Scale-invariant parton model

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We generalize the assumptions of the parton model to include the possibility of scale-invariant short-distance behavior. A physical picture is used to motivate our formal equations. Bjorken scaling is violated in a systematic way. We compute the asymptotic properties of νW_2 and show that its moments are power-behaved functions of Q^2. Arguments are presented for a nonzero value of σ_S/σ_T in the deep-inelastic limit. The asymptotic behavior of electromagnetic form factors is found to be $(Q^2)^{-a}$, where the constant a is related to properties of νW_2.

I. INTRODUCTION

The parton model[1] was originally intended as an intuitive guide to computations in quantum field theory. The crucial assumption of the parton model is that for small times and lengths the partons may be treated as freely moving constituents. Technically this is achieved if the parton-parton interactions are as soft as those in a super-renormalizable field theory. However, from the beginning it was realized that there are no such field theories in four dimensions, so it was necessary to invent transverse-momentum cutoffs to implement the parton-model field theoretically.[2] Since renormalizable field theories have dimensionless coupling constants, there is no time and length scale beyond which interactions can be ignored.

The purpose of the present article is to extend the intuitive parton model beyond the domain of super-renormalizable theories. In particular, we propose a parton picture to describe scale-invariant interactions.[3] Our approach is inspired by the Wilson-Kadanoff theory of scaling phenomena and Polyakov's similarity hypothesis for strong interactions.[4]

This article is organized as follows. In Sec. II we introduce the scale-invariant parton model. Bjorken scaling is violated in a systematic way. The moments of νW_2 are calculated and are shown to be power-behaved in Q^2. The ratio σ_S/σ_T should be finite in the deep-inelastic region. In Sec. III the electromagnetic form factor is studied by use of generalized Drell-Yan relations. The predicted asymptotic behavior has the form $(Q^2)^{-a}$ where the constant a is related to the moments of νW_2. Section IV contains discussion and concluding remarks.

We emphasize that the consistency between scale-invariant quantum field theory and our intuitive methods is at present a conjecture.

II. MODEL

Following Wilson[5] we assume that matter organizes itself into clusters. For example, molecules are made of atoms which are made of nuclei which are made of nucleons, etc. Each cluster is characterized by a certain size and time scale. The relation between these size scales appears to be accidental. However, as smaller and smaller scales are resolved in high-energy physics regularities may emerge. One interesting possibility suggested by renormalization-group studies of field theory[6] is that the connections between adjacent size scales become universal.

We begin by considering time and length scales of ordinary hadrons, 10^{-13} cm. Denote ordinary hadrons as $N=0$ clusters. We assume these clusters may be described as composites of $N=1$ clusters.[7] The description of these clusters will be carried out in an infinite-momentum frame (I.M.F.) in terms of multicluster wave functions depending on transverse positions and longitudinal fractions.

$$\psi_{N=0}(\eta_1, X_1; \eta_2, X_2; \dots)$$

is the amplitude to find clusters of type $N=1$ with longitudinal fractions η_i and transverse positions X_i in a cluster of type $N=0$.[8] In general, the wave function of a cluster of type N is a function of the fractions and positions of its constituent clusters of type $N+1$. In other words, our picture is that clusters are made of smaller clusters which are made of smaller clusters, etc. (See Fig. 1.) The picture suggested by Wilson and Polyakov (and Kadanoff in the context of critical phenomena)[4] is that the ratio of size scales $R_{N+1}/R_N = \Lambda^{-1}$ becomes independent of N (universal) for large N. The intuitive ideas are easiest to visualize when the transition between length scales is relatively sharp and the ratio of neighboring length scales Λ very far from unity. However, we believe that

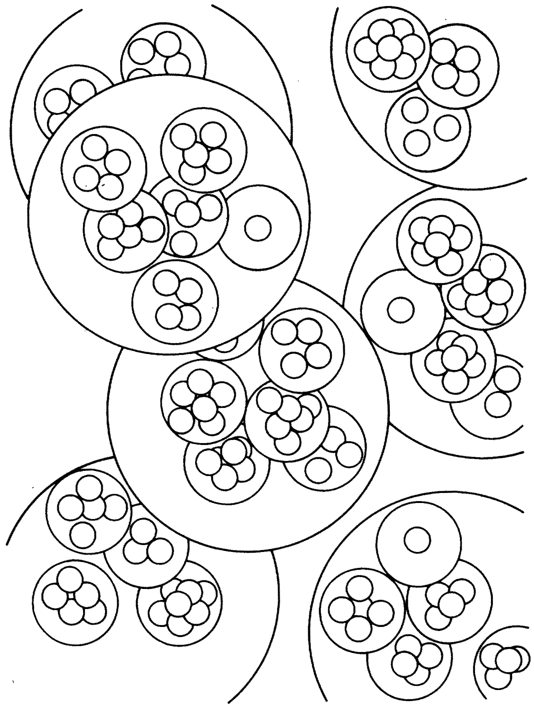

FIG. 1. Scale-invariant cluster distribution.

