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PHYSICAL REVIEW D VOLUME 9, NUMBER 2 15 JANUARY 1974

Callan-Symanzik equation and asymptotic behavior in field theory: form factors*

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(Received 1 October 1973)

We show that even for some external lines on the mass shell, the procedure of dropping the mass-insertion term in the Callan-Symanzik equation is justified for the form factor at high squared momentum transfer in a certain class of models. This provides a very quick method of summing leading contributions in perturbation theory, as well as summing the next-to-leading terms.

INTRODUCTION

In the Lagrangian formulation of quantum field theory, because of the singular behavior of products of operators at short distance there are anomalies in the Ward identities, compared to naive ones. For example, the Callan-Symanzik equation[1] is the correct Ward identity for broken scale invariance in perturbation theory. Another aspect which has been emphasized, mainly by Symanzik,[2,3] is that this equation can be used to estimate the asymptotic behavior of Green's functions.

In general, the usefulness of this equation may be limited due to the following reasons:

(a) We are ignorant with respect to the mass-insertion term.

(b) The parameters which appear in this equation are unknown.

(c) Even if we know something about (a) and (b), we need to face the problem of the solution of this equation.

In spite of these restrictions, there are situations in which our knowledge of the asymptotic behavior of Green's functions can be improved or some results from the perturbation theory can easily be reproduced, using this equation.

Let us consider, for example, the asymptotic behavior in momentum space of Green's functions in such a configuration that no partial sum of external momenta can be zero (except for the overall energy-momentum conservation), or be on the light cone, i.e., the situation of so-called nonexceptional momenta. When all external variables are very far from the mass shell and Euclidean (all $p_i^2 \to -\infty$), from the usual arguments on power counting[4] the inhomogeneous term can be dropped,[1-3] and we are left with a homogeneous partial-differential equation of first order governing such a limit of Green's functions. In this case we get a nice answer to the question (a). The questions (b) and (c) are considered in Refs. 2, 3, and 5. We will comment on these points later on.

How much further can we go beyond asymptotic nonexceptional Euclidean momenta? According to the suggestion of Symanzik,[3] short-distance and light-cone expansions provide some clues for the calculation of asymptotic Green's functions. Based on this idea, Hasslacher, Mueller, and Christ[6] considered the Bjorken limit of the structure functions of electroproduction. The restriction (a) was circumvented by considering the Callan-Symanzik equations satisfied by singular functions appearing in the light-cone expansion of the product of two currents. Some results on summing leading contributions[7] were easily reproduced for the pseudoscalar theory and the gluon model. The impossibility of getting concrete answers beyond the results of summing leading contributions is related to the restrictions (b) and (c).

There have been very few attempts to use Callan-Symanzik equations for the study of asymptotic behavior of Green's functions for exceptional[6] and/or Minkowskian momenta. This paper intends to be a modest step in this direction.

The simplest Green function in which asymptotic Minkowskian momenta are involved is the three-point vertex function[8] of Fig. 1.

The aim of this paper is to show that for certain models, for p and p' on the mass shell but $q^2 \to -\infty$, we get in perturbation theory the result that the inhomogeneous term (mass insertion) is of order $(q^2)^{-1}[\ln(-q^2/m^2)]^N$, where N is an integer, while the vertex function itself has only logarithmic contributions. So in the limit of high momentum transfer we have a homogeneous Callan-Symanzik equation for the vertex function in perturbation theory.

In Sec. I we discuss in detail the vertex function of the simplest renormalizable model: $\lambda\phi^3$ in six dimensions. We show for various orders (Appendix A complements Sec. I) that as far as logarithmic terms are concerned (not only the leading terms), the mass-insertion term is negligible as

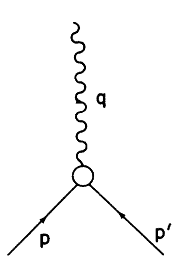

FIG. 1. The three-point Green's function.

compared to the vertex functions, from which follows the homogeneous Callan-Symanzik equation. The solution of this equation is exhibited and its connection with summing logarithmic terms in perturbation theory is discussed.

The results for other models are discussed in Sec. II. We show explicitly why the neglect of the mass insertion is unjustified for the gluon model and why (explicitly only in lowest order), it is justified for the pseudoscalar model. The results of Applequist and Primack[9] on summing the leading contributions are easily reproduced. It is argued that the nonphysical behavior so obtained cannot be improved by summing other nonleading logarithms.

Section III is devoted to the investigation of the validity of neglecting the mass-insertion term for other Green's functions and scattering amplitudes. The role of this equation as a constraint in perturbation theory is used to easily show why the $\lambda\phi^4$ model does not exhibit Regge behavior in perturbation theory.[10]

We end with Sec. IV, which summarizes our conclusions.

I. RESULTS IN A SIMPLE MODEL[11]

The simplest vertex function with three external legs in renormalizable models is that of $\lambda\phi^3$ in six dimensions. The virtue of this model clearly is that we do not have to worry about complications due to spinors.

In Appendix A we describe the method used throughout this paper. Our intention in presenting this technique is twofold: to make this paper self-contained and to provide a very useful method for the analysis of the asymptotic behavior of Green's functions and scattering amplitudes involving only massive particles in perturbation theory. For our purposes, this method turns out to be very economical since it permits us to handle simultaneously the vertex function and mass insertion. For a more complete and systematic discussion of the method we refer to Ref. 12. From now on we assume that the reader is familiar with the concepts and formulas explained in Appendix A and Ref. 12.

The $\lambda\phi^3$ model is described by the Lagrangian density

$$\mathcal{L}(\partial_\mu\phi(x), \phi(x)) = \tfrac{1}{2}[\partial_\mu\phi(x)][\partial^\mu\phi(x)] - \tfrac{1}{2}m^2\phi^2(x) - \frac{\lambda}{3!}\phi^3(x)\,.$$

The Feynman rules for $\lambda\phi^3$ in six dimensions are as for the same model in four dimensions, except for small changes. For each internal loop we need to introduce a factor of $(2\pi)^{-6}$ and the propagator is defined by (taking $\lambda = 1$)

$$\Delta_F(\lambda, p) = \frac{i}{(m^2 - i\epsilon - p^\mu g_{\mu\nu} p^\nu)\lambda}$$
$$= \frac{i e^{\lambda \pi i/2}}{\Gamma(\lambda)}$$
$$\times \int_0^\infty d\alpha\, \alpha^{\lambda-1} \exp[i\alpha(p^\mu g_{\mu\nu} p^\nu - m^2 + i\epsilon)]\,. \tag{1.1}$$

We note that for $\lambda = 1$ in some closed-loop calculations we will have problems with divergences. However, we can subtract consistently at all external-momenta zeros, using the Bogoliubov-Parasiuk-Hepp-Zimmermann[13] (BPHZ) scheme of renormalization in $\sigma + 1$ dimensions. The metric operator g which appears in (1.1) is

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -\underline{1}_\sigma \end{pmatrix}. \tag{1.2}$$

To the lowest, nontrivial order the graph contributing to the vertex function is shown in Fig. 2(a), while a typical mass insertion appears in Fig. 2(b).

