# Chiral magnetism (or magnetohadrochironics)

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In the infinite-momentum frame a hadron may be viewed as a one-dimensional stringlike structure composed of many constituents. We consider the dynamics of such a system and show how a spontaneous breakdown of chiral symmetry may occur. The effect is very similar to ferromagnetism. We formulate the theory of the distribution of quantum numbers in hadrons and establish the relation between this theory, chiral magnetism, soft-pion theorems, Regge behavior, and duality. In particular we show how the Harari-Gilman-Weinberg theory of chiral representation mixing follows from an approximation analogous to the treatment of a magnetic impurity in a ferromagnetic system. The first stage of approximation, when applied to a quark-parton system leads to a $\bar{q}q$ meson model. We demonstrate the need for spin-orbit coupling in the infinite-momentum quark system. Higher approximations are expected to lead to exotic states. A necessary consequence of our theory of spontaneous chiral-symmetry breakdown is the existence of a Pomeron-like vacuum trajectory with unit intercept. The natural order of magnitude of high-energy meson-meson total cross sections turns out to be $f_\pi^{-2} = 10$ mb, as conjectured by Pagels. A specific model included in an appendix yields $\sigma_{\text{tot}}(\pi\pi) = 2f_\pi^{-2}$. We do not explicitly deal with strangeness, or $SU_3$.

## I. INTRODUCTION

The spontaneous breakdown of chiral symmetry in hadron dynamics is generally studied as a vacuum phenomenon.[1] Because of an instability of the chirally invariant vacuum, the real vacuum is "aligned" into a chirally asymmetric configuration.

On the other hand an approach to quantum field theory exists in which the properties of the vacuum state are not relevant. This is the parton or constituent approach formulated in the infinite-momentum frame.[2] A number of investigations have indicated that in this frame the vacuum may be regarded as the structureless Fock-space vacuum. Hadrons may be described as nonrelativistic collections of constituents (partons). In this framework the spontaneous symmetry breakdown must be attributed to the properties of the hadron's wave function and not to the vacuum.[3]

The essential ingredient required for a spontaneous symmetry breakdown in a composite system is the existence of a divergent number of constituents. Indeed the SLAC-MIT experiments have already provided evidence in favor of a divergent sea of low-momentum partons.[4]

In this paper we idealize the parton dynamics through the use of the string or one-dimensional chain model of hadrons.[5] In Sec. II we explain in a very qualitative and intuitive manner, the connection between this model and spontaneous breakdown. In Sec. III we review the pertinent facts about the string model. Section IV introduces specific assumptions about currents and charges. Section V derives the connection between currents and Regge parameters. The chiral spontaneous breakdown is formulated in Sec. VI. A magnetic analogy is used to help visualize the ideas. We also derive soft-pion theorems from our assumptions.

In Sec. VII we consider the quark model as a specific realization of chiral magnetism. We demonstrate the need for spin-orbit coupling in the parton-parton interactions. An approximation scheme for computing the low-lying hadronic states and their pionic decays is introduced. In particular we show how representation-mixing schemes such as those of Gilman and Harari[6] and of Casher and Susskind[7] may be understood. In Sec. VIII we formulate some connections between the currents in previous sections and the coupling of the Pomeron trajectory. We also discuss the importance and relevance of exotic trajectories. Sec. IX is summary and conclusions.

In Appendix A we define the mathematical meaning of certain limits which occur throughout. Appendixes B and C provide concrete models of our assumptions and prove their consistency.

## II. INTUITIVE DISCUSSION

### A. The Parton Chain

According to Feynman and Bjorken and Paschos,[4] the results of deep-inelastic electroproduction can be summarized in terms of a distribution function which defines the number of partons per unit longitudinal momentum fraction $\eta$ and behaves like

$$\eta \to 0: \frac{dN}{d\eta} \sim \frac{1}{\eta}. \tag{2.1}$$

In analyzing the infinite many-body system implied by (2.1) we shall encounter questions which concern the longitudinal motions of the partons and those which concern transverse motions. As we have learned over the past few years the transverse motions of partons have a nonrelativistic (Galilean) behavior which is very helpful in drawing analogies and intuitions from nonrelativistic quantum mechanics. The dynamics of the longitudinal motions are more complicated. Since the questions considered in this paper concern the distributions of charges and currents in the transverse plane we shall simplify the longitudinal dynamics. For our purposes we can imagine that each parton has a definite nonfluctuating $c$-number longitudinal fraction $\eta$. The most important property of the infinite Feynman-Bjorken sea which follows from (1) is that near $\eta = 0$ the ratio of neighboring $\eta$'s is universal,

$$\frac{\eta_{j+1}}{\eta_j} = \lambda < 1 . \tag{2.2}$$

It is convenient to introduce the rapidity variable $y = \ln\eta$. The partons are then uniformly distributed over the negative half of the $y$ axis with a uniform density.

In analyzing the transverse motions of the system we will introduce the second simplifying assumption—that each parton couples to a few near neighbors on the rapidity axis. The center of the resulting string is occupied by the "hard" partons of large $\eta$. As we proceed away from the center the longitudinal fraction decreases according to Eq. (2.2). These assumptions are related in a well-known way to the dual resonance model.[2]

In what follows we will often take an extreme view, namely that each parton interacts only with its two nearest neighbors. This assumption is only introduced for notational convenience and plays no essential role.

### B. The Hamiltonian

In the infinite-momentum frame the Hamiltonian is identified with the so-called transverse mass squared.[2]

$$i\partial_\tau = H = P^2 + M^2 . \tag{2.3}$$

Here $\tau$ is the dilated time variable of the infinite-momentum frame (IMF), $P$ is the momentum in the transverse $(X, Y)$ plane, and $M$ is the rest mass of the system. We assume $H$ may be expressed as a function of the parton degrees of freedom. More specifically we assume it is a sum of terms, each containing the degrees of freedom of a neighboring pair of partons. In writing $H$ as such a sum we are constrained by longitudinal boost invariance[2] which requires the energy of a subsystem with fraction $\eta$ to be proportional to $\eta^{-1}$. Thus, labeling the sites by an index $(i)$,

$$H = \sum_i \frac{U(\zeta_i, \zeta_{i+1})}{\eta_i + \eta_{i+1}} . \tag{2.4}$$

Among the degrees of freedom $\zeta$ we have transverse position $X$, helicity $\sigma_z$, isospin $\tau^\alpha$, baryon number $B$, and whatever else is required by theory and experiment.

Another extreme assumption which we use throughout is that $H$ may be expressed as a sum of an orbital term depending only on transverse momentum and position, and a spin-isospin whatever-else part. We refer to this assumption as spin-orbit decoupling. We shall see later that this assumption is too strong if the individual partons are ordinary spin-$\frac{1}{2}$ isospin-$\frac{1}{2}$ fermions. However, it is not essential that the lattice sites in our model refer to individual partons. We can relax our assumptions by allowing the sites $(i)$ to describe pairs or even clusters of partons and allow $X(i)$ to describe the center of mass of such a cluster. We can then simply require that the clusters are chosen so that $X$ satisfies spin-orbit decoupling. Although a strong assumption is involved in claiming this to be possible, it has some experimental support. Firstly, the slopes of Regge trajectories appear to be independent of quantum numbers. This means that the orbital excitation energy required to increase $L$ by one unit is independent of the spin, isospin, baryon content of the hadron. Secondly, the transverse-momentum distributions of produced hadrons appear to be roughly independent of particle type. From this we learn that the transverse forces, acting on a chunk of matter, may not be sensitive to its quantum numbers.

### C. Energy Scales

The hadronic string differs from the usual one-dimensional systems studied in many-body theory in that the explicit factor $\eta_i^{-1}$ makes the Hamiltonian inhomogeneous. If the factor $\eta^{-1}$ were absent then the local equations of motion would be invariant under translations of the rapidity: $y \to y + y_0$. This corresponds to the transformation $\eta \to \alpha\eta$ and describes part of the action of a longitudinal boost.[2] As it is, the factor $(\eta_i + \eta_{i+1})^{-1}$ destroys this invariance of $H$. However, the equations of motion have another invariance involving $y$ translations. Consider the Heisenberg equation of motion for some parton quantity $\zeta_j$,

$$i\partial_\tau \zeta_j = [\zeta_j, H]$$
$$= -(\eta_{j-1}+\eta_j)^{-1}[U(j-1,j), \zeta_j] + (j \to j+1). \quad (2.5)$$

Since $\eta_j/\eta_{j+1}$ is universal for low $\eta$, we may write

$$\partial_\tau \zeta_j = \frac{F[\zeta_{j-1}, \zeta_j, \zeta_{j+1}]}{\eta_j} . \quad (2.6)$$

Evidently Eq. (2.6) is invariant under simultaneous rescaling of $\eta$ and $\tau$. Thus the invariance of the hadronic chain is

$$\eta \to \alpha\eta , \quad \tau \to \alpha\tau . \quad (2.7)$$

This will be recognized as the full transformation law for longitudinal momentum and IMF time under a longitudinal boost.[2]

The increase of energy scales as we proceed toward $\eta = 0$ has important consequences for chiral magnetism. Let us denote the ground state of the chain by $|G\rangle$. We will consider all hadrons to be excited states of $|G\rangle$. Because of the factor $\eta^{-1}$ in $H$ we see that as we approach $\eta = 0$ the energy scale increases to infinity. Therefore the energy needed to disturb the parton configuration near $\eta = 0$ is enormous. This implies that near $\eta = 0$ all hadrons look the same and are indistinguishable from $|G\rangle$.

A more quantitative statement may be made regarding the region of the $\eta$ axis involved in the excitation of states with mass squared $\lesssim \nu$. From the form of $H$ it is clear that the region $\eta \ll \nu^{-1}$ will remain "frozen" and only the partons with $\eta \gtrsim \nu^{-1}$ are significantly excited in these states. Restricting ourselves to the study of states with $M^2 < \nu$, we define the region $\eta \ll \nu^{-1}$ to be the *frozen sea* and the remaining excitable partons to be the *valence* system. In an approximate sense the degrees of freedom of the frozen sea may be eliminated from $H$ by replacing them by their ground-state expectation values. The effective Hamiltonian for states with $M^2 < \nu$ involves only a finite system of valence partons. However, as we increase the energy of the states we wish to study, the number of partons in the valence system increases (logarithmically with $\nu$).

Now, it may happen that the ground state of the infinite one-dimensional chain is chirally asymmetric throughout its length although the interparton forces are chirally symmetric. Roughly speaking the envisioned situation is similar to that of a one-dimensional Heisenberg ferromagnet in which energy considerations favor the alignment of spins along the chain. In the chiral case we expect that some chiral 4-vector [transforming under the $(\frac{1}{2}, \frac{1}{2})$ representation[8]] is aligned. Because of the $\eta^{-1}$ factor in $H$ the alignment of the elementary "magnets" becomes more and more difficult to break as $\eta \to 0$. If such a phenomenon occurs then the effective Hamiltonian for the valence system will have the form

$$H = H_0 + H_1 , \quad (2.8)$$

where $H_0$ represents the interaction of the valence partons among themselves and $H_1$ represents the interaction of the valence system with the frozen "magnetized" sea. Because the sea is chirally aligned $H_1$ will break the chiral symmetry of $H_0$.

A good analogy for this effect is to consider the valence system to be like a magnetic impurity in a ferromagnetic lattice. In the absence of the lattice the impurity Hamiltonian $H_0$ is rotationally symmetric. We may also add that the lattice dynamics and the lattice-impurity interaction are rotationally invariant. However, the ground state of the lattice is magnetized, say in the 3 direction. Therefore there will be an effective symmetry-breaking term in the Hamiltonian of the impurity of the form

$$H_1 = \mu_{\text{imp}} \cdot B_{\text{lat}} = \mu_3 |B| , \quad (2.9)$$

where $B$ is the external magnetic field of the lattice and $\mu_3$ is the 3rd component of the impurity magnetic-moment operator. The effect of $H_1$ is to split and mix the rotational multiplets which diagonalize $H_0$.

$H_1$ will also cause the chiral charge (angular momentum in the magnetic analogy) of the valence system to become nonconserved. Changes of the chiral charge will react back on the sea (lattice) and create a source of magnonlike chiral waves. These waves will propagate toward $\eta = 0$, carrying the chiral charge lost by the valence system. Upon arriving at $\eta = 0$ they will materialize as soft pions. The proportionality constant between the loss of chiral charge and the soft[8]-pion amplitude is the pion decay constant $f_\pi$. Thus[1]

$$i\partial_\tau Q_5^\alpha = \tfrac{1}{2} f_\pi T_{\pi\alpha} , \quad (2.10)$$

where $T_\pi$ is the emission amplitude for a pion carrying zero longitudinal and transverse momentum.

The purpose of this paper is to give a mathematical formulation of these ideas.

## III. THE STRING MODEL

We assume that a hadron in the infinite-momentum frame is a stringlike collection of partons[5] parametrized by a string variable $\theta$ which varies from 0 to $\pi$. The number of partons contained in the string element $d\theta$ localized near $\theta$ is

$$\frac{dN}{d\theta} = (\lambda_0 \sin\theta)^{-1}, \tag{3.1}$$

where $\lambda_0$ is a dimensionless constant. The density of longitudinal momentum is uniform on the $\theta$ axis so that the longitudinal fraction per parton near $\theta$ is

$$\eta(\theta) = \frac{\lambda_0}{\pi} \sin\theta. \tag{3.2}$$

Clearly near $\theta = 0, \pi$ we may approximate $\eta(\theta)$ by

$$\begin{aligned} &\theta \to 0: \ \eta \simeq \frac{\lambda_0}{\pi}\theta, \\ &\theta \to \pi: \ \eta \simeq \frac{\lambda_0}{\pi}(\pi - \theta). \end{aligned} \tag{3.3}$$

Thus near the ends we find an accumulation of low-momentum partons. The number of partons with longitudinal fraction between $\eta$ and $\eta + d\eta$ (for low $\eta$) is

$$\frac{dN}{d\eta} = 2\frac{dN}{d\theta}\frac{d\theta}{d\eta}, \tag{3.4}$$

where the factor 2 accounts for the two ends of the string. Thus

$$\eta \to 0: \ \frac{dN}{d\eta} \simeq 2(\lambda_0 \eta)^{-1}. \tag{3.5}$$

The transverse position of a parton near $\theta$ is called $X(\theta)$ and the part of $H$ governing the orbital motions is

$$H_{\text{orb}} = \sum_j \left( \frac{P_j^{\,2}}{\eta_j} + g\frac{(X_j - X_{j+1})^2}{\eta_j + \eta_{j+1}} \right). \tag{3.6}$$

We refer the reader to Ref. 5 for a discussion of (3.6) and its approximation by a continuum system. It is found that the equations of motion, boundary conditions, and normal-mode expansion of $X(\theta)$ are

$$(\partial_\tau^{\,2} - \partial_\theta^{\,2}) X(\theta, \tau) = 0, \tag{3.7}$$

$$\partial_\theta X|_{\theta=0,\pi} = 0, \tag{3.8}$$

$$X(\theta, \tau) = X_{\text{c.m.}}(\tau) + i\sqrt{2}\sum_l \frac{\cos l\theta}{\sqrt{l}}[a_l^-(\tau) - a_l^+(\tau)], \tag{3.9}$$

where the various constants are absorbed into the choice of mass scale (namely 1 = Regge slope).

It should be noticed that the replacement of the discrete index $i$ by the continuous parameter $\theta$ and of discrete equations by differential equations is an idealization applicable only to wavelengths much larger than the spacing of lattice points on the $\theta$ axis. A simple procedure which accounts for the discrete character of the hadron is to cut off those normal modes with wavelength smaller than the spacing. Thus we define the cutoff $l_{\max}$:

$$l_{\max} \approx (\lambda_0 \sin\theta)^{-1}. \tag{3.10}$$

We now summarize the main features which characterize the orbital motions:

(1) The mean-square transverse distance of partons at $\theta$ from the center of mass of the hadron satisfies

$$\begin{aligned} \langle [X(\theta) - X_{\text{c.m.}}]^2 \rangle &= 4\sum_l^{l_{\max}} \frac{\cos^2 l\theta}{l} \\ &\approx -4\ln(\sin\theta) \\ &\approx -4\ln\eta. \end{aligned} \tag{3.11}$$

Thus the partons of low $\eta$ are found at large distances and those of large $\eta$ are found near the center of mass. We have described this effect in detail elsewhere.[2] Here we only remark that the logarithmic increase in the size of the low-$\eta$ parton cloud accounts for Regge behavior and poles in form factors.

