrm{Abs}\, T(\lambda) = \mathrm{Abs}\, T(\lambda = 1) + \sum_{n=1}^{\infty} (\lambda - 1)^n [(\mathrm{Abs}\, T(\lambda = 1))^n + (\mathrm{Abs}\, T(\lambda = 1))^{n+1}]\,. \qquad (2.22)$$

These relations are also exact, independent of the details of the interaction. For completeness, we write the integrals (2.19) and (2.20) out in full:

$$\left(\lambda \frac{\partial}{\partial\lambda} - 1\right) T_2(q,p,p';\lambda) = -\tfrac{1}{2} i \int \frac{d^4p''}{(2\pi)^4}\, T(q,p,p'';\lambda)\, T(q,-p'',p';\lambda)\, \Delta(p'' + \tfrac{1}{2}q)\Delta(p'' - \tfrac{1}{2}q)\,, \qquad (2.23)$$

$$\left(\lambda \frac{\partial}{\partial\lambda} - 1\right) \mathrm{Abs}\, T_2(q,p,p';\lambda) = \int \frac{d^4p''}{(2\pi)^4}\, \mathrm{Abs}\, T(q,p,p'';\lambda)\, \mathrm{Abs}\, T(q,-p'',p';\lambda)\, \Delta(p'' + \tfrac{1}{2}q)\Delta(p'' - \tfrac{1}{2}q)\,. \qquad (2.24)$$

Equation (2.24) has been derived and studied by ABFST in the multiperipheral approximation, where λ is simply the coupling constant squared, and in the forward direction $t = q^2 = 0$. We see that this equation is not at all limited to the multiperipheral case, but can be established in general. Of course λ no longer corresponds to a fundamental coupling constant in the general case. ABFST find, in their model, that far-off-mass-shell behavior in (2.24) is unimportant. With strong damping of the external masses, the asymptotic behavior of (2.24) can be gotten quite simply. We will show how this comes about, but first we state precisely what we mean by proposition 4 of the Introduction.

We have proposed that T_2 be produced multiperipherally. By this we mean exactly the following to leading order in s:

$$\mathrm{Abs}\, T_2(s,t,u_1,u_2,u_3,u_4;\lambda) = \beta_2(t,u_1,u_2,u_3,u_4;\lambda) s^{\alpha(t,\lambda)} + \text{nonleading terms}, \qquad (2.25)$$

$$\frac{\partial}{\partial\lambda}[\mathrm{Abs}\, T_2(s,t,u_1,u_2,u_3,u_4;\lambda)] = \frac{\partial}{\partial\lambda}[\beta_2(t,u_1,u_2,u_3,u_4;\lambda) s^{\alpha(t,\lambda)}] + \text{nonleading terms}\,, \qquad (2.26)$$

and also

$$\left.\frac{\partial\alpha(t,\lambda)}{\partial\lambda}\right|_{\lambda=1} \neq 0\,, \qquad (2.27)$$

except at possibly a discrete number of points. With these assumptions, we find to leading order in s

$$\left.\left(\lambda \frac{\partial}{\partial\lambda} - 1\right) \mathrm{Abs}\, T_2(q,p,p';\lambda)\right|_{\lambda=1} = \ln s \left.\frac{\partial\alpha(t,\lambda)}{\partial\lambda}\right|_{\lambda=1} \beta_2(t) s^{\alpha(t)}\,. \qquad (2.28)$$

Equation (2.24) thus becomes, to leading order in s,

$$\mathrm{Abs}\, T_2(q,p,p') = \left(\left.\frac{\partial\alpha(t,\lambda)}{\partial\lambda}\right|_{\lambda=1}\right)^{-1} \frac{1}{\ln s} \int \frac{d^4p''}{(2\pi)^4}\, \mathrm{Abs}\, T(q,p,p'')\mathrm{Abs}\, T(q,-p'',p')\, \Delta(p'' - \tfrac{1}{2}q)\Delta(p'' + \tfrac{1}{2}q)\,. \qquad (2.29)$$

We emphasize that the full amplitude Abs T is not necessarily produced multiperipherally; in fact this is not true in the cases we shall consider. This is because we do not rule out the possibility that the irreducible part Abs$I$ contributes asymptotically to the amplitude. This is the case in Refs. 2 and 3, where both Abs T_2 and Abs$I$ violate the Froissart bound separately, but cancel to leading order in s to yield a sum which satisfies the Froissart bound and is positive in the forward direction.

Since T_2 is produced multiperipherally, it just corresponds to simple Regge pole exchange. We assume that T_2 has positive signature under crossing as is always assumed for diffractive scattering. Then T_2 is given by

$$\begin{aligned} T_2(s,t) &= \frac{e^{i\pi\alpha(t)/2}}{2\sin(\pi\alpha(t)/2)} \operatorname{Abs} T_2(s,t) \\ &= \frac{e^{i\pi\alpha(t)/2}}{2\sin(\pi\alpha(t)/2)} \left(\frac{\partial\alpha(t,\lambda)}{\partial\lambda}\right)^{-1} \\ &\quad\times \frac{1}{\ln s}(\operatorname{Abs} T \times_t \operatorname{Abs} T)\,. \end{aligned} \tag{2.30}$$

In Sec. III we will take (2.30) as the Born term in an eikonal expansion for T.

