Correspondence of partial-wave and impact-parameter representations*

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An alternative but much simpler proof is given that a recently developed high-energy expansion accomplishes the mapping between the partial-wave amplitude and the Fourier-Bessel impact-parameter amplitude. The high-energy expansion is also shown to converge to the exact result for Coulomb scattering at all energies and scattering angles. A Regge pole in the partial-wave amplitude maps to a cut in the Fourier-Bessel impact-parameter amplitude, i.e., the existence of poles in the impact-parameter amplitude is dubious.

A recent article in this journal[1] developed an infinite-series expansion for the impact-parameter amplitude $S_F(b)$ related to the scattering amplitude by the Fourier-Bessel transformation,

$$f(\theta)=(K/i)\int_0^\infty db\, bJ_0(qb)[S_F(b)-1] \ . \tag{1}$$

K is the wave number, b is the impact parameter, and $q=2K\sin(\frac{1}{2}\theta)$ is the momentum transfer. The amplitude $S_F(b)$ was defined in relation to the function $S(b)$ which interpolates the partial-wave phase shifts

$$S((l+\tfrac{1}{2})/K)\equiv\exp(2i\delta_l)$$

by the relation

$$bS_F(b)=\mathcal{O}\left(b,\frac{b}{db}\right)bS(b) \tag{2}$$

with the differential operator $\mathcal{O}$ given by an infinite series:

$$\mathcal{O}\left(b,\frac{d}{db}\right)=\sum_{k=0}^\infty\frac{1}{(2k)!}\,b_k\left(-\tfrac{1}{2}b\frac{d}{db}\right)\left(\frac{1}{K}\frac{d}{db}\right)^{2k} . \tag{3}$$

At a fixed impact parameter b, $\mathcal{O}$ generates an expansion in powers of K^{-2}. The polynomials $b_k(x)$ in (3) are defined in terms of generalized Bernoulli polynomials by $b_k(x)\equiv B_{2k}^{(2x)}(x)$ and these are generated by the series

$$\left(\frac{t}{2\sinh(\frac{1}{2}t)}\right)^{2x}=\sum_{k=0}^\infty b_k(x)\frac{t^{2k}}{(2k)!}\ ,\ |t|<2\pi\ . \tag{4}$$

The series expansion defined by (2) and (3) is of interest because it provides unambiguous corrections to the Glauber eikonal approximation[2] which is the leading contribution to $S_F(b)$ as $K\to\infty$.

The present paper develops a simple *a posteriori* proof of the mapping implied by (2) and also illustrates the convergence of the infinite series. In Table I we list some forms for $S_F(b)$ together with the corresponding $S(b)$. All of our results follow from line 1 of the table which can be proved in a straightforward manner from Eqs. (2), (3), and (4) for $|z|<2\pi$ and then, by analytic continuation, for all z.

Given line 1 of the table, one can show that whenever the partial-wave amplitude can be expressed as a Laplace transform

$$S(b)-1=\int_{x_0}^\infty dx\ T(x)e^{-Kbx}\quad(x_0\geq 0)\,. \tag{5}$$

The Fourier-Bessel amplitude can be expressed as the related integral

$$\begin{aligned}S_F(b)-1&=\int_{x_0}^\infty dx\cosh(\tfrac{1}{2}x)\,T(x)\\&\qquad\times\exp[-2Kb\sinh(\tfrac{1}{2}x)]\\&=\int_{y_0}^\infty dy\,T(2\sinh^{-1}(\tfrac{1}{2}y))e^{-Kby},\\ y_0&=2\sinh(\tfrac{1}{2}x_0)\,.\end{aligned} \tag{6}$$

Standard integral representations of the modified Bessel function $K_0(2Kbz)$ and the associated Legendre function of the second kind $Q_{Kb-1/2}(1+2z^2)$ can be related as in (5) and (6) with

$$\cosh x_0=1+2z^2$$

and

$$T(x)=[2(\cosh x-\cosh x_0)]^{-1/2}$$

and this proves line 2 of the table.

Now given line 2 of the table one immediately verifies that line 3 of the table follows. In line 3, $A(s,t)$ is the spectral function from the Mandelstam representation and $\tilde{s}$ is a kinematical factor (see Ref. 1 for details). The relevant point is that the form for $S(b)$ in line 3 is the (exact) Froissart-Gribov interpolation of the partial-wave amplitude[3] and the form for $S_F(b)$ is the (exact) Fourier-Bessel amplitude of Blankenbecler and Goldberger.[4] This proves in a more transparent manner than given in Ref. 1 that Eqs. (2) and (3) per-

TABLE I. Examples of Fourier-Bessel–partial-wave correspondence.