(2) The theory is invariant under the so-called Möbius mappings.[5,9] These are defined by first "Euclideanizing" the equations of motion by using an imaginary time[10] $\tau' = i\tau$. In the $(\tau', \theta)$ space the Möbius mappings are the conformal mappings which leave the strip $0 < \theta < \pi$, $-\infty < \tau' < \infty$ invariant. Among these mappings is the class which leaves the point $\tau' = \theta = 0$ fixed. Locally, near the fixed point, these mappings are dilations of the $(\tau', \theta)$ space. Thus the action of these mappings is

$$\begin{aligned} &\tau' \to \alpha\tau', \\ &\theta \to \alpha\theta \ \text{ for } \theta \to 0, \\ \text{or} \\ &\tau \to \alpha\tau, \\ &\theta \to \alpha\theta. \end{aligned} \tag{3.12}$$

Using (3.3) we find that these Möbius mappings have the same effect on the low-$\eta$ partons as the longitudinal boosts in (2.7). The remaining set of Möbius mappings is connected with invariance under the rest of the Lorentz group and particularly the notorious angular conditions. However they will play almost no role in our theory.

(3) For later reference we list two formulas. The vertex operator for the absorption of a transverse momentum $Q$ by a parton at $\theta$ is

$$T(Q,\theta)=e^{iQ\cdot X(\theta)}$$
$$=e^{iQ\cdot X_{\text{c.m.}}}\exp\left(\sqrt{2}\,Q\cdot\sum_l \frac{\cos l\theta}{\sqrt{l}}(a_l^+-a_l^-)\right). \tag{3.13}$$

To compute matrix elements of $T(Q)$ it is necessary to normal order the expression. For this purpose we use the formula

$$e^{-[A,B]/2}e^Ae^B=e^{A+B}$$

to get

$$T(Q,\theta)=e^{iQ\cdot X_{\text{c.m.}}}\exp\left(-Q^2\sum\frac{\cos^2 l\theta}{l}\right)$$
$$\times\exp\left(\sqrt{2}\,Q\cdot\sum\frac{\cos l\theta}{\sqrt{l}}\,a_l^+\right)$$
$$\times\exp\left(-\sqrt{2}\,Q\cdot\sum\frac{\cos l\theta}{\sqrt{l}}\,a_l^-\right). \tag{3.14}$$

The factor $\exp(iQ\cdot X_{\text{c.m.}})$ is trivial and gives a $\delta$ function for over-all transverse-momentum conservation. The remaining factor will be written

$$\exp\left(-q^2\sum\frac{\cos^2 l\theta}{l}\right):e^{iQ\cdot X(\theta)}: \tag{3.15}$$

The operator $:\exp[iQ\cdot X(\theta)]:$ is well behaved and has regular nonzero matrix elements at $\theta=0,\pi$. The factor

$$\exp\left(-Q^2\sum\frac{\cos^2 l\theta}{l}\right)$$

is the only factor which is sensitive to the cutoff procedure. Cutting off the divergent sum at $l_{\max}$ gives

$$\sum_l\frac{\cos^2 l\theta}{l}\approx-\ln(\sqrt{\lambda_0}\sin\theta)$$

Thus

$$T_{fi}(Q,\theta)=\delta^{(2)}(P_i+Q-P_f)(\sqrt{\lambda_0}\sin\theta)^{Q^2}:e^{iQ\cdot X(\theta)}:. \tag{3.16}$$

Equation (3.16) will play an important role in studying hadronic form factors.

The last formula we quote is useful in deriving dual amplitudes from the string model.[5] It reads

$$i\langle O_f|(e^{-iQ_f\cdot X_{\text{c.m.}}(0)}:e^{-iQ_f\cdot X(0,0)}:)(e^{iQ_i\cdot X_{\text{c.m.}}(\tau)}:e^{iQ_i\cdot X(0,\tau)}:)d\tau|O_i\rangle=\delta^{(2)}(P_f+Q_f-P_i-Q_i)u^{-s+M_i^2-1}(1-u)^{-2Q_f\cdot Q_i}du, \tag{3.17}$$

where $|O_i\rangle$ and $|O_f\rangle$ are orbitally unexcited hadrons of transverse momentum $P_i$ and $P_f$,

$$u=e^{i\tau},$$

and

$$s=M_f^2-Q_f^2-2P_f\cdot Q_f$$
$$=M_i^2-Q_i^2-2P_i\cdot Q_i.$$

We also may define $s$ as the usual Mandelstam invariant.

## IV. CURRENTS AND DENSITIES

### A. Currents on the String

Apart from the orbital degrees of freedom we shall equip each parton with a set of discrete quantum numbers $\{\xi\}$ consisting of helicity ($\sigma_z$), isospin ($\tau^\alpha$), and baryon number ($b$). The baryon number $b$ will be given the values $\pm1$ for partons and antipartons, respectively. Strangeness will be ignored throughout.

Assume $H$ contains a term which is nearest neighbor coupled in the discrete quantities $\{\xi\}$. Assume also that this term is independent of the orbital degrees of freedom. (See discussion in Sec. IIB.) The discrete part of $H$ will be called $H_\xi$,

$$H_\xi=\sum_j\frac{V(\xi_j,\xi_{j+1})}{\eta_j+\eta_{j+1}}. \tag{4.1}$$

The vector and axial-vector charge densities *on the string* are defined by averaging these charges over small intervals of $\theta$. Thus

$$\rho^\alpha=\frac{1}{\Delta\theta}\sum_{\Delta\theta}\tfrac{1}{2}\tau^\alpha(j),$$
$$\rho_5^\alpha=\frac{1}{\Delta\theta}\sum_{\Delta\theta}\tfrac{1}{2}\tau_5^\alpha(j), \tag{4.2}$$

where $\tau_5^\alpha$ is that combination of the operators $\xi$ which describes the axial charge of a single parton. For example in the usual spin-$\frac{1}{2}$, isospin-$\frac{1}{2}$ quark model[11]

$$\tau_5^\alpha(j)=b(j)\sigma_z(j)\tau^\alpha(j). \tag{4.3}$$

It is evident that $\rho^\alpha$ and $\rho_5^\alpha$ satisfy $SU_2\times SU_2$ commutation relations:

$$[\rho^\alpha(\theta),\rho^\beta(\theta')]=i\epsilon^{\alpha\beta\gamma}\rho^\gamma(\theta)\delta(\theta-\theta'),$$
$$[\rho^\alpha(\theta),\rho_5^\beta(\theta')]=i\epsilon^{\alpha\beta\gamma}\rho_5^\gamma(\theta)\delta(\theta-\theta'), \tag{4.4}$$
$$[\rho_5^\alpha(\theta),\rho_5^\beta(\theta')]=i\epsilon^{\alpha\beta\gamma}\rho^\gamma(\theta)\delta(\theta-\theta').$$

Let us temporarily assume that the parton-parton interaction is exactly invariant under $SU_2 \times SU_2$. In this case the Heisenberg equations for $\rho^\alpha$ and $\rho_5^\alpha$ will have the form of continuity equations. For example the equations for $\tau^\alpha$ and $\tau_5^\alpha$ will have the form

$$\tfrac{1}{2}\tau^\alpha(j) = -\left[\frac{J^\alpha(j,j+1)}{\eta_j+\eta_{j+1}} - \frac{J^\alpha(j-1,j)}{\eta_{j-1}+\eta_j}\right], \tag{4.5}$$

where $J^\alpha(j,j+1)$ depends on $\xi_j$ and $\xi_{j+1}$. Averaging over small intervals and defining

$$J^\alpha(\theta) = \frac{J^\alpha(j,j+1)}{\eta_j+\eta_{j+1}},$$

leads to

$$\begin{aligned}&\partial_\tau \rho^\alpha(\theta,\tau) + \partial_\theta J^\alpha(\theta,\tau) = 0,\\ &\partial_\tau \rho_5^\alpha(\theta,\tau) + \partial_\theta J_5^\alpha(\theta,\tau) = 0.\end{aligned} \tag{4.6}$$

We shall group $\rho$ and $J$ into a 2-vector in the $(\theta, \tau)$ space:

$$\rho^\alpha \equiv V_\tau^\alpha, \quad J^\alpha \equiv V_\theta^\alpha. \tag{4.7a}$$

Similarly

$$\rho_5^\alpha \equiv A_\tau^\alpha, \quad J_5^\alpha \equiv A_\theta^\alpha. \tag{4.7b}$$

When not distinguishing between vector and axial-vector quantities we will denote the currents and densities by

$$(\mathcal{J}_\tau, \mathcal{J}_\theta).$$

The symbol $\nabla$ will be used to indicate derivatives with respect to $(\theta, \tau)$:

$$\nabla \equiv (\partial_\tau, \partial_\theta).$$

Thus (4.6) is written

$$\nabla \cdot V = 0, \tag{4.8}$$

$$\nabla \cdot A = 0, \tag{4.9}$$

or

$$\nabla \cdot \mathcal{J} = 0. \tag{4.10}$$

The concept of a spontaneous symmetry breakdown entails the existence of an instability of the symmetric ground state under small perturbations which explicitly violate the symmetry. For the ferromagnet the small perturbation could be a weak external magnetic field which serves to define the direction of magnetization. Of course any explicit symmetry violation will cause an asymmetry of the ground state of a system. The special feature of spontaneous symmetry breakdown is that the asymmetry persists even as the perturbation tends to zero.

In the real world the role of the external magnetic field is replaced by the small pion mass. Accordingly we expect the exact interparton forces to violate chiral invariance by terms of order $m_\pi^2$. Equation (4.9) will be modified by the presence of a source term proportional to $m_\pi^2$,

$$\nabla \cdot A^\alpha = c\Phi^\alpha(\theta,\tau), \quad c \sim m_\pi^2. \tag{4.11}$$

Here $\Phi$ is a local operator on the string which remains finite as $c \to 0$.

The continuity equation does not tell the entire story even when $\mathcal{J}$ is conserved. Boundary conditions on $\mathcal{J}_\theta$ must be specified in order to know whether charges may be lost at the endpoints $\theta = 0, \pi$. The vector current is expected to satisfy[3]

$$V_\theta|_{0,\pi} = 0. \tag{4.12}$$

We shall see that as long as $c \neq 0$, $A_\theta|_{0,\pi}$ vanishes. However in the limit $c \to 0$ this condition cannot be maintained. In fact we shall see that spontaneous symmetry breakdown implies an instability with respect to leakage of current across the "ends" of the hadron.[3] However as long as $c \neq 0$ we may write (see Appendixes B and C for examples)

$$c \neq 0: \; A_\theta^\alpha|_{0,\pi} = 0. \tag{4.13}$$

### B. Currents in Space-Time

We shall now connect the string currents $\mathcal{J}(\theta,\tau)$ with the corresponding space-time current operators $j(x)$. We restrict ourselves to purely transverse-momentum-transfer matrix elements in the infinite-momentum frame. This is because we have been treating the longitudinal momenta of partons as fixed $c$ numbers, and the absorption of a longitudinal momentum would obviously excite these degrees of freedom: We therefore define the longitudinally integrated currents in the IMF,[13]

$$\begin{aligned}&j_\tau(X,\tau) = \int d\mathfrak{z}\; j_0(\mathfrak{z}, X, t),\\ &j_\perp(X,\tau) = \int d\mathfrak{z}\; e^{\omega} j_\perp(\mathfrak{z}, X, t),\end{aligned} \tag{4.14}$$

where $j_\mu$ are the 4-dimensional space-time current components and $\omega$ is the hyperbolic angle which describes the boost to the IMF. The matrix elements of $j_\tau$ and $j_\perp$ vanish unless the spatial momentum transfer is purely transverse.

The charge density operators $v_\tau(X,\tau)$ and $a_\tau(X,\tau)$ are given by a simple summation over the contributing partons:

$$\begin{aligned}&v_\tau^\alpha(X,\tau) = \sum_l \tfrac{1}{2}\tau_l^\alpha(\tau)\delta^{(2)}(X_l(\tau) - X),\\ &a_\tau^\alpha(X,\tau) = \sum_l \tfrac{1}{2}\tau_{5l}^\alpha(\tau)\delta^{(2)}(X_l(\tau) - X),\end{aligned} \tag{4.15}$$

where $X$ is a $c$-number field position and $X_l(\tau)$ is the $q$-number position of the $l$th parton.

Alternatively we may Fourier transform (4.15) to get

$$v_\tau^\alpha(Q,\tau)=\sum_l \tfrac{1}{2}\tau_l^\alpha(\tau)\exp[iQ\cdot X_l(\tau)],$$
$$a_\tau^\alpha(Q,\tau)=\sum_l \tfrac{1}{2}\tau_{5l}^\alpha(\tau)\exp[iQ\cdot X_l(\tau)]. \tag{4.16}$$

The continuum version of (4.16) is

$$j_\tau(Q,\tau)=\int_0^\pi d\theta\,\mathcal{J}_\tau(\theta,\tau)\exp[iQ\cdot X(\theta,\tau)]. \tag{4.17}$$

Next consider the transverse components of $v$. In the IMF the continuity equation becomes[12]

$$\partial_\tau v_\tau+\partial_\perp\cdot v_\perp=0. \tag{4.18}$$

We use the notation $\partial_\perp=(\partial_x,\partial_y)$. In momentum space,

$$\partial_\tau v_\tau+iQ\cdot v_\perp=0. \tag{4.19}$$

From (4.17 and (4.10) we obtain

$$\partial_\tau v_\tau=-\int_0^\pi d\theta\,e^{iQ\cdot X(\theta)}\partial_\theta V_\theta+\int_0^\pi d\theta\,V_\tau\tfrac{1}{2}\{iQ\cdot\partial_\tau X,e^{iQ\cdot X}\}, \tag{4.20}$$

where we have used

$$\partial_\tau e^{iQ\cdot X}=\{iQ\cdot P\eta^{-1},e^{iQ\cdot X}\}=\tfrac{1}{2}\{iQ\cdot\partial_\tau X,e^{iQ\cdot X}\}.$$

Integrating the first term on the right-hand side of (4.20) by parts and using (4.12) gives

$$\frac{i}{2}\int_0^\pi d\theta\,V_\theta\{Q\cdot\partial_\theta X,e^{iQ\cdot X}\}. \tag{4.21}$$

Thus

$$\partial_\tau v_\tau=\tfrac{1}{2}iQ\cdot\int d\theta\,V\cdot\{\nabla X,e^{iQ\cdot X}\}$$

and we may identify

$$v_\perp(Q,\tau)=\tfrac{1}{2}\int d\theta\,V(\tau)\cdot\{\nabla X(\theta,\tau),e^{iQ\cdot X(\theta,\tau)}\}. \tag{4.22}$$

Obviously we may add any term of the form $\tilde{v}_\perp$, where

$$\tilde{v}_x=iQ_yM(Q,\tau),$$
$$\tilde{v}_y=-iQ_xM(Q,\tau). \tag{4.23}$$

The term in Eq. (4.22) may be called the convective part of the current and the term in (4.23) the spin or magnetic current. To specify the operator $M$ we can assume that $v$ couples to the electromagnetic field minimally. Then if the partons are scalar bosons we find $M=0$, while for spin-$\frac{1}{2}$ fermions $M$ is a magnetic-moment-like operator:

$$M_{1/2}=\sum_j \eta_j^{-1}\tfrac{1}{2}\tau(j)\sigma_z(j)e^{iQ\cdot X(j)}. \tag{4.24}$$

For our purposes it will be sufficient to study the convective part of the current so that (4.22) may be used.