III. THE EIKONAL EXPANSION

In this section we will assume that the full amplitude T is generated by the eikonal iteration of T_2 given by (2.30). We are striving for a self-consistent model, and therefore until we have a solution to our equations, we have no idea how good an assumption this is. We do not have a set of Feynman graphs to represent T_2, and so we cannot take the approach of simply adding up Feynman graphs to see whether eikonalization occurs. It will turn out that the solution we find to our equations for T_2 is known to eikonalize, and this will reinforce our assumption.

The eikonal form is well defined, independent of the nature of the Born term or eikonal function as it is sometimes called. It takes its simplest form in impact-parameter space, as, for example, in Ref. 3. Let us define the following Fourier transforms:

$$T(s,b) = \int \frac{d^2q}{(2\pi)^2} e^{i\vec{q}\cdot\vec{b}}\, T(s,-\vec{q}^{\,2})\,, \tag{3.1}$$

$$T_2(s,b) = \int \frac{d^2q}{(2\pi)^2} e^{i\vec{q}\cdot\vec{b}}\, T_2(s,-\vec{q}^{\,2})\,. \tag{3.2}$$

Our assumption is then

$$T(s,b) = 2is\left[1 - \exp\left(\frac{i}{2s}\, T_2(s,b)\right)\right]. \tag{3.3}$$

In momentum space this becomes

$$T(s,-\vec{q}^{\,2}) = 2is\sum_{n=1}^{\infty}\frac{(-1)}{n!}\int\left[\prod_{j=1}^{n}\frac{d^2q_j}{(2\pi)^2}\,\frac{i}{2s}\,T_2(s,-\vec{q}_j^{\,2})\right] \times (2\pi)^2\delta^2\left(\vec{q} - \sum_{i=1}^{n}\vec{q}_i\right). \tag{3.4}$$

Putting the result (2.30) into this equation yields

$$T(s,-\vec{q}^{\,2}) = 2is\sum_{n=1}^{\infty}\frac{(-1)}{n!}\int\left[\prod_{j=1}^{n}\frac{d^2q_j}{(2\pi)^2}\,\frac{i}{2s}\,\frac{e^{i\pi\alpha(-\vec{q}_j^{\,2})/2}}{2\sin(\pi\alpha(-\vec{q}_j^{\,2})/2)}\left(\frac{\partial\alpha(-\vec{q}_j^{\,2},\lambda)}{\partial\lambda}\bigg|_{\lambda=1}\right)^{-1} \times \frac{1}{\ln s}\left[\operatorname{Abs} T(-\vec{q}_j^{\,2}) \times_t \operatorname{Abs} T(-\vec{q}_j^{\,2})\right]\right](2\pi)^2\delta^2\left(\vec{q} - \sum_{i=1}^{n}\vec{q}_i\right). \tag{3.5}$$

Written out in full this is

$$\begin{aligned} T(s,-\vec{q}^{\,2}) = 2is\sum_{n=1}^{\infty}\frac{(-1)}{n!}\int\Bigg[\prod_{j=1}^{n}\frac{d^2q_j}{(2\pi)^2}\,\frac{i}{2s}\,\frac{e^{i\pi\alpha(-\vec{q}_j^{\,2})/2}}{2\sin(\pi\alpha(-\vec{q}_j^{\,2})/2)}\left(\frac{\partial\alpha(-\vec{q}_j^{\,2},\lambda)}{\partial\lambda}\bigg|_{\lambda=1}\right)^{-1} \\ \times\frac{1}{\ln s}\int\frac{d^4p''}{(2\pi)^4}\operatorname{Abs} T(q_j,p,p'')\operatorname{Abs} T(q_j,-p'',p')\Delta(p''-\tfrac{1}{2}q_j)\Delta(p''+\tfrac{1}{2}q_j)\Bigg] \\ \times(2\pi)^2\delta^2\left(\vec{q} - \sum_{i=1}^{n}\vec{q}_i\right), \end{aligned} \tag{3.6}$$

where $q_j = (0, \vec{q}_j, 0)$.

This equation involves only the full amplitude up to functions of t. Therefore it can be used to study the energy dependence of the elastic scattering amplitude. Both I and T_2 have dropped out of the picture. We thus have the simple and pleasing result that the elastic amplitude satisfies a self-consistent integral equation involving no other amplitudes. Unfortunately, as the reader can see, this equation is highly nonlinear and quite complicated. We are interested in the energy dependence of $T(s,t)$ in this paper. This is the first problem which must be tackled and also this is where the greatest interest is at present.

Equation (3.6), as we see it, has quite a lot in common with the model of Frautschi and Margolis.[7] In this model the Born term of the eikonal expansion is taken to be a Regge pole with slope $\alpha'(t=0) = 0.95\ \mathrm{GeV}^{-2}$. The output acts effectively as a Regge pole with $\alpha(t=0)=1$, but with a smaller slope. We think of this as one step towards the solution of (3.6). Our equation tells us that we should take the output of this calculation, iterate it once in the t channel, divide by $\ln s$, multiply by assorted functions of t, and re-eikonalize in the s channel to get a new output. We should go through these steps infinitely many times, and if this procedure converges, then we will have a solution to (3.6). Moreover, our solution can be expected to satisfy both the t-channel Bethe-Salpeter equation and s-channel unitarity. In Sec. IV we consider the ABFST integral Abs $T\times_t$ Abs T. We will then exhibit a solution to (3.6).