scale-invariant field theory solutions will actually smoothly extrapolate between the length scales used here, and will not invalidate our rougher approach. Renormalization group investigations suggest that the dynamics of clusters of type N can be described by a Hamiltonian H_N which does not explicitly refer to the coordinates of smaller clusters ($N+1$, say). We assume that the Hamiltonian H_N binds clusters of type $N+1$ of transverse size R_{N+1} into clusters of type N and size R_N. Similarly we assume that the transverse distance between clusters of type $N+1$ within a cluster of type N is of order R_N. Using the same Hamiltonian we may also obtain the interaction between clusters of type N in the same way that interatomic forces may be derived from the Coulomb forces between electrons and nuclei. Thus, we derive an operation which gives H_{N-1} in terms of H_N. Renormalizable field theories have the property that the operation $H_N \to H_{N-1}$ becomes independent of N for N sufficiently large. We introduce the scale transformation[9]

$$\begin{aligned} &\tau \to \Lambda^2 \tau, \\ &X \to \Lambda X, \\ &\eta \to \eta, \\ &N \to N+1, \end{aligned} \tag{1}$$

where τ is the infinite-momentum time, X is the transverse coordinate, and η is the dimensionless longitudinal fraction variable.[8] Evidently the operation connecting H_N to H_{N+1} will be invariant if the important length scales of the theory disappear at short distances. It is convenient to define dimensionless coordinates

$$Y_N = X/R_N \tag{2}$$

and a dimensionless Hamiltonian $h_N(Y_N)$ which describes the motion of the clusters of type N on the rescaled time axis $\tau/R_N{}^2$. The operation $H_N \to H_{N-1}$ reads

$$h_{N-1}(Y) = T\{h_N(Y)\}. \tag{3}$$

The Hamiltonians $h_N(Y)$ should be local field-theoretic Hamiltonians in the infinite-momentum frame, the $N+1$ type clusters playing the role of bare quanta. The only exception to this is that for lengths smaller than R_{N+1} the Hamiltonians should be cut off. This does not mean that our theory is a cutoff theory. What it does mean is that to correctly study distances smaller than R_{N+1} we must proceed to the next scale and use H_{N+1}. By construction H_{N+1} is almost equivalent to H_N for lengths larger than R_{N+1} (but awkward) but is also a correct description for lengths between R_{N+1} and R_{N+2}.

The assumption of asymptotic scale invariance is that the sequence $\{h_N\}$ approaches a finite fixed point for large N.[6] So, for very large N, h_N can be replaced by a universal Hamiltonian h. Therefore, $\psi_N(Y,\eta)$ becomes independent of N in the same limit. This means that the structure of clusters becomes universal in terms of rescaled positions and times. Renormalizability alone does not require the existence of a fixed point in Eq. (3). It is possible that the recursion formula generates Hamiltonians which wander to infinity, tend to limit cycles, etc.[6] The special significance of the approach to a fixed point is that the solution to the scale-invariant equation (3) is also scale-invariant.

Consider an experiment which probes the structure of the hadron of size R_0. Let the wavelength λ of the probe be less than R_0 but greater than the first cluster size R_1. We are then justified in applying the usual ideas of the parton model replacing the partons by clusters of type 1. It is not legitimate to use clusters of type 2, 3, etc. here, because the probe cannot resolve the distance between these smaller clusters. Now, let the wavelength λ of the probe decrease so that smaller sizes in the hadron can be resolved. Clusters of type 1 are now irrelevant, because they no longer appear pointlike to the external probe. If λ is much less than R_1, but much larger than R_2, the parton description then becomes

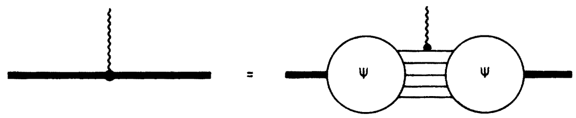

FIG. 2. Computation of g_{N+1} in terms of g_N. The thick (thin) lines represent clusters of type N (type $N+1$). The bubbles stand for wave functions.

legitimate using clusters of type 2. Thus our guiding principle will be the parton model with the additional rule that the relevant distribution function is that of clusters one size smaller than the wavelength of the external probe. This implies a relation between the type N of clusters resolved and the wavelength λ of the probe,

$$\lambda = \text{const} \times R_N = \xi \Lambda^{-N},$$

where ξ is a constant. Since $\lambda = Q^{-1}$,

$$N = [\ln(Q/\xi)]/\ln\Lambda. \tag{4}$$

Typically scale-invariant field theories produce matrix elements which vary as a power of the length scale of the clusters being probed. As an example, consider the absorption amplitude of an external scalar current by a cluster of type N. Suppose the momentum transfers Q are smaller than R_N^{-1}. The matrix element is approximately a constant g_N since the clusters of type N can be treated as pointlike. g_N itself can be calculated in terms of g_{N+1}. We assume that the current couples additively to the constituents (clusters of type $N+1$) of the Nth cluster, so that schematically (Fig. 2),

$$g_N = g_{N+1} \langle \psi_N | O | \psi_N \rangle,$$

$$g_N = g_{N+1}\, c_{N+1},$$

where O is the operator coupling the probe to the clusters.[8] In a scale-invariant limit, where $\psi_N(Y)$ becomes independent of N, we find that c_{N+1} becomes independent of N and this relation can be iterated producing

$$g_N = c^N g_0.$$

Recalling that R_N varies as Λ^{-N}, we obtain a power-law dependence of g_N on R_N,

$$g_N \sim (R_N)^{-(\ln c)/\ln\Lambda}. \tag{5}$$

Thus the strength of the coupling to an external current varies as a power of the cluster size R_N. A special case in which the coupling remains constant with decreasing scale is given by a conserved charge which enters a current algebra. In this case the clusters at the Nth level must form a representation of the current algebra. The scale-invariance assumption forces the algebraic properties of the clusters to become independent of N. Thus the couplings of clusters to external charges is normalized by the group structure and must be independent of N. For convenience we suppose that the clusters have charge ± 1. Our formulas can easily be extended to more reasonable cases such as quark quantum numbers.