If $\Gamma^{(n)}$ is the renormalized vertex function in nth order of perturbation theory and $\Delta\Gamma^{(n)}$ the corresponding mass insertion, we have

$$(\Delta)\Gamma^{(2)}(q, p, p') = \frac{(-i\lambda)^3}{2!\,(2\pi)^6}\left(\frac{\pi^3}{i}\right)\int_0^1\int_0^1 \frac{dt_1 dt_2}{(1+t_1+t_2)^3}\,(M)$$
$$\times \left\{\ln\left[1 - \frac{q^2 t_1 t_2 + p'^2 t_2 + p^2 t_1}{(m^2 - i\epsilon)(1+t_1+t_2)^2}\right] + \ln\left[1 - \frac{q^2 t_1 + p'^2 t_1 t_2 + p^2 t_2}{(m^2 - i\epsilon)(1+t_1+t_2)^2}\right]\right.$$
$$\left. + \ln\left[1 - \frac{q^2 t_2 + p'^2 t_2 + p^2 t_1 t_2}{(m^2 - i\epsilon)(1+t_1+t_2)^2}\right]\right\}. \tag{1.3}$$

In (1.3) M is the mass-insertion operator $m^2\partial/\partial m^2$. The permutations correspond to various scalings in Feynman α parameters. For example the first term in (1.3) corresponds to the E family of Fig. 2(c). This scaling is (the integration over t_G has already been performed; see Appendix A)

$$\alpha_3 = t_G\,,$$
$$\alpha_2 = t_G t_2\,,$$
$$\alpha_1 = t_G t_1\,.$$

When analyzing the asymptotic behavior of Γ and $\Delta\Gamma$ for $q^2 \to -\infty$, p^2 and p'^2 on shell, we can make the approximation (which does not alter the asymptotic behavior) $(1 + t_1 + t_2) \to 1$ *and* neglect the "external masses" $p^2 = p'^2 \to 0$. With this approximation

$$\Gamma^{(2)} \sim \text{const} \int_0^1\int_0^1 dt_1 dt_2$$
$$\times \left[\ln\left(1 - \frac{q^2}{m^2} t_1 t_2\right) + \ln\left(1 - \frac{q^2}{m^2} t_1\right) + \ln\left(1 - \frac{q^2}{m^2} t_2\right)\right]$$
$$\sim \text{const}' \times \ln\left(-\frac{q^2}{m^2}\right), \tag{1.4a}$$

$$\Delta\Gamma^{(2)} \sim \text{const} \int_0^1\int_0^1 dt_1 dt_2$$
$$\times \left[\frac{1}{1 - (q^2/m^2)t_1 t_2} + \frac{1}{1 - (q^2/m^2)t_1} + \frac{1}{1 - (q^2/m^2)t_2}\right]$$
$$\sim \text{const}'' \times \frac{1}{q^2} \ln^2\left(-\frac{q^2}{m^2}\right) + O\left(\frac{1}{q^2} \ln\left(-\frac{q^2}{m^2}\right)\right). \tag{1.4b}$$

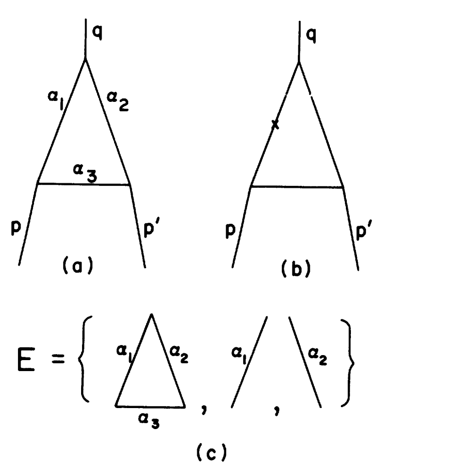

FIG. 2. (a) The vertex function in second order of perturbation theory, (b) one of its mass insertions, and (c) one particular E family.

In Appendix A we present explicit calculations for all fourth-order graphs, as well as ladder graphs in all orders of perturbation theory. The approximations used there are essentially the same as explained above. Here we give the results: For ladder graphs in all orders and all graphs in fourth-order (at this order we have also considered nonleading contributions), the result is that

$$\Gamma^{(2n)}(q,p,p') \underset{q^2\to-\infty}{\sim} C_1\left[\ln\left(-\frac{q^2}{m^2}\right)\right]^\alpha + O\left(\left[\ln\left(-\frac{q^2}{m^2}\right)\right]^{\alpha-1}\right), \tag{1.5a}$$

$$\Delta\Gamma^{(2n)}(q,p,p') \underset{q^2\to-\infty}{\sim} C_2\frac{1}{q^2}\left[\ln\left(-\frac{q^2}{m^2}\right)\right]^\beta + O\left(\frac{1}{q^2}\left[\ln\left(-\frac{q^2}{m^2}\right)\right]^{\beta-1}\right), \tag{1.5b}$$

where α and β are positive integers and C_1 and C_2 are constants. For the leading terms $\alpha=n$ and $\beta=n+1$.

As will be discussed in Appendix A, the important point for any order is that for the computation of $\Gamma^{(n)}$ we have, after subtraction, integrals of logarithms. Referring to (A2), we see that these integrals come from graphs with superficial degree of divergence zero. But for $\Delta\Gamma^{(n)}$ we have integrals of $[1-(q^2/m^2)(\text{polynomial in } t_j)]^{-1}$, corresponding to superficial degree of divergence -1, from which our results follows. From another point of view, interestingly enough, what happens is that despite the Minkowskian nature of the external momenta, the power counting[4] is working. At first sight one might consider this result not totally unexpected, for one could naively argue that (working) in the infinite-momentum frame in the limit considered, all external variables have at least one large component and hence can be scaled with a large parameter. For any vector "a" we introduce the infinite-momentum-frame variable $a=(a^+,\vec{a},a^-)$, $a^\pm=(a_0\pm a_5)$, and $\vec{a}=(a_1,a_2,a_3,a_4)$ and scalar product $a\cdot b=\frac{1}{2}(a^+b^-+a^-b^+)-\vec{a}\cdot\vec{b}$. In terms of infinite-momentum variables[14] we have

$$\begin{aligned} q &= Q(1,\vec{0},-1),\\ p &= Q\left(\frac{p^2}{Q^2},\vec{0},1\right)+O(1/Q^3,\vec{0},1/Q),\\ p' &= Q(1,\vec{0},p'^2/Q^2)+O(1/Q,\vec{0},1/Q^3),\\ q^2 &= -Q^2. \end{aligned} \tag{1.6}$$

However, the fact that being able to scale all variables with a large parameter, with nonexceptional momenta, is not sufficient to ensure the validity of power-counting arguments will be shown in Sec. II with explicit examples. We postpone the discussions to Sec. II.