IV. THE ABFST INTEGRAL

We wish to study the integral on the right-hand side of (2.24). This problem has been essentially solved by ABFST in the forward direction $q^2=0$. In this section we generalize their results to the nonforward direction. For completeness we will quote some of their results. First, we make the following change of variables:

$$W = \int \frac{d^4p''}{(2\pi)^4} \operatorname{Abs} T(q,p,p'') \operatorname{Abs} T(q,-p'',p') \times \Delta((p''-\tfrac{1}{2}q)^2)\,\Delta((p''+\tfrac{1}{2}q)^2)$$

$$= \int ds_0\,ds'\,du_1'\,du_2' \operatorname{Abs} T(s_0,t,\mu^2,\mu^2,-u_1',-u_2') \times \operatorname{Abs} T(s',t,-u_1'-u_2',\mu^2,\mu^2) \times \Delta(-u_1')\Delta(-u_2')k(s,s_0,s',t,u_1',u_2')\,, \quad (4.1)$$

where k is given by

$$k(s,s_0,s',t,u_1',u_2') = \frac{1}{(2\pi)^4}\int d^4p''\,\delta(s_0-(p+p'')^2)\,\delta(s'-(p'-p'')^2) \times \delta(u_1'+(p''-\tfrac{1}{2}q)^2)\,\delta(u_2'+(p''+\tfrac{1}{2}q)^2)$$

$$= \frac{1}{8(2\pi)^4}\,\frac{1}{s}\,\frac{\theta(J)}{\sqrt{J}}\,, \quad (4.2)$$

and J is given by

$$J = \begin{vmatrix} -\frac{1}{2}t & \frac{1}{2}(u_1'-u_2') & 0 & 0 \\ \frac{1}{2}(u_1'-u_2') & u_1'+u_2'+\frac{1}{2}t & \frac{1}{2}(-2\mu^2+u_1'+u_2')+s_0 & s'-\mu^2+\frac{1}{2}(u_1'+u_2') \\ 0 & \frac{1}{2}(-2\mu^2+u_1'+u_2'+t)+s_0 & -2\mu^2+\frac{1}{2}t & s-2\mu^2 \\ 0 & s'-\mu^2+\frac{1}{2}(u_1'+u_2'+t) & s-2\mu^2+\frac{1}{2}t & 2\mu^2-\frac{1}{2}t \end{vmatrix} \quad (4.3)$$

for the external particles on shell.

The phase space in the integral (4.1) coming from the integration over the subenergies s_0 and s' grows logarithmically with s if the masses $-u_1'$ and $-u_2'$ are held close to the mass shell by damping of the various factors in the integrand. This logarithmic growth comes from the rapidity available to the virtual pions in the integral. The region of phase space where the inequalities $s_0/s<\epsilon$, $s'/s<\epsilon$, and $\epsilon s_0, \epsilon s' >$ (all masses) are satisfied also has phase space which grows as $\ln s$ for any positive ϵ. The region where they are not satisfied makes up the edge of the rapidity phase space when the masses are held close to the mass shell. The phase space here is constant as s increases. Therefore, for large s, virtually all of the phase space comes from the region where the inequalities are satisfied.

If we take these inequalities as true throughout the entire integration, the mistake we will make will be down by a factor $\ln s$ from the leading part of the integral, as long as the integrand is not on the average $\ln s$ bigger in the region where the inequalities are not satisfied. This does not happen if Abs T grows as a power of s greater than -1 (modulo logs). ABFST have considered this case carefully and their results can be gotten by taking our inequalities to be true inside the entire region of integration. So long as Abs T is polynomially bounded in s and grows faster than $1/s$, this procedure will give the correct leading behavior (unless the integration over the masses gives zero for the leading term).

We are interested in amplitudes which satisfy the finite-mass hypothesis and which are polynomially bounded in s. We ignore the possibility that the mass integration gives zero, and so we are free to take our inequalities as true inside the entire integral for W. Therefore, we need only keep the leading term of Abs T in the integrand. We can choose ϵ to be a small number and therefore neglect terms of order ϵ compared to 1. We emphasize that as far as the leading behavior of the integral (4.1) is concerned this is not an approximation.

Taking the inequalities

$$s \gg s' \gg \text{all masses}, \quad s \gg s_0 \gg \text{all masses} \tag{4.4}$$

to be true inside the integral (4.1), we find

$$W = \frac{1}{8(2\pi)^4}\,\frac{1}{s}\int_{4\mu^2}^{s} ds_0 ds' \int_0^\infty du_1' du_2' \,\text{Abs}\,T(s_0, t, \mu^2, \mu^2, -u_1', -u_2')\,\text{Abs}\,T(s', t, -u_1', -u_2', \mu^2, \mu^2)$$
$$\times \Delta(-u_1')\Delta(-u_2')\,\frac{\theta(H(s_0 s'/s, |t|))}{(H(s_0 s'/s, |t|))^{1/2}}\,, \tag{4.5}$$

where

$$H(s_0 s'/s, |t|) = -\left(\frac{u_1' - u_2'}{2}\right)^2 + |t|\left(\frac{u_1' + u_2'}{2}\right) - \frac{|t|^2}{4} - |t|\,\frac{s_0 s'}{s}\ . \tag{4.6}$$