The convective part of the axial-vector current may be defined by

$$a_\tau(Q,\tau)=\int d\theta\,A_\tau(\theta,\tau)e^{iQ\cdot X(\theta,\tau)},$$
$$a_\perp(Q,\tau)=\tfrac{1}{2}\int d\theta\,A(\theta,\tau)\cdot\{\nabla X(\theta,\tau),e^{iQ\cdot X(\theta,\tau)}\}. \tag{4.25}$$

Evaluating $\partial_\tau a_\tau+iQ\cdot a_\perp$ by integration by parts we find

$$\partial_\tau a_\tau+iQ\cdot a_\perp=\int d\theta\,\nabla\cdot A e^{iQ\cdot X}.$$

Using (4.11) the continuity equation for $a$ becomes

$$\partial_\tau a_\tau+iQ\cdot a_\perp=c\int d\theta\,\Phi(\theta,\tau)e^{iQ\cdot X(\theta,\tau)}. \tag{4.26}$$

## V. CURRENTS AND RESIDUES

### A. Definition of Operator Dimensions

In this section we derive the relations between local distributions, spectrum of states and Regge parameters. We define a local dynamical variable $F(\theta)$ as one which is built out of the degrees of freedom which pertain to the partons at or near $\theta$. These operators may be assigned a transformation law under the transformation defined by (3.12). These transformations are closely related to longitudinal boosts of the low-$\eta$ partons. Thus these transformation laws will be chosen so as to reflect the boost properties of the parton degrees of freedom.

For example, the transverse position operator, the helicity and isospin will transform according to:

$$X(\theta)\to X(\alpha\theta),$$
$$P(\theta)\to P(\alpha\theta),$$
$$\sigma(\theta)\to\sigma(\alpha\theta), \tag{5.1}$$
$$\tau(\theta)\to\tau(\alpha\theta),$$
$$\tau_5(\theta)\to\tau_5(\alpha\theta).$$

The corresponding densities however, will transform with an extra power of $\alpha$ namely,

$$\mathcal{J}_\tau(\theta)d\theta\to\mathcal{J}_\tau(\alpha\theta)d(\alpha\theta)$$

or (5.2)

$$\mathcal{J}_\tau(\theta)\to\alpha\mathcal{J}_\tau(\alpha\theta).$$

Similarly, the fluxes will acquire the same power of $\alpha$, because of the explicit factor $\eta^{-1}$ in Eq. (4.5):

$$\mathcal{J}_\theta(\theta) \to \alpha \mathcal{J}_\theta(\alpha\theta). \tag{5.3}$$

The continuity equation for the axial charge dictates the transformation law of the source $\Phi$:

$$\Phi(\theta) \to \alpha^2 \Phi(\alpha\theta). \tag{5.4}$$

This indicates that $\Phi(\theta)$ is of the form

$$\Phi(\theta) = \frac{\varphi(\theta)}{\eta^2(\theta)}, \tag{5.5}$$

where $\varphi(\theta)$ depends on local parton degrees of freedom such as those in (5.1). All the operators which we consider will transform according to

$$F_i(\theta) \to \alpha^{-d_i} F_i(\alpha\theta), \tag{5.6}$$

and the power $d_i$ will be called the dimension of $F_i$. These dimensions should not be confused with the field-theoretic naive dimensions which refer to space-time dilations.

Another important property of $F(\theta)$ is the behavior of its matrix elements as $\sin\theta \to 0$. This behavior is dynamical and cannot be deduced from the dimension. For example we consider the creation of an isospin-one state by flipping the isospin of a parton in an isospin-zero state. The flipped parton will react on its neighbors and cause the isospin to be distributed over the chain. The precise distribution of isospin in stationary states will be controlled by the interparton dynamics. Generally we will assume that each local variable $F_i(\theta)$ tends to a power law near $\theta = 0, \pi$ (see Appendix A):

$$F_i(\theta) = (\sin\theta)^{\gamma_i} \hat{F}_i(\theta), \tag{5.7}$$

where $\hat{F}_i(\theta)$ is an operator whose matrix elements are generally finite nonzero numbers for $\theta = 0, \pi$. We call $\hat{F}(\theta)$ the residue of $F(\theta)$ and define its dimension

$$\hat{d}_i = d_i - \gamma_i. \tag{5.8}$$

An an example we consider the operator $\exp[iQ\cdot X(\theta)]$. Formally this operator has dimension zero. As we have seen,

$$e^{iQ\cdot X(\theta)} \approx (\sin\theta)^{Q_2} \!:\! e^{iQ\cdot X(\theta)} \!:\! e^{iQ\cdot X_{\text{c.m.}}},$$

where $:\exp[iQ\cdot X(\theta)]:$ is a finite operator near $\theta = 0, \pi$. Evidently $e^{iQ\cdot X_{\text{c.m.}}} : e^{iQ\cdot X(\theta)}:$ is the residue of the operator and the power $\gamma$ is identified as $Q^2$. The dimension of $: e^{iQ\cdot X}:$ is thus $\hat{d}(e^{iQ\cdot X}) = -Q^2$.

### B. Vertices

Next consider the coupling of a local external field to the parton system. In general the coupling will be to one of the quantities $F_i(\theta)$. For example, in Sec. IV we have seen how electromagnetic fields couple to the hadron. Let us suppose the external field is a space-time tensor with any number of transverse indices and a number $L$ of longitudinal or time-like indices. For example the quantities $j_\perp$ have $L=0$ while $j_\tau$ has $L=1$. Scalar fields always have $L=0$.

Longitudinal boost invariance requires[13] the vertex to have dimension $L-1$

$$T(Q) = \sum_l F(l)\eta_l^{-1+L-d} e^{iQ\cdot X(l)}, \tag{5.9}$$

where $F(l)$ has dimension $d$.

The continuum analog of (5.9) is

$$T(Q) = \int d\theta\, F(\theta)(\sin\theta)^{-2+L-d} e^{iQ\cdot X(\theta)}. \tag{5.10}$$

The dimension of the vertex is therefore insured to be $(L-1)$. Equations (4.17), (4.22), (4.24), (4.25), and (4.26) are all examples of (5.9) and (5.10). In (4.17) we are coupling to an external time-component of a vector potential so that $L=1$. Clearly $F$ is given by $\mathcal{J}_\tau$ which has $d=-1$. Hence (5.10) is satisfied. For the transverse currents $L=0$ and $F = \mathcal{J}\cdot\nabla X$ which has $d=-2$. Finally in (4.26) we consider the form for a field which couples to the divergence of the axial-vector current. Since the divergence is a scalar, $L=0$. Equation (5.10) is satisfied since $\Phi$ has dimension $-2$.

We shall now consider the singularities in $T(Q)$ as a function of $Q^2$. These singularities will be identified with the spectrum of hadrons which have the quantum numbers of the external field. For the vector current this includes the $\rho$ meson and for the axial-vector current, the $\pi$ and $A_1$ mesons. We work with Eq. (5.10) and make the substitution indicated by (5.7) and (5.8).

$$T(Q) = \int d\theta (\sin\theta)^{-2+L-d+\gamma+Q^2} \times \hat{F}(\theta) : e^{iQ\cdot X(\theta)} : e^{iQ\cdot X_{\text{c.m.}}}. \tag{5.11}$$

Since $\hat{F}$ and $: e^{iQ\cdot X}:$ have regular finite matrix elements, the singularities of $T(Q)$ must arise from the factor $(\sin\theta)^{-2+L-d+\gamma+Q^2}$. Clearly a pole will occur from divergences of the integral near $\sin\theta = 0$, when $-2+L-d+\gamma+Q^2 = -1$. Thus the position of the lowest mass singularity in $T(Q)$ is

$$Q^2 = d - \gamma - L + 1$$

or (5.12)

$$m^2 = L - 1 - \hat{d}.$$

The residue of the pole is easily seen to be

$$[\hat{F}(0) : e^{iQ\cdot X(0)} : + \hat{F}(\pi) : e^{iQ\cdot X(\pi)} :] e^{iQ\cdot X_{\text{c.m.}}}. \tag{5.13}$$

Thus we see that the on-mass-shell coupling matrices for hadrons are functions of the degrees of freedom of the ends of the hadronic string.

### C. Regge Residues

We next consider an amplitude $A+i\to B+j$, where $A$, $i$, $B$, $j$ are hadrons. $A$ and $B$ are arbitrary except that for simplicity we assume they are orbitally unexcited. The hadrons $i$ and $j$ are coupled to the operators

$$:e^{iQ_i\cdot X}:\hat{F}_i\,,\quad :e^{iQ_j\cdot X}:\hat{F}_j$$

as in Eq. (5.13). We consider the process[10] where $i$ is absorbed at $(0,\tau)$ and $j$ is subsequently emitted at point $(0,0)$. This is given by

$$T^{BA}_{ji}(\tau)=\langle B|:e^{-iQ_j\cdot X(0,0)}:e^{-iQ_j\cdot X_{\text{c.m.}}(0)}\hat{F}_j(0,0):e^{iQ_i\cdot X(0,\tau)}:e^{iQ_i\cdot X_{\text{c.m.}}(\tau)}\hat{F}_i(0,\tau)|A\rangle\,. \tag{5.14}$$

We shall consider only the case in which $F_i$ and $F_j$ are independent of orbital degrees of freedom. Then the expression in (5.14) factorizes into an orbital and internal factor. Using (3.17) and integrating over $\tau$ we get

$$T^{BA}_{ji}=\int_0^1 du\,u^{-s-1+M_A^2}(1-u)^{2Q_i\cdot Q_j}\times\langle B|\hat{F}_j(00)\hat{F}_i(0\tau)|A\rangle u\equiv e^{i\tau}\,. \tag{5.15}$$

Let us next use (5.12) to write

$$\begin{aligned}2Q_i\cdot Q_j&=(Q_i+Q_j)^2-Q_i^2-Q_j^2\\&=-t+L_i+L_j-\hat{d}_i-\hat{d}_j\,.\end{aligned}$$

Then $T$ is given by

$$T^{BA}_{ji}=\int_0^1 du\,u^{-s-1+M_A^2}(1-u)^{-t+L_i+L_j-\hat{d}_i-\hat{d}_j-2}\times\langle B|\hat{F}_j(00)\hat{F}_i(0\tau)|A\rangle\,. \tag{5.16}$$

To extract the Regge parameters from (5.16) we consider the high-energy limit in which $s\to\infty$ with $t$ fixed. Since $u=e^{i\tau}$ it is evident that the important region of integration is $\tau\sim s^{-1}$.

The behavior of $T$ for $s\to\infty$ will therefore depend on the behavior of $\langle B|\hat{F}_j(00)\hat{F}_i(0\tau)|A\rangle$ as $\tau\to 0$. Let us suppose this matrix element behaves as $\tau^a$. Then the integrand behaves like

$$u^{-s}(1-u)^{-t+L_i+L_j-\hat{d}_i-\hat{d}_j+a-2}$$

and the form of the amplitude for $s\to\infty$ is $\beta(t)s^{\alpha(t)}$ with

$$\alpha(t)=(1+\hat{d}_i+\hat{d}_j-L_i-L_j-a)+t\,. \tag{5.17}$$

The amplitude will be proportional to the numerical coefficient of $\tau^a$ in the small-$\tau$ expansion of $\langle B|\hat{F}_j(00)\hat{F}_i(0\tau)|A\rangle$. Thus in order to properly understand the high-energy behavior of $T$ we must develop a theory for $F(0)F(\tau)$ for small $\tau$.

### D. The Operator Residue Expansion

Following ideas developed by Wilson in quantum field theory we propose that the residues $\hat{F}$ satisfy an operator-product expansion when $\tau\to 0$. The motivation for Wilson's expansion is the possible dilatation invariance of quantum field theory at small distances. The analogous symmetry which motivates our expansion is the local dilatation invariance indicated in Eq. (2.7) and (3.12). We now write the expansion:

$$\hat{F}_j(00)\hat{F}_i(0\tau)=\sum_k c_{jik}\hat{F}_k(00)\tau^{\hat{d}_i+\hat{d}_j-\hat{d}_k}\,. \tag{5.18}$$

The summation is over all $\hat{F}_k$'s which are not forbidden by quantum-number considerations. In general the unit operator may be included among the $\hat{F}_k$'s in which case $\hat{d}_k$ is zero.

Let us now insert (5.18) into (5.16) to obtain

$$T^{BA}_{ji}\underset{s\to\infty}{\sim}\sum\int du\,u^{-s}(1-u)^{-t-2-\hat{d}_k+L_i+L_j}\times c_{jik}\langle B|\hat{F}_k(00)|A\rangle\,. \tag{5.19}$$

Evidently the high-energy behavior is

$$\begin{aligned}&T\to\beta^{BA}_{jik}s^{\alpha_k(t)-L_i-L_j}\,,\\&\alpha_k(t)=(1+\hat{d}_k)+t\,,\\&\beta^{BA}_{jik}=c_{jik}\langle B|\hat{F}_k(00)|A\rangle\beta(t)\,,\end{aligned} \tag{5.20}$$

where $\beta(t)$ is independent of $A$, $B$, $i$, $j$ and may be computed from (5.19).

Evidently (5.20) describes the coupling of a factorizable Regge pole with intercept

$$\alpha_k(0)=1+\hat{d}_k\,. \tag{5.21}$$

Let us apply these arguments to the important case of the isospin distribution $V^\alpha_\tau(\theta)$. The particle which couples to isospin is the $\rho$ meson whose intercept is $\alpha_\rho(0)=\frac{1}{2}$. Using (5.21), we find $\hat{d}(V_\tau)=-\frac{1}{2}$.

Since $V$ has dimension $-1$ Eq. (5.8) gives

$$\gamma(V_\tau)=-\tfrac{1}{2}\,. \tag{5.22}$$

This indicates that the isospin per unit $\theta$ (or $\eta$) varies as $\theta^{-1/2}$ (or $\eta^{-1/2}$). Equivalently the isospin per parton varies as $\eta^{1/2}$. This fact has important phenomenological consequences for deep-inelastic scattering and multiparticle production.[2]

These connections between local currents on the string-like hadron and Regge trajectories may be a powerful tool for exposing the relations between current algebra and Regge theory. In Sec. VI we show how chiral symmetry may be formulated as a theory of string currents and residue operators.

## VI. CHIRAL SYMMETRY AS A SPONTANEOUSLY BROKEN SYMMETRY

### A. Spontaneous Symmetry Breakdown

We assume that the interparton forces are invariant or almost invariant under transformations generated by the chiral charges $(\frac{1}{2}\tau_\alpha, \frac{1}{2}\tau_5^\alpha)$. Ordinarily, for a system of a finite number of degrees of freedom this implies that the ground state is either a singlet or a member of a degenerate chiral multiplet of finite multiplicity. For a system of infinitely many degrees of freedom, a spontaneous breakdown may occur. This happens when energy considerations favor a nonzero expectation value for some noninvariant quantity throughout the system. For example the finite-energy states of a ferromagnet have a nonvanishing magnetization $\langle\sigma\rangle$ throughout most of the system. This is true in spite of the fact that no direction is favored by the local spin-spin interactions.

The group of interest[14] in strong-interaction physics is $SU_2\times SU_2$. This group is locally isomorphic to the group of 4-dimensional rotations $(O_4)$ and for our purposes they are identical. The representations of $SU_2\times SU_2$ that will occur in our study are tensors of $O_4$. Thus we will speak of scalars, vectors, tensors etc. under the chiral group.

We assume that it is possible to form a chiral 4-vector with components $(\varphi^\alpha(\theta), \varphi^4(\theta))$ from the local parton variables near $\theta$. We also assume that $\varphi$ is invariant under spatial rotations about the longitudinal direction in the infinite-momentum frame. (This is to avoid a spontaneous breakdown of rotational symmetry.) The $\varphi$'s are to be thought of as combinations of the $\xi(l)$'s and therefore have dimension zero in the sense of Sec. V. Our fundamental assumption is that the spontaneous breakdown is due to a nonzero expectation value of $\varphi^4(\theta)$ for the enormous number of partons of low $\eta$. Thus $\langle\varphi^4(\theta)\rangle \not\to 0$ when $\sin\theta\to 0$. The other three components of $\varphi$ are assumed to tend to zero in order to avoid a breakdown of isospin symmetry. The similarity with ferromagnetism suggests that we call the local degrees of freedom from which $\varphi$ is formed by the name "chiral magnets."

We shall next argue that the alignment of chiral magnets near $\theta=0,\pi$ causes a breakdown of the boundary conditions (4.13).[3] Thus we will find that indefinite quantities of chiral charge may be lost across the "wee" parton ends of the hadron. At this point the reader should consult Appendix A for the precise meaning of boundary conditions near $\theta=0,\pi$.

