(1972).
[24]See, e.g., K. Fujikawa, B. W. Lee, and A. I. Sanda, Phys. Rev. D 6, 2923 (1972).
[25]G. Barbarino *et al.*, Nuovo Cimento Lett. 3, 689 (1972); C. Bacci *et al.*, Phys. Lett. 38B, 551 (1972).
[26]S. Gasiorowicz and D. A. Geffen, Rev. Mod. Phys. 41, 531 (1969).
[27]See, e.g., S. Mandelstam, *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser *et al*. (MIT Press, Cambridge, Mass., 1970). See also J. L. Gervais and A. Neveu, Nucl. Phys. B46, 381 (1972).

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Galilean subdynamics and the dual resonance model*

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Dynamics in Minkowski space is discussed in terms of an eight-parameter extended Galilei group, a subgroup of the Poincaré group. This method—Galilean subdynamics—is developed and discussed in detail and applied to the construction of two explicit models having four-point amplitudes of the Veneziano type. There are no difficulties with unphysical states.

I. INTRODUCTION

The present paper is an application of the "method of Galilean subdynamics," to the problem of constructing a consistent basis for the dual resonance model.

Galilean subdynamics, developed in connection with our interpretation of Dirac's positive-energy wave equation,[1,2] is another attempt to describe relativistic dynamics. Dynamics in Minkowski space has been conventionally described as the change with time of a configuration given at one instant of time in a particular reference frame in Minkowski space. There have always been difficulties with this approach for both classical theory and quantum theory. One of the difficulties is the proper description of a composite particle. As a suitable reference frame, one might choose, for example, a rest frame of the particle. If the constituents of the particle move slowly then one can use nonrelativistic mechanics for the description—at least approximately. In the general case, however, this clearly does not work, and it seems unlikely that proceeding in this way one can ever find a relativistic description of a hadron.

Much more successful have been attempts at an over-all Minkowski space-time viewpoint. At the classical level such theories do allow one to describe, and to develop models for, composite systems.[3] At the quantum level there are still difficulties despite the remarkable results of, say, the dual resonance model within the S-matrix framework.

There exists a potentially important alternative to this over-all space-time viewpoint. For classical physics this alternative was introduced by Dirac[4] and designated the *front form* of dynamics. Dirac proposed to consider a family of parallel tangent spaces to the light cone instead of the usual family of parallel spaces at various instants of time (called by Dirac the *instant form* of dynamics). Superficially, for classical theory, it is not clear that the front form of dynamics is any better than the instant form of dynamics. However, the quantum version of the front form shows an important distinction from the classical case. This follows from the fact that of the three coordinates in the front—x_+, x_1, x_2, (with x_- being the coordinate specifying the front)—the coordinate x_+, unlike x_1 and x_2, has the nature of a time, that is, the momentum conjugate to x_+ (denoted by P_-) has a spectrum confined to the open positive half-line. It is accordingly not permitted in the quantum version to assume a kinematics based upon specifying a point within the front. A way out of this fundamental difficulty is afforded by the fact that there exists an eight-parameter subgroup of the Poincaré group which adjoins the operator P_- to the seven generators that leave the front invariant. This subgroup has the group structure of nonrelativistic (Galilean) dynamics in two space dimensions, together with a scaling operator. The momentum operators P_- and P_+ play the role of Galilean Hamiltonian and Galilean mass, respectively, and inherently possess the proper (positive) spectrum. We call this structure "Galilean subdynamics," or the "Galilean subworld." Unlike the instant subworld (associated with a six-parameter subgroup) the Galilean subworld contains dynamics within itself which properly fits into the dynamics of the Poincaré world. For the dynamics within this Galilean subworld one can use nonrelativistic quantum mechanics; in a previous paper[2] we showed that this dynamics allowed one to describe a composite particle lying on a linear Regge trajectory as a bound system within the Galilean subworld. The various excited levels of this bound system (the structure involves two harmonic oscillators) manifest themselves in Minkowski space as particles with mass and spin related by $(\text{mass})^2$ = linear function of spin. That all these particles are excitations of a single structure is a meaningful statement which can be demonstrated in the Galilean subworld, but not in Minkowski space.

In Sec. II, we discuss Galilean subdynamics in detail, developing the results of Ref. 2 from a new, and simpler, point of view. This section contains several new results essential for the generalizations necessary to incorporate the quantized structures of the dual resonance model.

The dual resonance model was initiated by Veneziano in 1968 (Ref. 5) when he exhibited an explicit four-point amplitude incorporating duality and crossing symmetry with all particles lying on Regge trajectories. It was an essential characteristic of the model that the Regge trajectories were straight lines $m^2 \sim J$, with a common slope parameter; such indefinitely rising trajectories necessarily imply resonances of zero width.[6] In a sense the associated unitarity difficulty could be disregarded, by viewing the model as a new type of zeroth-order approximation to a self-consistent S-matrix theory, to be corrected in higher approximations.[6]

Subsequent developments and generalizations of the dual resonance model, for example, to the N-point amplitude, have enhanced the appeal of the model by demonstrating general factorization properties[7] and agreement with multiperipheralism concepts. These generalizations have been characterized by impressive technical virtuosity in extending the original model mathematically,[8] and by physical insight, seeking for example to base the model on the quantization of a relativistic string in Minkowski space.[9]

Despite the many successes of these generalizations, the dual resonance model still suffers from serious physical defects: all current versions of the model (in Minkowski space) *inherently involve the existence of particles with spacelike momenta ("tachyons") and/or negative-metric particles ("ghosts").*[10] While one may reasonably postpone the known unitarity limitations of the model (as the next step in an approximation procedure say), it is not possible to ignore inconsistencies.

In Secs. III and IV we use the method of Galilean subdynamics to demonstrate, in two explicit cases, a foundation for the dual resonance model (DRM), which eliminates, inherently, all difficulties connected with unphysical particles.

The simpler case, in Sec. III, generalizes the pair of oscillators of the new Dirac equation to a denumerably infinite sequence of pairs. The model is a quantized "string" with two modes of oscillation, but "visible" as such only in the Galilean subworld. Such a model automatically has the correct propagator for the DRM, and, as we demonstrate, permits one to obtain a Veneziano four-point amplitude for spinless particles. The generalization to N-point amplitudes is not clear; this difficulty is a property of the model (as we discuss in that section).

In Sec. IV we construct a model which possesses dual resonance N-point amplitudes for spinless particles. This model is a quantized version of a two-dimensional surface ("spherical rubber sheet"), if viewed in the Galilean subworld. Such a structure has many attractive features, for example, it appears to offer the possibility of incorporating spin into the external particles of the dual resonance model.

II. GALILEAN SUBDYNAMICS

Although one purpose of the present section is to review Dirac's positive-energy wave equation and to explain its generalization,[2] we shall not simply repeat the contents of Ref. 2. Instead we shall develop these results from an entirely different point of view, which not only is considerably simpler, but makes evident the remarkable connection of this construction to Wigner's original treatment of the Poincaré group.[11] We utilize this new viewpoint to develop the concept of Galilean subdynamics in a way both simpler, and more detailed, than originally.

Let us begin by recalling that Dirac's (generalized) new relativistic wave equation[1] describes a composite particle, having intrinsic spin j and mass m, whose internal structure is based upon two harmonic oscillators (π_1, ξ_1) and (π_2, ξ_2), where π_i and ξ_i are the usual dimensionless momenta and coordinates $[\xi_i = (\omega\mu/\hbar)^{1/2} q_i,\ \pi_i = (\hbar\omega\mu)^{-1/2} p_i]$. We also use the (non-Hermitian) operators $2^{1/2}\bar{a}_i \equiv i\pi_i + \xi_i$ and $2^{1/2} a_i \equiv -i\pi_i + \xi_i$ which obey the commutation rule $[\bar{a}_i, a_j] = \delta_{ij}$, all other commutators vanishing.

Our procedure will be to construct a basis over the Minkowski space variables $\{x_\mu\}$ and the harmonic-oscillator coordinates ξ_i, carrying an irreducible representation (irrep) realizing Poincaré group generators.[12] These generators are taken to be

$$P_\mu \equiv (\hbar/i)\partial_\mu ,$$
$$M_{\mu\nu} \equiv \epsilon_{\mu\nu\lambda\sigma} x_\lambda P_\sigma + S_{\mu\nu} , \tag{1}$$

where $S_{\mu\nu}$ are generators of a Lorentz group realization based upon the two harmonic oscillators of the internal structure. It is essential to the success of this construction that the internal structure actually supports a realization of the symplectic group Sp(2, 2) whose ten generators are given by[13]

$$\begin{aligned}
\{J\}:\ & J_1 = \tfrac{1}{2}(a_1\bar{a}_2 + a_2\bar{a}_1) , \\
& J_2 = \tfrac{1}{2}(a_1\bar{a}_1 - a_2\bar{a}_2) , \\
& J_3 = \tfrac{1}{2}i(a_1\bar{a}_2 - a_2\bar{a}_1) , \\
\{K\}:\ & K_1 = \tfrac{1}{4}(a_1{}^2 - a_2{}^2 + \bar{a}_1{}^2 - \bar{a}_2{}^2) , \\
& K_2 = -\tfrac{1}{2}(a_1a_2 + \bar{a}_1\bar{a}_2) , \\
& K_3 = \tfrac{1}{4}i(a_1{}^2 + a_2{}^2 - \bar{a}_1{}^2 - \bar{a}_2{}^2) , \\
\{V\}:\ & V_1 = \tfrac{1}{4}i(\bar{a}_1{}^2 - \bar{a}_2{}^2 + a_2{}^2 - a_1{}^2) , \\
& V_2 = \tfrac{1}{2}i(a_1a_2 - \bar{a}_1\bar{a}_2) , \\
& V_3 = \tfrac{1}{4}(a_1{}^2 + a_2{}^2 + \bar{a}_2{}^2 + \bar{a}_1{}^2) , \\
& V_0 = \tfrac{1}{2}(a_1\bar{a}_1 + a_2\bar{a}_2 + 1) .
\end{aligned} \tag{2}$$

This complicated choice of coordinates for the Sp(2, 2) operators is necessary to accord with the conventional choice of front coordinates: $P_\pm = P_0 \pm P_3$. [The necessity to choose a relative orientation (for Minkowski vs spin space) is discussed in connection with Eq. (48).]