Putting this expression into (3.6) we find

$$T(s, -\vec{q}^{\,2}) = 2is\sum_{n=1}^\infty \frac{(-1)}{n!}\int\Bigg[\prod_{j=1}^{n}\frac{d^2 q_j}{(2\pi)^2}\,\frac{i}{2s}\,\frac{e^{i\pi\alpha(-\vec{q}_j^{\,2})/2}}{2\sin(\pi\alpha(-\vec{q}_j^{\,2})/2)}\left(\frac{\partial\alpha(-\vec{q}_j^{\,2},\lambda)}{\partial\lambda}\bigg|_{\lambda=1}\right)^{-1}$$
$$\times \frac{1}{\ln s}\,\frac{1}{8(2\pi)^4}\,\frac{1}{s}\int_{4\mu^2}^{s} ds_0 ds' \int_0^\infty du_1' du_2' \Delta(-u_1')\Delta(-u_2')$$
$$\times \text{Abs}\,T(s_0, -\vec{q}_j^{\,2}, \mu^2, \mu^2, -u_1', -u_2')$$
$$\times \text{Abs}\,T(s', -\vec{q}_j^{\,2}, -u_1', -u_2', \mu^2, \mu^2)$$
$$\times \frac{\theta(H(s_0 s'/s, \vec{q}_j^{\,2}))}{(H(s_0 s'/s, \vec{q}_j^{\,2}))^{1/2}}\Bigg](2\pi)^2\delta^2\left(\vec{q} - \sum_{i=1}^{n}\vec{q}_i\right). \tag{4.7}$$

Let us try to find a Regge-pole solution to (4.7). This will at least give us a feeling for the kinematics of the equation, and in fact will lead to the only solution we have found. The possibility of solutions with cuts as well as poles competing for the leading behavior of the amplitude is incredibly complicated. It is our guess that no such solutions exist with constant cross sections, but if they do they are guaranteed to be orders of magnitude more complex than the one we shall find.

We study the integral for the case $\text{Abs}\,T(s, t) = \beta_1(t)s^{\alpha_1(t)}$. This yields

$$W = \frac{1}{8(2\pi)^4}\,\frac{1}{s}\int_{4\mu^2}^{s} ds_0 ds' \int_0^\infty du_1' du_2'\,\beta_1(t, \mu^2, \mu^2, -u_1', -u_2') s_0^{\alpha_1(t)}\,\beta_1(t, -u_1', -u_2', \mu^2, \mu^2) s'^{\alpha_1(t)}$$
$$\times \Delta(-u_1')\Delta(-u_2')\,\frac{\theta(H(s_0 s'/s, |t|))}{(H(s_0 s'/s, |t|))^{1/2}}\ . \tag{4.8}$$

Making the change of variables $s_0 \to p = s_0 s'/s$, we find

$$W = \frac{1}{8(2\pi)^4}\,\frac{1}{s}\int_{4\mu^2}^{s} ds' \int_{4\mu^2 s'/s}^{s'} dp \int_0^\infty du_1' du_2'\,\frac{s}{s'}\,s^{\alpha_1(t)} p^{\alpha_1(t)}\,\beta_1(t, -u_1', -u_2', \mu^2, \mu^2)\,\beta_1(t, \mu^2, \mu^2, -u_1', -u_2')$$
$$\times \Delta(-u_1')\Delta(-u_2')\frac{\theta(H(p, |t|))}{(H(p, |t|))^{1/2}}\ . \tag{4.9}$$

Because of the finite-mass hypothesis, the integrand of (4.9) is strongly damped in p. s' can be taken to be much greater than p without affecting the leading behavior. Thus the upper limit on p can be set to infinity. The lower limit on p is of order s'/s, which can be taken arbitrarily small.

Therefore we can take the lower limit on the p integration to be zero. This yields

$$W=\frac{1}{8(2\pi)^4}\, s^{\alpha_1(t)} \int_{4\mu^2}^{s} \frac{ds'}{s'} \int_0^\infty dp\, J_1(p,t)\ , \quad (4.10)$$

with

$$J_1(p,t)=p^{\alpha_1(t)}\int_0^\infty du_1' du_2' \beta_1(t,\mu^2,\mu^2,-u_1',-u_2') \times \beta_1(t,-u_1',-u_2',\mu^2,\mu^2) \times \Delta(-u_1')\Delta(-u_2')\frac{\theta(H(p,|t|))}{(H(p,|t|))^{1/2}}\ . \quad (4.11)$$

To leading order in s, (4.10) is

$$W=(\ln s)s^{\alpha_1(t)}\frac{1}{8(2\pi)^4}\int_0^\infty dp\, J_1(p,t)\ . \quad (4.12)$$

Let us now put this result into (4.7) and see if we can find any solutions.