To prove that $\lim A_\theta\neq 0$ we resort to an artifice of allowing the hadronic string to interact with a second system whose states form a finite representation of $SU_2\times SU_2$. For definiteness we will allow the extra chiral magnet to be described by a Dirac matrix representation of $SU_2\times SU_2$. The extra chiral magnet will be coupled to the degrees of freedom at point $\theta_0$ where $\theta_0$ is chosen very near zero. Thus the local energy scale for disturbances near $\theta_0$ is extremely large. (See II C.) However, we shall couple the extra chiral magnet with a coupling constant $g\sim 1$ (as opposed to $g\sim\theta_0^{-1}$) so that it be incapable of significantly exciting high-energy excitations.

The coupling of the extra chiral magnet $(\gamma^\alpha, \gamma^4)$ to the degrees of freedom near $\theta_0$ will be

$$g(\varphi^\alpha\gamma^\alpha+\varphi^4\gamma^4)\,. \tag{6.1}$$

Since (6.1) is chirally symmetric the total chiral charge should be conserved.

Now let us assume that $\langle\varphi^4(\theta_0)\rangle\neq 0$ and $\langle\varphi^\alpha(\theta_0)\rangle\approx 0$ for the low-lying energy levels. (Recall that the coupling is not strong enough to appreciably excite states with energy $\sim\theta_0^{-1}$.) The effective coupling can then be approximated by

$$g\langle\varphi^4\rangle\gamma^4\,. \tag{6.2}$$

Next consider the equation of motion for the chiral charge, $q^\alpha$, of the extra magnet. Since $q^\alpha=\frac{1}{2}i\gamma^4\gamma^\alpha$

$$\partial_\tau q^\alpha=\frac{g}{i}\langle\varphi^4\rangle[\tfrac{1}{2}i\gamma^4\gamma^\alpha, \gamma^4]$$
$$=-g\langle\varphi^4\rangle\gamma^\alpha\,, \tag{6.3}$$

$$\partial_\tau{}^2 q^\alpha=g^2\langle\varphi^4\rangle^2 q^\alpha\,. \tag{6.4}$$

Clearly the extra chiral magnet precesses in the external field of the partons near $\theta_0$. The periodic increase and decrease of $q^\alpha$ must be compensated by a periodic flux of chiral charge into and out of the point $\theta_0$. Indeed, the conservation of charge near $\theta_0$ reads

$$-A_\theta^\alpha\big|_{\theta_0-\epsilon}^{\theta_0+\epsilon}=\partial_\tau q^\alpha\,. \tag{6.5}$$

so that we find that a low-energy perturbation at the point $\theta_0$ can cause a current to flow with magnitude of order $\langle\varphi^4\rangle$.

Since $\varphi^4$ does not tend to zero for $\sin\theta\to 0$ we discover that the low-energy matrix elements of $A_\theta$ cannot in general tend to zero. This phenomenon indicates that the system is subject to a spontaneous current loss which can be written in the form

$$\partial_\tau Q_5^\alpha = A_\theta^\alpha|_\pi^0 \neq 0\,. \tag{6.6}$$

The reason for the existence of a nonvanishing $\dot{Q}_5^\alpha$ in a symmetric theory is that we have not really accounted for all the degrees of freedom by considering partons with $\eta > 0$. As emphasized by Feynman, the existence of a $\eta^{-1}$ parton distribution means that no matter how much momentum we give to a hadron, it always has a tail of finite-momentum partons. These partons can interact and get mixed up with the virtual pairs in the vacuum. Thus the vacuum may be regarded as a source or sink for the chiral charge which disappears into the end points at $\theta = 0, \pi$. However, as we shall see, it is not necessary to know the details of this source and all the relevant information is contained in the parton-parton interactions.

We would also like to point out that according to Feynman the $\eta^{-1}$ parton distribution is connected with the constancy of hadronic cross sections at high energy. We may speculate then that the existence of constant cross sections is a prerequisite for spontaneous symmetry breakdown. In Sec. VIII we will in fact see that $\langle\varphi^4\rangle$ should be proportional to the asymptotic cross section for $\pi\pi$ scattering.

### B. The Behavior of $\varphi^\alpha$

It is helpful to introduce a small explicit chiral-symmetry breaking into the Hamiltonian. The asymmetry may eventually be allowed to tend to zero or it may be retained to describe the small pion mass. The perturbation is analogous to an external magnetic field imposed on a ferromagnet. We write

$$\begin{aligned} H &= H_{\text{symmetric}} + h\,, \\ h &= c\sum_j \frac{\varphi^4(j)}{\eta(j)} \\ &= c\int \frac{d\theta}{\sin^2\theta}\,\varphi^4(\theta), \end{aligned} \tag{6.7}$$

where $c$ is a small symmetry-breaking parameter.

Next consider the time derivative of the total axial charge $Q_5^\alpha$,

$$\begin{aligned} \partial_\tau Q_5^\alpha &= \int d\theta\, \partial_\tau A_\tau^\alpha(\theta) \\ &= -i[Q_5^\alpha, h]\,. \end{aligned} \tag{6.8}$$

Since $\varphi^4(\theta)$ is a component of a chiral 4-vector, $-i[Q_5^\alpha, \varphi^4] = \varphi^\alpha$. Thus

$$\partial_\tau Q_5^\alpha = \int \frac{d\theta}{\sin^2\theta}\,\varphi^\alpha(\theta)\,. \tag{6.9}$$

Evidently we may identify the source $c\Phi^\alpha$ of (4.11) with

$$c\Phi^\alpha = \frac{\varphi^\alpha}{\sin^2\theta}\ . \tag{6.10}$$

Let us now write

$$\Phi^\alpha = (\sin\theta)^\gamma \hat{\Phi}^\alpha\,, \tag{6.11}$$

so that (6.9) becomes

$$\partial_\tau Q_5^\alpha = c\int d\theta\,(\sin\theta)^\gamma\,\hat{\Phi}^\alpha\,. \tag{6.12}$$

In order that the matrix elements of $\dot{Q}_5$ be finite the constant $\gamma$ must satisfy

$$\gamma > -1 \tag{6.13}$$

as long as $c \neq 0$. However, as $c \to 0$ Eq. (6.6) indicates that $\dot{Q}_5$ should remain finite. It is easily seen that this may occur *if and only if* $\gamma$ *tends to* $-1$ *linearly with* $c$. Thus we write

$$\gamma = -1 + \mu(c) \tag{6.14}$$

and require $\mu$ to tend linearly to zero with $c$. For convenience we normalize $(\varphi^\alpha, \varphi^4)$ so that $\mu = c$ for small $c$.

Equation (6.12) then becomes

$$\partial_\tau Q_5^\alpha = \mu\int d\theta\,(\sin\theta)^{-1+\mu}\,\hat{\Phi}^\alpha(\theta)\,. \tag{6.15}$$

To take the symmetry limit we note that

$$\lim_{\mu\to 0}\ \mu(\sin\theta)^{\mu-1} = \delta(\sin\theta)\,. \tag{6.16}$$

Thus in the limit

$$\partial_\tau Q_5^\alpha = \hat{\Phi}^\alpha(0) + \hat{\Phi}^\alpha(\pi)\,. \tag{6.17}$$

From (6.6) we make the identification

$$\begin{aligned} A_\theta^\alpha(\theta = 0) &= \hat{\Phi}^\alpha(0)\,, \\ A_\theta^\alpha(\theta = \pi) &= -\Phi^\alpha(0)\,. \end{aligned} \tag{6.18}$$

### C. The Pion

The spontaneous breakdown of chiral symmetry entails the existence of a massless pion which plays the role of a Goldstone-Nambu boson. To see how our parton dynamics yields this result consider the space-time divergence of the axial-vector current. From Eq. (4.26) we find

$$\langle f\,|\partial_\mu a^\mu|i\rangle = \mu\left\langle f\middle|\int d\theta\, e^{iQ\cdot X(\theta)}\,\Phi^\alpha(\theta)\middle|i\right\rangle .$$

Now since $\Phi^\alpha$ has dimension $-2$ and $\partial_\mu a^\mu$ is a Lorentz scalar Eq. (5.12) indicates that the matrix element has poles at $-Q^2 = \gamma + 1$. Using (6.14) we find

$$m_\pi^2 = \mu \, .$$

Thus in the limit of chiral symmetry we find $m_\pi^2 \to 0$.

### D. Soft-Pion Emission

According to Nambu,[1] any change in the chiral charge of a fast moving hadron is accompanied by the emission of soft pions.[8] To see why this is so we consider the residue of the pion pole in (6.15). The residue is

$$c\hat{\Phi}(0) : e^{iQ\cdot X(0)} + c\hat{\Phi}(\pi) : e^{iQ\cdot X(\pi)} : \, .$$

The pole residue must factor into two factors; one expressing the coupling of the axial-vector current to a pion and the other being the on-shell coupling of the pion to the hadronic system. The coupling of the pion to the axial-vector current is

$$\tfrac{1}{2} m_\pi^2 f_\pi \quad (f_\pi = 190 \text{ MeV}).$$

The coupling matrix of pions to hadrons we call $T_\pi(Q)$. Thus,

$$\tfrac{1}{2} i f_\pi T_{\pi\alpha}(Q) = \hat{\Phi}^\alpha(0) : e^{iQ\cdot X(0)} : + \hat{\Phi}^\alpha(\pi) : e^{iQ\cdot X(\pi)} : = A_\theta^\alpha e^{iQ\cdot X}\big|_\pi^0 \, . \tag{6.19}$$

In deriving (6.19) we have used $m_\pi^2 = c$.

Now, allow $Q \to 0$ so that

$$\tfrac{1}{2} i f_\pi T_\pi \;(Q=0) = \hat{\Phi}(0) + \hat{\Phi}(\pi) \, .$$

Using (6.17),

$$T_{\pi\alpha}(Q=0) = \frac{2}{if_\pi} \partial_\tau Q_5^\alpha \, . \tag{6.20}$$

Thus we see that the emission of pions always accompanies a change in $Q_5$.

One final expression for pion emission amplitudes is obtained by taking matrix elements of (6.20) and using $i\dot{Q}_5 = [Q_5, H]$. Since the eigenvalues of $H$ are transverse mass-squared we get

$$\langle f | T_{\pi\alpha}(0) | i\rangle = -2 f_\pi^{-1} (m_i^2 - m_f^2) \langle f | Q_5^\alpha | i \rangle \, . \tag{6.21}$$

Equation (6.21) is a generalization of the Goldberger-Treiman relation.

The Adler-Weisberger-Weinberg soft-pion theorems may also be easily proved. We briefly sketch the argument. Consider an amplitude ${}_{\text{out}}\langle f, \pi^\alpha | i, \pi^\beta\rangle_{\text{in}}$, where $|f\rangle$ and $|i\rangle$ are members of a common isospin multiplet. The amplitude is proportional to

$$T_{\alpha\beta} = \frac{1}{i}\left(\frac{2}{f_\pi}\right)^2 \int_{-\infty}^{\infty} d\tau \langle f | T(T_\alpha(0, -Q_\alpha) T_\beta(\tau, Q_\beta)) | i \rangle, \quad T_\alpha(\tau, Q) \equiv A_\theta^\alpha(\tau) : e^{iQ\cdot X(\tau)} : \big|_\pi^0 \, , \tag{6.22}$$

where we have used (6.19) to express the on-shell pion vertices. The symbol $T$ indicates chronological ordering.

The amplitude is expanded in powers of $Q_\alpha$ and $Q_\beta$ and after some algebra and partial $\tau$ integrations we obtain $T_{\alpha\beta}$ to first order in $Q_\alpha$ and $Q_\beta$,

$$T_{\alpha\beta} \simeq \frac{1}{i} 4 f_\pi^{-2} (Q_\alpha + Q_\beta)\cdot(P_f + P_i) \langle f | [Q_5^\alpha, Q_5^\beta] | i \rangle \, . \tag{6.23}$$

Since $[Q_5^\alpha, Q_5^\beta]$ is the total isospin operator we replace

$$\langle f | [Q_5^\alpha, Q_5^\beta] | i \rangle$$

by

$$i\epsilon^{\alpha\beta\gamma} I^\gamma$$

to obtain Weinberg's form of the Adler-Weisberger relation.

### E. Cutoff Procedure

Although spontaneous breakdown is associated with infinite systems, it is possible to make approximations using finite systems if they are correctly constructed. The trick is to replace the effect of the eliminated degrees of freedom by a suitable external symmetry-breaking field. Consider, for example, some problem involving a particular small portion of a ferromagnet. Let us divide the infinite ferromagnet into a finite region $\Omega_1$ containing the region of interest and a second part $\Omega_2$ consisting of the rest of the ferromagnet. Assuming the interactions are near-neighbor, the effect of $\Omega_2$ on $\Omega_1$ is mainly to align the spins near the boundary of $\Omega_1$. Our point is that to a good approximation, the same effect can be achieved by replacing $\Omega_2$ by a magnetic field near the boundary of $\Omega_1$.

Let us consider an approximation to the hadronic chain in which we "chop off" the ends, thus leaving a finite number of partons. We will remove those partons with $\sin\theta < \epsilon$. This procedure should allow a reasonable description of states with $M^2$ less than some value $\nu(\epsilon)$. As we have argued in Sec. II C, we expect

$$\nu(\epsilon) \sim \frac{1}{\epsilon} \, . \tag{6.24}$$

Write the Hamiltonian as

$$H = H(>\epsilon) + H(<\epsilon) + H(\epsilon), \quad (6.25)$$

where $H(>\epsilon)$ involves only those degrees of freedom with $\sin\theta > \epsilon$, $H(<\epsilon)$ involves the region $\sin\theta < \epsilon$, and $H(\epsilon)$ is the interaction between the two regions.

For those states with $m^2 \ll \nu$ the region $\sin\theta < \epsilon$ remains frozen in its ground state since the energy needed to excite this region is $\gtrsim \nu$. Thus we may replace $H(<\epsilon)$ by an additive constant. We may also replace $H(\epsilon)$ by an effective interaction in which the degrees of freedom ($\sin\theta < \epsilon$) do not explicitly appear. In general they will be replaced by a few ground-state expectation values including $\langle\varphi^4\rangle$ (see Appendixes A and B). Furthermore, since the configuration of the subsystem ($\sin\theta < \epsilon$) is chirally asymmetric, the effective interaction between the two regions will break the chiral symmetry of the valence ($\sin\theta \geq \epsilon$) system. Thus we may write

$$H_{\text{eff}} = H_a + H_b, \quad (6.26)$$

where $H_a$ is chirally symmetric, and $H_b$ breaks the symmetry. Both terms include only the degrees of freedom ($\sin\theta \geq \epsilon$), but $H_b$ includes only those at $\sin\theta \simeq \epsilon$.

The symmetry-breaking operator $H_b$ gives rise to a nonconservation of the axial charge in the region ($\sin\theta \geq \epsilon$). In order to properly approximate the physics of the complete chain, the time derivative of this portion of the axial charge should satisfy

$$\partial_\tau Q_5 \approx A(\epsilon) - A(\pi - \epsilon). \quad (6.27)$$

The left-hand side of (6.27) is defined by the low-energy ($\ll \epsilon^{-1}$) matrix elements of the truncated system. The right-hand side is defined by corresponding matrix elements of the untruncated system. The equality supplies the constraint on the operator $H_{\text{eff}}$ which insures the validity of the approximation procedure. (The sequence of approximations in which $\epsilon \to 0$ is an elaborate form of the self-consistent-field approximation.[15]) The procedure we have defined works well when the ratio of neighboring longitudinal fractions $\eta_{j+1}/\eta_j = \lambda$ is small. To further deal with (6.27) we use (6.17) and the continuity of $\hat{\Phi}$ and $A_\theta$ to replace the right-hand side by

$$\partial_\tau Q_5 = \hat{\Phi}(\epsilon) + \hat{\Phi}(\pi - \epsilon). \quad (6.28)$$

Thus $H_b$ must satisfy

$$i[Q_5^\alpha, H_b] \approx \hat{\Phi}^\alpha(\epsilon) + \hat{\Phi}^\alpha(\pi - \epsilon) = \frac{\varphi^\alpha(\epsilon)}{\epsilon} + \frac{\varphi^\alpha(\pi - \epsilon)}{\epsilon}, \quad (6.29)$$

where we have used (6.10) and (6.11) to relate $\varphi$ and $\hat{\Phi}$.