The Lorentz group generators $S_{\mu\nu}$ are the subset: $S_{ij} = \epsilon_{ijk}J_k$, $S_{i4} = K_i$. (To be precise, let us remark that the operators $\{J, K, V\}$ are Hermitian with respect to the inner product based on the vacuum ket $|0\rangle$ defined by $\bar{a}_i|0\rangle = 0$.)

The ingenious feature of Dirac's construction is that the finite-dimensional (nonunitary) representations $(n, 0)$ of the group Sp(2, 2) also play a role. The simplest nontrivial case is that for (1, 0); this has the four-dimensional basis, denoted by Q:

$$Q \equiv \begin{bmatrix} a_1 \\ a_2 \\ \bar{a}_2 \\ -\bar{a}_1 \end{bmatrix} . \tag{3}$$

Under the action of the Sp(2, 2) generators Θ we find that

$$[\Theta, Q] = \tilde{\Theta} Q , \tag{4}$$

where the $\tilde{\Theta}$ are numerical 4×4 matrices. The structure represented by Eq. (4) is a mapping from the boson *operator* form of the generators to a *numerical* matrix form of the generators. This mapping preserves commutation relations but not the Hermitian character of the two realizations. Let us prove this remark, particularly since the precise nature of this mapping is easily confused.

We must show that commutators are preserved, hence we consider the commutator

$$\begin{aligned}
&[\Theta_\alpha, [\Theta_\beta, Q]] - [\Theta_\beta, [\Theta_\alpha, Q]] \\
&\quad = [[\Theta_\alpha, \Theta_\beta], Q] \quad \text{by the Jacobi relation,} \\
&\quad = [\Theta_\alpha, \tilde{\Theta}_\beta Q] - [\Theta_\beta, \tilde{\Theta}_\alpha Q] \quad \text{by Eq. (4)} \\
&\quad = [\tilde{\Theta}_\beta\tilde{\Theta}_\alpha - \tilde{\Theta}_\alpha\tilde{\Theta}_\beta] Q \quad \text{since the } \tilde{\Theta} \text{ are numerical and commute with } \Theta.
\end{aligned} \tag{5}$$

Applying Eq. (4) once again, we see that

$$\begin{aligned}
\tilde{\Theta}_{[\alpha,\beta]} &= \tilde{\Theta}_\beta\tilde{\Theta}_\alpha - \tilde{\Theta}_\alpha\tilde{\Theta}_\beta \\
&\equiv \text{transpose of } [\Theta_\alpha, \Theta_\beta] .
\end{aligned} \tag{6}$$

This result means that the matrices $\tilde{\Theta}_\alpha$ are the *transposed matrices* of that finite-dimensional representation of Sp(2, 2) carried by Q. We will see below that this "left-handed" nature, intrinsic to this mapping, Eq. (4), is essential for the physics.

Using this mapping we can now determine how the basis Q transforms under a finite Lorentz transformation, denoted by Λ.

$$\Lambda:\ Q \to Q' \equiv e^{i\Lambda\cdot M} Q e^{-i\Lambda\cdot M} = e^{i\Lambda\cdot S} Q e^{-i\Lambda\cdot S} . \tag{7}$$

Using the Baker-Campbell-Hausdorff rule, we find

$$Q' = \sum_k \frac{1}{k!} [i\Lambda\cdot S, Q]_{(k)} , \tag{8}$$

where $[\Theta, Q]_{(k)}$ denotes the kth multiple commutator.

Introducing the mapping defined by Eq. (4) we find that

$$[i\Lambda\cdot S, Q]_{(k)} = (i\Lambda\cdot\tilde{S})^k Q ,$$

so that we obtain the result

$$\Lambda:\ Q \to Q' = e^{i\Lambda\cdot\tilde{S}} Q . \tag{9}$$

This result shows that the boson operator basis Q transforms as expected, under a general Lorentz transformation Λ; what is less obvious is that (1) the commutation relations for the transformed bosons are *unchanged*, and (2) the vacuum ket $|0\rangle$ *has changed*.

Consider the first remark. The elements of the basis vector Q may be considered individually as operators. As such, the transformation generated by Λ is *unitary*, when expressed in the form

$$\Lambda:\ Q_i \to Q_i' \equiv U(\Lambda)Q_i U^{-1}(\Lambda) = \begin{bmatrix} a_1(\Lambda) \\ a_2(\Lambda) \\ \bar{a}_2(\Lambda) \\ -\bar{a}_1(\Lambda) \end{bmatrix}.$$

In this form, the invariance of the commutation relations is obvious.

Next consider the ket $|0\rangle$. This ket is defined by the two conditions: $\bar{a}_i|0\rangle = 0$. It is invariant for those Lorentz transformations which leave the time axis invariant; that is, the rotations leaving the unit 4-vector (0, 0, 0, 1) invariant. We must include this information in the notation for the ket; accordingly we denote $|0\rangle$ by $|0;\mathfrak{u}_0\rangle$, where $\mathfrak{u}$ denotes a unit 4-vector (proper velocity) and $\mathfrak{u}_0$ the vector (0, 0, 0, 1). Under the Lorentz transformation Λ, this ket becomes

$$e^{i\Lambda\cdot S}|0;\mathfrak{u}_0\rangle \equiv |0;\mathfrak{u}_\Lambda\rangle,$$

where $\mathfrak{u}_\Lambda$ is the unit vector $\Lambda\mathfrak{u}_0$. Clearly one has

$$\bar{a}_i(\Lambda)|0;\mathfrak{u}_\Lambda\rangle = 0 .$$

We seek now to interpret the meaning of the ket vector $|0;\mathfrak{u}_\Lambda\rangle$, where $\mathfrak{u}_\Lambda$ denotes an arbitrary unit 4-vector with positive time component.

To do so, let us find the wave function, $\langle\xi_1\xi_2|0;\mathfrak{u}_\Lambda\rangle$, using bra vectors adapted to the (ξ_1, ξ_2) basis for the internal space. One could find this wave function directly by using

$$|0;\mathfrak{u}_\Lambda\rangle \equiv e^{i\Lambda\cdot S}|0;\mathfrak{u}_0\rangle,$$

so that

$$\langle\xi_1\xi_2|0;\mathfrak{u}_\Lambda\rangle = \langle\xi_1\xi_2|e^{i\Lambda\cdot S}|0;\mathfrak{u}_0\rangle. \tag{10}$$

This direct calculation is surprisingly complicated. An alternative—equivalent but far easier—procedure utilizes the two differential equations

$$\langle\xi_1\xi_2|\bar{a}_i(\Lambda)|0;\mathfrak{u}_\Lambda\rangle = 0 . \tag{11}$$

Let Λ denote a general boost, parametrized by $(\chi\theta\varphi)$, where $\tanh\chi = v/c$, and $(\theta\varphi)$ specify the direction of the three-vector v. It follows from Eq. (9) that the transformed boson operators $\bar{a}_i(\Lambda)$ have the explicit form

$$\begin{aligned}\bar{a}_1(\chi\theta\varphi) &= \cosh(\tfrac{1}{2}\chi)\bar{a}_1 \\ &\quad + i\sinh(\tfrac{1}{2}\chi)(-\sin\theta\, e^{-i\phi}a_1 + \cos\theta\, a_2), \\ \bar{a}_2(\chi\theta\varphi) &= \cosh(\tfrac{1}{2}\chi)\bar{a}_2 \\ &\quad + i\sinh(\tfrac{1}{2}\chi)(\cos\theta\, a_1 + \sin\theta\, e^{i\phi}a_2).\end{aligned} \tag{12}$$

Solving the two differential equations, (11), leads to the desired wave function:

$$\langle\xi_1\xi_2|0;\mathfrak{u}_\Lambda\rangle = \pi(\cosh\chi + \sinh\chi\sin\theta\sin\varphi)^{-1/2}\exp\{-(2\cosh\chi + 2\sinh\chi\sin\theta\sin\varphi)^{-1}[(\xi_1^2+\xi_2^2) + 2i\sinh\chi\cos\theta\,\xi_1\xi_2 + i\sinh\chi\sin\theta\cos\varphi(\xi_2^2 - \xi_1^2)]\}. \tag{13}$$

The $(\chi\theta\varphi)$ parameters specify an arbitrary boost; an alternative parametrization uses the 4-vector p/m, having unit length. In terms of this velocity, and choosing a symplectic (front) [14] coordinate frame $(A\cdot B = -A_1B_1 - A_2B_2 + A_+B_- + A_-B_+)$, one finds a much more understandable form for the wave function:

$$\langle\xi_1\xi_2;0;p/m\rangle = \left(\frac{m}{\pi p_+}\right)^{1/2}\exp\left\{-\left(\frac{m}{2p_+}\right)\left[(\xi_1^2+\xi_2^2) - \frac{2ip_2}{m}\xi_1\xi_2 + \frac{ip_1}{m}(\xi_2^2 - \xi_1^2)\right]\right\}. \tag{14}$$

Since the boson operators all commute with the (Minkowski space) momentum operators P, we are allowed to multiply the wave functions of Eq. (14) by the momentum eigenfunction $\exp(ip\cdot x/\hbar)$, without affecting the internal space solution. Thus we obtain the wave function:

$$\langle\xi_1\xi_2|0;p\rangle = \exp\left[\frac{i}{\hbar}(p\cdot x)\right]\langle\xi_1\xi_2|0;p/m\rangle. \tag{15}$$

This wave function is precisely the solution to Dirac's new equation, and corresponds to a (composite) particle having intrinsic spin zero, four-momentum p, and mass $m = (p\cdot p)^{1/2}$.

Our method of constructing this solution is now clear: we have simply Lorentz-transformed the *rest-frame harmonic-oscillator solution:*

$$\langle\xi_1\xi_2|0;(000m)\rangle = \pi^{-1/2}\exp(-imt/\hbar) \times\exp[-\tfrac{1}{2}(\xi_1^2+\xi_2^2)]. \tag{16}$$

(Note that the harmonic-oscillator part of the wave function is independent of the value of the mass m; the mass dependence comes only from the momentum exponential.)