V. THE FIXED-POLE SOLUTION

It follows from (2.30) and (4.12) that

$$\alpha_1(t)=\alpha(t)\ , \quad (5.1)$$

so that T_2 and T have the same energy dependence. Using this result in our eikonal expansion (4.7) we find

$$\beta_1(-\vec{q}^{\,2})s^{\alpha(-\vec{q}^{\,2})}\frac{e^{i\pi\alpha(-\vec{q}^{\,2})/2}}{2\sin(\pi\alpha(-\vec{q}^{\,2})/2)} = 2is\sum_{n=1}^{\infty}\frac{(-1)}{n!}\int\left[\prod_{j=1}^{n}\frac{d^2q_j}{(2\pi)^2}\,\tfrac{1}{2}i\,\frac{e^{i\pi\alpha(-\vec{q}_j^{\,2})/2}}{2\sin(\pi\alpha(-\vec{q}_j^{\,2})/2)}\left(\frac{\partial\alpha(-\vec{q}_j^{\,2},\lambda)}{\partial\lambda}\bigg|_{\lambda=1}\right)^{-1} s^{\alpha(-\vec{q}_j^{\,2})-1}\frac{1}{8(2\pi)^4}\int_0^\infty dp\, J_1(p,-\vec{q}_j^{\,2})\right] \times(2\pi)^2\delta^2\left(\vec{q}-\sum_{i=1}^{n}\vec{q}_i\right)\ . \quad (5.2)$$

If $\alpha(0)<1$, then only the first term of the eikonal series of (5.2) matters, and the asymptotic energy dependence of the amplitude is determined by a simple multiperipheral equation. In this case the total cross section must go to zero. We are not interested in this case here since it has been studied extensively already and since multiple scattering plays no role.

Let us therefore look for a solution to (5.2) with $\alpha(0)=1$. We have found only one solution of this form and this is

$$\alpha(-\vec{q}^{\,2})=1 \quad (5.3)$$

for all $\vec{q}^{\,2}$. It is difficult to imagine other solutions with $\alpha(0)=1$. If $\alpha(t)$ has a nonzero slope, then higher terms in (5.2) have cuts in the angular momentum plane. Thus any solution with a finite-slope Regge pole must include cuts also. If one tries to include cuts and study this possibility, our equation becomes an extremely complicated mess. We cannot rigorously rule this case out, but if one believes at all in simplicity in nature and in the validity of our equation, then the possibility of nonzero slope Pomerons with intercept 1 can be laid to rest. When (5.3) is put into (5.2), the s dependence of the equation completely drops out and we are left with a geometrical bootstrap equation for the residue function $\beta_1(t)$:

$$i\beta_1(-\vec{q}^{\,2})=-4i\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int\left[\prod_{j=1}^{n}\frac{d^2q_j}{(2\pi)^2}\,\tfrac{1}{4}\left(\frac{\partial\alpha(-\vec{q}_j^{\,2},\lambda)}{\partial\lambda}\bigg|_{\lambda=1}\right)^{-1}\frac{1}{8(2\pi)^4}\int_0^\infty dp\, J_1(p,-\vec{q}_j^{\,2})\right](2\pi)^2\delta^2\left(\vec{q}-\sum_{i=1}^{n}\vec{q}_i\right)\ . \quad (5.4)$$

We see that our equation has the remarkably simple solution

$$T(s,t)=if(t)s\ . \quad (5.5)$$

This is the same form that T would have if a vector particle were exchanged in the t channel and the eikonal approximation made for multiple scattering. Logical paradoxes arise if one tries to attribute this behavior to the exchange of a real particle in the t channel. It is well known that the Bethe-Salpeter equation is capable of producing Regge poles. These poles produce real bound states at values of t for which $\alpha(t)$ is a right-signature integer. In our model the Bethe-Salpeter equation is forced to produce a fixed pole at $J=1$ which never rises through a right-signature point to produce a bound state. Through the optical theorem [Eq. (2.1)] we see that (5.5) predicts a constant total cross section at asymptotic energies. It also predicts a nonshrinking form for $\partial\sigma/\partial t$.

It is well known in field theory that multiple vector-particle exchange leads to a simple eikonal form for the scattering amplitude. Because of this result we have good reason to believe that the exchange of a fixed pole will also eikonalize. This reinforces the self-consistency of our model.

We can understand the Frautschi-Margolis result qualitatively in the following way. They found the output of their eikonal procedure to have a smaller effective slope than their input, both with an intercept 1. If they were next to iterate in the t channel and divide by $\ln s$, as our equation suggests, and then eikonalize this new object they would get an output with a still smaller effective slope. Doing these steps infinitely many times, they would converge to a fixed singularity at $J=1$, which is our solution (5.5).

The solution of our equation supports the droplet model of Chou, Wu, and Yang.[8] In this model, two hadrons scattering at high energies probe each other with a massive vector particle. The Born term of the eikonal expansion is just the convolution of their hadronic densities in impact-parameter space. It becomes a product of form factors in momentum space. Paradoxes arise here also if one tries to attribute this behavior to the exchange of a vector particle which can be produced in the laboratory. We see that enforcing t-channel and s-channel unitarity simultaneously leads in a natural and plausible way to this picture. Chou, Wu, and Yang go on to assume that the hadronic form factors are the same as the electromagnetic form factors, and this last assumption fits differential cross-section data quite well although it is hard to see how this comes about. We have no reason to believe that these form factors are the same in our model, but we will keep this possibility in mind.

We have argued for our equation in the region $t<0$. It might be possible to continue it analytically up to the elastic threshold at $t=4\mu^2$. Beyond this point the situation is not clear since additional constraints not contained in the eikonal approximation may become important. It is well known that problems arise with an amplitude that goes like $sf(t)$ above t-channel elastic threshold. The only known way to resolve the situation is to have a shielding cut with very special properties appear in this region.[9] We have not been successful in continuing our model to this region, so we must assume for the present that such a cut does appear.