Let us write

$$H_b = \frac{\varphi^4(\epsilon) + \varphi^4(\pi - \epsilon)}{\epsilon} + H'. \quad (6.30)$$

Since $\varphi^4$ is the 4th component of a chiral 4-vector $i[\varphi^4, Q_5^\alpha] = \varphi^\alpha$, so that $H'$ must commute with $Q_5^\alpha$. We therefore absorb $H'$ and $H_a$ into the chirally symmetric term $H_0$:

$$H_{\text{eff}} = H_0 + \frac{\varphi^4(\epsilon) + \varphi^4(\pi - \epsilon)}{\epsilon}. \quad (6.31)$$

Thus we see that we can account for the aligning forces exerted by the frozen sea by the presence of a chirally asymmetric term in $H_{\text{eff}}$.

### F. Representation-Mixing Schemes

Let us suppose that we have solved for the eigenvectors of $H_0$. These eigenvectors will obviously form $SU_2 \times SU_2$ multiplets. We shall label an eigenvector by a multiplet ($A$) and an index ($i$) which distinguishes states within the multiplet. Thus a typical eigenvector of $H_0$ is written $|A, i\rangle_0$. Since in the cutoff model only a finite number of partons are present the multiplets are finite dimensional.

We can estimate the size of the multiplets which will be required to account for the levels with $m^2 < \epsilon^{-1}$. The partons involved in the excitation of these states are found in the interval $\sin\theta > \epsilon$ and their total number is

$$N(\epsilon) \simeq \int_\epsilon^{\pi-\epsilon} \frac{d\theta}{\lambda_0 \sin\theta} \sim -\ln\epsilon. \quad (6.32)$$

Each parton is an $n$-dimensional representation of $SU_2 \times SU_2$ so that the largest representation that can be built has multiplicity

$$n^{N(\epsilon)} \approx n^{-\ln\epsilon} = \epsilon^{-\ln n}. \quad (6.33)$$

Thus as we try to understand the chiral properties of increasingly heavy states, the representations we will need to describe these states will grow as a power of the mass.

This does not mean that it is the heavy states that we will have to assign to large multiplets. On the contrary, *it is the light states which will belong to the largest representations.* Consider for example the ground state or some very low lying excitation. Every time we add a pair of partons to the end of the chain we increase the size of the representation that describes the ground state. This is because the added partons are not in a singlet but are aligned with a nonvanishing $\langle\varphi^4\rangle$. This situation is similar to what happens if we consider larger and larger portions of a ferromagnet. Here, even in the ground state, the angular momentum of the subsystem grows.

On the other hand the small representations correspond to configurations in which chiral magnets are antialigned and therefore must have large energy. Thus we come to the surprising conclusion that at each level of approximation, the largest multiplets correspond to low-energy states while the smallest multiplets have the largest energy. Furthermore, there is no absolute meaning to saying that a particular hadron is a superposition of some particular chiral multiplets since adding partons to the description will change the multiplets involved.

Nevertheless there is a meaning to the representation content of a hadron. Let us consider a hadron with mass squared $=\nu \ll \epsilon^{-1}$. From what we have said previously, we can expect the state vector to approximately factorize

$$|\psi\rangle \approx |\psi_v\rangle |\psi_s\rangle \,, \tag{6.34}$$

where $\psi_v$ describes the partons with $\sin\theta > 1/\nu$ and $\psi_s$ describes the ground-state configuration of partons with $\epsilon^{-1} < \sin\theta < \nu^{-1}$. The chiral content of $\psi_v$ may consist of a few finite representations of $SU_2 \times SU_2$. However the representations describing $\psi_s$ increase with decreasing $\epsilon$. In a sense the representations included in $\psi_v$ are a minimal description of the hadron $|\psi\rangle$.

Let us return now to the eigenvectors of $H_0$ which we have labeled $|Ai\rangle_0$. These of course are not the actual eigenfunctions of $H_{\text{eff}}$ because of the symmetry-breaking term in (6.31). In general the physical eigenvectors of $H_{\text{eff}}$ are related to those of $H_0$ by a unitary transformation which will mix and split the multiplets. Thus we define physical eigenvectors by

$$|Ai\rangle = V|Ai\rangle_0 \,. \tag{6.35}$$

The matrix $V$ contains a great deal of useful physics. Suppose, for example, we are computing the width of a transition $(Ai) \to (Bj) + \pi$. According to (6.20) we have

$$\langle Bj | T_\pi(0) | Ai\rangle = \frac{2}{f_\pi}\,(m_{Bj}{}^2 - m_{Ai}{}^2)\langle Bj | Q_5 | Ai\rangle . \tag{6.36}$$

Since we know the action of $Q_5$ on the states $|Ai\rangle_0$ and not $|Ai\rangle$ it is essential to expand the physical states in $SU_2 \times SU_2$ multiplets. Using (6.35) for this purpose we get[16]

$$\langle Bj | T_\pi | Ai\rangle = \frac{2(m_{Bj}{}^2 - m_{Ai}{}^2)}{f_\pi}\, {}_0\langle Bj | V^{-1} Q_5 V | Ai\rangle_0 . \tag{6.37}$$

The amplitudes $T_\pi$ $(Q=0)$ for all helicity states are sufficient for computing the relevant decay rate according to[6,7],

$$\Gamma(A \to B\pi) = \frac{8}{\pi f_\pi{}^2}\,\frac{1}{2J_A+1}\sum_\lambda \langle B|Q_5(\lambda)|A\rangle q_{\text{c.m.}}{}^3 \,, \tag{6.38}$$

where $\lambda$ is the helicity and $J_A$ the spin of $A$.

## VII. QUARK MODELS

### A. The Need for Spin-Orbit Coupling

In this section two questions are considered. We first discuss the possibility of realizing our assumptions within the conventional quark-parton model[17] without spin-orbit coupling. We find that we cannot. In fact it is necessary to introduce some spin-orbit coupling in order to produce a spontaneous breakdown of the desired kind, or the model must include quarks and pseudoquarks (quarks of opposite parity).

The second question involves the possibility of extracting useful information without solving the dynamics of the entire hadronic string. We suggest that it may only be necessary to study the dynamics of a few quarks of maximal $\eta$ to understand the pattern of low-lying hadrons. Our approximation, when applied to a specific model, naturally leads to the Gilman-Harari[6] theory of chiral representation mixing.

### B. A No-Go Theorem

The space of states of a single quark (ignoring orbital motions and strangeness) is described in terms of isospin and spin indices acted upon by $\tau$ and $\sigma$ matrices. The $\tau$ matrices represent the isospin, and $\sigma_z$ the helicity or spin along the longitudinal axis. The matrices $\sigma_x$ and $\sigma_y$ are helicity-flip operators which form a transverse 2-vector.

Another equivalent representation is to use a Dirac-like spinor space to describe the $SU_2 \times SU_2$ properties of each quark. We make the identifications

$$\begin{aligned}
&\gamma^\alpha = \begin{pmatrix} 0 & \tau^\alpha \\ \tau^\alpha & 0 \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},\\
&\pm\sigma_z = \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4 ,\\
&\tau^\alpha = \frac{1}{2i}\,\epsilon^{\alpha\beta\gamma}\gamma^\beta\gamma^\gamma ,\\
&\pm\sigma_z\tau^\alpha = \tau_5^\alpha = \frac{1}{i}\,\gamma^4\gamma^\alpha ,
\end{aligned} \tag{7.1}$$

where the plus (minus) applies to quarks (antiquarks). In describing a whole string of quarks we build an infinite product of such Dirac spaces with $\gamma$ matrices for each quark (or antiquark).

Let us now consider the possible objects which can be used to describe $(\varphi^\alpha, \varphi^4)$. The conditions we require are

(1) $(\varphi^\alpha, \varphi^4)$ is a chiral 4-vector function of the $\gamma$'s near $\theta$.

(2) $(\varphi^\alpha, \varphi^4)$ commutes with the total angular momentum about the $z$ axis, $J_z$. If this were not so then a nonvanishing $\langle\varphi^4\rangle$ would spontaneously break rotational invariance. For similar reasons $\varphi^4$ should commute with reflections about a transverse axis and charge conjugation.

(3) A less fundamental restriction follows from the assumption of no spin orbit coupling. This requires $\varphi$ to be built from the internal degrees of freedom $(\tau, \sigma)$ without dependence on orbital variables. This is the only assumption which can reasonably be questioned.

It follows from (2) and (3) that $\varphi$ commutes with the sum of all $\sigma_z$'s,

$$[\varphi, \sum \sigma_z] = 0$$

or (7.2)

$$[\varphi, \sum \gamma_5] = 0\,.$$

*Theorem:* There does not exist a 4-vector function of the $\gamma$'s which satisfies (7.2). This includes operators which are formed from the degrees of freedom of several partons.

*Proof:* Suppose such a $\varphi$ existed. Then it would have to commute with $\prod_i \gamma_5(i)$. This is because $i\gamma^5 = \exp(\frac{1}{2} i\pi\gamma_5)$, so that

$$\prod_l i\gamma_5(l) = \exp[\tfrac{1}{2} i\pi \sum \gamma_5(l)]\,. \tag{7.3}$$

Thus anything which commutes with $\sum\gamma_5$ also commutes with $\prod\gamma_5$. Now for a single $\gamma$ space, $\gamma_5$ is the unitary operator representing that rotation (in the 4-dimensional space) which reflects all coordinates. Therefore in the product space this rotation is represented by $\prod\gamma_5$. Thus it follows that the unitary operator $\prod\gamma_5$ reverses the sign of every 4-vector function of the $\gamma$'s.

$$\left(\prod\gamma_5\right)^{-1} \varphi^{\alpha,4}\left(\prod\gamma_5\right) = -\varphi^{\alpha,4}\,. \tag{7.4}$$

This completes the proof that no $(\varphi^\alpha, \varphi^4)$ can commute with $\sum\sigma_z$.

One way out is to enlarge the space of states of a single quark so as to include quarks of positive and negative parity. In our opinion this would probably add too many unwanted low-lying states to the hadron spectrum. We shall therefore adopt the spin-orbit alternative.

It is not altogether clear how large the spin-orbit coupling must be in the underlying parton-parton interaction. What is clear is that $\varphi^4$ must be composed of spin-isospin quantities together with orbital variables. Thus it is certain that the effective Hamiltonian in (6.31) for the cutoff system is spin-orbit coupled. In other words any model which represents the low-lying hadrons as finite collections of quarks must contain spin-orbit coupling in the IMF.

### C. A Meson Model

We proceed by assuming that the elementary systems which behave with little or no spin-orbit coupling are clusters of partons. More explicitly we will assume the $\theta$ axis is populated by $q\bar{q}$ pairs. Within a pair we allow spin orbit coupling but we assume that it is absent or weak between pairs. Thus the elementary chiral magnets are $q\bar{q}$ pairs whose orbital excitations are now included as internal degrees of freedom.

We assume that the total longitudinal fraction of a chiral magnet is given by $\lambda_0 \sin\theta/\pi = \eta(\theta)$. The internal orbital wave function of the pair is written

$$\begin{aligned} &\psi(X_q - X_{\bar{q}}, \eta_q/\eta_{\bar{q}}), \\ &\eta_{\bar{q}} + \eta_q = \eta(\theta)\,. \end{aligned} \tag{7.5}$$

We shall further simplify the orbital motions by allowing only four possible orbital states, all others being assumed to have much higher energy. We label the states $|\uparrow\rangle$, $|\downarrow\rangle$, $|z\rangle$, $|s\rangle$.

$|\uparrow\rangle$ and $|\downarrow\rangle$: These states have ±1 unit of orbital helicity (orbital angular momentum about the $z$ axis).

$$\begin{aligned} &L_z|\uparrow\rangle = |\uparrow\rangle\,, \\ &L_z|\downarrow\rangle = -|\downarrow\rangle\,. \end{aligned} \tag{7.6}$$

They are assumed to be symmetric under the interchange of $\eta_q$ and $\eta_{\bar{q}}$.

$|z\rangle$: This state has $L_z = 0$,

$$L_z|z\rangle = 0\,. \tag{7.7}$$

It is assumed to be antisymmetric under $\eta_q \leftrightarrow \eta_{\bar{q}}$.

$|s\rangle$: This state also carries $L_z = 0$

$$L_z|s\rangle = 0\,. \tag{7.8}$$

It is assumed symmetric under $\eta_q \leftrightarrow \eta_{\bar{q}}$.

Our next assumption should really only be correct if the ratio of neighboring energy scales (ratios of $\eta$'s) is large. Namely, we assume that the lowest-energy mesons can be treated by cutting off the entire chain except for one chiral magnet near $\theta = \frac{1}{2}\pi$. The rest of the chain is then replaced by a symmetry-breaking interaction according to the prescription of Sec. VI E.

According to Sec. VI E the first step is to construct the eigenvectors of $H_0$ which form irreducible chiral multiplets of definite charge conjugation, parity and helicity. We shall denote the states of a $q\bar{q}$ system as follows:

TABLE I. Breakdown of $q\bar{q}$ states into $SU_2 \times SU_2$ multiplets for the helicity-0 sector.

| Chiral property | Name | Configuration |
|---|---|---|
| $X_\mu$ | $\pi_0$ | $(1/\sqrt{2})(\uparrow\uparrow\Big\downarrow - \downarrow\downarrow\Big\uparrow)T$ |
| | $\sigma_0$ | $(1/\sqrt{2})(\uparrow\uparrow\Big\downarrow + \downarrow\downarrow\Big\uparrow)S$ |
| $X_\mu$ | $(\pi_N)_0$ | $(1/\sqrt{2})(\uparrow\uparrow\Big\downarrow + \downarrow\downarrow\Big\uparrow)T$ |
| | $'\eta'_0$ | $(1/\sqrt{2})(\uparrow\uparrow\Big\downarrow - \downarrow\downarrow\Big\uparrow)S$ |
| $X_{\mu\nu}$ | $(A_1)_0$ | $(1/\sqrt{2})(\uparrow\downarrow - \downarrow\uparrow)sT$ |
| | $(\rho)_0$ | $(1/\sqrt{2})(\uparrow\downarrow + \downarrow\uparrow)sT$ |
| $X_{\mu\nu}$ | $(A_2)_0$ | $(1/\sqrt{2})(\uparrow\downarrow + \downarrow\uparrow)zT$ |
| | $B_0$ | $(1/\sqrt{2})(\uparrow\downarrow - \downarrow\uparrow)zT$ |
| $X$ | $f_0$ | $(1/\sqrt{2})(\uparrow\downarrow + \downarrow\uparrow)zS$ |
| $X$ | $C_0$ | $(1/\sqrt{2})(\uparrow\downarrow - \downarrow\uparrow)zS$ |
| $X$ | $\omega_0$ | $(1/\sqrt{2})(\uparrow\downarrow + \uparrow\downarrow)sS$ |
| $X$ | $D_0$ | $(1/\sqrt{2})(\uparrow\downarrow - \downarrow\uparrow)sS$ |

(1) orbital: The orbital state is either $|\Big\uparrow\rangle$, $|\Big\downarrow\rangle$, $|z\rangle$, or $|s\rangle$.

(2) spin: The spin content of the $q\bar{q}$ system is denoted by small arrows indicating the eigenvalues of $\sigma_z$. Thus there are 4 spin states given by $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ with spin-helicity 1, −1, 0, 0, respectively.

(3) isospin: The isospin configuration of a pair is either triplet ($T$) or singlet ($S$).

Thus typical states are labeled $(\uparrow\uparrow\Big\uparrow)T$, $(\downarrow\uparrow\Big\downarrow)S$, $(\uparrow\downarrow z)T$ etc. We shall also indicate the transformation properties of a multiplet by a symbol $X$, $X_\mu$, or $X_{\mu\nu}$. These symbols denote chiral singlets, 4-vectors and antisymmetric tensors. In Tables I, II, and III we give the eigenvectors of $H_0$ (chiral multiplets of definite helicity, parity, and $C$). They are named according to the quantum numbers of their real mesonic counterparts. In some cases there is ambiguity. For example the $\pi$ and $A_1$ labels can be interchanged in the helicity-zero sector. Since however, the physical eigenvectors will be superpositions of these states there is no particular need to resolve these ambiguities.