Before discussing deep-inelastic electroproduction we shall comment about the general nature of the amplitudes $\psi_N(Y, \eta)$. Since $h_N(Y)$ is similar to a field-theoretic Hamiltonian in the I.M.F., the ψ are also similar to those in a (cutoff) field theory. The state of a cluster of type N will be a finite normalizable superposition of states with different number of type $N+1$ clusters. In general the superposition will begin with one cluster of type $N+1$ with a probability Z less than unity. The constant Z is the finite wave-function renormalization constant of the cutoff theory. The remaining multicluster terms should have wave functions similar to those in naive parton models with transverse cutoff.

We will now discuss the deep-inelastic electroproduction structure function νW_2. According to the *naive* pointlike parton model

$$\nu W_2(Q^2, \nu) = F_2(Q^2/2\nu) \tag{6}$$

measures the longitudinal-momentum distribution of the charged partons in the target[1,10]

$$F_2(Q^2/2\nu) = \eta \frac{dN}{d\eta}\bigg|_{\eta = Q^2/2\nu}, \tag{7}$$

where $dN/d\eta$ is the number of charged partons having longitudinal fraction η.

We will now apply this argument in the case of the scale-invariant parton model. At each value of Q^2 the structure function $F_2(\eta, N)$ measures the longitudinal-momentum distribution of the clusters of type $N \sim \ln Q^2$. So, to pass from one scale to the next we need to know how the clusters of type $N+1$ are distributed in the clusters of type N. To do this we introduce a function $f_{N+1,N}(\beta, \eta)/(\beta/\eta)$ which gives the probability per unit β/η to find a cluster of type $N+1$ and longitudinal fraction β in a cluster of type N and longitudinal fraction η. Longitudinal boost invariance requires that $f_{N+1,N}$ depends only on the ratio β/η. Then the distribution of clusters of type $N+1$ having longitudinal fraction β satisfies the equation

$$\frac{F_2(\beta, N+1)}{\beta} = \int_\beta^1 \frac{f_{N+1,N}(\beta/\eta)}{(\beta/\eta)} \frac{F_2(\eta, N)}{\eta} \frac{d\eta}{\eta}. \tag{8}$$

It is consistent with Eq. (8) to define the functions $F_2(\eta, N)$ and $f_{N+1,N}(\alpha)$ to vanish when their arguments η and α do not lie in the region from zero to unity. Then Eq. (8) can be written,

$$F_2(\beta, N+1) = \int_0^\infty f_{N+1,N}(\beta/\eta) F_2(\eta, N) \frac{d\eta}{\eta}.$$

The assumption of asymptotic scale invariance requires that $f_{N+1,N}(\beta/\eta)$ becomes independent of N for N large. In this limit the function $f_{N+1,N}(\beta/\eta)$ can be replaced by a single function $f(\beta/\eta)$. It is convenient to rewrite Eq. (8) in terms of the rapidity variable,

$$y = \ln\eta \tag{9}$$

so that

$$F_2(y, N+1) = \int_{-\infty}^{\infty} f(y-y') F_2(y', N)\, dy'. \tag{10}$$

This equation can be solved by Laplace transform: Define the αth moment of $F_2(\eta, N+1)$,

$$M_\alpha(N) = \int_0^1 \eta^\alpha F_2(\eta, N) \frac{d\eta}{\eta}. \tag{11}$$

In terms of rapidity the moment $M_\alpha(N)$ becomes

$$M_\alpha(N) = \int_{-\infty}^{\infty} e^{\alpha y} F_2(y, N)\, dy. \tag{12}$$

Substituting into Eq. (10) we have a relation between moments of the type $N+1$ and N,

$$M_\alpha(N+1) = m_\alpha M_\alpha(N), \tag{13}$$

where m_α is the αth moment of the kernel f,

$$m_\alpha = \int_{-\infty}^{\infty} e^{\alpha y} f(y)\, dy. \tag{14}$$

Equation (13) is a scale-invariant relation satisfied by the moments of the structure function νW_2. It must be supplemented with boundary conditions at the small-N level which describes the large scale character of hadronic structure. These boundary conditions break the scale invariance because they refer to a particular length scale. Choosing boundary conditions[11]

$$M_\alpha(N=0) = M_\alpha,$$

where M_α are the moments of νW_2 in the first scaling region $(1 < Q^2 < \Lambda^2)$, Eq. (13) can be solved,

$$M_\alpha(N) = (m_\alpha)^N M_\alpha. \tag{15}$$

Using the relation between the momentum transfer Q^2 and the scale N given in Eq. (4), Eq. (15) implies

$$M_\alpha(Q^2) = (m_\alpha)^{[\ln(Q^2/\xi^2)]/\ln\Lambda^2} M_\alpha,$$
$$\int_0^1 \eta^\alpha \nu W_2 \frac{d\eta}{\eta} = \left(\frac{Q^2}{\xi^2}\right)^{-d_\alpha} M_\alpha, \tag{16}$$

where

$$d_\alpha = -[\ln(m_\alpha)]/\ln\Lambda^2.$$

In other words, the moments of νW_2 are power-behaved in Q^2 with α-dependent powers. Scaling laws of this type (for integer α) have been obtained from field-theory studies by Polyakov, Mack, and others.[12]