From the explicit results (1.5a) and (1.5b) we see that the mass-insertion term is negligible as compared to the vertex function. So we can write the Callan-Symanzik equation in the form

$$\left(m^2\frac{\partial}{\partial m^2}+\beta(\lambda)\frac{\partial}{\partial\lambda}-3\gamma(\lambda)\right)\Gamma(q^2,m^2)=0. \tag{1.7}$$

From dimensional-analysis arguments, we have

$$\left(-q^2\frac{\partial}{\partial q^2}+\beta(\lambda)\frac{\partial}{\partial\lambda}-3\gamma(\lambda)\right)\Gamma(q^2,m^2)=0. \tag{1.8}$$

The importance of Eq. (1.8) is that we have a homogeneous partial-differential equation of first order for the vertex function in a physical limit. However, the importance of this result is diminished for two reasons: (1) Our model is not realistic. (2) We have solved (partially) only the problem (a) mentioned in the Introduction. Using the Callan-Symanzik equation for two- and three-point vertex functions and renormalization conditions, the parameters β and γ can be expressed in terms of $(\partial/\partial p^2)\Delta\Gamma_2(0,0)$ and $\Delta\Gamma_3(0,0,0)$ (mass insertions for the two- and three-point vertex functions). These expressions gives us a hint about how to compute the parameters β and γ: perturbation theory. In perturbation theory we have

$$\begin{aligned} \beta &= b_0\lambda^3+b_1\lambda^5+\cdots,\\ \gamma &= c_0\lambda^2+c_1\lambda^4+\cdots \end{aligned} \tag{1.9}$$

explicitly:

$$\begin{aligned} b_0 &= -\tfrac{3}{8}(4\pi)^{-3},\\ c_0 &= \tfrac{1}{24}(4\pi)^{-3}. \end{aligned} \tag{1.10}$$

The linear partial-differential equation (1.8) is solved by the standard method of characteristics. For a more complete discussion we refer to Refs. 5 and 2. We present here just the main results. The general solution of Eq. (1.8) is

$$\Gamma(q^2,m^2)=\phi\left(\ln Q^2+\int_{\lambda_0}^{\lambda}d\lambda'\frac{1}{\beta(\lambda')}\right)\times\exp\left[3\int_{\lambda_0}^{\lambda}d\lambda'\frac{\gamma(\lambda')}{\beta(\lambda')}\right]. \tag{1.11}$$

As expected, from the nature of the partial-differential equation (1.8) the solution for $\Gamma(q^2,m^2)$ involves a function ϕ, which is fixed by initial data on some noncharacteristic curve. [This is problem (c) mentioned in the Introduction.] The asymptotic behavior of the vertex function can be extracted without knowledge of ϕ only if additional hypotheses (considered below) are made. For this purpose we transform the expression (1.11) to

$$\Gamma(q^2,m^2)=\exp\left[-3\int_{\lambda}^{\bar\lambda(Q)}d\lambda'\frac{\gamma(\lambda')}{\beta(\lambda')}\right]\times\Gamma_{\text{nonasy}}(q^2,m^2,\bar\lambda(Q)), \tag{1.12}$$

where

$$\frac{d\bar{\lambda}(Q)}{dQ}=\frac{2}{Q}\beta(\bar{\lambda}(Q)), \tag{1.13a}$$

$$\bar{\lambda}(1)=\lambda\,. \tag{1.13b}$$

Here Γ_{nonasy} means the value of Γ for nonasymptotic q^2, i.e., $Q=1$ in (1.6). From now on Q is dimensionless ($m=1$). Again let us remind ourselves that (1.12) does not exhibit too much progress.

However, under the hypothesis that the "invariant charge" $\bar{\lambda}(Q)$ is small, we will take for $\Gamma_{\text{nonasy}}(q,p,p',m^2,\bar{\lambda}(Q))$ the value of the trivial graph, i.e., $-i\bar{\lambda}(Q)$. As we will show below, the hypothesis of $\bar{\lambda}(Q)$ being very small can be implemented in this model even for very high values of λ (the physical coupling constant) in the limit $Q\to\infty$.

Taking for β and γ the expansions (1.9), and for Γ_{nonasy} the value $-i\bar{\lambda}(Q)$, we have from (1.12)

$$\begin{aligned}\Gamma(q^2,m^2)&=\exp\left\{-3\left[\frac{c_0}{b_0}\ln\left(\frac{\bar{\lambda}(Q)}{\lambda}\right)+\frac{1}{3}\left(\frac{c_1}{c_0}-\frac{b_1}{b_0}\right)[\bar{\lambda}^3(Q)-\lambda^3]+\cdots\right]\right\}[-i\bar{\lambda}(Q)]\\&=-i\lambda\left(\frac{\bar{\lambda}(Q)}{\lambda}\right)^{1-3c_0/b_0}\exp\left[-\left(\frac{c_1}{c_0}-\frac{b_1}{b_0}\right)[\bar{\lambda}^3(Q)-\lambda^3]+\cdots\right]\,.\end{aligned} \tag{1.14}$$

The expression (1.14) corresponds to the sum of leading terms in perturbation theory when we take the lowest-order values for β and γ. The sum of next-to-leading terms corresponds to the values in next order in β and γ (c_1, b_1), and so on. Substituting the lowest order in (1.13a) with the condition (1.13b) we get

$$\bar{\lambda}(Q)=\frac{\lambda}{(1-2b_0\lambda^2\ln Q^2)^{1/2}}\,. \tag{1.15}$$

With the values in (1.10) for b_0 and c_0, we have for the sum of leading logarithms in perturbation theory

$$\Gamma(Q^2)\underset{Q^2\to\infty}{\sim}\lambda(1+\tfrac{3}{4}(4\pi)^{-3}\lambda^2\ln Q^2)^{-2/3}\,. \tag{1.16}$$

Another advantage of this approach is that, besides easily summing the leading contributions, we can do better. We could sum the next-to-leading terms, computing c_1 and b_1 and taking for β and γ the term next to the lowest order. However, just by inspection of (1.14) we see that by keeping only a finite number of nonleading contributions (in the above sense), a more realistic asymptotic power behavior for Γ is never obtained.