Our next task is to find a suitable choice for the operator $\varphi^4$. The constraints on $\varphi^4$ are the same as in Sec. VII B except that Eq. (7.2) is replaced by

$$[\varphi^4, J_z] = 0, \tag{7.9}$$

where $J_z$ includes the orbital as well as spin helicity. In our case (7.9) becomes

$$[\varphi^4, \sigma_z(\bar{q}) + \sigma_z(q) + 2L_z] = 0. \tag{7.10}$$

In fact there are 12 independent 4-vectors which satisfy (7.10) and the parity, charge conjugation requirements. In order to conveniently express them we will define a few more operators which act on the orbital degrees of freedom,

$$\begin{aligned} L^+ &= |z\rangle\langle \Big\downarrow| + |\Big\uparrow\rangle\langle z|, \quad L^- = (L^+)^\dagger \\ B^+ &= |s\rangle\langle \Big\downarrow| + |\Big\uparrow\rangle\langle s|, \quad B^- = (B^+)^\dagger \end{aligned} \tag{7.11}$$

also define the combinations

$$\begin{aligned} U\cdot V &= U^+V^- + U^-V^+, \\ \frac{1}{i}U\times V &= U^+V^- - U^-V^+. \end{aligned} \tag{7.12}$$

The 12 operators which are candidates for $\varphi^4$ are then:

$$\begin{aligned} &(\sigma_q - \sigma_{\bar{q}})\times B, \\ &(\sigma_q - \sigma_{\bar{q}})\times B\tau_q\cdot\tau_{\bar{q}}, \\ &(\sigma_q - \sigma_{\bar{q}})\cdot BJ_z, \\ &(\sigma_q - \sigma_{\bar{q}})\cdot BJ_z\tau_q\cdot\tau_{\bar{q}}, \\ &(\sigma_q - \sigma_{\bar{q}})\times BJ_z^{\,2}, \\ &(\sigma_q - \sigma_{\bar{q}})\times BJ_z^{\,2}\tau_q\cdot\tau_{\bar{q}}, \end{aligned} \tag{7.13}$$

$$B \to L, \quad \cdot \leftrightarrow \times. \tag{7.14}$$

We have previously[7] made an analysis of meson transitions using (6.37) and assuming that the chirally symmetric breaking term in $H_{\rm eff}$ is a linear superposition of the operators in (7.13) and (7.14). The results of our analysis included those of Gilman and Harari.[6] Furthermore our study indicated that the largest term in the symmetry-breaking part of $H_{\rm eff}$ is probably

$$c(\sigma_q - \sigma_{\bar{q}})\times B = c\varphi^4, \tag{7.15}$$

where $c$ is a constant.

We shall illustrate the use of (7.15) by applying it to the $\pi$, $A_1$, $\rho$, $\sigma$ system in the helicity-zero

TABLE II. $SU_2 \times SU_2$ multiplets for helicity-1 sector.

| Chiral property | Name | Configuration |
|---|---|---|
| $X_\mu$ | $\rho_0$ | $(\uparrow\uparrow)sT$ |
| | $\omega_0$ | $(\uparrow\uparrow)sS$ |
| $X_\mu$ | $(A_2)_0$ | $(\uparrow\uparrow)zT$ |
| | $(f)_0$ | $(\uparrow\uparrow)zS$ |
| $X_{\mu\nu}$ | $(A_1)_0$ | $(1/\sqrt{2})(\uparrow\downarrow + \downarrow\uparrow)\Big\uparrow T$ |
| | $(B)_0$ | $(1/\sqrt{2})(\uparrow\downarrow - \downarrow\uparrow)\Big\uparrow T$ |
| $X$ | $D_0$ | $(1/\sqrt{2})(\uparrow\downarrow + \downarrow\uparrow)\Big\uparrow S$ |
| $X$ | $C$ | $(1/\sqrt{2})(\uparrow\downarrow - \downarrow\uparrow)\Big\uparrow S$ |

TABLE III. $SU_2 \times SU_2$ multiplets for helicity-2 sector.

| Chiral property | Name | Configuration |
|---|---|---|
| $X_\mu$ | $(A_2)_0$ | $(\uparrow\uparrow)\Big\uparrow T$ |
| | $f_0$ | $(\uparrow\uparrow)\Big\uparrow S$ |

sector to derive the Gilman-Harari model. From Table I we see that $\rho$ and $A_1$ form a chiral tensor and that $(\pi^\alpha, \sigma)$ form a chiral 4-vector. The eigenvalues of $H_0$ (called $m_0^2$) must satisfy

$$m_0^2(A_1) = m_0^2(\rho),\qquad m_0^2(\pi) = m_0^2(\sigma). \tag{7.16}$$

Now inspection of the operator in (7.15) shows that $|\rho\rangle_0$ and $|\sigma\rangle_0$ are eigenvectors with eigenvalue zero. Thus the symmetry-breaking term will leave $|\rho\rangle_0$ and $|\sigma\rangle_0$ as eigenvectors and $m_0^2(\rho)$, $m_0^2(\sigma)$ as eigenvalues. Harari and Gilman assume that the $\sigma$ and $\rho$ are experimentally degenerate with $M^2$ of order 0.5 $\text{GeV}^2$. From this it follows that the eigenvalues $m_0^2(\rho)$, $m_0^2(\sigma)$, $m_0^2(A_1)$, $m_0^2(\pi)$ are all equal.

Now let us turn on the symmetry-breaking part of $H_{\text{eff}}$ given by (7.15). It is easy to see that

$$\varphi^4|A_1\rangle_0 = |\pi\rangle_0,\qquad \varphi^4|\pi\rangle_0 = |A_1\rangle_0. \tag{7.17}$$

Thus $c\varphi^4$ is merely an off-diagonal matrix in the $(\pi, A_1)$ subspace and will induce $(\pi, A_1)$ mixing. The physical eigenvectors of $H_{\text{eff}}$ are

$$\begin{aligned} |\pi\rangle &= \frac{1}{\sqrt{2}}(|\pi\rangle_0 - |A_1\rangle_0),\\ |A_1\rangle &= \frac{1}{\sqrt{2}}(|A_1\rangle_0 + |\pi\rangle_0),\\ |\rho\rangle &= |\rho\rangle_0,\\ |\sigma\rangle &= |\sigma\rangle_0 \end{aligned} \tag{7.18}$$

and the eigenvalues are

$$\begin{aligned} m_\pi^2 &= m_0^2 - c,\\ m_{A_1}^2 &= m_0^2 + c,\\ m_\rho^2 &= m_0^2 = m_\sigma^2. \end{aligned} \tag{7.19}$$

Thus we see that $\pi$ and $A_1$ are split from $(\rho, \sigma)$ by equal amounts. Using schematic masses

$$m_\rho^2 = m_\sigma^2 = \tfrac{1}{2},\qquad m_\pi^2 = 0,\quad m_{A_1}^2 = 1,$$

we find

$$m_0^2 = \tfrac{1}{2},\quad c = \tfrac{1}{2}. \tag{7.20}$$

Harari and Gilman also compute the decay rates $\sigma \to \pi\pi$, $\rho \to \pi\pi$, and $A_1 \to \rho\pi$ by evaluating the matrix elements

$$\langle\pi|Q_5|\sigma\rangle,\quad \langle\pi|Q_5|\rho\rangle,\quad \langle\rho|Q_5|A_1\rangle$$

and using

$$T_{fi} = \frac{2}{f_\pi}(m_f^2 - m_i^2)\langle f|Q_5|i\rangle.$$

The predicted rates are in impressive agreement with experiment.

The analysis has been extended[7] to include the remaining mesons indicated in Table I. We find that small admixtures of the operators in (7.13) and (7.14) are required. Good general agreement with experimental widths is obtained except for the $D$ and $E$ widths which seem to come out too large. Since the $D$ and especially the $E$ are rather heavy it is likely that we cannot maintain the approximation that only one chiral magnet is involved in their excitation.

It should be noted that every step of our procedure would be required in computing the properties of an impurity in a ferromagnet. The computation would begin with the eigenvectors and eigenvalues of the free impurity in the absence of the ferromagnetic system. Rotational configuration mixing would then be induced by the magnetic field of the lattice. Finally we would compute the lifetime of the excited impurity states by assuming an amplitude for spin wave emission. The amplitude would be proportional to the time derivative of the angular momentum of the impurity.

## VIII. SPONTANEOUS BREAKING AND THE POMERON

In Sec. VI we hinted about a possible connection between spontaneous symmetry breaking and the constancy of asymptotic cross sections. In this section we will derive the connection. Our view of the Pomeron is an extreme one and therefore requires discussion.

In the usual approach to duality it is assumed that only the nonexotic channels are populated by resonant structures. The Regge trajectories built from these nonexotic resonances are also assumed nonexotic. On the other hand the high-energy constant cross sections are built from a nonresonant background which can be exotic. Now unfortunately mathematical models of duality do not support such a view. Instead they stubbornly insist on producing an ordinary even signatured trajectory with $\alpha(0) = 1$ in addition to lower lying $\rho$'s, $\pi$'s, etc. This suggests to our mind an even more extreme approximation in which all of Hilbert space including exotic states is filled with narrow resonances.

This view is also indicated by the parton-string model. Given a system of many partons it is only reasonable that its excited states will include quantum number excitations.

On the other hand we have argued that the structure of $H$ is such that only a small number of partons is important at low energies. In particular, if the ratio of longitudinal fractions of neighbors is small then the energy required to excite 4-partons will be much larger than the 2-parton energies. This could easily explain the absence of prominent low-lying exotics. In any case we suspect that it is somewhat a matter of taste and preference whether to populate all Hilbert space with resonant levels describing the average level density or to classify states as resonant and background.

The advantage of a pure resonant model is simplicity and solvability and we feel that it is significant that in such models, the level structure is always just sufficient to build an even-signatured trajectory with $\alpha(0)=1$.

The disadvantage is that phenomenologically it seems to be a badly contaminated or distorted approximation. Thus phenomenologically on the positive side we have:

(1) $\alpha_P(0)=1$. This seems required by a pure dual resonance interpretation.

(2) Factorization of the Pomeron vertex supports a Regge-pole interpretation.

On the negative side:

(a) $\alpha_P'(0)$ appears to be $\frac{1}{2}$ or smaller instead of equaling unity.

(b) Neither the exotic or vacuumlike resonances required by a dual interpretation seems to exist in a prominent way.

In what follows it will be seen that our assumptions necessarily lead to the existence of an even-signatured, $\alpha(0)=1$ trajectory and that its coupling is of similar order of magnitude to the observed Pomeron.

Consider the scattering of a pion off a target hadron $|A\rangle$. Using the vertex given in (6.19) we encounter expressions like

$$\frac{4}{f_\pi^2}\int_{-\infty}^{0} d\tau \langle A|\hat{\Phi}^\alpha(0,0):e^{-iQ_\alpha\cdot X(0,0)}:\hat{\Phi}^\beta(0,\tau) \times :e^{iQ_\beta\cdot X(0,\tau)}:|A\rangle e^{i\tau(m_A^2-s)}. \quad (8.1)$$

This term describes the emission and absorption of the pion from the $\theta=0$ end of the target hadron. In addition a similar term

$$\alpha \leftrightarrow \beta, \quad s \leftrightarrow u \quad (8.2)$$

describes a process in which the time sequence of emission and absorption are interchanged. Two more terms describe the emission and absorption from the $\theta=\pi$ end of the target.

We may also consider processes in which one pion interacts at $\theta=0$ and the other at $\theta=\pi$. However such processes are not important at high energies and may be ignored for our purposes.

As usual the orbital part of (8.1) is given by $(m_\pi^2=0)$,

$$e^{i\tau(m_A^2-s)}(1-e^{i\tau})^{-t}. \quad (8.3)$$

To compute the amplitude for $s\to\infty$ we require the value of

$$\langle A|\hat{\Phi}^\alpha(0,0)\hat{\Phi}^\beta(0,\tau)|A\rangle \quad (8.4)$$

for $\tau \sim s^{-1}$. For this purpose we shall use the operator-product expansion in Eq. (5.18). We assume the expansion begins with a singular $c$-number term followed by less singular $I=0$ and 1 operators. Exotic contributions to the product, if they are present at all, should be less singular as $\tau\to 0$. Thus

$$\hat{\Phi}^\alpha(0,0)\hat{\Phi}^\beta(0,\tau) \simeq \frac{\kappa}{\tau^2}\delta^{\alpha\beta}+\cdots. \quad (8.5)$$

We have used the fact that the dimension of $\hat{\Phi}$ $[=\varphi(\sin\theta)^{-1}]$ is $-1$. The contribution of (8.1) then behaves like

$$\delta^{\alpha\beta}\frac{4\kappa}{f_\pi^2}\int_{\tau\sim s^{-1}}^{0} d\tau\, e^{-i\tau s}(1-e^{i\tau})^{-t}\tau^{-2}. \quad (8.6)$$

For $t\simeq 0$ this behaves like

$$\delta^{\alpha\beta}\frac{4\kappa}{f_\pi^2}\frac{(-s)^{1+t}}{t}. \quad (8.7)$$

We may double this in order to account for the interaction at $\theta=\pi$.

The term (8.2) is obtained by interchanging $(\alpha\leftrightarrow\beta)$ and $s\leftrightarrow u$. Thus we get

$$\delta^{\alpha\beta}\frac{8\kappa}{f_\pi^2}[(-s)^{1+t}+(-u)^{1+t}]\frac{1}{t}. \quad (8.8)$$

For large $s$ we may replace $u$ by $(-s)$ to get

$$\delta^{\alpha\beta}\frac{8\kappa}{f_\pi^2}\frac{s^{1+t}}{t}(1-e^{i\pi t}) \simeq -i\delta^{\alpha\beta}\frac{8\pi\kappa}{f_\pi^2}s^{1+t} \quad (8.9)$$

which by the optical theorem gives

$$s\to\infty:\ \sigma_{\rm tot}\to 8\pi\kappa f_\pi^{-2}. \quad (8.10)$$

We shall next show that the constant $\kappa$ is proportional to the symmetry-breaking parameter $\langle\varphi^4\rangle$. To prove this we require a slight generalization of (8.5) in which one of the operators $\Phi$ is evaluated slightly away from $\theta=0$. Our expansion will be written for the chronologically ordered operator:

$$T(\hat{\Phi}^\alpha(0,0)\hat{\Phi}^\beta(\theta,\tau)) \simeq \delta^{\alpha\beta}\frac{\kappa}{\tau^2-\theta^2}F(\theta/\tau)+\cdots. \quad (8.11)$$

This is the most general form consistent with $\hat{d}_\Phi=-1$. The function $F(\theta/\tau)$ must equal unity for $\theta=0$ in order to agree with (8.5).

Now it can be shown that if the equations of motion possess the full Möbius invariance[9] necessary for duality and crossing symmetry then $F(\theta/\tau)$ must be independent of $\theta/\tau$. Thus, assuming Möbius invariance,

$$T(\hat{\Phi}^\alpha(0,0)\hat{\Phi}^\beta(\theta,\tau)) \simeq \delta^{\alpha\beta}\frac{\kappa}{\tau^2-\theta^2}+\cdots . \quad (8.12)$$

Now using (6.17) allows (8.12) to be written in the form

$$T(\dot{Q}_5^\alpha(0)\hat{\Phi}^\beta(\theta,\tau)) - T(\hat{\Phi}^\alpha(\pi,0)\hat{\Phi}^\beta(\theta,\tau)) \simeq \frac{\delta^{\alpha\beta}\kappa}{\tau^2-\theta^2}+\cdots . \quad (8.13)$$

Since $\hat{\Phi}^\alpha(\pi,0)\hat{\Phi}^\beta(\theta,\tau)$ is not expected to be singular when $\tau\to 0$ and $\theta\to 0$ we ignore it. Integrating (8.13) over $\tau$ and retaining only the most singular pieces as $\theta\to 0$ gives

$$[Q_5^\alpha, \hat{\Phi}^\beta(\theta,0)] = \delta^{\alpha\beta}\kappa\int\frac{d\tau}{\tau^2-\theta^2} = \delta^{\alpha\beta}\frac{\pi\kappa}{\theta} . \quad (8.14)$$

The commutator on the left-hand side of (8.14) can be computed since $\hat{\Phi}^\beta$ is assumed to transform as a chiral 4-vector. Using (6.10) we thus get

$$[Q_5^\alpha, \hat{\Phi}^\beta(\theta)] = \frac{\varphi^4(\theta)}{\sin\theta}\delta^{\alpha\beta} \quad (8.15)$$

and

$$\frac{\pi\kappa\delta^{\alpha\beta}}{\theta} \simeq \frac{\langle\varphi^4(\theta)\rangle}{\sin\theta}\delta^{\alpha\beta} . \quad (8.16)$$

Thus in the limit $\theta\to 0$,

$$\pi\kappa = \langle\varphi^4\rangle . \quad (8.17)$$

Combining (8.17) and (8.10) yields

$$\sigma_{\text{tot}} \to \frac{8\langle\varphi^4\rangle}{f_\pi^{\;2}} . \quad (8.18)$$

Thus the asymptotic total cross sections should be of order $f_\pi^{-2}$ with a proportionality factor $8\langle\varphi^4\rangle$. We see that the existence of a spontaneous breakdown requires a nonvanishing high-energy cross section. Numerically we would expect $\langle\varphi^4\rangle$ to be of order unity since it is built of spin and spinlike quantities. In fact if we suppose $\sigma_{\pi\pi}\simeq 20$ mb then we find $\langle\varphi^4\rangle\simeq\frac{1}{4}$ (as found in the model of Appendix C).