In qualitative terms the Q^2 dependence of these moments means that as Q^2 increases and smaller clusters are resolved, the structure function νW_2 should shift into the low-η region as depicted in Figs. 3(a) and 3(b). This follows because as Q^2 increases the contribution of a given cluster (type N) will be replaced by several clusters (type $N+1$) each of smaller longitudinal momentum. As νW_2 changes with Q^2 however, the area under the curve should remain constant. This follows from longitudinal-momentum conservation—the sum of the longitudinal momenta of the $N+1$ clusters in a cluster of type N should be the longitudinal momentum of the Nth cluster,

$$\int f(\eta)\, d\eta = 1. \tag{17}$$

Therefore, $m_{(\alpha=1)} = 1$ so from Eq. (13) it is clear that $M_{(\alpha=1)}(N)$ is independent of N and

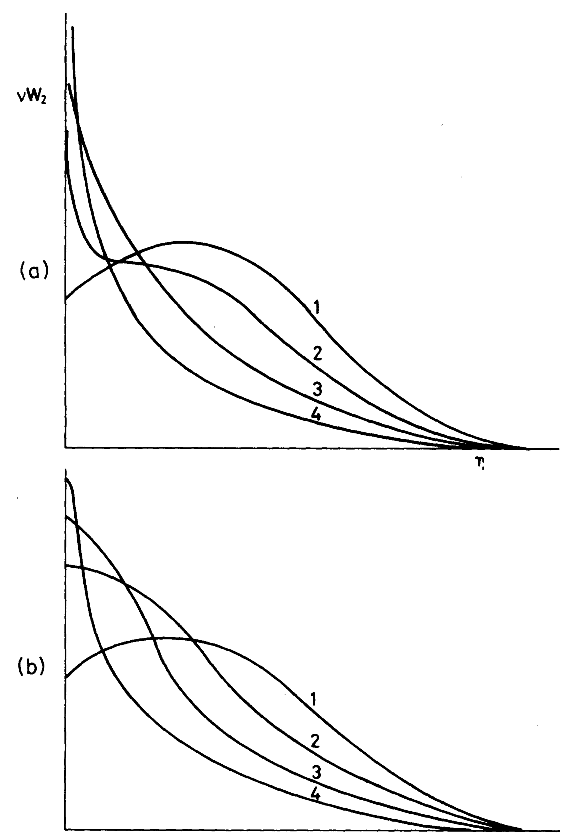

FIG. 3. (a) The function νW_2 assuming $f(\eta=0) \neq 0$. The different curves represent successive orders of magnitude for Q^2. (b) Same as (a) except $f(\eta=0)=0$.

$$\int_0^1 \nu W_2\, d\eta = \text{const.} \tag{18}$$

We will now study the Q^2 dependence of νW_2 assuming various properties of the kernel f. Suppose, for example, that $f(y)$ has a nonzero value c as $y \to -\infty$. The αth moment of f will then behave as

$$m_\alpha \sim c/\alpha \tag{19}$$

as α tends toward zero. Experimentally νW_2 tends to a constant[11] as $y \to -\infty$ so that M_α also behaves as α^{-1}. Equation (15) then implies that

$$M_\alpha(N) \sim \left(\frac{c}{\alpha}\right)^N M_\alpha \sim \frac{c^N}{\alpha^{N+1}}\,. \tag{20}$$

Inverting Eq. (12) gives the structure function as y tends to $-\infty$,

$$F_2(y, N) \sim |y|^{N-1} c^N/\Gamma(N+1). \tag{21}$$

In terms of Q^2 and $\eta = Q^2/2\nu$, we have

$$F_2(Q^2/2\nu, Q^2) \sim \frac{c[c\ln(Q^2/2\nu)]^{[\ln(Q^2/\xi^2)]/\ln\Lambda^2-1}}{\Gamma([\ln(Q^2/\xi^2)]/\ln\Lambda^2+1)} \tag{22}$$

for $Q^2/2\nu$ sufficiently small [Fig. 3(a)].