Before ending this section, I want to comment on the approximation $\Gamma_{\text{nonasy}}\cong\bar{\lambda}(Q)$, which in this model is justified from (1.15) since b_0 is negative. In this model the physical coupling constant does not need to be weak in order for the "invariant charge" to be small for $Q\to\infty$. The same thing is not true for the other renormalizable models such as the gluon model, pseudoscalar theory, quantum electrodynamics (QED), and $\lambda\phi^4$ theory.

II. FORM FACTORS[8]: OTHER MODELS

Now we try to extend this approach for other models. The simplest one would be the vertex function of the superrenormalizable model $\lambda\phi^3$ in four dimensions. The naive argument fails in this case. In the second order

$$(\Delta)\Gamma(q,p,p',m^2)\propto\int d^4k(M)\times\frac{1}{[(p-k)^2-m^2][(p'-k)^2-m^2](k^2-m^2)}\,. \tag{2.1}$$

From (2.1) it follows that for $q^2\to-\infty$

$$\Gamma(q^2,m^2)\sim\frac{1}{q^2}\left[\ln\left(-\frac{q^2}{m^2}\right)\right]^2,$$

$$\Delta\Gamma(q^2,m^2)\sim\frac{1}{q^2}\ln\left(-\frac{q^2}{m^2}\right)\,.$$

However, this gives a hint to understanding why this approach is valid in six dimensions, but fails in four dimensions. The point is that in general, even for models involving fermions, the logarithmic contribution comes from asymptotic regions of integration in momentum space (ultraviolet logarithms), or from infrared regions[14] of integration (infrared logarithms). When only the ultraviolet region of integration plays an important role, we see [using (1.6)] that as far as Γ and $\Delta\Gamma$ are concerned, power-counting arguments still work. This is the case in six dimensions. However, in four dimensions we also have infrared logarithms[14] and naive power-counting arguments do not work. This hint leads us to the possibility of extending this approach to other models. The gluon model is a model in which we have ultraviolet and infrared logarithms, while in the pseudoscalar model we have the same situation as in the model of $\lambda\phi^3$ in six dimensions.[9]

Clearly these remarks are valid, as long as the leading logarithms are concerned. For example, in the model considered in Sec. I, if we consider terms which go as $(1/q^2)$, the logarithms multiplying these terms no longer will be only ultraviolet. We will show now in second order that our approach works in the pseudoscalar theory, but fails in the gluon model. The consequences are then analyzed.

For both models we can define

$$\Gamma_\mu^{(5)}(q,p,p')=\bar{u}(p)\Lambda_\mu^{(5)}(q,p,p')u(p'). \tag{2.2}$$

The superscript (5) will denote the pseudoscalar theory.

In second order the relevant contributions to $(\Delta)\Lambda_\mu^{(5)}$, $(\Delta)\Lambda_\mu$ are proportional to $\gamma^5(\Delta)I_\mu\gamma^5$, $\gamma^\nu(\Delta)I_\mu\gamma_\nu$, respectively, where the integral $(\Delta)I_\mu$ is given by (μ is the mass of pion or the mass of gluon)

$$\begin{aligned}(\Delta)I^\nu &\equiv \int d^4k\,(M')\frac{(\not p-\not k+m)\gamma^\nu(\not p'-\not k+m)}{[(p-k)^2-m^2][(p'-k)^2-m^2][k^2-\mu^2]}\\
&=\frac{\pi^2}{i}\int_0^\infty\!\int_0^\infty\!\int_0^\infty \frac{d\alpha_1 d\alpha_2 d\alpha_3}{(\alpha_1+\alpha_2+\alpha_3)^2}(M')\exp\{i[\alpha_1\alpha_3 p^2+\alpha_2\alpha_3 p'^2+\alpha_1\alpha_2 q^2+(\alpha_1+\alpha_2)(-m^2+i\epsilon)+\alpha_3(-\mu^2+i\epsilon)]\}\\
&\quad\times\left\{\not p\gamma^\nu\not p'+\frac{(\alpha_1\not p+\alpha_2\not p')\gamma^\nu(\alpha_1\not p+\alpha_2\not p')}{(\alpha_1+\alpha_2+\alpha_3)^2}+\frac{-(\alpha_1\not p+\alpha_2\not p')\gamma^\nu\not p-\not p\gamma^\nu(\alpha_1\not p+\alpha_2\not p')}{\alpha_1+\alpha_2+\alpha_3}-\frac{i\gamma^\nu}{\alpha_1+\alpha_2+\alpha_3}\right.\\
&\quad\left.+m^2\gamma^\nu+m\left[\gamma^\nu\left(\not p'+\frac{-\alpha_1\not p-\alpha_2\not p'}{\alpha_1+\alpha_2+\alpha_3}\right)+\left(\not p+\frac{-\alpha_1\not p-\alpha_2\not p'}{\alpha_1+\alpha_2+\alpha_3}\right)\gamma^\nu\right]\right\}.\end{aligned} \tag{2.3}$$

M' is the mass-insertion operator $m^2\partial/\partial m^2+\mu^2\partial/\partial\mu^2$.

(2.3) displays the difference in the case of the gluon model and pseudoscalar theory. Consider the "eikonal term"[14,15] [The first term in the large curly brackets of (2.3)]. From the Dirac equations, $(\not p'+m)u(p')=0$, $\bar{u}(p)(\not p+m)=0$, we see that the "eikonal term" does not play any role in the pseudoscalar theory. This is true, since γ^5 does not reverse the order of the product $\not p\gamma^\nu\not p'$. But in the gluon model, since $\gamma_\nu\not p\gamma^\mu\not p'\gamma^\nu=\not p'\gamma^\mu\not p+\cdots$ the order of $\not p$ and $\not p'$ is reversed, giving a very important role to the eikonal term. In terms of integrals, the effective superficial degree of divergence of the gluon model is -2 (to this order), making the theory not too much different from $\lambda\phi^3$ in four dimensions. While in the pseudoscalar theory the effective superficial degree of divergence is zero and hence this theory is very similar to the $\lambda\phi^3$ model in six dimensions. These observations are true, insofar as the leading logarithms are concerned, when $-q^2\to\infty$ but are no longer valid for other contributions.