Once we have obtained this insight into the remarkably simple structure underlying Dirac's result, it is quite easy to write out the generalization to nonzero spin. First note that in the rest frame, the generators for the spin, J_i in Eq. (2), all vanish when applied to Dirac's wave function, (16), since $J_i|0\rangle=0$. To obtain, in the rest frame, a state vector having spin j, projection μ, one clearly uses the ket vectors:

$$|j\mu\rangle \equiv \frac{(a_1)^{j+\mu}(a_2)^{j-\mu}}{[(j+\mu)!(j-\mu)!]^{1/2}}\,|0\rangle\ . \tag{17}$$

It is obvious now, that in a general frame, obtained from the original frame by Λ, we obtain the transformed result:

$$|j\mu;\Lambda\rangle=[(j+\mu)!(j-\mu)!]^{-1/2}\times[a_1(\Lambda)]^{j+\mu}[a_2(\Lambda)]^{j-\mu}\,|0;\mathfrak{u}_\Lambda\rangle\ . \tag{18}$$

The corresponding wave function, having sharp momentum $\vec{p}$ is accordingly given by

$$\langle\xi_1\xi_2|j\mu;p\rangle=\exp\left(\frac{i}{\hbar}p\cdot x\right)\langle\xi_1\xi_2|j\mu;p/m\rangle\ . \tag{19}$$

Let us now verify directly that this result constitutes an elegant and simple construction for a Poincaré irrep with the invariant labels m and j.

(a) The wave function Eq. (19) possesses sharp four-momentum p, with $p\cdot p=m^2$. Proof: Operate with the momentum generators $\vec{P}$ noting that the internal part of the wave function is unaffected.

(b) The wave function Eq. (19) transforms as a $(2j+1)$-dimensional irrep under all general Lorentz transformations which leave the 4-vector p-invariant. Proof: Consider the spin-0 case first. Under a general Lorentz transformation Λ, we find that

$$\Lambda:\ \exp(ip\cdot x/\hbar)\rightarrow\exp(i\Lambda p\cdot x/\hbar)\ ,$$

and that for the internal wave function

$$e^{i\Lambda\cdot S_{\mathrm{op}}}\langle\xi_1\xi_2|0;p/m\rangle=\langle\xi_1\xi_2|e^{i\Lambda\cdot S}e^{i\Lambda(P/m)\cdot S}|\,0;0001\rangle=\langle\xi_1\xi_2|0;\Lambda p/m\rangle\ . \tag{20}$$

(The notation S_{op} is to denote that these generators are to be written as differential operators in ξ and $\partial/\partial\xi$.)

It follows that under a Lorentz transformation that leaves p invariant, the Dirac solution is indeed invariant, as required for spin zero.

For the case where the spin is j, we note first that the operators $J_i(\Lambda)$ defined by

$$\begin{aligned}J_1'&=\tfrac{1}{2}[a_1(\Lambda)\bar{a}_2(\Lambda)+a_2(\Lambda)\bar{a}_1(\Lambda)]\ ,\\ J_2'&=(J_+')^+\\ &=\tfrac{1}{2}[a_1(\Lambda)\bar{a}_1(\Lambda)-a_2(\Lambda)\bar{a}_2(\Lambda)]\ ,\\ J_3'&=\tfrac{1}{2}i[a_1(\Lambda)\bar{a}_2(\Lambda)-a_2(\Lambda)\bar{a}_1(\Lambda)]\ ,\end{aligned} \tag{21}$$

clearly leave the ket $|0;\mathfrak{u}_\Lambda\rangle$ invariant. These operators are, in fact, the generators of the stability group ("little group") of the Lorentz (unit) vector $\mathfrak{u}_\Lambda=p/m$. [This follows from the fact that $J_i'=U(\Lambda)J_iU^{-1}(\Lambda)$, so that J_i' is just the transformed version of the rest-frame stability group, SU(2).]

The desired result now follows from the construction of Eq. (19), since the $a_i(\Lambda)$ used in this construction carry an irrep(j, m) under the J_i'.

Note that this discussion shows quite directly that Eq. (19) possesses the two Poincaré invariants: $P^2\rightarrow m^2$ and $W^2\rightarrow m^2j(j+1)$. [For the second invariant simply note that $W^2=(m\vec{J}')^2$ in a general frame.]

Let us now complete this discussion by giving the generalized new Dirac equation, and discussing briefly how it relates to the previous discussion. The two key points are that (1) the column vectors Q, for the n quanta, carry an irrep$(n, 0)$ of Sp(2, 2) having $\binom{n+3}{3}$ dimensions and (2) the matrices $\tilde{V}_\mu$ defined by the map $[V_\mu, Q]=\tilde{V}_\mu Q$ transform under general Poincaré transformations as a 4-vector. Thus we may form the invariant operator $\tilde{V}\cdot P$ and define the covariant wave equation (for spin $s=\frac{1}{2}n$)

$$\mathcal{P}_n(\tilde{V}\cdot P/m)Q\psi=0\ , \tag{22}$$

where the polynomial $\mathcal{P}_s$ is defined by

$$\mathcal{P}_n(\tilde{V}\cdot P/m)\equiv\prod_{k=-n}^{n}[\tilde{V}\cdot P/m-(2k+1)]\ . \tag{23}$$

The content of these equations (whose properties are proven in detail in Ref. 2) is most easily seen from Dirac's spin-zero example. For this case, the matrices $\tilde{V}_\mu$ are the transpose of Dirac matrices, so that

$$\{\tilde{V}\}=(i\rho_2\vec{\sigma},\rho_3)\equiv\{\Gamma_\mu\}\ . \tag{24}$$

Thus, for spin-zero one finds

$$(\Gamma_\mu\partial^\mu-mc/\hbar)\begin{bmatrix}a_1\\ a_2\\ \bar{a}_2\\ -\bar{a}_1\end{bmatrix}\psi=0\ , \tag{25}$$

that is, a set of four simultaneous first-order equations. In the rest frame, these equations reduce to an obvious form: $\bar{a}_i\psi=0$. The solution in a general frame is simply Eq. (15), as given previously. The solutions for general spin follow similarly.

Returning to our previous direct construction of the Poincaré irreps, we may summarize our results in this way. One has an internal space over two oscillator variables $(\xi_1\xi_2)$; for every 4-vector p/m (of unit length), one has an associated ket $|0;p/m\rangle$ having no quanta. The internal oscillators carry no momentum; the boson operators $a_i(p/m)$ create quanta. To every state of N quanta, there corresponds a Poincaré state of spin $\frac{1}{2}N$. The mass associated with this Poincaré state is arbitrary, and is determined solely by the momentum part of the wave function, $\exp[(i/\hbar)p\cdot x]$.

Thus in a very literal sense, this construction corresponds exactly to the concept of a "relativistic harmonic oscillator" each of whose states of excitation correspond to Poincaré particle states.[15]

From this point of view, the generalized new Dirac equation is simply a conventional way, using the language of covariant wave equations, to state a result much more easily grasped from the harmonic-oscillator viewpoint, which constructs the Poincaré representations directly.

Let us now demonstrate that the harmonic oscillator view-point can be made even more literal, and can lead to the concept of a subdynamics. The possibility of a subdynamics stems from the fact that the Poincaré group possesses an eight-parameter subgroup generated by the operators

$$\mathfrak{F}=\{P_i\,,P_+,P_-,J_{12}=J_3,K_{-i}\,;M_{03}\}\quad(i=1,2) \tag{26}$$

which obey the commutation rules:

(a) (2+1) Galilei group generators[16]*:* $\mathcal{G}$

$$\begin{aligned}
&[J_3,P_i]=i\epsilon_{3ij}P_j\,,\\
&[J_3,K_{-i}]=i\epsilon_{3ij}K_{-j}\,,\\
&[J_3,P_-]=0\,,\\
&[K_{-i}\,,K_{-j}]=0\,,\\
&[K_{-i}\,,P_j]=i\delta_{ij}P_+,\\
&[K_{-i}\,,P_-]=2iP_i\,,\\
&[P_i\,,P_j]=0\,,\\
&[P_i\,,P_-]=0\,.
\end{aligned} \tag{27a}$$

(b) Mass generator: P_+

$$[P_+,\mathcal{G}]=0\,. \tag{27b}$$

(c) Scaling generator: M_{03}

$$\begin{aligned}
&[M_{03},J_3]=0\,,\\
&[M_{03},K_{-i}]=iK_{-i}\,,\\
&[M_{03},P_i]=0\,,\\
&[M_{03},P_\pm]=\pm iP_\pm\,.
\end{aligned} \tag{27c}$$

These commutation rules can be recognized as the commutation relations of an extended Galilei group $\mathcal{G}$ (in two spatial dimensions) together with a dilation (scaling) operator $D\equiv M_{03}$. For this interpretation one must identify the operator P_- as the Galilei group Hamiltonian ($H_{\mathcal{G}}$) and the operator P_+ as the *mass* operator $M_{\mathcal{G}}$ for the Galilei group.[17-19]

We will seek solutions to this Galilean structure, which exploit the fact that these generators correspond to nonrelativistic quantum mechanics of interacting particles, in a two-dimensional plane. Corresponding to this interpretation we take the center-of-mass momenta, P_1 and P_2, to be sharp: $P_i\to p_i$. The operator P_+, in the Galilean plane corresponds to the total mass, which because of the scaling generator M_{03}, has continuous eigenvalues $p_+>0$. We will also take this generator to be sharp: $P_+\to p_+>0$. The associated wave function is then

$$\exp\left[\frac{i}{\hbar}\,(p_1x_1+p_2x_2-p_+x_-)\right]. \tag{28}$$

For the Galilean Hamiltonian we take the operator P_- to obey the subsidiary relation

$$P_-\psi=(i\hbar\partial/\partial x_+)\psi\,, \tag{29}$$

and then specify P_- to have the form

$$P_-\to H_{\mathcal{G}}=H_{\text{c.m.}}+H_{\text{internal}}\,,$$

where

$$\begin{aligned}
H_{\text{c.m.}}&=\frac{P_1^{\,2}+P_2^{\,2}}{2P_+}\,,\\
H_{\text{int}}(\vec{\mathfrak{u}})&=\frac{m_0^{\,2}}{2P_+}[a_1(\Lambda)\bar{a}_1(\Lambda)+a_2(\Lambda)\bar{a}_2(\Lambda)+1]\,,
\end{aligned} \tag{30}$$

and Λ is *any* Lorentz transformation which transforms $\mathfrak{u}_0$ into $\mathfrak{u}$. Note that H_{int} does not depend on which Λ one chooses to carry $\mathfrak{u}_0$ into $\mathfrak{u}$; that is, H_{int} is invariant under Lorentz transformations which leave $\mathfrak{u}$ invariant.