Perhaps the most interesting consequence of this line of reasoning is that the scale of strong-interaction cross sections[18] is set by $f_\pi^{-2}=10.7$ mb. Two other possible length scales are available, neither of which seems a reasonable candidate. The first is the pion mass which is numerically satisfactory. However it seems unreasonable to us that $\sigma\to\infty$ as $m_\pi^{\;2}\to 0$. The other scale is the Regge slope which would require cross sections of order 0.5 mb.

## IX. SUMMARY AND CONCLUSIONS

In the preceding we have attempted to present a unified approach to hadronic phenomena associated with the structure of low-lying hadrons and forward high-energy collisions.

Our approach was motivated by the need to synthesize the following:

(1) Bjorken scaling of the deep-inelastic electroproduction structure functions.

(2) Quark-model phenomenology.

Property (1) suggests that hadrons are composed of a large number of pointlike (on the scale of hadronic physics) partons. The usual quark model on the other hand, implies that only a small number (2 for mesons, 3 for baryons) of partons are involved in the low-lying excitation spectrum. The remaining infinity is thus to be identified with the "frozen sea."

(3) Constant total cross sections, the Feynman scaling of inclusive spectra, and the observed flattening of $\nu W_2(\omega\to\infty)$. These properties imply the $d\eta/\eta$ distribution of partons.

(4) Limited ($\leq 0.3$ GeV) transverse momentum of produced hadrons.

(5) Regge behavior and duality.

(4) and (5) supply the main evidence[19] for the stringlike structure of hadrons.

(6) Current algebra, partial conservation of axial-vector current (PCAC), and their saturation by finite-dimensional representation-mixing schemes.

These indicate a chiral alignment of the "frozen sea," which is manifested as an "external" $SU_2\times SU_2$-breaking field acting on the valence partons.

The dominant configuration of the hadronic IMF wave function thus describes a string of partons, wiggling in the transverse plane. The string parameter corresponds to the average longitudinal fraction carried by the parton, and provides an energy scale of internal motions which rises exponentially as we move away from the spatial center of the hadron. The phenomenon of "chiral magnetism" realizes the spontaneous breakdown of $SU_2\times SU_2$ and causes the appearance of the chiral "*order parameter*" $\langle\varphi^4\rangle$. The latter is the agency through which information is transmitted to and from the active central partons. In particular, the order parameter provides the link between low-energy pionic amplitudes and high-energy total cross sections and sets their scale at $f_\pi^{-2}$.

Our treatment is evidently incomplete and some important questions have been ignored. In particular, no attempt was made to deal with the dynamics of the longitudinal momentum. The $c$-number approximation we have used is of course inconsistent with Lorentz invariance and makes it impossible to treat processes which involve longitudinal momentum transfers. A second subject which was ignored is $SU_3$ structure. We believe that an extension from $SU_2$ to $SU_3$ involves more than a trivial renaming of indices and the addition of a $\lambda$ -quark mass term. In fact, such problems as the instability of objects with the quantum numbers of one quark, and the $\eta-\eta'$ mixing would have to be faced. We also note that our treatment of the Pomeron cannot be the full story. There is obviously much more to understand about the Pomeron and the related issue of exotic states.

Finally, we wish to point out that even when these issues will have been satisfactorily settled, there will remain the unsolved fundamental problem of "deriving" the model from a relativistic quantum field theory. We emphasize that this is an entirely open problem and involves much more than the selection and iteration of a class of diagrams. Presumably something like a proof that one-dimensional configurations dominate the path integrals of the appropriate quantum field theory is needed.


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## APPENDIX A

The aim of this Appendix is to formulate some of the general properties possessed by near-neighbor coupled systems with an increasing energy scale. In particular we discuss: (a) the systematic construction of the low-energy states and (b) the complementary issue of the behavior of local quantities as $\eta \to 0$. Some of the statements made here are illustrated by a soluble model in Appendix B.

A parton at the $j$th site will be equipped with an $l$-dimensional state space and its associated operator algebra represented by the dynamical variables $\zeta_a(j)$, $a=1,\ldots,l^2$. It will also prove convenient to introduce a notation for "quasilocal" operators,

$$\zeta_a(j)\zeta_b(j+1)\cdots \equiv \zeta_{ab\ldots}(j,j+1,\ldots)\,. \qquad \text{(A1)}$$

More generally, the set of operators associated with a cluster $(j,\ldots,j+C)$ will be designated $\zeta_r(j)$. The Hamiltonian as usual is

$$H=\sum_{j=0}^{\infty}\lambda^{-j}U[\zeta(j),\zeta(j+1)]\,, \qquad \text{(A2)}$$

where

$$U[\zeta,\zeta']=u_{ab}\zeta_a\zeta'_b\,. \qquad \text{(A3)}$$

The basic assumption used in the following is that the ratio $\lambda$ which sets the local scale of energy is sufficiently small. By this we mean that the eigenvalues of

$$\sum_{j=0}^{N-1}\lambda^{-j}U(j,j+1)$$

are small compared to those of $\lambda^{-N}U(N,N+1)$, so that the $N$th parton cannot be appreciably excited by the rest of the chain.

(a) We now turn to the problem of finding the spectrum of low-lying states and their associated transition amplitudes. Specifically, suppose we are interested in the states whose energy is bounded by

$$\epsilon<\lambda^{-\nu}\,. \qquad \text{(A4)}$$

The structure of the Hamiltonian indicates that if we want to compute the properties of these states to order $\lambda^n$ it should suffice to cut off the chain at

$$N(\nu,n)\approx\nu+n\,. \qquad \text{(A5)}$$

In fact, the problem may be approached by defining an effective Hamiltonian[15] $H_{n\nu}[\zeta(1)\ldots\zeta(\nu)]$ which acts only on the first $\nu$ partons. The eigenvalues and eigenvectors of $H_{n\nu}$, when supplemented by renormalization factors are sufficient for computing all amplitudes to the desired order. $H$ can be built in steps by first diagonalizing $\lambda^{-N}U(N,N+1)$, then treating $\lambda^{-(N-1)}U[\zeta_{N-1},\zeta_N]$ as a perturbation, and so on, till the $\nu$th site has been reached. This results in a pattern of split and resplit levels as in Fig. 1. As we are interested only in the levels whose energy is smaller than $\lambda^{-\nu}$, until the $\nu$th site has been reached, only the ground-state splitting need be considered.

We exhibit without proof[15] the effective $H$ to order $\lambda^2$,

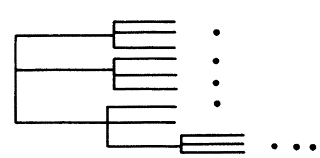

FIG. 1. Splitting and resplitting of the levels of a chain with a geometrically increasing energy scale.

$$H_{\rm eff}=\lambda^{-(N-1)}\Big\{\langle G(N)|\zeta_a(N)|G(N)\rangle u_{ab}\zeta_b(N-1)$$
$$+\langle G(N)|\zeta_a(N)|\gamma(N)\rangle\frac{\lambda^{-(N-1)}}{E_G-E_\gamma}\langle\gamma(N)|\zeta_{a'}(N)|G(N)\rangle u_{ab}u_{a'b'}\zeta_b(N-1)\zeta_{b'}(N-1)+\zeta_a(N-1)u_{ab}\zeta_b(N-2)\Big\}. \tag{A6}$$

Here $G(N)$ and $\gamma(N)$ are the ground state and excited states of $\lambda^{-N}U[\zeta(N),\zeta(N+1)]$. We remark that for simple systems the operators $\zeta_a$ in general transform in a definite way under the symmetry group of $U$. Thus, if $|G\rangle$ is a singlet, then the first term vanishes. Moreover, the second term is a simple one-particle invariant operator which in many cases is just a $c$ number. We thus see that the leading term in a symmetric system will be the third term, which in fact means that the chain might as well have been cut off at the $(N-1)$ site. In other words, the structure of the low-lying states will be independent of the rest of the chain. On the other hand, if $G$ is not a singlet and $u_{ab}\langle G|\zeta_b|G\rangle\neq 0$ for some $a$, then the first term of (A6) will supply the "external field" which polarizes the low-lying states.

(b) We now turn to the opposite problem, namely, that of deriving the behavior of low-energy matrix elements of the operators $\zeta_r(j)$ defined in (A1), as $j\to\infty$.

Consider the Heisenberg equation for $\zeta_a(j)$,

$$i\partial_\tau\zeta_a(j)=\lambda^{-j}f_{abb'}\zeta_{bb'}(j,j+1)+\lambda^{-(j-1)}g_{abb'}\zeta_{bb'}(j-1,j). \tag{A7}$$

where $f$ and $g$ are determined by commuting $\zeta$ with $U$. By commuting $\zeta_{ab}(j,j+1)$ etc. with $H$, a sequence of equations is generated which involves longer and longer operator chains $\zeta_{a\ldots}(j\ldots)$. In general the operators $\{\zeta_r(j)\}$ pertaining to a cluster will satisfy,

$$\lambda^j i\partial_\tau\zeta_r(j)=K_{rs}\zeta_s(j)+F_{rs}\zeta_s(j+1)+G_{rs}\zeta_s(j-1). \tag{A8}$$

The crucial point is that due to the progressive energy scale, Eq. (A8) will in most cases be effectively cut off to leading order in $\lambda$ after a finite number of steps $C$. To see this, note that to first order in $\lambda$ the low-energy state $|\epsilon\rangle$ can be represented by

$$|\epsilon\rangle\simeq\chi_\nu\psi_\epsilon(j<\nu), \tag{A9}$$

where $\chi_\nu$ is the ground state of the subchain $(N,\ldots,\nu)$, $N$ being the terminal cutoff point and $\lambda^{-\nu}>\epsilon$. Thus, if in Eq. (A8) every term can be broken into a product,

$$F_{a_0\ldots a_n}\zeta_{a_0\ldots a_n}(j,\ldots,j-n)=[F^{(1)}_{a_0\ldots a_C}\zeta_{a_0\ldots a_C}(j,\ldots,j-C)]\times[F^{(2)}_{a_{C+1}\ldots a_n}\zeta_{a_{C+1}\ldots a_n}(j-C-1,\ldots,j-n)] \tag{A10}$$

such that the first factor on the right-hand side has a nonvanishing ground-state expectation value, then to first order in $\lambda$ this factor may be replaced by a $c$ number,

$$[F^{(1)}_{\{a\}}\zeta_{\{a\}}(j,\ldots,j-C)]\to\langle\chi_\nu|[\ ]|\chi_\nu\rangle. \tag{A11}$$

This procedure cuts off (A8) and effectively turns the matrices $F,G,K$ into finite-dimensional ones. We now remark that when sandwiched between low-energy states the left-hand side of Eq. (A8) may be neglected so that the equation turns into an energy-independent recursion relation for the matrix elements

$$\zeta_r(j)=T_{rs}(\lambda)\zeta_s(j-1), \tag{A12}$$

where $T(\lambda)$ is a transfer matrix. The behavior of $\zeta(j)$ as $j\to\infty$ will be determined by the eigenvalues $t(\lambda)$ of $T$. We thus arrive at

$$j\to\infty:\ \ \zeta_r(j)\sim\hat{\zeta}_r[t(\lambda)]^j\equiv\hat{\zeta}_r\lambda^{j\gamma}. \tag{A13}$$

The operators $\hat{\zeta}$ *do* depend on the low-energy structure and should be determined by matching the recursion relation (A12) with the values of $\langle\epsilon|\zeta(j)|\epsilon'\rangle$ for $\lambda^{-j}\sim\epsilon,\epsilon'$, thus providing Eq. (A12) with boundary conditions.

One other consequence of the cutoff Eq. (A8) is that for short times $\tau\lesssim\lambda^j\approx\lambda_0\sin\theta$, the behavior of the cluster $(j,\ldots,j-C)$ is effectively independent of its neighbors. In other words, the algebra formed by the operators $\{\zeta_r(j,\tau)\}$ for $\tau\lesssim\lambda^j$ is a closed finite linear algebra of dimension $\sim l^{2C}$. Evidently this property is the origin of the operator-product expansion introduced in Sec. V C. The coefficients $c_{ijk}$ of Eq. (5.18) are determined by the transfer matrix $T$.

Finally, we remark that to first order in $\lambda$, Eq.

(A12) simulates a differential equation if $\lambda^j$ is treated as a continuous parameter. This is the origin of the "continuum" notation used in the text. To summarize, the considerations used in the text concerning operators $\xi(\theta)$ are meaningful only when used for the leading terms of low-energy matrix elements when $\sin\theta \to 0$.

## APPENDIX B: THE FERROMAGNETIC MODEL

The model of a ferromagnetic hadron involves a set of fictitious spinlike variables which we label $\sigma_1$, $\sigma_2$, $\sigma_3$. The $\sigma$'s are Pauli matrices but should not be associated with real spin or isospin. The internal $SU_2$ symmetry of the system provides a simple example of a spontaneous breakdown. Thus consider the one-dimensional ferromagnetic Hamiltonian:

$$H = -\frac{g}{2}\sum_j \frac{\sigma(j)\cdot\sigma(j+1)}{\eta_j}\,, \tag{B1}$$

where

$$\sigma\cdot\sigma \equiv \sigma_1\sigma_1 + \sigma_2\sigma_2 + \sigma_3\sigma_3\,.$$

We shall be interested in the properties of the system as $\eta \to 0$ (or $j\to\infty$). As usual [Eq. (2.2)] we assume that ratios of neighboring $\eta$'s become constant and thus

$$\eta_j \to \lambda^j \tag{B2}$$

with $\lambda < 1$.

We allow $j$ to range between unity (fast partons with large $\eta$) and $\infty$ (wee partons). This is a slight departure from the usual string model which has two wee ends but will suffice for our purposes.

The Hamiltonian (B1) is rotationally symmetric but its ground state is totally aligned. The direction of alignment is arbitrary and will be chosen to be the 3 direction. We do this by introducing a small symmetry-breaking term of the form

$$\delta H = c\sum \frac{\sigma_3(j)}{\eta_j} \tag{B3}$$

[See (6.7)]. The ground state is thus

$$|G\rangle = |\downarrow,\downarrow,\downarrow,\ldots\downarrow,\ldots\rangle\,. \tag{B4}$$

The symmetry-breaking parameter analogous to $\varphi^4$ is $\sigma_3$. Since the ground state is independent of $c$ we have

$$\langle\sigma_3\rangle = 1 \tag{B5}$$

even in the symmetry limit $c\to 0$.

Consider the Heisenberg equation for

$$\sigma_-(j) = \tfrac{1}{2}[\sigma_1(j) - i\sigma_2(j)]\,,$$

$$\begin{aligned} i\dot\sigma_-(j) &= [\sigma_-(j), H+\delta H] \\ &= -i[J_-(j,j+1) - J_-(j-1,j)] + \frac{2c}{\eta_j}\sigma_-(j)\,, \end{aligned} \tag{B6}$$

where $J_-(j,j+1)$ is the flux of the $(-)$ component of spin:

$$J_-(j,j+1) = \frac{ig}{\eta_j}[\sigma_-(j)\sigma_3(j+1) - \sigma_-(j+1)\sigma_3(j)]\,. \tag{B7}$$

Equation (B6) is the discrete form of the continuity equation for $\sigma_-$. The equation is of the form (4.5) with an extra source term. It may be approximated by a continuity equation of the form (4.11) with the source term defined by

$$c\Phi_- \propto -2ic\frac{\sigma_-(j)}{\eta(j)^2}\,. \tag{B8}$$

We will use (B6) to find the eigenfunctions and energy eigenvalues of the one spin-wave excitations. We define the energy of the state $|G\rangle$ to be zero by subtracting a constant from $H$. We shall consider a one spin-wave excitation with energy $\epsilon$,

$$|\epsilon\rangle = \sum_j a_j\sigma_+(j)|G\rangle\,.$$

Substituting in (B6) gives

$$\begin{aligned} \epsilon a(j) &= \frac{2c}{\eta_j}a(j) + \frac{1}{\eta_j}[a(j) - a(j+1)] \\ &\quad - \frac{1}{\eta_{j-1}}[a(j-1) - a(j)]\,. \end{aligned} \tag{B9}$$

(The constant $g$ was absorbed into the scale of $\epsilon$ and $c$.)