Introducing the scaling explained in Sec. I and Appendix A, from (2.3) we find the asymptotic behavior (in second order; omitting irrelevant constants)

$$\begin{aligned}&\Lambda^\nu(q^2,m^2,\mu^2)\sim\gamma^\nu\left[\ln\left(-\frac{q^2}{\mu^2}\right)\right]^2,\\
&\Delta(\Lambda^\nu)\sim\gamma^\nu\ln\left(-\frac{q^2}{\mu^2}\right),\end{aligned} \tag{2.4a}$$

$$\begin{aligned}&\Lambda_5^\nu(q^2,m^2,\mu^2)\sim\gamma^\nu\ln\left(-\frac{q^2}{m^2}\right),\\
&\Delta(\Lambda_5^\nu)\sim\frac{\gamma^\nu}{q^2}\left[\ln\left(-\frac{q^2}{m^2}\right)\right]^2,\end{aligned} \tag{2.4b}$$

In what follows, we are considering only those graphs for which $\lambda\phi^4$ does not play any role. So, we consider $\beta_\lambda(g,\lambda)=0$ and $\gamma_\lambda(g,\lambda)=0$ (no renormalization of the bare coupling constant λ). From these considerations and the explicit result (2.4b), which can be extended to isospin, we have for the proton form factor in the pseudoscalar model

$$\left(m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}+\beta'(g)\frac{\partial}{\partial g}-\gamma'\right)F_{1p}(q^2,m^2,\mu^2)=0, \tag{2.5}$$

where g is the π-N coupling constant.

We note that the procedure of dropping the mass-insertion is justified only for the F_1 form factor.[8] However, the F_2 form factor is $O((1/q^2)F_1)$.

The coefficients β' and γ' in perturbation theory have the expansions

$$\beta'=b_0g^3+b_1g^5+\cdots, \tag{2.6a}$$

$$\gamma'=c_0g^2+c_1g^4+\cdots. \tag{2.6b}$$

Explicitly, we have (see Appendix B)

	b_0	c_0
U(1)	$\frac{5}{32\pi^2}$	$\frac{1}{32\pi^2}$
SU(2)	$\frac{5}{32\pi^2}$	$\frac{3}{32\pi^2}$

(2.7)

U(1) corresponds to the usual π^0-p Yukawa theory.

In perturbation theory the asymptotic behavior of the form factor F_1 depends only on the ratio q^2/m^2. This statement corresponds to the absence of infrared singularities when the pion mass goes to zero for logarithmic contributions. Since this is the case,[9] the solution for (2.5) goes straightforwardly along the lines discussed in Sec. I. The asymptotic behavior for the proton form factor is [we are considering SU(2) and U(1) at the same time]

$$F_{1p}(Q^2) \cong e\left(\frac{\bar{g}(Q)}{g}\right)^{-c_0/b_0} \times \exp\left[-\frac{1}{3}\left(\frac{c_1}{c_0}-\frac{b_1}{b_0}\right)[\bar{g}^3(Q)-g^3]+\cdots\right]. \tag{2.8}$$

Let us now consider the pion-nucleon form factor, which will be denoted from now on by $\mathfrak{F}_{\pi NN}$. Again, the inhomogeneous term can be dropped, leading us to the equation for $\mathfrak{F}_{\pi NN}$:

$$\left(m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}+\beta(g)\frac{\partial}{\partial g}-2\gamma_1-\gamma_2\right) \times \mathfrak{F}_{\pi NN}(q^2,m^2,\mu^2,g)=0\,. \tag{2.9}$$

γ_1 and γ_2 are the anomalous contributions to the dimensions of the elementary fields.

Using analogous expansions for γ_1, γ_2 as we did for γ' in (2.6b), and from similar considerations of the solution of the equation (2.5), the asymptotic behavior for the pion-nucleon form factor is

$$\mathfrak{F}_{\pi NN}(Q^2)=g\left(\frac{\bar{g}(Q)}{g}\right)^{1-(2c_0^{(1)}+c_0^{(2)})/b_0} \times \exp\left\{-\frac{1}{3}\left(\frac{2c_1^{(1)}+c_1^{(2)}}{2c_0^{(1)}+c_0^{(2)}}-\frac{b_1}{b_0}\right) \times[\bar{g}^3(Q)-g^3]+\cdots\right\}. \tag{2.10}$$

In Appendix B we calculate the constant $b_0-2c_0^{(1)}-c_0^{(2)}$ in the U(1) model and SU(n).[16] Explicitly, we have [the constants S_1 and S_2 will be defined later in (2.16)–(2.18)]

	b_0	$b_0-2c_0^{(1)}-c_0^{(2)}$	
U(1)	$\frac{5}{32\pi^2}$	$\frac{1}{16\pi^2}$	
SU(2)	$\frac{5}{32\pi^2}$	$\frac{-1}{16\pi^2}$	(2.11)
SU(n)	$\frac{2S_1+S_2+2S_3}{32\pi^2}$	$\frac{S_1}{16\pi^2}$	

We can compare our solution with the results of Appelquist and Primack[9] on the sum of leading logarithms, taking for $\beta, \gamma', \gamma_1, \gamma_2$ in (2.5) and (2.9) the lowest-order contribution, with b_0, c_0, and $b_0-2c_0^{(1)}-c_0^{(2)}$ given by (2.7) and (2.11). Substituting these constants in (2.8) and (2.10), with the "invariant charge"

$$\bar{g}(Q^2)=g\left(1-\frac{5}{16\pi^2}g^2\ln Q^2\right)^{-1/2}, \tag{2.12}$$

we get for the U(1) Yukawa model

$$F_{1p}(Q^2)=e\left(1-\frac{5g^2}{16\pi^2}\ln Q^2\right)^{1/10}, \tag{2.13a}$$

$$\mathfrak{F}_{\pi NN}(Q^2)=g\left(1-\frac{5g^2}{16\pi^2}\ln Q^2\right)^{-1/5}, \tag{2.13b}$$

while for the isospin pseudoscalar theory, we get

$$F_{1p}(Q^2)=e\left(1-\frac{5g^2}{16\pi^2}\ln Q^2\right)^{3/10}, \tag{2.14a}$$

$$\mathfrak{F}_{\pi NN}(Q^2)=g\left(1-\frac{5g^2}{16\pi^2}\ln Q^2\right)^{1/5}, \tag{2.14b}$$

in agreement with the results in Ref. 9. As pointed out in Sec. I, our method allows us to go beyond summing leading contributions. The point is that no improvement in terms of getting power behavior can be achieved by going to the next-to-leading logarithms (taking the terms in c_1, $c_1^{(1)}$, $c_1^{(2)}$, and b_1 for γ', γ_1, γ_2, and β). Besides this, the whole scheme of perturbation theory turns out (for this model) to be inconsistent for strong interactions. We can see this from (2.12) since in order for the invariant charge to be small, the coupling constant needs to be weak for large Q, making the strong interactions weaker for high momentum transfer.