It is useful to express this internal Hamiltonian in terms of the original boson operators a_i. Using the explicit form, Eq. (12), for $a_i(\Lambda)$ one finds (after some calculation) the form

$$\begin{aligned}
H_{\text{int}}=(m_0^{\,2}/2P_+)\{(\mathfrak{u}_+)[(\pi_1-A_1)^2+(\pi_2-A_2)^2]\\
+(1/\mathfrak{u}_+)(\xi_1^{\,2}+\xi_2^{\,2})\}\,,
\end{aligned} \tag{31}$$

where the A_i are defined to be

$$\begin{aligned}
A_1&=\frac{\mathfrak{u}_2\xi_2-\mathfrak{u}_1\xi_1}{\mathfrak{u}_+}\,,\\
A_2&=\frac{\mathfrak{u}_1\xi_2+\mathfrak{u}_2\xi_1}{\mathfrak{u}_+}\,.
\end{aligned} \tag{32}$$

This form for the internal Hamiltonian has certain unusual, but essential, features which we wish to discuss. First, note that the velocity $\mathfrak{u}_+$ acts as a *scale factor* in determining the "size" of π_i and ξ_i. The generator for this transformation is K_3 which obeys the commutation rules

$$\begin{aligned}
&[K_3,\pi_i]=+\tfrac{1}{2}\pi_i\,,\\
&[K_3,\xi_i]=-\tfrac{1}{2}\xi_i
\end{aligned} \tag{33}$$

so that π_i and ξ_i scale oppositely under K_3 (as is necessary for the Heisenberg commutator to be properly scale-invariant). Thus we see that the scaling transformation, for a finite boost denoted by $\mathfrak{U}_+$, scales the internal coordinates:

$$\mathfrak{U}_+ \text{ boost: } \pi_i \to (\mathfrak{U}_+)^{1/2}\pi_i \ , \tag{34}$$

$$\xi_i \to (\mathfrak{U}_+)^{-1/2}\xi_i \ . \tag{35}$$

The second unusual feature of this Galilean Hamiltonian is that through the coupling via the "vector potential" A_i, the internal wave function is a function of the velocity $\mathfrak{U}_1$ and $\mathfrak{U}_2$, which one would expect to be parallel to the momentum p_1 and p_2 of the center-of-mass Galilean motion. Note, however, that the *eigenvalues* of the operator in brackets, $\{\cdots\}$ in Eq. (31), are nonetheless independent of the 4-velocity $\vec{\mathfrak{U}}$.

These unusual features of the Galilean Hamiltonian pose a problem as to the logic (and consistency) of the subdynamical approach. Clearly one can choose arbitrary eigenvalues for the three commuting momentum operators P_1, P_2, P_+; but these three data do *not* suffice to determine the 4-velocity $\mathfrak{U}$, for one still lacks the mass value (implied by P_-) which must be obtained from the Hamiltonian itself. Although the structure is indeed self-consistent, let us simplify matters by avoiding this direct procedure, and following an alternative path. Consider this same Galilean Hamiltonian to be given by

$$H_{\mathcal{G}} = H(\chi\theta\varphi) = \frac{P_1^2+P_2^2}{2P_+} + \left(\frac{m_0^2}{2P_+}\right)[a_1(\chi\theta\varphi)\bar{a}_1(\chi\theta\varphi) + a_2(\chi\theta\varphi)\bar{a}_2(\chi\theta\varphi)+1] \ ,$$

where $(\chi\theta\varphi)$ now specifies a *fixed* 4-velocity $\mathfrak{U}$:

$$(\sinh\chi\cos\theta,\ \sinh\chi\sin\theta\cos\varphi,\ \sinh\chi\sin\theta\sin\varphi,\ \cosh\chi). \tag{36}$$

The Galilean world is to contain besides this Hamiltonian, $H(\chi\theta\varphi)$, the seven additional generators:

$$\{P_+, P_1, P_2;\ L_{-1}, L_{-2}, L_{03};\ M_{12} = L_{12} + J_{12}(\mathfrak{U})\} \ . \tag{37}$$

Note that only in the last generator, M_{12}, have we introduced an operator acting on the internal space, and this operator is an explicit function of $\chi\theta\varphi$, that is, $J_{12} = J_{12}(\mathfrak{U})$.

It is easily verified that these operators, together with $H(\chi\theta\varphi)$, close on the commutation relations of the Galilei group plus scaling.

Let us give the eigenfunction belonging to a Galilei group representation for which the five operators: P_+, P_1, P_2, $H(\chi\theta\varphi)$, and J_{12} (spin) have been brought to diagonal form. Denoting these eigenvalues by

$$(P_+, P_1, P_2, H = P_-) \to p \ , \quad J_{12} \to M \ , \tag{38a}$$

we find for the wave function

$$\psi(p, JM; \chi\theta\varphi) = e^{ip\cdot x}\left\langle \xi_1\xi_2 \left| \frac{[a_1(\chi\theta\varphi)]^{J+M}[a_2(\chi\theta\varphi)]^{J-M}}{[(J+M)!(J-M)!]^{1/2}} \right| 0; \chi\theta\varphi \right\rangle \ . \tag{38b}$$

(Note that the label J is determined by the number of quanta: $N+1 = 2J+1$.)

From the Galilean Hamiltonian we find

$$H_{\mathcal{G}} \to p_- = \frac{1}{2p_+}[p_1^2 + p_2^2 + m_0^2(N+1)]; \tag{39a}$$

hence, one obtains the relations

$$2p_+p_- - p_1^2 - p_2^2 \equiv p\cdot p \equiv m^2 = m_0^2(N+1) \ . \tag{39b}$$

Thus for arbitrary values of the parameters $(\chi\theta\varphi)$, we have solutions defined in a Hilbert space labeled by these parameters. The momentum eigenvalue p is however *independent* of these parameters.

The problem is now to take this solution, from the Galilean world, into the Poincaré world. Does this solution, Eq. (38b) belong to an irrep of $\mathcal{P}$?

Clearly the solution, Eq. (38b), does possess the Poincaré invariant $p^2 \to m^2$. Consider then the second Poincaré invariant. We already know that the stability group for p is generated by the spin operators

$$J_1' = \tfrac{1}{2}[a_1(p/m)\bar{a}_2(p/m) + a_2(p/m)\bar{a}_1(p/m)] \ ,$$
$$J_2' = \tfrac{1}{2}[a_1(p/m)\bar{a}_1(p/m) - a_2(p/m)\bar{a}_2(p/m)] \ , \tag{21'}$$
$$J_3' = \tfrac{1}{2}i[a_1(p/m)\bar{a}_2(p/m) - a_2(p/m)\bar{a}_1(p/m)] \ .$$

There are now three possibilities:

(a) We may deny that the operators J_i', above, belong to our Poincaré generators. (Thus we must also take $M_{12} = L_{12}$, as the Galilean rotation operator.)

Clearly the little group is now trivially represented, so that $W^2 \to 0$.

That is, our Galilean solutions lead to particles of spin *zero*, but mass $m^2 = m_0^2(N+1)$, where N

=number of quanta.

(b) We may take the little-group generators to be J_i', *but agree that to obtain a Poincaré* irrep *we choose the unit 4-vector* $\mathfrak{u}(\chi\theta\varphi)$ *to be identical to* p/m. Thus we orient the two vectors p and $\mathfrak{u}$ to be parallel, and agree that all Lorentz transformations are henceforth to be generated by $M_{\mu\nu} = L_{\mu\nu} + J_{\mu\nu}$. [Note that $J_{\mu\nu}$ is defined in fixed frame, $(\chi\theta\varphi)=0$, cf. Eq. (2).]

It follows now that the little group is generated by J_i', and hence $W^2 \to m^2 J(J+1)$, where $J=\frac{1}{2}N$ (N being the number of quanta).

This is the self-consistent solution to our original Galilean problem, Eqs. (31) and (32), whose internal Hamiltonian depends parametrically through $\mathfrak{u}$ on the 4-velocity p/m operator. This is the solution which we gave in Ref. 2. (We discuss the self-consistency in more detail below.)

Staunton[20] has carried out a purely algebraic construction for the process sketched above. He found the two possibilities (a) and (b). The concept of a Galilean subdynamics had been hinted at earlier, notably by Bardakci and Halpern,[21] and by Susskind.[22] Staunton's construction, and particularly his discussion, clearly indicates the critical necessity for a complete construction; these earlier attempts, in his view, would lead only to possibility (a).

(c) We indicated a third possibility exists. This would be to allow p and $\mathfrak{u}$ to remain in a fixed, nonparallel, orientation, and to use the Poincaré generators of possibility (b). Since the basis $\langle\xi_1\xi_2|0,\mathfrak{u}\rangle$ is part of an infinite dimensional Sp(2, 2) representation generated by $\{J, V, K\}$, we can reexpress the vector $\langle\xi_1\xi_2|0,\mathfrak{u}\rangle$, in a basis tailored to p/m. Since this transformation is infinite-dimensional we would obtain an unlimited number of Poincaré irreps all of the same mass as the vector p.

We exclude this possibility, (c), by the requirement of self-consistency ($p/m=\mathfrak{u}$). Possibility (a) is excluded by requiring Poincaré generators which act on the internal variables.

Let us return now to the question of self-consistency for the structure using possibility (b), above. The logic goes this way: We solve a Galilean problem for every fixed numerical value of the 4-velocity $\mathfrak{u}$. We then identify the 4-velocity $\mathfrak{u}$ with the 4-velocity p/m (p and m being uniquely defined for each Galilean eigensolution) and then assert that each such eigensolution properly belongs to the Poincaré world.