Using $\eta_j = \lambda^j$ ($\lambda<1$), we may solve (B9) in the region $j\to\infty$ by a power behavior

$$a(j) \sim \eta_j{}^\gamma = \lambda^{j\gamma}\,. \tag{B10}$$

We find an equation for $\gamma$ which is independent of $\epsilon$:

$$2c + [1-\lambda^{\gamma-1}](1-\lambda) = 0\,. \tag{B11}$$

For $c\to 0$, we find

$$1-\gamma = \frac{2c}{(1-\lambda)\ln\lambda} < 0\,. \tag{B12}$$

The coefficients $a(j)$ control the behavior of matrix elements of local quantities analogous to those studied in Sec. V. In particular the matrix elements of $J_-$, $\sigma_-$, $\sigma_3$ and $\Phi_-$ are:

$$\langle G|\sigma_-(j)|\theta\rangle = a(j)\,, \tag{B13}$$

$$\langle G|J_-(j,j+1)|\epsilon\rangle = \frac{i}{\eta_j}[a(j) - a(j+1)]\,, \tag{B14}$$

$$\langle\epsilon|\sigma_3(j)|\epsilon\rangle = 1 - |a(j)^2|\,, \tag{B15}$$

$$\langle G|\Phi_-(j)|\epsilon\rangle = -2i\,\frac{a(j)}{\eta_j{}^2}\,. \tag{B16}$$

We find that as $\eta \to 0$,

$$\sigma_3(\eta) \sim 1\,, \tag{B17}$$

$$\sigma_-(\eta) \sim \eta^{\gamma}\,, \tag{B18}$$

$$J_-(\eta) \sim \eta^{\gamma-1}\,, \tag{B19}$$

$$\Phi_-(\eta) \sim \eta^{\gamma-2}\,, \tag{B20}$$

Note that in the symmetry limit Eq. (B12) gives $\gamma \to 1$ linearly with $c$. Thus $J_-$ tends to a nonzero limit as $\eta \to 0$ [compare with (6.6)]. Moreover (B14) insures that as $c \to 0$ $J_-(\eta \to 0) = \hat{\Phi}_-$, where $\hat{\Phi}_-$ is the coefficient of $\eta^{\gamma-2}$ in (B20) [compare Eq. (6.15)].

Let us next illustrate the cutoff procedure defined in Sec. VIE and used in Sec. VII. We shall work in the symmetry limit $c = 0$.

Consider the recursion relation (B9) for the amplitudes $a(j)$. Rewrite Eq. (B9) in the form

$$a(j+1) = a(j)(1 + \lambda - \epsilon\lambda^{j}) - \lambda a(j-1)\,.$$

Let us concentrate on the region where $\epsilon\lambda^j \ll \lambda$. (This is the region we termed the frozen sea in the text.) We may evidently ignore $\epsilon\lambda^j$ to a first approximation and we are thus left with an $\epsilon$ -independent linear recursion relation which determines the properties of local quantities in this region. Note that to this approximation Eq. (B9) is satisfied by $a(j) = C\lambda^j$, where $C$ is an arbitrary constant. The constant $C$ should be determined by continuing the equation to small $j$'s and normalizing the state $|\epsilon\rangle$ according to

$$\sum |a(j)|^2 = 1\,. \tag{B21}$$

$C$ will thus depend on the energy $\epsilon$. In order to determine the energy eigenvalues $\epsilon$ a cutoff procedure should be formulated. Imagine terminating the chain at the $N$th site. The eigenvalue condition for $\epsilon$ is just the determinantal equation which results from Eq. (B9). We thus get $\epsilon = \lim_{N\to\infty}\epsilon_N$, where $\epsilon_N$ solves

$$\mathrm{Det}\begin{bmatrix} 1+\lambda-\epsilon\lambda^{N} & \lambda & & \cdots & \\ 1 & 1+\lambda-\epsilon\lambda^{N-1} & & \cdots & \\ & & \ddots & & \\ \vdots & & & & \lambda \\ & & & 1 & 1+\lambda-\epsilon \end{bmatrix} \equiv \Delta_N(\epsilon) = 0\,. \tag{B22}$$

For $\lambda \ll 1$ it is natural to expand $\epsilon$ in powers of $\lambda$. In order to compute $\epsilon$ up to order $\lambda^n$, the terms $\epsilon\lambda^j$ for $j > n$ can be neglected in the expansion of $\Delta_N$. It is easily verified that this corresponds to truncating $\Delta_N$ at the site which contains $\epsilon\lambda^n$; thus

$$\Delta_n(\epsilon) = 0 \tag{B23}$$

gives $\epsilon$ correctly to order $\lambda^n$. The sequence of equations $\Delta_n = 0$ corresponds to the eigenvalue equations for a sequence of finite chains of length $n$ whose *last spin is acted upon by an external "magnetic field"* of strength $\frac{1}{2}\lambda^{-n}$. This statement may be verified by inspecting the recursion relations which follow from the Hamiltonian:

$$H_n = H_0 + \frac{\sigma_3(n)}{2\lambda^n}\,, \tag{B24}$$

where $H_0$ is the unperturbed Heisenberg Hamiltonian for $n$ spins. Evidently, once $\epsilon$ has been computed to order $\lambda^n$ it may be substituted into the recursion relation for the $a(j)$'s which will thus also be computed to this order. In particular, the value of $\langle G|J_-(n, n+1)|\epsilon\rangle$ and the associated $\partial_T\langle G|\sum\sigma_-(j)|\epsilon\rangle$ will have been correctly evaluated to this order. Note that the value of the "external field" is just $\frac{1}{2}\lambda^{-j}\langle G|\sigma_3|G\rangle$, and may be interpreted as the effective field which the "frozen sea" exerts on the "valence system."

To illustrate the ideas of Sec. VII, let us apply the cutoff procedure to the first two spins. If $\lambda \ll 1$, this is an adequate approximation to the first four energy levels. We thus have

$$H_{\mathrm{eff}} = -\tfrac{1}{2}\sigma(1)\cdot\sigma(2) + \frac{1}{2\lambda}\sigma_3(2) - \langle G|H_{\mathrm{eff}}|G\rangle\,. \tag{B25}$$

The eigenstates and their energies are (for $\lambda \ll 1$):

$$\begin{aligned} |G\rangle &= |\downarrow\downarrow\rangle &&: 0, \\ |\epsilon\rangle &= |t\rangle + (1-\lambda)|s\rangle &&: 1, \\ |a\rangle &= |\uparrow\uparrow\rangle &&: \lambda^{-1}, \\ |b\rangle &= |t\rangle - (1+\lambda)|s\rangle &&: \lambda^{-1}+1, \end{aligned} \tag{B26}$$

where $|t\rangle$ and $|s\rangle$ are the triplet and singlet combinations of vanishing total $\sigma_3$. It is evident that to order 1, the splitting between the two low-lying states $|G\rangle$ and $|\epsilon\rangle$ could have been gotten by considering the cutoff problem of *one* spin in an external field of strength $1/2\lambda^0$, namely,

$$H_{\text{eff}} = \tfrac{1}{2}\sigma_3(1). \tag{B27}$$

The eigenstates and eigenvalues of (B27) are of course the state $|G\rangle$ and $|\epsilon\rangle$ when $\lambda$ is neglected. The results summarized by Eq. (B29) are the analogs of the procedure used in Sec. VII for the $q\bar{q}$ system. Note that the coefficient $(1-\lambda)$ supplies here the mixing angle between the singlet and triplet representations of the total spin $\sigma(1) + \sigma(2)$.

Finally we may compute $\langle G|\dot{\sigma}_-(1) + \dot{\sigma}_-(2)|\epsilon\rangle$ to order $\lambda^0$ and compare with the values of $J_-(1, 2)$. It is readily verified that to order $\lambda^0$ both are given by

$$\begin{aligned} \partial_\tau \left\langle G \middle| \sum_{j=1}^{2} \sigma_-(j) \middle| \epsilon \right\rangle &= -i\sqrt{2} \\ &= -i\langle G|[\sigma_-(1)\sigma_3(2) - \sigma_3(1)\sigma_-(2)]|\epsilon\rangle . \end{aligned} \tag{B28}$$

Note that by treating $\sigma_3(2)$ in Eq. (B30) as a $c$ number and fixing its value at unity, the results of the single-spin problem are recovered, which proves the consistency of the approximation procedure to this order. Moreover, the order $\lambda^0$ results would evidently not change by including more spins, so that the single spin Hamiltonian (B27) in fact approximates well the properties of the first pair of low-lying states as long as order $\lambda$ effects are neglected.

## APPENDIX C: AN $SU_2 \times SU_2$ MODEL

We consider a model of $SU_2 \times SU_2$ currents due to the authors and J. Kogut.[20] The model can be formulated as the smoothed continuum limit of a discrete system similar to the ferromagnetic model. We shall give the continuum formulation which is based on the use of anticommuting Fermi fields on the $\theta$ axis. It should be recalled that in one dimension, any Fermi field may be reformulated in terms of commuting spinlike degrees of freedom by using the Jordan-Wigner trick.[21]

Define the 2-dimensional Dirac matrices

$$\gamma_\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_\theta = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \alpha = \gamma_\tau\gamma_\theta \tag{C1}$$

and Dirac field $\psi(\theta)$ satisfying the Möbius-invariant Dirac equation

$$\gamma \cdot \nabla = \frac{ic}{\sin\theta}\,\psi . \tag{C2}$$

The fields $\psi$ also carry an isospin index acted on by $\tau$ matrices. We shall introduce two such $I = \frac{1}{2}$ Dirac fields called $\psi^A$ and $\psi^B$ which have opposite relative parity. The equations satisfied by $\psi^{A,B}$ are of the form (C2):

$$\begin{aligned} \left(\gamma \cdot \nabla - \frac{ic^A}{\sin\theta}\right)\psi^A &= 0, \\ \left(\gamma \cdot \nabla - \frac{ic^B}{\sin\theta}\right)\psi^B &= 0. \end{aligned} \tag{C3}$$

Multiplication of (C2) by $\gamma \cdot \nabla$ gives

$$(\partial_\tau{}^2 - \partial_\theta{}^2)\psi = -\frac{c(c + i\gamma_\theta \cos\theta)}{\sin^2\theta}\,\psi . \tag{C4}$$

Requiring a solution of the form

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} e^{-i\omega\tau}$$

implies

$$\begin{aligned} (\omega^2 + \partial_\theta{}^2)\psi_1 &= \frac{c(c + \cos\theta)}{\sin^2\theta}\,\psi_1 , \\ (\omega^2 + \partial_\theta{}^2)\psi_2 &= \frac{c(c - \cos\theta)}{\sin^2\theta}\,\psi_2 . \end{aligned} \tag{C5}$$

Under reflection of the $\theta$ interval about $\theta = \frac{1}{2}\pi$ the equations for $\psi_1$ and $\psi_2$ are interchanged.

Near $\theta = 0$ the solutions behave like

$$\begin{aligned} \psi_1 &\sim \theta^{-c} , \\ \psi_2 &\sim \theta^{1-c} . \end{aligned} \tag{C6}$$

Near $\theta = \pi$ the solutions behave like

$$\begin{aligned} \psi_1 &\sim (\pi - \theta)^{1-c} , \\ \psi_2 &\sim (\pi - \theta)^{-c} . \end{aligned} \tag{C7}$$

Explicit solutions can be constructed in terms of Bessel functions. The eigenfrequencies turn out to be

$$\omega = \tfrac{1}{2} - c,\ \tfrac{3}{2} - c,\ \tfrac{5}{2} - c, \ldots . \tag{C8}$$

Equations (C6), (C7), (C8) may be applied directly to the fields $\psi^A$, $\psi^B$ by substituting $c^A = c$ and $c^B = c$. Thus

$$\psi_1^A \to \theta^{-c_A},\ (\pi-\theta)^{1-c_A},$$
$$\psi_2^A \to \theta^{1-c_A},\ (\pi-\theta)^{-c_A},$$
$$\psi_1^B \to \theta^{-c_B},\ (\pi-\theta)^{1-c_B}, \tag{C9}$$
$$\psi_2^B \to \theta^{1-c_B},\ (\pi-\theta)^{-c_B}.$$

The isospin currents are defined by

$$V_\tau^\alpha = \overline{\psi}^A\gamma_\tau \tfrac{1}{2}\tau^\alpha\psi^A + \overline{\psi}^B\gamma_\tau\tfrac{1}{2}\tau^\alpha\psi^B,$$
$$V_\theta^\alpha = \overline{\psi}^A\gamma_\theta \tfrac{1}{2}\tau^\alpha\psi^A + \psi^B\gamma_\theta\tfrac{1}{2}\tau^\alpha\psi^B, \tag{C10}$$

$$A_\tau^\alpha = \overline{\psi}^A\alpha\gamma_\tau\tfrac{1}{2}\tau^\alpha\psi^B + A \leftrightarrow B,$$
$$A_\theta^\alpha = \overline{\psi}^A\alpha\gamma_\theta\tfrac{1}{2}\tau^\alpha\psi^B + A \leftrightarrow B. \tag{C11}$$

The components of the currents obviously satisfy $SU_2 \times SU_2$ commutation relations. The vector current is exactly conserved and $A$ satisfies the continuity equation:

$$\nabla\cdot A^\alpha = c\Phi^\alpha, \tag{C12}$$

where

$$c = -\tfrac{1}{2}(c_A + c_B), \tag{C13}$$

$$\Phi^\alpha = \frac{-i}{\sin\theta}(\overline{\psi}^A\alpha\tau^\alpha\psi^B + A \leftrightarrow B). \tag{C14}$$

By explicitly commuting $\Phi$ with $A_\tau$ we find that $\Phi^\alpha$ is a member of a chiral 4-vector with 4th component

$$\Phi^4 = \frac{i}{\sin\theta}(\overline{\psi}^A\psi + \overline{\psi}^B\psi^B). \tag{C15}$$

Using (C9) we find

$$c\Phi^\alpha \underset{\sin\theta\to 0}{\sim} c(\sin\theta)^{c-1}\hat{\Phi}^\alpha \tag{C16}$$

[see Eq. (6.11)].

$$A_\theta \underset{\sin\theta\to 0}{\sim} (\sin\theta)^c \hat{A}_\theta . \tag{C17}$$

Using the explicit form of the operators it is easily found that in the limit $c \to 0$

$$\hat{A}_\theta(0) = \hat{\Phi}(0),$$
$$\hat{A}_\theta(\pi) = -\hat{\Phi}(\pi). \tag{C18}$$

Finally the position of the $\rho$ pole may be adjusted by varying $c_A$. Assuming the symmetry limit $c_A + c_B = 0$ we find that by setting $c_A = \frac{1}{4}$ the vector current satisfies

$$V_\tau \sim (\sin\theta)^{-1/2}\hat{V}_\tau$$

which according to (5.12) is equivalent to $m_\rho^2 = \frac{1}{2}$.

Explicit computation shows that

$$\Phi^4 = (\overline{\psi}^A\psi^A + \overline{\psi}^B\psi^B)\frac{1}{i\sin\theta}$$

has a $c$-number piece which behaves like $(\sin\theta)^{-2}$ near $\sin\theta \to 0$. Thus defining the dimension-zero operator $\varphi^4 = \sin^2\theta\Phi^4$ we obtain a nonvanishing ground-state value for $\langle\varphi^4\rangle$. $\langle\varphi^4\rangle$ turns out to be $\frac{1}{4}$. Using (8.18), we obtain

$$\sigma_{\pi\pi} \underset{s\to\infty}{\longrightarrow} 2f_\pi^{-2} = 21.4 \text{ mb}.$$

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