Another interesting possibility suggested by low-order graphs in scalar field theories is that $f(\eta)$ vanishes as a power of η near $\eta = 0$,

$$f(\eta) \sim \eta^p \quad (\eta \approx 0). \tag{23}$$

Then the moments of $f(\eta)$ near $\alpha = 0$ are

$$m_\alpha \sim \frac{1}{\alpha - p} \tag{24}$$

and the moments of the structure function become

$$M_\alpha(N) \sim \left(\frac{1}{\alpha - p}\right)^N \frac{1}{\alpha}\,, \tag{25}$$

where we have taken $M_\alpha \sim \alpha^{-1}$ again. For α small enough Eq. (25) reduces to

$$M_\alpha(N) \sim \left(\frac{1}{p}\right)^N \frac{1}{\alpha} \tag{26}$$

so

$$F_2(\eta, Q^2) \sim (p)^{-[\ln(Q^2/\xi^2)]/\ln\Lambda^2}$$

or

$$F_2(\eta, Q^2) \sim (Q^2/\xi^2)^{-\ln p/\ln\Lambda^2} \tag{27}$$

at $\eta = 0$ [Fig. 3(b)].

It is also interesting to understand the consequences of particular assumptions about $f(y)$ for y near zero (leading-parton effect[8]). Assume that each N-type cluster has a probability Z to have only one "bare" type $N+1$ cluster carrying all its longitudinal momentum. This will contribute a term $Z\delta(y)$ to $f(y)$. The remaining multiparticle contributions to $f(y)$ are likely to contribute a term which tends to zero near $y = 0$ as a power of y. (This is characteristic of low-order graphs.) We also assume $F_2(y, N=0)$ vanishes as a power near $y = 0$,

$$\begin{aligned} &f(y) = Z\delta(y) + dy^\delta, \\ &F_2(y, N=0) \sim y^\gamma. \end{aligned} \tag{28}$$

Experiment indicates γ is near three.[11]

The Laplace transforms of $F_2(y, N=0)$ and $f(y)$ behave as

$$\begin{aligned} &M_\alpha(N=0) \sim \alpha^{-\gamma-1}, \\ &m_\alpha = Z + d\alpha^{-\delta-1} \end{aligned} \tag{29}$$

for $\alpha \to \infty$. Substituting into Eq. (15) we determine $M_\alpha(N)$ as $\alpha \to \infty$,

$$M_\alpha(N) \sim \alpha^{-\gamma-1} Z^N. \tag{30}$$

Or as a function of Q^2,

$$\begin{aligned} M_\alpha(Q^2) &\sim \alpha^{-\gamma-1} Z^{[\ln(Q^2/\xi^2)]/\ln\Lambda^2} \\ &\sim \alpha^{-\gamma-1}\left(\frac{Q^2}{\xi^2}\right)^{(\ln Z)/\ln\Lambda^2} \quad (\alpha \to \infty). \end{aligned} \tag{31}$$

The constants Z and γ will be related to form factors in a later discussion.

By inverting Eq. (12) and using Eq. (31) we find

$$F_2(\eta, Q^2) \to (1-\eta)^\gamma \left(\frac{Q^2}{\xi^2}\right)^{(\ln Z)/\ln\Lambda^2}$$

General constraints on the power indices d_α of the moments of νW_2 follow from the positivity of the kernel f (recall that f is a probability distribution). This implies that the moments of f, m_α, cannot vanish faster than an exponential

$$m_\alpha > e^{-\mu\alpha},$$

where μ is some positive constant. Using Eq. (16) gives

$$d_\alpha < \mu\alpha$$

for large α. A lower bound on d_α can also be obtained. Namely, d_α must be larger than some positive constant. This follows from observing that if $d_\alpha \to 0$ for large α, then $m_\alpha \to 1$ for large α. But then $f(\eta)$ must have a δ-function singularity at $\eta = 1$ with normalization unity. Equation (17) and positivity would then require $f(\eta) = \delta(\eta - 1)$. We discard this option.[13] Therefore, the constraints on d_α become

$$0 < d_\alpha < \mu \cdot \alpha \quad (\alpha \text{ large}). \tag{32}$$

Another measurable quantity of interest is the ratio of the cross sections for absorbing longitudinal and transverse virtual photons, σ_S/σ_T. In the usual parton model based on spin-$\frac{1}{2}$ charged constitutents this ratio tends to zero in the deep-inelastic region. If the average transverse-momentum fluctuations of the partons is κ, then[14]

$$\frac{\sigma_S}{\sigma_T}\approx\frac{\kappa^2}{Q^2}\,.$$

In the scale-invariant parton model one expects that σ_S/σ_T tends to a nonzero constant. This follows from the fact that the transverse momentum of the probed clusters should increase proportionally to the inverse of their sizes R_N^{-1}. But R_N^{-1} grows as Q, so that the ratio κ^2/Q^2 remains roughly constant as Q^2 grows.

III. ELECTROMAGNETIC FORM FACTOR

We now discuss the Q^2 behavior of hadronic form factors. This discussion will be presented in three parts. First, the simple Drell-Yan relation[15] for the naive parton model will be reviewed, then deviations coming from the structure of the first cluster will be estimated and finally a formula for the truly asymptotic behavior ($\ln Q^2 \gg 1$) of the form factor will be obtained.