Note particularly that the eigenvalues $m = m_0 \times (N+1)^{1/2}$ are independent of $\mathfrak{u}$, so that the identification $\mathfrak{u} = p/m$ is simultaneously valid for the set of solutions $\{N\}$. What is even more remarkable is that we can identify $\mathfrak{u}$ with P/m, as an operator relation in the Galilean Hamiltonian; that this procedure is also valid establishes the self-consistency of this approach to Galilean subdynamics. This fact must now be demonstrated.[23]

If the operator P/m is to imply the 4-velocity $\mathfrak{u}$, as an operator relation, then it follows that under Galilean transformations both $\mathfrak{u}$ and P/m must transform together. Thus we must adjoin spin transformations to the Galilean space-time transformations; that is, the appropriate generators are now

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} \ , \tag{40}$$

replacing the analogous generators in Eq. (37). (It follows that the $S_{\mu\nu}$ are now defined in a fixed frame, just as for the $L_{\mu\nu}$.) The Galilean Hamiltonian has now the form

$$H_G = \frac{1}{2P_+}[P_1{}^2 + P_2{}^2 + m_0{}^2 V_0(P/m)] \ , \tag{41a}$$

where

$$V_0(P/m) \equiv a_1(P/m)\bar{a}_1(P/m) + a_2(P/m)\bar{a}_2(P/m) + 1 \ . \tag{41b}$$

(Observe that the explicit form for $V_0(P/m)$, given in Eq. (31), shows that only P_+, P_1, and P_2 appear as operators in $V_0(P/m)$. Observe moreover that under the subset of Lorentz transformations belonging to the Galilean world the transformed P_+', P_1', P_2' do *not* get mixed with $P_- = H_G$. [This can be seen from the commutation relations, Eq. (27).] Hence the shorthand notation $V_0(P/m)$ does not imply any circularity (i.e., H_G involved in its own definition). This notation is quite convenient since we may formally consider all Lorentz transformations, knowing that the desired Galilean subset is correct.)[24]

Let us now examine the Galilean commutation relations. We know already that P_+, P_1, P_2, together with the subset of the $M_{\mu\nu}$, obey Galilean commutation relations; the problem reduces only to verifying that H_G obeys Galilean commutation relations with these operators. The task can be reduced further, since it is sufficient to demonstrate that $V_0(P/m)$ is invariant under all Galilean generators (excepting H_G). Clearly $V_0(P/m)$ is invariant under P_+, P_1, and P_2. We will now show that $V_0(P/m)$ is invariant under all $M_{\mu\nu}$.

Consider the Lorentz transformation $U_S(\Lambda)$ generated by the spin operators $S_{\mu\nu}$; we have

$$U_S(\Lambda) \equiv \exp[i\varphi_{\mu\nu}(\Lambda)S_{\mu\nu}] \ .$$

One verifies directly that we have

$$U_S:\ V_0(P/m) \to U_S(\Lambda)V_0(P/m)U_S{}^{-1}(\Lambda) = V_0(\Lambda P/m) \ . \tag{42}$$

Next consider the same Lorentz transformation (Λ) but generated now by the Minkowski-space operators $L_{\mu\nu}$. We define

$$U_L(\Lambda) \equiv \exp[i\varphi_{\mu\nu}(\Lambda)L_{\mu\nu}] \ .$$

Let us use an equivalent form for $V_0(P/m)$, that is,

$$V_0(P/m) = U_S(B_{P/m})(a_1\bar{a}_1 + a_2\bar{a}_2 + 1)\, U_S^{-1}(B_{P/m}) \ . \tag{43}$$

Since the operator $(a_1\bar{a}_1 + a_2\bar{a}_2 + 1)$ is rotationally invariant the transformed operator depends only on the boost, $B_{P/m}$, which takes the rest frame vector (0001) into P/m. (Strictly this is a numerical relation, but by abuse of notation we extend it to the operator P/m.) The desired transformation by U_L is

$$U_L(\Lambda):\ \ V_0(P/m) \to U_L(\Lambda)\, U_S(B_{P/m})(a_1\bar{a}_1 + a_2\bar{a}_2 + 1) \times U_S^{-1}(B_{P/m})\, U_L^{-1}(\Lambda) \ . \tag{44}$$

The operator $(a_1\bar{a}_1 + a_2\bar{a}_2 + 1)$ is invariant under all $U_L(\Lambda)$. Hence we may replace this operator by

$$U_L^{-1}(\Lambda)(a_1\bar{a}_1 + a_2\bar{a}_2 + 1)\, U_L(\Lambda)$$

and then consider the transformation $\mathcal{T}$ generated by

$$\mathcal{T} \equiv U_L(\Lambda)\, U_S(B_{P/m})\, U_L^{-1}(\Lambda) \ . \tag{45}$$

The operators $L_{\mu\nu}$ act on the operator P/m to transform it *contragrediently*. Since the transformation $\mathcal{T}$ is unitary this can only mean that $P/m \to \Lambda^{-1}P/m$. We have shown thereby that

$$U_L(\Lambda):\ \ V_0(P/m) \to V_0(\Lambda^{-1}P/m) \ , \tag{46}$$

and hence under the combined transformation

$$U_M(\Lambda) = U_L(\Lambda)\, U_S(\Lambda) = U_S(\Lambda)\, U_L(\Lambda)$$

we obtain the desired result:

$$U_M(\Lambda):\ \ V_0(P/m) \to V_0(P/m) \ , \tag{47}$$

that is, $V_0(P/m)$ is formally invariant under all Lorentz transformations, and invariant under all Lorentz transformations belonging to the Galilean world. We conclude that the identification of P/m and $\mathfrak{u}$ as an operator relation leads to a consistent Galilean problem, as asserted.

This rather involved discussion has produced one new result. If one examines the argument, it is clear that the proof goes through unchanged if we replace $V_0 = a_1\bar{a}_1 + a_2\bar{a}_2 + 1$ by any positive, nonzero, function $f(V_0)$. In this way, one may obtain, if desired, a more general mass-spin relation: $m^2 = m_0^2 f(2J+1)$.

Let us discuss one rather puzzling point here. In contrast to the Poincaré problem the Galilean problem—by virtue of the identification P/m with $\mathfrak{u}$—implicitly specifies a particular relative orientation of the spin and space-time coordinate systems. We may see this most easily by examining the harmonic oscillator part of the wave function $\langle \xi_1\xi_2 | 0; p/m\rangle$. [One could equally well use the operator $V_0(P/m)$.] If we define the operator $L_{\mu\nu}$ to act on the p/m, then $\langle \xi_1\xi_2 | 0; p/m\rangle$, just as for $V_0(P/m)$, must be invariant under $M_{\mu\nu}$. Consider the operator M_{12}, and take $L_{12} = x_1P_2 - x_2P_1$. Then the invariance under M_{12} requires that

$$J_{12} = -\tfrac{1}{2} i (a_2\bar{a}_1 - a_1\bar{a}_2) \ , \tag{48}$$

which is a different choice of orientation than customary. Similar arguments account for the specific choices given in Eq. (2).

We conclude by emphasizing that it is no great feat to obtain any desired sequence of Poincaré irreps with mass arbitrarily related to spin; one can do this by fiat as one wishes. The point is that by such a construction—viewed in the Poincaré world—one learns no structural information whatsoever. Thus there is nothing remarkable, Poincaré-wise, in obtaining the sequence: $m^2 = m_0^2 \times (2J+1)$. What is remarkable in the construction discussed above is that this whole set of Poincaré irreps, viewed in the Galilean subworld, form a coherent set of states generated from a single (Galilean) Hamiltonian. (The fact that this is possible, for all masses and spins, is a strict result that the condition, $p/m = \mathfrak{u}$, is *independent of mass*.)

Let us note, as first shown by Staunton,[20] that this construction involves a single Hilbert space only in the Galilean subworld—in the Poincaré world the structure fragments into separate Hilbert spaces for each mass and spin. This is a remarkable property, which may help explain an old paradox of particle physics.[25]

III. CONSTRUCTION OF THE VENEZIANO FOUR-POINT AMPLITUDE FOR A STRINGLIKE MODEL

The purpose of the present section is to detail an explicit construction—within the framework of Galilean subdynamics—of the 4-point Veneziano amplitude[5] (Euler's beta function) which underlies the whole approach of the dual resonance model. Galilean subdynamics is a much more rigid framework than that of the DRM (we shall point out the differences below) and, although there are heuristic elements in our construction, the requirement of strict self-consistency is sufficiently strong that the construction could have failed, and the fact that it did not, is nontrivial information.[26]

The most expeditious way to approach the DRM is to use the operator formalism[7] of Fubini and

Veneziano. In this approach, one identifies two structural elements in the set of multiperipheral diagrams:

(a) a propagator to which one ascribes an associated operator, $D(s_i)$, where the Mandelstam invariant is $s_i = (\sum_{j=0}^{i} p_j)^2$;

(b) a vertex to which one associates an operator $V(p_i)$, with p_i denoting the four-momentum of the spinless particle (with common mass p_i^2) entering (or leaving) the vertex.

Using these two elements one can associate to every multiperipheral diagram a specific matrix element,

$$M(p_1 \cdots p_{N+1}) \equiv \langle 0 | V(p_N) \cdots V(p_3) D(s_2) V(p_2) | 0 \rangle. \tag{49}$$

Note that the momentum p_1 is associated with the ket $|0\rangle$, and the final momentum p_{N+1} is associated with the bra $\langle 0|$. (The condition $\sum_{i=1}^{N+1} p_i = 0$ is assumed.)

The specification of the two operators $D(s)$ and $V(p)$ thus suffices to identify a given model, which must then, of course, be proved to possess the desired "dual resonance" properties. Since we wish to employ the physical picture of a Galilean subdynamics these two operators should ideally be uniquely determined by the underlying physics of this assumed structure.

Before entering on this discussion, let us first dispose of some simpler points. What reference frame shall be used for the Galilean front? There is a special frame distinguished in all two-body reactions: the scattering plane formed by p_1 and p_2. *This plane we shall identify as the "transverse plane,"* (p_1, p_2), *of the Galilean dynamics*. The two symplectic times $\tau_\pm$ (conjugate to $P_\mp$) are then "perpendicular" to the scattering plane. Let us remark: (1) that this frame is well-defined even for multiperipheral final states and (2) that this reference frame is *not* the one used in the "infinite-momentum frame" construction.[17]

The Galilean subworld thus comprises the two spatial dimensions of the scattering plane, the temporal dimension τ_+ conjugate to the Galilean Hamiltonian H_S, and the "time" τ_- conjugate to the momentum P_+, which plays the role of Galilean mass.

It is physically appealing to attempt now to identify the propagator and vertex operators directly from the scattering process as it takes place in the Galilean subworld. Such an attempt is almost certainly premature, since our understanding of Galilean subdynamics is, so far, limited entirely to structures underlying the "one-body" Poincaré problem. One can see this quite directly, in that it is an essential element in our construction that *the Galilean constituents never are found at infinite separations* (=discrete spectrum). Thus scattering is precluded *ab initio*.