Gribov and Pomeranchuk[10] have proved, under certain conditions, that Regge poles must factorize. Their proof breaks down in the presence of the above-mentioned shielding cut. In general, the residue of fixed poles need not factorize.

We do have some reason to believe that the Born term (T_2) does factorize. This is because it is assumed to be produced multiperipherally and such production tends to yield this property. The geometric picture of Chou, Wu, and Yang also demands that the Born term factorize, and we feel that our integral equation is closely related to their picture. If the Born term does factorize, the complexity of the multiple scattering terms in our model makes it unlikely that the full amplitude does also. Factorization of the full amplitude would violate some Pomeron decoupling theorems.[11]

Writing (5.4) out in full we find

$$\beta_1(-\vec{q}^{\,2}) = -4i\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int\Bigg[\prod_{j=1}^{n}\frac{d^2q_j}{(2\pi)^2}\,\tfrac{1}{4}\left(\frac{\partial\alpha(-\vec{q}_j^{\,2},\lambda)}{\partial\lambda}\bigg|_{\lambda=1}\right)^{-1}$$
$$\times\frac{1}{8(2\pi)^4}\int_0^\infty dpp\int_0^\infty du_1'du_2'\beta_1(-\vec{q}_j^{\,2},\mu^2,\mu^2,-u_1',-u_2')$$
$$\times\,\beta_1(-\vec{q}_j^{\,2},-u_1',-u_2',\mu^2,\mu^2)\Delta(-u_1')\Delta(-u_2')\frac{\theta(H(p,\vec{q}_j^{\,2}))}{(H(p,\vec{q}_j^{\,2}))^{1/2}}\Bigg]$$
$$\times(2\pi)^2\delta^2\left(\vec{q}-\sum_{i=1}^{n}\vec{q}_i\right)\,. \qquad (5.6)$$

Our interpretation of the fixed-pole solution of our equation is the following. t-channel and s-channel unitarity play equal roles in determining the asymptotic form of the scattering amplitude. The t-channel Bethe-Salpeter equation and the s-channel eikonal expansion balance each other at asymptotic energies, yielding a fixed pole in the J plane at $J=1$. Looked at in this way, it is not an accident if total cross sections are constant asymptotically; it is just a simple way for nature to satisfy her own rules.

VI. OTHER SOLUTIONS

We have not found any solutions to (4.7) which have growing total cross sections. The reason for this is that if Abs T satisfies the Froissart-bound off-shell and is strongly damped in the external masses, then T_2, if it is produced multiperipherally in the way we have assumed here, can grow no faster than s^1.[6] This is insufficient to yield growing total cross sections if T_2 is taken as the Born term of the eikonal expansion.

Let us give up the assumption that T_2 is produced multiperipherally. We showed in Ref. 6, with an assumption of polynomial boundedness, that in this case $T_2(s, t) < s^{1+\epsilon}$, with ϵ an arbitrary positive number. It follows from this that T_2 satisfies the bound

$$|T_2(s, t)| < \beta(t)s(\ln s)^{\lambda(t)} \tag{6.1}$$

for some $\beta(t)$ and $\lambda(t)$, with $\lambda(t)$ finite for all t. If T_2 does not get arbitrarily large as $t \to -\infty$ with $|t|/s \ll 1$, then we can find a constant m such that

$$|T_2(s, t)| < \beta(t)s(\ln s)^m \tag{6.2}$$

for all t. Let us take T_2 as strong as possible:

$$T_2(s, t) = A(t)s(\ln s)^m . \tag{6.3}$$

Taking the Fourier transform of this we find

$$T_2(s, b) = A(b)s(\ln s)^m . \tag{6.4}$$

We now insert (6.4) into our expression for $T(s, b)$ [Eq. (3.3)]:

$$T(s, b) = 2is\{1 - \exp[\tfrac{1}{2}iA(b)(\ln s)^m]\} . \tag{6.5}$$

$T_2(s, t)$ has a cut from $t = 4\mu^2$ to ∞. It follows from this that $A(b)$, for large b, goes at best as

$$A(b) \approx Ae^{-2\mu b} . \tag{6.6}$$

$T(s, b)$ goes to zero for large b. In particular

$$T(s, b) \approx 0 , \tag{6.7}$$

when

$$|A(b)|(\ln s)^m \ll 1 . \tag{6.8}$$

$T(s, b)$ is damped quickly for b larger than $b_{\max}$ given by

$$b_{\max} = \frac{m \ln(\ln s)}{2\mu} . \tag{6.9}$$

We also have the optical theorem

$$\sigma_{\text{tot}} = \frac{1}{s}\int d^2b \, \text{Im}\, T(s, b) . \tag{6.10}$$

The largest that $\text{Im}\, T(s, b)$ can be is $4s$. Putting this into (6.10) and integrating out to $b_{\max}$ we have

$$\sigma_{\text{tot}} = 4\pi\left(\frac{m \ln(\ln s)}{2\mu}\right)^2 . \tag{6.11}$$

Therefore, if the Froissart bound is satisfied off shell, the scattering amplitude is given by the eikonal of a two-particle reducible term, off-shell behavior is strongly damped, and the assumption of polynomial boundedness of Ref. 6 is correct, then total cross sections must satisfy[12]

$$\sigma_{\text{tot}} \leq c[\ln(\ln s)]^2 . \tag{6.12}$$

This is considerably short of Froissart-bound saturation, which occurs when $\sigma_{\text{tot}} = (\pi/\mu^2)(\ln s)^2$.