The basis of the Drell-Yan relation within the context of the *naive* conventional parton model is the assumed boundedness of the transverse-momentum fluctuations of the partons. A hadron in the infinite-momentum frame is viewed as a system of partons in phase space with transverse momenta contained within an interval of size κ as in Fig. 4(a). If a virtual photon of transverse momentum $Q \gg \kappa$ is absorbed by a parton of longitudinal fraction η, then in general the resulting parton distribution has negligible probability to compose a stable hadron, since it cannot be confined to a strip of width κ [Fig. 4(b)]. However, if the struck parton carries almost all the longitudinal fraction of the hadron as in Fig. 5(a), then the resulting distribution can form an outgoing hadron. The requirement that the resulting partons are confined to a strip of width κ as shown in Fig. 5(b), implies that the longitudinal fraction of the struck parton satisfies

$$\eta > 1-\kappa/Q\,.$$

The form factor $G_0(Q^2)$ at large Q^2 is proportional to the probability to find a parton in this range. Therefore,

$$G_0(Q^2)\sim\int_{1-\kappa/Q}^{1}F_2(\eta,N=0)\,d\eta. \tag{33}$$

Assuming that $F_2(\eta,N=0)$ vanishes as a power of $(1-\eta)$ near $\eta=1$, we obtain

$$G_0(Q^2)\sim\int_{1-\kappa/Q}^{1}(1-\eta)^{\gamma}\,d\eta\sim\left(\frac{\kappa}{Q}\right)^{\gamma+1} \tag{34}$$

which is the Drell-Yan relation.[16]

Now increase the momentum Q until it lies in the second scaling region ($\Lambda<Q<\Lambda^2$). The photon is now absorbed by a cluster of type $N=2$ which is within a particular cluster of type $N=1$. In

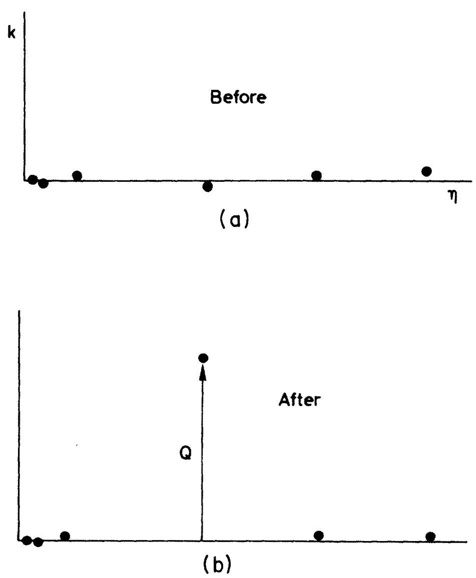

FIG. 4. (a) Typical parton distributions in a hadron. (b) Parton distribution after absorption of a deep-inelastic photon.

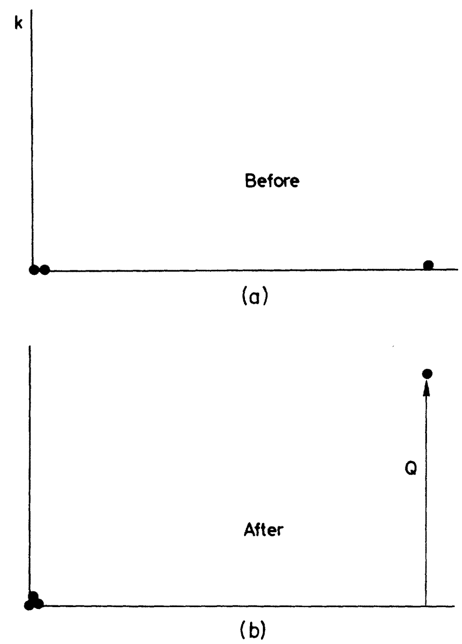

FIG. 5. (a) Parton distribution in a hadron during a "leading-parton fluctuation." (b) Distribution after the leading parton absorbs a deep-inelastic photon.

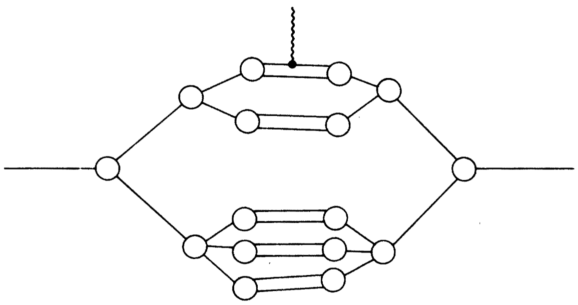

FIG. 6. Form-factor calculation described in text.

order that the hadron not be broken we require that the struck cluster of type $N=2$ absorb the photon's momentum without breaking up its parent cluster of type 1, and that the cluster of type 1 project back into the hadron wave function. (See Fig. 6.) Thus, in addition to the first Drell-Yan factor controlling the absorption of the cluster of type 1 into the hadron's wave function, there is a second factor describing the composite structure of the cluster of type 1. This second factor is obtained by applying the Drell-Yan argument to the cluster of type 1

$$\int_{1-\Lambda\kappa/Q}^{1} f_{2,1}(\eta)\,d\eta \equiv g(Q^2/\Lambda\kappa), \tag{35}$$

where $\Lambda\kappa$ is the momentum fluctuation of type two clusters. Thus, the form factor in the second scaling region is roughly given by

$$G_1(Q^2)=G_0(Q^2)\,g(Q^2/\Lambda\kappa). \tag{36}$$

An iteration of this argument gives $G_N(Q^2)$ in the Nth scaling region,

$$G_N(Q^2)=G_0(Q^2)\,g(Q^2/\Lambda\kappa)\,g(Q^2/\Lambda^2\kappa)\cdots g(Q^2/\Lambda^N\kappa). \tag{37}$$