We shall accordingly adopt the following picture for the two-body reaction. A detailed description of the temporal development of the scattering (in the Galilean subworld) is not yet feasible, and we shall schematize the collision defined by the initial state at $\tau_+ \to -\infty$ to have occurred and to have formed a "compound hadronic state"; this composite structure then makes a transition, at $\tau_+ \to +\infty$, to the final state. Thus at all times, except $\tau_+ = \pm\infty$, the colliding hadrons are united into a single excited hadronic structure. This view guarantees a rudimentary form of duality, in that whatever channel one examines, the resonances are always those of the same hadronic structure.

This schematization amounts to asserting that the propagator is to be uniquely and self-consistently determined by the Galilean Hamiltonian, whereas the vertex operator is to be chosen heuristically, to produce agreement with the 4-point Veneziano amplitude.

A. The propagation operator

The dual resonance model (string model) suggests that the spectrum of eigenstates that must be used in the Galilean subdynamics is that of a quantized string. This spectrum is easily achieved: We simply replace the pair of oscillators a_1, a_2 by a denumerably infinite set of boson pairs $\{b_1^{(n)}, b_2^{(n)}\}$.

One technical point arises: The entire set of boson operators must all be aligned with the same 4-velocity $\mathfrak{u}$. Since we will be dealing exclusively with state vectors defined by excitation quantum numbers, that is,

$$|\{l_\alpha\}\rangle = \prod_{i=1}^{\infty} \left(\frac{[b_1^{(i)}(P/m)]^{l_{i,1}} [b_2^{(i)}(P/m)]^{l_{i,2}}}{[l_{i,1}!\, l_{i,2}!]^{1/2}} \right) |0; P/m\rangle \tag{50}$$

[where it is understood that only a finite number of l_α's differ from zero] then, strictly speaking, the ket vector on the left-hand side depends on the 4-velocity, but we shall suppress this information, since it plays no role in the following.

The Galilean Hamiltonian for this model (once again suppressing $\mathfrak{u}$) is given by

$$H_{\text{internal}} = \frac{|\text{constant}|}{2P_+} + \frac{m_0^2}{2P_+} \sum_{n=1}^{\infty} n(b_1^{(n)} \bar{b}_1^{(n)} + b_2^{(n)} \bar{b}_2^{(n)}), \tag{51}$$

where we have used the freedom to shift the origin of H_S by the (positive) constant shown.

The propagator is, as mentioned, determined by the Galilean subdynamics, and is necessarily the operator $(H_{\text{int}} - E)^{-1}$, where H_{int} is given by Eq. (51).

The operator, H_{int}, is not invariant under the scaling operation in the Galilean subworld and it is convenient to redefine it to be scale-invariant. Thus we use $2P_+H_{\text{int}} \equiv \mathcal{H}$, which is scale-invariant, and effectively only multiplies the propagator by a scaling factor. Using the fact that

$$2P_+H_{\text{int}} = 2P_+P_- - P_1^{\,2} - P_2^{\,2}$$
$$= m^2 \equiv s\ ,$$

we see that *the scaling-invariant propagator in the Galilean world actually involves the Mandelstam variable s:*

$$D(s) = (\mathcal{H} - s)^{-1}\ . \qquad (52)$$

Introducing the usual integral representation for $D(s)$ we obtain

$$D(s) = \int_0^1 dx\, x^{\mathcal{H}-s-1}\ , \qquad (53)$$

where

$$H = \alpha_0 + (m_0)^2 \sum_n (n)(b_1^{(n)}\bar{b}_1^{(n)} + b_2^{(n)}\bar{b}_2^{(n)})\ ,$$

and both α_0 and m_0 are arbitrary positive, nonzero constants. (Strictly we should divide $\mathcal{H}$ by a dimensional quantity, but the usage above is customary.)

One can best appreciate the significance of this result by considering the leading trajectory: $N_1 = l_{1,1} + l_{1,2} \neq 0$; $N_i = 0$, $i > 1$. For this case, it follows [from Sec. II] that

(a) the spin $J = \frac{1}{2}N_1$;

(b) the states are nondegenerate (except for the spatial orientation degeneracy);

(c) the mass of the Poincaré state is given by

$$m^2 = \alpha_0 + (m_0^{\,2})(2J)\ .$$

The construction of this propagator for the present stringlike model is therefore completely straightforward. The simplicity of the construction should not blind one to the fact that the construction is nontrivial; in particular, one sees that

(a) the scale-invariant *Galilean* Hamiltonian is correctly related to the Poincaré-invariant Mandelstam variable;

(b) all states of the Galilean subworld correspond both to poles of the scale-invariant propagator and to physical Poincaré states;

(c) the physical Poincaré states lie on parallel linear trajectories, with a nondegenerate leading trajectory;

(d) the relation between mass and intrinsic spin for states on the trajectories is a strict consequence of the model (in sharp contrast to the *ad hoc* relation imposed in the DRM).

B. The vertex operator

The construction of a satisfactory vertex operator is considerably more involved than the propagator construction. This is to be expected since the vertex operator has to "mock up" the (unknown) details of the scattering process, whereas the propagator either meshed properly with the Galilean and Poincaré physical pictures, or failed entirely.

In the DRM the vertex operator is chosen as the generator of a coherent state vector having a well-defined four-momentum; in symbols,

$$V(p) = \exp\left(-p_\mu \sum_n a_\mu^{(n)}/\sqrt{n}\right) \exp\left(p_\mu \sum_n \bar{a}_\mu^{(n)}/\sqrt{n}\right). \qquad (54)$$

A direct transcription of this form into the present model fails, since:

(a) There are only two indices available for the scalar product, and hence only the momentum p_i can be coupled $(i = 1,2)$.

(b) Even if difficulty (a) were brushed aside, the structure would still be unsatisfactory, *since it would imply that zero-spin particles could excite spinor resonances.* (This follows from the fact that the boson operators $b_i^{(n)}$ carry spin$\frac{1}{2}$).

While it might be possible to invent a lang:age to disguise this latter difficulty—calling it, say, excitation "off the spin shell"—it remains nonetheless serious and clearly contradictory to the structure of the Galilean subworld. In fact, angular momentum considerations are not only essential, but one of the basic advantages inherent to the whole picture of a Galilean substructure.

To circumvent difficulty (b) it is essential, for consistency of the model, that the coherent states produced by the vertex operator *involve only integer spins.* Thus the operators $a_\mu^{(n)}$ must be replaced by operators *bilinear* in our (spin-$\frac{1}{2}$) bosons.

To solve difficulty (a) one realizes that a bilinear boson operator—if *distinct* bosons (index n) are involved—covers precisely four states, orthogonal in all products of *arbitrary* degree. (Contrary to what might be expected the same boson b_i taken twice covers but two states in this required sense.)

Thus the resolution of difficulty (b) can resolve (a) as well, and we have a four-dimensional basis—with correct angular momentum properties—available for defining a *formal* four-vector product. There are many ways to achieve a structure having the desired properties. Perhaps the simplest is to double the basis, but this—besides being inelegant—introduces additional degeneracy. A neater (though far from unique) way to achieve the goal is to map the boson structure into itself, taking successive pairs of bosons in the sequence to define the vector basis.

Accordingly take $\mathcal{H}$, the scale-invariant internal Galilean Hamiltonian, to have the form

$$\mathcal{H} = \alpha_0 + (m_0)^2 \sum_{n=1}^{\infty} \left(\frac{2n+1}{4}\right)(b_1^{(n)}\bar{b}_1^{(n)} + b_2^{(n)}\bar{b}_2^{(n)}) , \tag{55}$$

and define the orthogonal basis,

$$\begin{aligned} V_j^{(n)} &= b_1^{(2n-1)} b_1^{(2n)}, \quad j=1 \\ &= b_2^{(2n-1)} b_2^{(2n)}, \quad j=2 \\ &= b_1^{(2n-1)} b_2^{(2n)}, \quad j=3 \\ &= b_2^{(2n-1)} b_1^{(2n)}, \quad j=0 . \end{aligned} \tag{56}$$

The use of the bilinear operators $V_i^{(n)}$, in place of the boson operators $b_i^{(n)}$, completely changes the nature of the "coherent states"; in particular, exponential functions no longer suffice and we anticipate this by defining a new function, $E(x)$:

$$E(x) \equiv \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{3/2}} \quad . \tag{57}$$

This function does not obey the exponential property: $e^x e^y = e^{x+y}$ and we must accordingly define the vertex function $V(\vec{p})$ to be an infinite product ($p_0 \equiv ip_4$)

$$V(p) \equiv A(p)[A(p)]^\dagger ,$$

with (58)

$$A(p) \equiv \prod_{n=1}^{\infty}\prod_{j=1}^{4} E(p_j V_j^{(n)}/\sqrt{n}\,) .$$

It will be noted that this vertex function is invariant, as required, under rotations about the (1, 2) axis in the Galilean plane. (Note that translations are not defined for the boson structure.)

C. Evaluation of the four-point amplitude

The evaluation of the four-point amplitude now follows directly from the rules of the DRM, using the specific D and V operators of paragraphs (a) and (b) above. We will carry out the calculation in detail.

The amplitude $\mathcal{A}$ is given by

$$\begin{aligned} \mathcal{A} &= \langle 0|V(p^{(3)})D(s)V(p^{(2)})|0\rangle , \\ \mathcal{A} &= \int_0^1 dx\, x^{\alpha_0 - s - 1}\langle 0|A^\dagger(p^{(3)})x^{\mathcal{H}}A(p^{(2)})|0\rangle . \end{aligned} \tag{59}$$

(Here the operators, $A^\dagger$ acting on $|0\rangle$ and A acting on $\langle 0|$ have been replaced by unity. Similarly m_0 has been taken to be unity.)