Another advantage of this approach is that it allows us to handle easily the asymptotic pion-nucleon $\mathfrak{F}_{\pi NN}$ form factor for the SU(n) pseudoscalar model. If the interaction Lagrangian is written in the form

$$\mathcal{L}_{\text{int}}(\underline{\phi},\partial_\mu\underline{\phi},\psi,\partial^\alpha\psi)=ig\bar{\psi}(x)\gamma^5\lambda^a\psi(x)\underline{\phi}_a-\frac{\lambda}{4!}(\underline{\phi}_a{}^2)^2, \tag{2.15}$$

Defining the constants S_1, S_2, S_3, which appear in (2.11), according to Zee[16] as

$$\sum_a\lambda^a\lambda^b\lambda^a=S_1\lambda^b, \tag{2.16}$$

$$\sum_a\lambda^a\lambda^a=S_2\underline{1}, \tag{2.17}$$

$$\mathrm{Tr}\lambda^a\lambda^b=S_3\delta^{ab}, \tag{2.18}$$

from the results (2.11) for SU(n), follows the result for the sum of leading contributions:

$$\mathfrak{F}_{\pi NN}(Q^2)=g\left(1-\frac{2S_1+S_2+2S_3}{32\pi^2}\,g^2\ln Q^2\right)^{-S_1/(2S_1+S_2+2S_3)}.$$

Clearly (2.13b) and (2.14b) are particular cases of (2.19), since for U(1) $S_1=1=S_2=S_3$ and for SU(2) $S_1=-1,\ S_2=3,\ S_3=2$.

III. CALLAN-SYMANZIK EQUATION AND ASYMPTOTIC BEHAVIOR

In this section we want to say a few words on the possibility of generalization of this procedure to other vertex functions and scattering amplitudes. In the opinion of the author, at least for various configurations such as Regge limit with t fixed and form factors, the superficial degree of divergence being zero plays an important role. We get this feeling by looking at expression (A1) in Appendix A. For three external lines this is equivalent to saying that the comparison of the vertex function and its mass insertion corresponds to comparing integrals of $\ln[m^2-q^2$ (polynomial in t_i)] with integrals of (same argument)$^{-1}$. From this follows our result that the mass-insertion term is negligible compared to the vertex function. In this respect, the nonvalidity of this simple approach for the gluon model is a consequence of the validity of the eikonal approximation[15] for this model. The eikonal approximation in form factors makes the effective superficial degree of divergence become smaller as we increase the order of perturbation theory, making the integrals not too much different from those encountered in the superrenormalizable $\lambda\phi^3$. Another process in which zero superficial degree of divergence is involved is π-π scattering. However, since for certain configurations exceptional momenta are involved, the analysis turns out to be more complicated.[17]

These comments do not rule out the possibility of using the Callan-Symanzik equation for estimating asymptotic behavior in perturbation theory; they only say that dropping the mass-insertion term only works in very special cases. This paper has enlarged somewhat the domain of validity of this procedure. Clearly for other cases we need to introduce a clever way of bypassing problem (a) of the Introduction, as was done in Ref. 6.

While the usefulness of the Callan-Symanzik equation is limited for the complete determination of the asymptotic behavior in perturbation theory of scattering amplitudes and Green's functions for nonexceptional momenta, it can be explored as a test of some conjectured asymptotic behavior in perturbation theory. In this way the Ward identity for scale-symmetry breaking can play an important role as a constraint which must be satisfied in perturbation theory. One such example is the conjectured Regge behavior in field theory. Even though the usual $\lambda\phi^3$ leads to Regge behavior in perturbation theory,[18] for π-π scattering, Bjorken and Wu[10] pointed out that $g\phi^4$ does not lead to such behavior. We show this more easily, using the Callan-Symanzik equation. The Regge behavior for π-π scattering implies, using dimensional-analysis arguments,

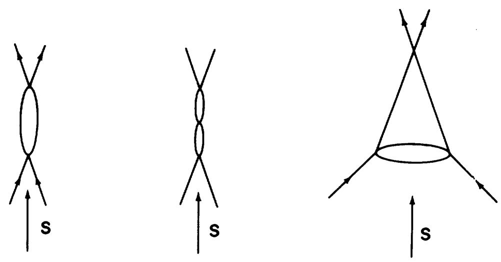

FIG. 3. The lowest-order contributions to the π-π scattering in the $\lambda\phi^4$ model.

$$A(s,t,m^2)\to\left(\frac{s}{m^2}\right)^{\alpha(t/m^2)}. \tag{3.1}$$

The Callan-Symanzik equation for $A(s,t,m^2)$ in (3.1) leads us to the relationship

$$\left\{\left[-t\frac{\partial}{\partial t}\alpha\left(\frac{t}{m^2}\right)+\beta(g)\frac{\partial}{\partial g}\alpha\left(\frac{t}{m^2}\right)\right]\ln\left(\frac{s}{m^2}\right)-\alpha\left(\frac{t}{m^2}\right)-4\gamma\right\}A(s,t,m^2)=\Delta A(s,t,m^2). \tag{3.2}$$

in the limit for $s\to\infty$ and t fixed. (3.2) says that $\Delta A(s,t,m^2)$ must have terms of the form $\ln(s/m^2)A(s,t,m^2)$.[19] Now, just looking at the lowest-order contributions, the graphs shown in Fig. 3, we see that $A\to(\ln s)^n$ (n integer) but $\Delta A\to(1/s)[\ln(s/m^2)]^{n'}$, violating the condition imposed by (3.2).

IV. CONCLUSIONS

We have shown that the use of the Callan-Symanzik equation may provide a very quick and precise method for summing leading contributions in perturbation theory; this is true also for form factors, when the squared momentum transfer is very large. Besides being a very quick method of avoiding lengthy calculations in perturbation theory, this approach provides a way to see that summing logarithms in perturbation theory cannot reproduce, in certain models (for which this method works), the physical asymptotic behavior of the form factor $F\simeq(q^2)^{-\alpha}$, with $\alpha\gtrsim 2$.