Using Eq. (28) in Eq. (35) gives $g(Q^2/\Lambda\kappa)\approx Z$ so that Eq. (38) gives

$$G_N(Q^2)\approx\left(\frac{Q}{\kappa}\right)^{-\gamma-1} Z^N. \tag{38}$$

From Eq. (4),

$$N=[\ln(Q/\xi)]/\ln\Lambda,$$

so the form factor behaves like

$$G(Q^2)\approx(Q)^{-\gamma-1}Z^{[\ln(Q/\xi)]/\ln\Lambda}$$
$$\approx(Q)^{-\gamma-1+(\ln Z)/\ln\Lambda}. \tag{39}$$

This calculation relates the properties of f and $F_2(N=0)$ to the asymptotic dependence of the form factor.

The parameter Z can be related to the anomalous dimension of a charge-carrying field. To see this suppose that the underlying field theory is cut off at some enormous

$$Q_{\text{cutoff}}=\xi\Lambda^{\bar N}\quad,$$

where $\bar N \gg 1$. Imagine describing the entire theory with the field-theoretic Hamiltonian $h_{\bar N+1}$. Now, consider the probability that a cluster of type 1 contains just one cluster of type $\bar N+1$. In our model this is the probability that a cluster of type 1 be a cluster of type 2, times the probability that a cluster of type 2 be a cluster of type 3, etc. The resulting wave-function renormalization constant, $Z^{\bar N}$, is

$$Z^{\bar N}=(Q_{\text{cutoff}}/\xi)^{(\ln Z)/\ln\Lambda}. \tag{40}$$

This relation shows how the wave-function renormalization constant of the charged field varies with the cutoff momentum. Equivalently, it defines the anomalous dimension of the field to be $(\ln Z)/2\ln\Lambda$.

Finally we observe that the form factor of nonconserved currents will contain an additional Q^2-dependent factor. This factor describes the variation of the coupling of the current to clusters of different sizes. Recall from Eq. (5) that this additional factor is of the form $e^{\text{const}\,N}$. Since $N\sim\ln Q^2$, the additional factor is seen to be power-behaved and would introduce an additional power of Q^2 into the form factor of the nonconserved current.

IV. DISCUSSION

We conclude with some comments concerning our assumptions and we present a list of our main results. Our first assumption was the existence of discrete scales. When a virtual photon's mass $\sqrt{Q^2}$ is between Λ^N and Λ^{N+1} we have assumed that conventional parton ideas can be applied. Strictly speaking, this requires two things: first, the photon's wavelength λ must be small enough to resolve the distance between clusters of type N (incoherent scattering); and second, the photon's wavelength must be large enough so that clusters of type N appear as points. Real field theories may not contain such abrupt transitions between scales. A better description of scale invariance would require a differential form of the recursive equations. We expect that these refinements will smoothly connect the discrete recursive equations used here. For example, the differential form of Eq. (8) should read

$$\frac{\partial F_2(\beta,\ln Q^2)}{\partial\ln Q^2}=\int_0^\infty \bar f(\beta/\eta)\,F_2(\eta,\ln Q^2)\,\frac{d\eta}{\eta}.$$

Our second main assumption is the existence of a fixed point in the transformation Eq. (3). This fixed point led to the scale invariance of the theory. Wilson[17] has emphasized that fixed points are just one of the possible behaviors to be expected. Another possibility is the existence of a limit cycle solution to Eq. (3). This case would be similar to the scale-invariant solution discussed here when averaged over cycles. Other possibilities include irregular wandering of the parameters of the Hamiltonian or a systematic approach to infinity of some parameters. Even if the scale-invariant solution is correct, we do not know at what level (what value of N) the repetitiveness begins. For example, if quarks exist, then the repetition cannot begin at the hadron level because of quantum numbers. It is possible that it begins at the quark level ($N=1$, say) with mini-quarks inside quarks, etc. It is also possible that it does not begin until many orders of magnitude have elapsed.

Another assumption we have used is that the electric charge of clusters is independent of N. Another possibility is that the charge decreases[18] asymptotically with increasing N. This behavior would necessarily violate current algebra at short distances, since the electric charge would then be fixed by the nonlinear algebra. There is, of course, no guarantee that current algebra is true for distances smaller than 10^{-14} cm.

The scale-invariant parton model may also be applied to the structure of final-state hadron distributions produced in deep-inelastic processes. In general, however, this requires dynamical assumptions beyond those discussed here. One application in this direction is due to Polyakov.[19] However, one qualitative feature of electroproduced final states may be independent of the detailed mechanisms and should be pointed out. For a given q^2 of the virtual photon, clusters of type $N=[\ln(Q/\xi)]/\ln\Lambda$ are struck and carry off the momentum of the photon. Clusters of type N are bound within clusters of type $N-1$ and therefore have transverse-momentum fluctuations characteristic of the size of the clusters of type $N-1$, i.e., ${R_{N-1}}^{-1}$. But ${R_{N-1}}^{-1}$ grows linearly with Q, so before and after the absorption of the photon, the struck cluster has momentum transverse to the photon direction of order Q. Thus, we expect that as q^2 increases (for fixed $q^2/2q\cdot p$) the transverse momentum (relative to the photon direction) of current fragments (carrying a fixed fraction of the photon's lab energy) should increase. This effect should accompany a breakdown of the Bjorken scaling law and therefore should not necessarily apply at the present values of Q^2 ($1<Q^2<10$ GeV^2).