In the standard way, one evaluates the action of $x^{\mathcal{H}}$ on the "coherent state" $A(p)$, and finds that

$$x^{\mathcal{H}}A(p)|0\rangle = \prod_{n,j} E(p_j^{(3)}x^{2n}\,V_j^{(n)}/\sqrt{n}\,)|0\rangle .$$

Next one evaluates the matrix element:

$$\begin{aligned} &\left\langle 0\left|A^\dagger(p^{(3)})\prod_{n,j} E(p_j^{(3)}x^{2n}V_j^{(n)}/\sqrt{n}\,)\right|0\right\rangle \\ &\quad = \prod_{\substack{nn'\\ jj'}}\prod_{kk'} (k!\,k'!)^{-3/2}\langle 0|(|p_{j'}^{(3)}V_{j'}^{(n')}/\sqrt{n'}\,|^\dagger)^{k'} \\ &\qquad \times (p_j^{(2)}V_j^{(n)}x^{2n}/\sqrt{n}\,)^k|0\rangle . \end{aligned}$$

For this one uses the result that

$$\langle 0|[(V_{j'}^{n'})^\dagger]^{k'}(V_j^{(n)})^k|0\rangle = \delta_n^{n'}\delta_j^{j'}\delta_k^{k'}(k!)^2 ,$$

and hence obtains the desired result:

$$\begin{aligned} \langle 0|\cdots|0\rangle &= \exp\left(p^{(2)}\cdot p^{(3)}\sum_{n=1}^{\infty}\frac{x^{2n}}{n}\right) \\ &= (1-x^2)^{-p^{(2)}\cdot p^{(3)}} . \end{aligned}$$

The scalar product $p^{(2)}\cdot p^{(3)}$ is related to the variable t by

$$\begin{aligned} t &\equiv (p^{(2)} + p^{(3)})^2 \\ &= 2\mu^2 + 2p^{(2)}\cdot p^{(3)} \ , \end{aligned}$$

(taking the colliding particles to have mass μ, or μm_0 in dimensional terms). Introducing this into the amplitude yields the result

$$\begin{aligned} \mathcal{A} &= \int_0^1 dx\, x^{\alpha_0 - s - 1}(1-x^2)^{-t/2+\mu^2} \\ &= \frac{1}{2}\int_0^1 du\, u^{(\alpha_0 - s)/2 - 1}(1-u)^{-t/2+\mu^2} . \end{aligned} \tag{60}$$

To obtain symmetry between the variables s and t, one must require that the (arbitrary positive) constant $\alpha_0 = 4\mu^2$.

It follows that the four-point amplitude is precisely of the desired Veneziano form, that is, a beta function:

$$\mathcal{A} = B(-\tfrac{1}{2}(s-\alpha_0);\, -\tfrac{1}{2}(t-\alpha_0)) . \tag{61}$$

This function has poles, in either the s or t channel, at the values

$$s = \alpha_0 + 2N, \quad \text{for } N = 0, 1, \ldots . \tag{62}$$

If one now examines the values of s determined by the scale-invariant Hamiltonian, Eq. (51), one sees that only an even number of excitations ($\sum_{i=1} N_i$ = even) are allowed for $s = \alpha_0 + 2N$. That is to say, the poles of the four-point amplitude, Eq. (61), correspond to integer-intrinsic-spin excitations.

The leading trajectory, *for these excitations*, corresponds to exciting $N_1 = N_2 = N$, $N_{i>2} = 0$; the corresponding angular momentum is $J = N$. (That this is the leading trajectory is a consequence of our choice of vertex function which excites pairs of bosons in order to yield only integral-angular-momentum intermediate states.)

The desire for symmetry between s and t led to a definite choice for $\alpha_0 = 4\mu^2$, where the common mass of each external particle is μm_0. One may interpret this result consistently as the least possible energy for the system $(=2\mu m_0)$ for the two colliding particles at rest at infinity. (This is not completely trivial, for if the hadronic system obeys an integral relationship in m^2, then the composite structure (formed of two such systems) cannot obey the same relation. Thus the freedom to choose α_0 in the Galilean substructure is essential to the consistency of the model.)

Let us summarize: We have accomplished our stated objective of constructing, self-consistently, a four-point amplitude for spinless particles precisely of the desired form, that is, a Veneziano amplitude symmetric in s and t with poles for only integer spins for the (excited) composite system.

The essential point about this construction is that it can be done consistently, not that it represents believable physics. The model, of itself, does imply amplitudes for *all* multiperipheral diagrams, but for the five-point amplitude, and beyond, they are not of the DRM form. They are in fact quite *unacceptable* amplitudes, since they are not necessarily functions of the Poincaré invariants $p^{(i)}\cdot p^{(j)}$, even though they possess the property (as is easily shown) *that they do factorize*. We will not discuss these general amplitudes further, since their construction is not our main purpose.

The failure to obtain an acceptable multiperipheral amplitude for $N>4$ is not as serious a blemish on the model as it might appear. The reason is that basically our model yields a propagator and the splitting process whereby the composite system becomes two hadrons is clearly *outside* our construction. We have an *ad hoc* way to get the Veneziano amplitude but no further.

IV. CONSTRUCTION OF N-POINT AMPLITUDES FOR A SPHERICAL RUBBER SHEET MODEL

The construction of a stringlike model in the Galilean subworld, discussed in Sec. III, was successful in the sense that it permitted of a Veneziano four-point amplitude consistent with all requirements of physical acceptability in Minkowski space. The model was unsatisfactory in that the postulated vertex operators did *not* allow of an N-point generalization, for $N>4$.

The failure of the N-point generalization is a direct consequence of the necessity to couple the momentum vector into a *bilinear* (vector) operator in the Galilean subworld. Thus the failure is a consequence of the basic fact that the boson structures in the Galilean subworld carry spin-$\frac{1}{2}$; it follows that this defect is inherent to the model and not a superficial flaw. Yet the spin-$\frac{1}{2}$ features of the construction are a fundamental advantage, indeed the strength of the whole Galilean approach.

This seeming dilemma can be circumvented if we preserve the Galilean substructure but use a field-theoretic mapping ("second quantization") that introduces *linear* boson operators for spin-1 structures. Expressed in different words, we postulate a field theory of a two-dimensional surface, instead of the previously used field theory of a linear string (having two modes of oscillation).

The desired field-theoretic structure now postulates that to each eigenket of the Galilean subworld,

$$|JM;\mathfrak{u}=p/m\rangle \equiv \frac{[a_1(p/m)]^{J+M}[a_2(p/m)]^{J-M}}{[(J+M)!(J-M)!]^{1/2}}|0;p/m\rangle\,, \tag{63}$$

there is associated a boson operator $b_{JM}(p/m)$ in a new Fock space which creates (symmetrically) for each application, a spin-J excitation all associated with the little group of p/m.

The operators $\mathcal{H}$ and J_{12} which defined the original one-particle spaces of the Galilean subworld, now take on the form

$$\mathcal{H} \to m_0{}^2 b_{00}(p/m)\bar{b}_{00}(p/m) + m_1{}^2\sum_{\substack{J>0\\M}}(2J)b_{JM}(p/m)\,\bar{b}_{JM}(p/m)\,, \tag{64}$$

$$J_k \to \sum_{J,M,M'}[J(J+1)]^{1/2}C^{J1J}_{MkM'}b_{JM'}\bar{b}_{JM}\,. \tag{65}$$

The eigenkets of this new Fock space are given by

$$|\{\alpha_{JM}\}\rangle \equiv \prod_{J,M}\frac{[b_{JM}(p/m)]^{\alpha_{JM}}}{[\alpha_{JM}!]^{1/2}}|0\rangle\,. \tag{66}$$

Note that for this space the vacuum ket $|0\rangle$ corresponds to the vacuum irrep of the Poincaré group, *not* to a spinless particle.

Using the operator $\mathcal{H}$ we see that (cf. Eq. (38), ff.) each state $\{\alpha_{JM}\}$ of this Fock space is an eigenstate of $\mathcal{H}$, and accordingly implies that the mass of the associated Poincaré state is

$$m^2(\{\alpha\}) = m_0{}^2\alpha_{00} + m_1{}^2\sum_{J,M}(2J)\alpha_{JM}\,. \tag{67}$$

The 3-component of the spin is diagonal on the state $\{\alpha_{JM}\}$ has the value

$$J_3 \to \sum_{JM} M\alpha_{JM}\,. \tag{68}$$

However the total spin is, in general, degenerate, and corresponds to the *vector* sum: $\sum_{JM}\vec{J}\alpha_{JM}$.

These results are in striking contrast to the analogous results obtained from the string model (Sec. III). Unlike the string model—where quanta of different excitation modes nonetheless all carry spin$\frac{1}{2}$—the quanta of the various modes of the rubber sheet carry different angular momenta. Thus the Jth mode having excitation energy $2Jm_1^2$, also carries angular momentum J. It follows from the results, Eq. (67) and (68), that the leading trajectory is highly degenerate with the degeneracy increasing with excitation.[27,28]

Let us turn now to the task of constructing a dual resonance model based on this rubber sheet structure. We take over the same viewpoint as discussed in the previous string model. That is: a two-body hadronic collision unites the two hadrons into a single excited hadronic structure, whose propagator is uniquely and self-consistently determined by the Galilean Hamiltonian in the subworld. By contrast, the formation and decay of the composite hadron are viewed as heuristic elements in the construction, to be approximated ("mocked up") by vertex functions chosen to produce agreement with the Veneziano amplitude.

A. The propagator

The propagator is determined by the Galilean subdynamics and, just as in the previous model, we choose—for convenience—to use the scale-invariant form for the Hamiltonian operator:

$$\mathcal{H} = m_0^2 + m_1^2 \sum_{JM} (2J) b_{JM}\bar{b}_{JM} . \tag{69}$$

[We suppress the four-velocity in Eq. (69), and henceforth, for ease of writing.]

We note, as before, that the scale-invariant propagator involves the Mandelstam variable s, so that

$$\text{Propagator} \equiv D(s) = \int_0^1 dx\, x^{\mathcal{H}-s-1} . \tag{70}$$

Aside from the different degeneracy structure, the trajectories are otherwise similar to those of the string model. In particular, the trajectories are straight lines with a common slope. [By using the freedom that the model splits into two disjoint structures, integer versus $\frac{1}{2}$-integer, we may choose a different slope (and intercept) for the fermionic hadrons.]

B. The vertex operator

The vertex operator in the DRM associates an external four-momentum (bearing mass μ) with the generator of a coherent state. The DRM vertex is formally Lorentz-invariant.

To proceed by analogy, the vertex we seek must be invariant to rotations in the Galilean plane, but Lorentz concepts are otherwise not operative. One faces an immediate difficulty: Only two-dimensional vectors exist in the Galilean plane, so how is one to map a four-vector onto this structure?

The task is not as futile as it might appear. Recall that the scaling operator, together with the two Galilean boosts, sufficed to determine a unique four-velocity, once we know the mass (as we do for the vertex). Recall too, that the space $(\xi_1\xi_2)$ over which our Poincaré irreps are realized is, in fact, a homogeneous space onto which an arbitrary four-velocity (boost) is mapped *nonlinearly*. [These remarks become clearer if one looks at the function $\langle \xi_1\xi_2|0;\mathfrak{u}\rangle$, Eq. (14), which demonstrates this mapping explicitly.]