These arguments also show why we think that the Born term should be two-pion reducible. If Feynman graphs which are two-pion irreducible are included, then these corrections must die at least as fast as $e^{-4\mu b}$ for large b, compared with $e^{-2\mu b}$ for T_2. This suggests that $b_{\max}$ and therefore σ_{tot} should be determined by T_2 alone.

VII. EQUATIONS FOR THE RESIDUE FUNCTIONS

We have tried studying Eq. (5.6) for $\beta_1(q^2)$, with the following assumptions:

(1) $\left.\dfrac{\partial \alpha(t, \lambda)}{\partial \lambda}\right|_{\lambda=1}$ independent of t,

(2) $\Delta(u) = \dfrac{1}{u - \mu^2}$.

Since $\alpha(t, \lambda)|_{\lambda=1}$ is independent of t, it is tempting to speculate that the derivative is also. This would be the case in the leading-log approximation $\alpha(t, \lambda) = 2\lambda - 1$, but it could be true in many other cases. If $\alpha(t, \lambda) = a(t)f(\lambda) - 1$, for example, then a must be independent of t. Using the free-pion propagator is of course a standard approximation.

As a next approximation, we assumed that the mass dependence of $\text{Abs}\, T$ introduces no new t dependence in $\text{Abs}\, T \times_t \text{Abs}\, T$, i.e.,

$$\text{Abs}\, T \times_t \text{Abs}\, T \propto \beta_1^2(q^2) s \ln s .$$

We have not been able to solve the equation in this case, and at this time we believe it has no solution. We now feel that the mass dependence is important and cannot be ignored in this way. Although (5.6) has the rather appealing interpretation of being a geometrical bootstrap for the hadronic density of the pions, especially when written in impact-parameter space, we do not understand off-shell effects well enough to solve it at present.

Therefore, let us go back to the much simpler multiperipheral equation

$$\begin{aligned}\left(\lambda\frac{\partial}{\partial\lambda} - 1\right) \text{Abs}\, T_2(\lambda)|_{\lambda=1} &= \text{Abs}\, T \times_t \text{Abs}\, T \\ &= (\text{Abs}\, I + \text{Abs}\, T_2) \times_t (\text{Abs}\, I + \text{Abs}\, T_2) ,\end{aligned} \tag{7.1}$$

where

$$\text{Abs}\, T_2(s, t; \lambda) = \beta_2(t, \lambda)s^{\alpha(t, \lambda)} + (\text{nonleading terms}) , \tag{7.2}$$

$$\alpha(t, 1) = 1 , \tag{7.3}$$

and

$$\left.\frac{\partial\alpha(t,\lambda)}{\partial\lambda}\right|_{\lambda=1}=c\ . \tag{7.4}$$

It follows that we can choose λ_2 so that

$$\alpha(t,\lambda_2)<1\ , \tag{7.5}$$

and this should be satisfied for λ_2 in some finite region. It follows from (7.3) and (7.5) that

$$\mathrm{Abs}\,T_2(s,t)-\frac{1}{\lambda_2^{\,2}}\mathrm{Abs}\,T_2(s,t;\lambda_2)=\mathrm{Abs}\,T_2(s,t) \tag{7.6}$$

to leading order in s. We also have

$$\mathrm{Abs}\,T_2(\lambda)=\sum_{n=2}^{\infty}\lambda^n(\mathrm{Abs}I)^n\ . \tag{7.7}$$

Putting this into (7.6) yields

$$\mathrm{Abs}\,T_2(s,t)=\sum_{n=3}^{\infty}(\mathrm{Abs}I)^n-\frac{1}{\lambda_2^{\,2}}\sum_{n=3}^{\infty}(\lambda_2)^n(\mathrm{Abs}I)^n \tag{7.8}$$

to leading order in s. Since the left-hand side of (7.8) does not depend on λ_2, the right-hand side cannot either to leading order in s and so

$$\mathrm{Abs}\,T_2(s,t)=\sum_{n=3}^{\infty}(\mathrm{Abs}I)^n+(\text{nonleading terms})\ . \tag{7.9}$$

Comparing this with (7.7) we see that $(\mathrm{Abs}I)^2$ cannot have the same s dependence as $\mathrm{Abs}\,T_2$, but must be smaller:

$$(\mathrm{Abs}I)^2\ll\mathrm{Abs}\,T_2\ . \tag{7.10}$$

It follows from (7.9) that

$$\mathrm{Abs}\,T_2=\mathrm{Abs}I\times_t\mathrm{Abs}\,T_2+(\text{nonleading terms}). \tag{7.11}$$

This is the linear multiperipheral equation for T_2. We see that the inhomogeneous term can be ignored. Using these results in (7.1) we find

$$c\ln s\,\mathrm{Abs}\,T_2=\mathrm{Abs}\,T_2\times_t\mathrm{Abs}\,T_2+(\text{nonleading terms})\ . \tag{7.12}$$