Fourier-Bessel amplitude	Partial-wave amplitude
$S_F(b) \equiv b^{-1}\Theta\left(b,\frac{d}{db}\right)S(b)$	$S(b)$
$\exp[-2Kb\sinh(\frac{1}{2}z)]$	$\exp(-Kbz)$
$K_0(2Kbz)$	$Q_{Kb-1/2}(1+2z^2)$
$1+\left(\frac{i}{\pi \tilde{s}}\right)\int_{t_0}^{\infty} dt\, A(s,t)K_0(bt^{1/2})$	$1+\left(\frac{i}{\pi \tilde{s}}\right)\int_{t_0}^{\infty} dt\, A(s,t)Q_{Kb-1/2}(1+t/2K^2)$
$1+\frac{\alpha-\alpha^*}{Kb}\left[1+2\alpha\int_0^{\infty} d\theta \exp(2\alpha\theta-2Kb\sinh\theta)\right]$	$\frac{Kb-\alpha^*}{Kb-\alpha}$

form the desired mapping based on analyticity of the scattering amplitude in the momentum-transfer variable t of Mandelstam.

A special case of physical interest is Coulomb scattering since that is the only case where $S(b)$ is exactly known, i.e., with the above notation

$$S(b)=e^{i\varphi}\frac{\Gamma(Kb+\frac{1}{2}+i\eta)}{\Gamma(Kb+\frac{1}{2}-i\eta)}, \tag{7}$$

where $\eta = Ze^2/\hbar v$ and $\varphi = -2\eta\ln(2Kr)$. The phase φ is the usual infinite part of the Coulomb phase shift. If we introduce an absorptive part to the Coulomb potential by means of the replacement $\eta \to \eta + i\epsilon$ (where ϵ is positive but arbitrarily small), then a Laplace transform representation holds as in (5) with $x_0 = 0$ and

$$T(x)=e^{-(1/2+i\eta)x}\frac{(1-e^{-x})^{-2i\eta-1}}{\Gamma(-2i\eta)}. \tag{8}$$

The corresponding Fourier-Bessel amplitude representing the sum in (2) is then determined from Eq. (6) to be

$$S_F(b)=e^{i\varphi}(Kb)^{2i\eta}$$
$$= b^{2i\eta}e^{-2i\eta\ln(2r)}. \tag{9}$$

This result coincides with the one which follows from the Glauber eikonal approximation without any corrections at all. It is well known to reproduce the exact Coulomb scattering amplitude and phase at all energies and angles when the integral of (1) is performed and the convergence parameter ϵ permitted to approach zero. Thus we conclude that the Fourier-Bessel expansion of (2) converges in the Coulomb case to the correct result. This fact shows that *all* corrections to the Glauber approximation must vanish in the Coulomb case as is indeed found in the leading orders from an eikonal expansion about the Glauber propagator.[5]

Note that the infinite series of Regge poles of (7) is replaced in (8) by a cut along the negative real axis in the complex b^2 plane. In this regard it is also of some interest to point out the Fourier-Bessel amplitude (see line 4 of the table) corresponding to a single Regge pole. The Fourier-Bessel amplitude involves an integral which is nonsingular for $\mathrm{Re}(b)>0$[6] showing that $b[S_F(b)-1]$, which appears in Eq. (1), has no singularities in the right-half b plane. In the complex b^2 plane, $b[S_F(b)-1]$ is analytic with the exception of a cut along the negative real axis as above, i.e., there are no poles. This fact casts considerable doubt on the use of poles in the impact-parameter amplitude since they cannot be associated with Regge poles in the partial-wave amplitude.

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[1]S. J. Wallace, Phys. Rev. D 8, 1846 (1973).

[2]R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al*. (Interscience, New York, 1959), Vol. I, p. 315.

[3]M. Froissart, invited paper at the LaJolla Conference on Weak and Strong Interactions, 1961 (unpublished); V. N. Gribov, Zh. Eksp. Teor. Fiz. 41, 677 (1961) [Sov. Phys.—JETP 14, 478 (1962)].

[4]R. Blankenbecler and M. L. Goldberger, Phys. Rev. 126, 766 (1962).

[5]S. J. Wallace, Ann. Phys. (N.Y.) 78, 190 (1973).

[6]The Regge pole corresponds to use of $x_0=0$ and $T(x) = (\alpha-\alpha^*)e^{\alpha x}$ in Eq. (5). In this event, the convergence of (5) requires $\mathrm{Re}(Kb-\alpha)>0$. However, the integral in Eq. (6) converges for $\mathrm{Re}(Kb)>0$ independent of α and thus provides the analytic continuation needed if $\mathrm{Re}(\alpha)>0$.