The scaling operator, in effect, adjusts the ratio of p_+ to p_-. Choosing this ratio to be unity is exactly the condition that $p_3=0$. But for two-body scattering the condition $p_3=0$ defines the scattering plane, which we have also defined to be the Galilean plane.

For two-body scattering processes therefore, we have precisely the desired freedom to map four-vectors ($p_3=0$) one-to-one into three-vectors. Because the underlying space $(\xi_1\xi_2)$ is symplectic, the three-vector carried by this space is isotropic; that is, it is a null vector in SO(2, 1) (as illustrated in the function $\langle \xi_1\xi_2|0;\mathfrak{u}\rangle$.)

With this motivation, let us now write out the vertex operator.

$$V(\vec{p}) = A(\vec{p})A(\vec{p})^\dagger , \tag{71a}$$

$$A(\vec{p}) \equiv \exp\left(\sum_{\substack{J=1,2,\ldots \\ M=1,0,-1 \\ \text{only}}} p_M b_{JM}/\sqrt{J}\right) . \tag{71b}$$

[Note that in this expression we are considering $\vec{p}$ to be the (complex) three-vector: $\vec{p}=(p_1, p_2, ip_0)$. Note also that we have suppressed the four-velocity reference-frame dependence of the operators b_{JM}, since this detail will automatically take care of itself (we are working in the Galilean subworld exclusively).]

One important consequence of the form of this vertex is apparent: only integer angular momenta couple to this vertex, and all spin-$\frac{1}{2}$ complications attendant to the string model have disappeared.

C. Evaluation of the four-point amplitude

Using the rules of the DRM we can evaluate quite easily the four-point amplitude. This amplitude is defined by

$$\mathcal{A}=\langle 0|V(\vec{p}^{(3)})D(s)V(\vec{p}^{(2)})|0\rangle$$
$$=\int_0^1 dx\, x^{\alpha_0-s-1}\langle 0|A^\dagger(\vec{p}^{(3)})\, x^{\mathfrak{X}}A(\vec{p}^{(2)})|0\rangle \ . \qquad (72)$$

[We have taken $m_1=1$ in this result, and set $(m_0/m_1)=\alpha_0$.]

The action of $x^{\mathfrak{X}}$ on a coherent state is a standard result. One finds

$$x^{\mathfrak{X}}A(\vec{p})|0\rangle=\exp\left(\sum_{J,M}p_M x^{2J}b_{JM}/\sqrt{J}\,|0\rangle\right) \ . \qquad (73)$$

Thus the matrix element reduces to an inner product of coherent states, that is,

$$\langle 0|A^\dagger(\vec{p}^{(3)})x^{\mathfrak{X}}A(\vec{p}^{(2)})|0\rangle=\exp\left[\vec{p}^{(3)}\cdot\vec{p}^{(2)}\sum_{J=1}\frac{x^{2J}}{J}\right]$$
$$=\exp[-\vec{p}^{(3)}\cdot\vec{p}^{(2)}\ln(1-x^2)]$$
$$=(1-x^2)^{-\vec{p}^{(3)}\cdot\vec{p}^{(2)}} \ . \qquad (74)$$

We have obtained the desired form, just as in our previous model, for the Veneziano function:

$$\mathcal{A}=\tfrac{1}{2}B(-\tfrac{1}{2}(s-\alpha_0);\ -\tfrac{1}{2}(t-\alpha_0)) \ . \qquad (75)$$

The previous discussion for this result can be taken over intact. It is worth noting that the intermediate states excited by the collision, properly have only integer spins.

D. The N-point amplitude

The vertex function and propagator in this "rubber sheet" model have been realized as closely analogous to those of the DRM; in particular, the vertex operator involves linear boson operators which generate true coherent states. It follows that the same abstract properties of the DRM techniques carry over to the present structure. The single formal difference is that we use the three-vector: (p_1,p_2,ip_0) to replace the actual four-vector p at each given vertex. The final answers thus involve inner products of the form: $p\cdot q=p_1q_1+p_2q_2-p_0q_0$. *For four-vectors lying in the scattering plane* ($p_3=0$), *this is exactly the same as the Lorentz-invariant product.*

Thus we obtain a formal N-point amplitude having exactly the properties (such as factorization, ...) characteristic of the dual resonance model itself, with the constraint that the general multiperipheral process lie in the scattering plane. This is not as restrictive as it sounds, for such a configuration always exists, and we may simply define the general case as the formal continuation of the planar amplitude.

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[1]P. A. M. Dirac, Proc. R. Soc. Lond. A322, 435 (1971); A328, 1 (1972).

[2]L. C. Biedenharn, M. Y. Han, and H. van Dam, Phys. Rev. D 8, 1735 (1973).

[3]P. Ramond, Phys. Rev. D 7, 449 (1973).

[4]P. A. M. Dirac, Rev. Mod. Phys. 21, 392 (1949).

[5]G. Veneziano, Nuovo Cimento 57A, 190 (1968).

[6]S. Mandelstam, in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser *et al.*, (MIT Press, Cambridge, Mass., 1971), Vol. I, p. 167.

[7]S. Fubini and G. Veneziano, Nuovo Cimento 67, 29 (1970).

[8]For a recent review see M. A. Virasoro, *Proceedings of the XVI International Conference on High Energy Physics, Chicago-Batavia, Ill., 1972,* edited by J. D. Jackson and A. Roberts (NAL, Batavia, Ill., 1973), Vol. 1, p. 460; J. H. Schwarz *ibid.* Vol. 1, p. 462. We have been particularly helped by the reviews of Mandelstam (Ref. 6), and by G. Veneziano in *Elementary Processes at High Energy,* edited by A. Zichichi (Academic, New York, 1971), and by V. Alessandrini, D. Amati, M. LeBellac, and D. Olive, Phys. Rep. 1C, 270 (1971).

[9]Y. Nambu, Lectures at the Copenhagen summer institute, 1970 (unpublished); L. N. Chang and F. Mansouri, Phys. Rev. D 5, 2535 (1972); D. Olive in *Proceedings of the XVI International Conference on High Energy Physics, Chicago-Batavia, Ill., 1972,* (Ref. 8), Vol. 1, p. 472; J. L. Gervais and B. Sakita, Phys. Rev. Lett. 30, 716 (1973). For a particularly clear survey see A. Minami, Prog. Theor. Phys. (to be published).

[10]H. D. I. Abarbanel, invited paper N. Y. Academy of Sciences Conference, March 1973 (unpublished); NAL Report No. NAL-CONF-73/17-THY (unpublished).

[11]E. P. Wigner, Ann. Math. 40, 149 (1939); see also A. S. Wightman, *Dispersion Relations and Elementary Particles,* edited by C. DeWitt and R. Omnès (Wiley, New York, 1961), p. 188.

[12]Dirac, and also Kihlberg, have emphasized the impor-

tance of the choice of carrier space for the representations of Ref. 11. See P. A. M. Dirac, J. Math. Phys. 4, 901 (1963); A. Kihlberg, Nuovo Cimento 53A, 592 (1968).

[13]See P. A. M. Dirac, Ref. 12.

[14]The use of such coordinates in quantum electrodynamics stems from F. Rohrlich, Acta Phys. Austriaca 32, 87 (1970); H. Leutwyler, in *Springer Tracts in Modern Physics*, edited by G. Höhler (Springer, Berlin, 1969), Vol. 50, p. 29. A particularly precise exposition is in H. Leutwyler, J. R. Klauder, and L. Streit, Nuovo Cimento 66A, 536 (1970).

[15]Compare with R. P. Feynman, M. Kinslinger, and F. Ravndal, Phys. Rev. D 3, 2706 (1971). See also Y. S. Kim and M. E. Noz, Phys. Rev. D 8, 3521 (1973).

[16]A survey may be found in L. O'Raifeartaigh, *Lecture Notes in Physics*, edited by V. Bargmann (Springer, New York, 1970).

[17]This eight-parameter group must be carefully distinguished from the similar, but logically distinct "infinite-momentum frame" (IMF) kinematic subgroup (Ref. 18). Physically the most important difference is in the lack of a scaling operator. This scaling operator plays a decisive role in Galilean subdynamics. Mathematically, the IMF is properly to be viewed as a contraction limit (Ref. 19).

[18]J. Kogut and L. Susskind, Phys. Rep. 8C, 75 (1973). [Note that this paper omits the scaling operator.]

[19]H. Bacry and N. P. Chang, Ann. Phys. (N.Y.) 47, 407 (1968).

[20]L. P. Staunton, Phys. Rev. D 8, 2446 (1973).

[21]K. Bardakci and M. B. Halpern, Phys. Rev. 176, 1686 (1968).

[22]L. Susskind, Phys. Rev. 165, 1535 (1968); 165, 1547 (1968).

[23]A more elegant demonstration can be achieved using the concept of homogeneous spaces.

[24]Note, however, that one cannot *actually* adjoin the transformations not in the Galilean subset. The reason is that such transformations would involve P_-, and hence H_G itself, which is not permissible (cf. Ref. 20).

[25]Isospin symmetry, and therefore also SU(3), imply coherent combinations of vectors in distinct superselection spaces; which is a contradiction. If the isospin symmetry existed only in the Galilean subworld, the contradiction would be removed.

[26]Our method differs from the approach of Casalbuoni, Gatto, and Longhi; see R. Casalbuoni, R. Gatto, and G. Longhi, Nuovo Cimento Lett. 2, 159 (1969). Compare also A. O. Barut and I. H. Duru, Trieste Report No. IC/72/116, 1972 (unpublished).

[27]The asymptotic properties of these trajectories are of interest. One finds that the degeneracy of the levels with mass m and maximal $J \propto (m)^{1/2}$ is asymptotically of the order of $\exp(bm)$, where b is a positive constant. Although highly degenerate, this leading trajectory is, asymptotically, not the most degenerate. The number of states having mass m, with no restrictions on J, is given asymptotically by

$$\text{No. of states} \sim \exp[(\text{constant})^2 \times m \ln m].$$

This differs only slightly (the $\ln m$ factor is omitted) from the asymptotic density obtained by Hagedorn (Ref. 28) and from that obtained in the DRM (string model) (Ref. 8).

[28]R. Hagedorn, Nuovo Cimento 56A, 1027 (1968).