Writing this out in full we have

$$s\beta_2(t)=\frac{1}{c\ln s}\frac{1}{8(2\pi)^4}s\ln s\times\int_0^\infty dp\,p\int_0^\infty du_1'du_2'\times\beta_2(t,\mu^2,\mu^2,-u_1',-u_2')\times\beta_2(t,-u_1',-u_2',\mu^2,\mu^2)\times\Delta(-u_1')\Delta(-u_2')\frac{\theta(H(p,|t|))}{(H(p,|t|))^{1/2}}\ . \tag{7.13}$$

$\beta_2(t)$, being the residue function for the Born term, should factorize for a geometrical interpretation:

$$\beta_2(t,u_1,u_2,u_3,u_4)=F(t,u_1,u_2)F(t,u_3,u_4)\ . \tag{7.14}$$

Putting this statement into (7.13) we have

$$c=\frac{1}{8(2\pi)^4}\times\int_0^\infty dp\,p\int_0^\infty du_1'du_2'\beta_2(t,-u_1',-u_2',-u_1',-u_2')\times\Delta(-u_1')\Delta(-u_2')\frac{\theta(H(p,|t|))}{(H(p,|t|))^{1/2}}\ . \tag{7.15}$$

This equation must be satisfied for all t and is therefore a strong sum rule for the Born term's residue function. Making a change of variables, with $t=-\vec{q}^{\,2}$, we find the more appealing form

$$c=\frac{1}{2(2\pi)^4}\times\int d^2q'\int_0^\infty dp\,p\frac{1}{(\vec{q}'^2+p+\mu^2)}\frac{1}{[(\vec{q}-\vec{q}')^2+p+\mu^2]}\times\beta_2(-\vec{q}^{\,2},-\vec{q}'^2-p,-(\vec{q}-\vec{q}')^2-p,-\vec{q}'^2-p,-(\vec{q}-\vec{q}')^2-p)\ . \tag{7.16}$$

Written in this form it is clear that p is the common transverse mass of the two virtual pions in the bilinear Bethe-Salpeter equation.

We have not been able to solve these equations for the residue function. We feel that an additional input is necessary to determine it completely. The main obstacle is a lack of knowledge of the off-shell behavior of this function. We see no way to avoid going off-shell in a model of this type and we apologize for these difficulties.

VIII. CONCLUSION

We have seen that if one takes seriously the postulate that the full elastic scattering amplitude is asymptotically the eikonal iteration of a t-channel two-particle reducible set of Feynman graphs, then the t-channel Bethe-Salpeter equation puts constraints on the form of the solution. Assuming that the Born term is produced multiperipherally enables one to get a closed integral equation for the full elastic scattering amplitude (by using the bilinear form of the Bethe-Salpeter equation). This equation suggests the formation of a fixed pole in the angular momentum plane at $J=1$, leading to behavior identical to the Chou, Wu, and Yang diffractive model.

If the Born term is not produced multiperipherally, but our other propositions are correct, and if this Born term falls off exponentially in impact parameter as one would expect in a theory with no massless particles, then, with an assumption of polynomial boundedness, other solutions can lead to total cross sections growing no faster than $[\ln(\ln s)]^2$.

Froissart-bound saturation can occur as in Refs. 2 and 3 if (a) the Froissart bound is not satisfied off the mass shell, (b) the scattering amplitude is not strongly damped off the mass shell, or (c) the assumption of polynomial boundedness in Ref. 6 is wrong. We feel though, in the light of our equation, that it is difficult to argue against constant total cross sections on the grounds that this would be an accident of nature.

ACKNOWLEDGMENTS

I thank Professor Lorella M. Jones for encouragement and many helpful discussions during the course of this work. I also thank Professor J. D. Sullivan for interesting conversations and criticism.

*This work was supported in part by the National Science Foundation, Grant No. NSF GP 25303.

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PHYSICAL REVIEW D VOLUME 9, NUMBER 3 1 FEBRUARY 1974

Model for p-p diffraction

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(Received 1 November 1973)

Previous work on π-π scattering is generalized to include the p-p case. A fixed-pole solution is found for the p-p scattering amplitude. After approximating the residue of this pole, we compare with high-energy data from the CERN ISR. The one-parameter model explains several aspects of these data.

I. INTRODUCTION

In a previous paper[1] we presented a model for π-π scattering at infinite energy. The basic assumption of this model is that the exact scattering amplitude is given by the eikonal iteration of some set of Feynman graphs which are two-pion reducible in the crossed channel. The eikonal approximation is a natural one for high-energy scattering since it is based on the intuitively appealing idea that two highly relativistic particles at some impact parameter b simply traverse classical trajectories through the region of interaction and are phase-shifted by an amount proportional to a potential. In a relativistic field theory this potential is assumed to be some set of Feynman graphs. One would expect the longest-range forces to be the most important ones which contribute to this potential, since these long-range forces determine the extent of the scattering region in b space, which in turn roughly determines the cross section. These ideas have been partially borne out in perturbation theory[2]; however, the final situation is not clear since in some models there is no unique classical path for the scattering particles,[3] and in others the inclusion of vertex corrections destroys the simple eikonal result.[4] Despite these complications it seems reasonable to assume that there do exist field theories which, when solved completely at asymptotic energies, have the ei-