We conclude with a short summary of the main results of our analysis:

1. Scaling laws for the moments of νW_2,

$$\int_0^1 \eta^n\,\nu W_2(Q^2,\nu)\,d\eta = L(n)(Q^2)^{-d_n}\,,$$

where $d_0=0$, $L(n)\sim n^{-\gamma-1}$, and $d_n\to(\ln Z)/\ln\Lambda^2$ as $n\to\infty$.

2. Nonzero ratio σ_S/σ_T in the deep-inelastic region.

3. Behavior of the asymptotic electromagnetic form factor of a hadron,

$$G(Q^2)\sim(Q)^{-a}\,,$$

where $a=\gamma+1-(\ln Z)/\ln\Lambda$. Associated with the structure of the form factor is the behavior of νW_2 near $\eta=1$,

$$\nu W_2\sim(1-\eta)^\gamma(Q)^{(\ln Z)/\ln\Lambda}.$$

4. Increase of average transverse momentum (relative to photon direction) of current fragments as Q^2 increases (at fixed $q^2/2p\cdot q$).

Note added in proof. Combining the discussion following Eq. (40) with Eq. (31) shows that $\lim_{n\to\infty} d_n$ equals the anomalous dimension of the charged field. This relation has also been obtained by G. Parisi, Nuovo Cimento Lett. 4, 777 (1972).

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[6] K. G. Wilson and J. Kogut, Phys. Rep. (to be published).

[7] $N = 0$ clusters are interpreted as hadrons, $N = 1$ clusters are today's partons, etc.

[8] J. Kogut and Leonard Susskind, Phys. Rep. (to be published).

[9] The transformation defined by Eq. (1) is a combination of a dilatation and a longitudinal boost. Since $\tau = (t+z)/\sqrt{2}$ and $\eta = (E + p_z)/\sqrt{2}$ the dilatation operation gives

$$\tau \to \Lambda\tau ,$$
$$X \to \Lambda X,$$
$$\eta \to \Lambda^{-1}\eta .$$

A longitudinal boost rescales τ and η,

$$\tau \to e^{\omega}\tau, \quad \eta \to e^{\omega}\eta$$

Choosing $e^{\omega} = \Lambda$ and combining the two transformations,

$$\tau \to \Lambda^2\tau ,$$
$$X \to \Lambda X,$$
$$\eta \to \eta .$$

[10] Our kinematic conventions are the following. In a deep-inelastic process q denotes the momentum of the virtual photon and p is the momentum of the target hadron. $q^2 = -Q^2 < 0$. $\nu = p \cdot q$. In the deep-inelastic limit Q^2 and ν approach infinity while their ratio is held fixed. Infinite-momentum frames are defined as in Ref. 8.

[11] Properly speaking, the boundary condition should be taken at a value of $N = N_0$ where scale invariance has become approximately valid. The extrapolation between N_0 and hadrons involves non-scale-invariant physics. The only modification in our analysis which follows from taking $N_0 \neq 0$ is a readjustment of the parameter γ appearing in Eq. (29).

[12] The fact that scale-invariant theories do not necessarily lead to Bjorken scaling for νW_2 was demonstrated in the context of perturbation theory by M. Kugler and S. Nussinov, Nucl. Phys. B28, 97 (1971), and many others. Scaling laws for the moments of νW_2 were suggested by A. M. Polyakov, Zh. Eksp. Teor. Fiz. 61, 1572 (1971) [Sov. Phys.—JETP 34, 1177 (1972)], on the basis of perturbation theory. Using scale invariance and Wilson's operator product expansion G. Mack, Nucl. Phys. B35, 592 (1971), also derived scaling laws for the moments and related them to the dimensions of the operators in the expansion. Unfortunately, this paper contains errors. See Y. Frishman, in *Proceedings of the XVI International Conference on High Energy Physics, Chicago-Batavia, Ill., 1972,* edited by J. D. Jackson and A. Roberts (NAL, Batavia, Ill., 1973).

[13] To our knowledge the only field theories where structure functions are δ functions at $\eta = 1$ are free. Since super-renormalizable theories are free at short distances, then $f(\eta) = \delta(\eta - 1)$ and Bjorken scaling occurs. It remains to be seen if realizations of scale invariance different from the one considered in this paper can sensibly lead to Bjorken scaling.

[14] See, for example, R. P. Feynman, *Photon-Hadron Interactions* (Benjamin, New York, 1972).

[15] S. D. Drell and Tung-Mow Yan, Phys. Rev. Lett. 24, 181 (1970).

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[17] K. G. Wilson, Phys. Rev. D 3, 1818 (1971).

[18] This possibility was suggested to us by A. Casher. It could provide a mechanism for avoiding an infinite neutron-proton mass difference.

[19] A. M. Polyakov, Zh. Eksp. Teor. Fiz. 60, 1572 (1971) [Sov. Phys.—JETP 33, 850 (1971)].