However, from a speculative point of view, this behavior is not ruled out from the quantum-field-theory scheme. Indeed, as long as the exact solution has negligible mass-insertion terms, such a behavior is obtained if the coupling constant is a zero of the Callan-Symanzik β function. In this case we have for the vertex function in the renormalizable $\lambda\phi^3$ model in six dimensions

$$\left(-q^2\frac{\partial}{\partial q^2}-\gamma\right)\Gamma(q^2,m^2)=0,$$

from which follows the asymptotic power behavior.

As explained in Sec. II, the validity of the procedure of dropping the mass insertion is limited to models for which the logarithmic contributions come from integrals with effective superficial degree of divergence zero.

For other vertex functions or for scattering amplitudes we only comment on the fact that though as yet the equation of Callan and Symanzik has not been useful for the *complete* characterization of these functions, the equation does impose important constraints which must be obeyed to arbitrary orders in perturbation theory. For example certain postulated asymptotic behavior, such as Regge behavior, can be tested in perturbation theory.

ACKNOWLEDGMENTS

The author wants to thank C. H. Woo, P. K. Mitter, and P. H. Weisz for many helpful discussions.

APPENDIX A

In this appendix we want to complement the results in lowest order in the renormalizable $\lambda\phi^3$ model. After the introduction of the method, we consider the fourth-order terms. The approximations (which do not change the asymptotic behavior) are explained for one graph. We work in the BPHZ [13] renormalization scheme.

We start by describing the method used in this paper. In order to analyze the asymptotic behavior, in a unified way, of the vertex function and the mass insertion (obviously the method is also very useful for the analysis of other Green's functions as well as scattering amplitudes), we make use of the method described in Ref. 12. In order to show its power,[20] we describe the essential formulas for our considerations for a scalar theory in $\sigma+1$ dimensions. Some small modifications are required when spinor and vector fields are introduced.

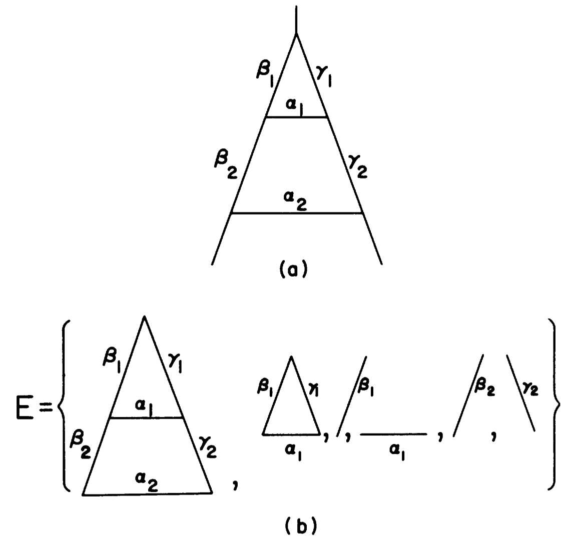

FIG. 4. (a) The ladder-graph contribution to the vertex function in fourth order. (b) One particular E family.

A generalized graph $\mathcal{G}(\underline{\lambda},\underline{p})$ in momentum space is defined by

$$\mathcal{G}(\underline{\lambda},\underline{p})=\prod_L \Delta_F(\lambda_i,p_i), \tag{A1}$$

where $\Delta_F(\lambda_i,p_i)$ is the generalized Feynman propagator in $\sigma+1$ dimensions [Eqs. (1.1), (1.2)] and L is set of lines of the graph.

The substitution of (1.1) in (A1) leads us, after "internal momentum" integration and according to the scaling introduced in Ref. 12, to

$$\mathcal{G}(\underline{\lambda},\underline{p})=C\Gamma(\Lambda-\mu(G))\sum_E\int_0^1\cdots\int_0^1\prod_{H\neq G} t_H{}^{\Lambda(H)-\mu(H)/2-1}\frac{1}{[E(t)]^{(\sigma+1)/2}}\left[m^2\left(\sum\beta_l\right)-\sum\frac{p^iF_{ij}p^j}{E(t)}\right]^{\mu(G)}. \tag{A2}$$

C is a constant irrelevant to our discussion since we are just comparing the asymptotic behavior of $\mathcal{G}$ and $\Delta\mathcal{G}$.

Following Speer,[12] E is a maximal family of nonoverlapping subgraphs of a graph G, each two-connected or consisting of a single line such that no union of two or more disjoint elements of E is two-connected. Two connected means that the subgraphs cannot be disconnected by revolving any vertex. H are elements of E and $G\in E$;

$$\Lambda(H)=\sum_{l\in L(H)}(\lambda_l-1), \tag{A3}$$

where $L(H)$ is the set of lines of H.

$\mu(H)$ [$\mu(G)$] is the superficial degree of divergence of the subgraph [graph]. For nonderivative coupling scalar theory in $\sigma+1$ dimensions μ is given by

$$\mu(H)=\sigma+1-\tfrac{1}{2}(\sigma-1)B-\sum n_i[\sigma+1-(\sigma-1)\tfrac{1}{2}\nu_{B_i}]\,. \tag{A4}$$

B is the number of external boson lines and, if $\mathcal{L}_I=\sum\mathcal{L}_i$, n_i is the number of ith-type vertices of $\mathcal{L}_i$. $E(t)$ and F_{ij} are obtained from the Symanzik functions by taking in evidence the powers of t_G in the scaling process, and

$$\beta_l=t_G^{-1}\alpha_l\,.$$

In order to illustrate the whole procedure we discuss the ladder graphs with explicit result in fourth order, corresponding to the graph of Fig. 4(a). From now on, we discard all irrelevant constants.

For the propagator, we take the usual one (corresponding to $\lambda_i=1$). This requires some subtractions in order to get a finite expression for the vertex function. Using the procedure for the renormalized integrand, we get (omitting constants)

$$\Gamma^{(4)}(q,p,p',m^2)=\int_0^1\frac{dt_1}{t_1}\int_0^1 dt_2\cdots dt_5\left\{\frac{\ln[1-\sum p_iF^{\Gamma}_{ij}p_j/m^2(\sum\beta_l)E(t_i)]}{[E(t_i)]^3}-\frac{\ln[1-\sum p_iF^{\Delta}_{ij}p_j/m^2(\sum\beta_l)E_1(t_i)E_2(t_i)]}{[E_1(t_i)E_2(t_i)]^3}\right\}, \tag{A5a}$$

$$\Delta\Gamma^{(4)}(q,p,p',m^2)=\int_0^1\frac{dt_1}{t_1}\int_0^1 dt_2\cdots dt_5\left\{\frac{E^{-3}(t_i)}{1-\sum p_iF^{\Gamma}_{ij}p_j/m^2(\sum\beta_l)E(t_i)}-\frac{[E_1(t_i)E_2(t_i)]^{-3}}{1-\sum p_iF^{\Delta}_{ij}p_j/E_1(t_i)E_2(t_i)m^2(\sum\beta_l)}\right\}, \tag{A5b}$$

where (the arrows correspond to approximations which do not alter the asymptotic behavior)

$$E_1(t_i)=1+t_2+t_3\to 1,\quad E_2(t_i)=1+t_4+t_5\to 1,\quad E(t_i)=t_1(1+t_2)+(E_1)(E_2)\to 1\,,\ \sum\beta_l=E_2+t_1E_1\to 1\,, \tag{A6}$$

$$\sum p_iF^{\Delta}_{ij}p_j=p^2t_4+p'^2t_5+q^2t_4t_5\to q^2t_4t_5\,, \tag{A7}$$

$$\begin{aligned}\sum p_iF^{\Gamma}_{ij}p_j&=q^2(t_1^2t_2t_3+t_1\{t_2E_2+t_3[t_4+t_2(t_4+t_5)]\}+t_4t_5E_1)+p'^2(t_1t_3+t_5E_1)+p^2(t_1t_2t_3+t_4E_1)\\&\to q^2(t_1^2t_2t_3+t_1\{t_2+t_3[t_4+t_2(t_4+t_5)]\}+t_4t_5).\end{aligned} \tag{A8}$$

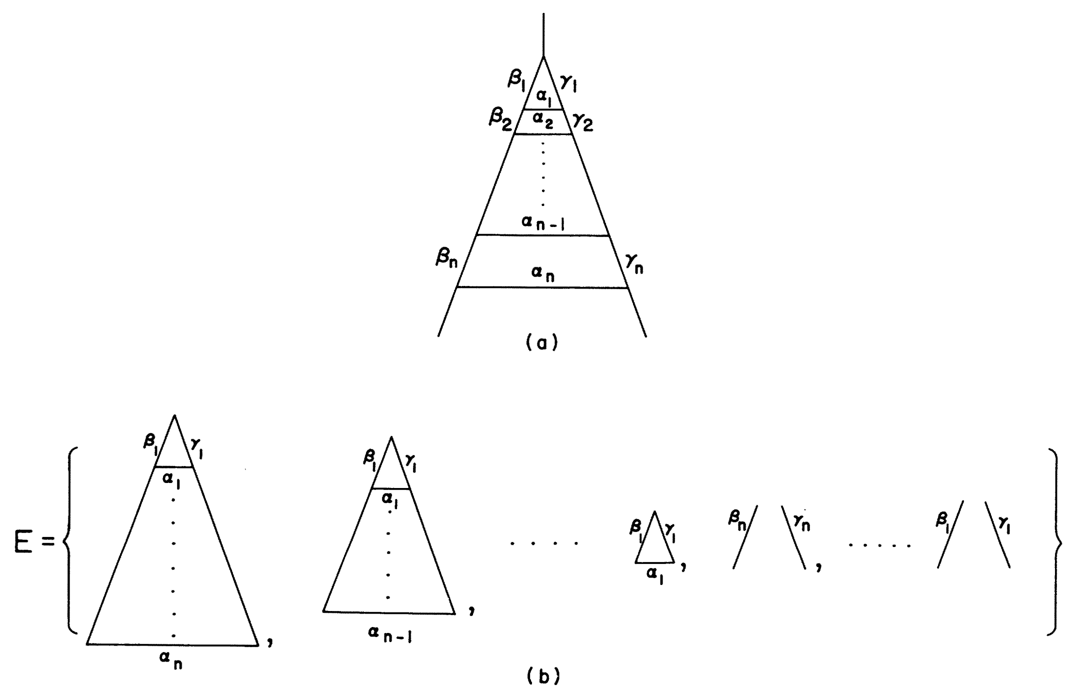

FIG. 5. (a) The ladder-graph contribution in the vertex function in $2n$th order. (b) One E family is explicitly exhibited.

The expressions in (A6)–(A8) are obtained from the scaling sketched in Fig. 4(b). Other scalings do not affect what we will say below. In the following we will just state our main results for all ladder graphs. All which we state below was verified explicitly in fourth and sixth order (including subtraction). We do not reproduce this laborious calculation in this paper in order to save space.

The results are: In order to analyze the behavior of leading contributions for $\Gamma^{(n)}$ and $\Delta\Gamma^{(n)}$ the subtraction terms do not alter the asymptotic behavior of these functions unless they are in the constants multiplying $[\ln(-q^2/m^2)]^n$ for $\Gamma^{(2n)}$ and $(1/q^2)\times[\ln(-q^2/m^2)]^N$ for $\Delta\Gamma^{(2n)}$.

For scalings corresponding to leading contributions [sketched in Fig. 4(b)], $\Gamma^{(2n)}(q,p,p',m^2)$ and $\Delta\Gamma^{(2n)}(q,p,p',m^2)$ can be written in the form

$$\Gamma^{(2n)}(q,p,p',m^2)=\int_0^1\frac{dt_1}{t_1}\frac{dt_2}{t_2}\cdots\frac{dt_{n-1}}{t_{n-1}}\int_0^1 dt_{\beta_1}\cdots dt_{\beta_n}\int_0^1 dt_{\gamma_1}\cdots dt_{\gamma_n}\left[\frac{1}{E^3(t,t_\beta,t_\gamma)}\ln\left(1-\frac{\sum p_iF^\Gamma_{ij}(t,t_\beta,t_\gamma)p_j}{m^2E(t,t_\beta,t_\gamma)}\right)+\text{subtraction terms}\right], \quad \text{(A9a)}$$

$$\Delta\Gamma^{(2n)}(q,p,p',m^2)=\int_0^1\frac{dt_1}{t_1}\frac{dt_2}{t_2}\cdots\frac{dt_{n-1}}{t_{n-1}}\int_0^1 dt_{\beta_1}\cdots dt_{\beta_n}\int_0^1 dt_{\gamma_1}\cdots dt_{\gamma_n}\left[\frac{E^{-3}(t,t_\beta,t_\gamma)}{1-\sum p_iF^\Gamma_{ij}p_j/m^2E(t,t_\beta,t_\gamma)}+\text{subtraction terms}\right]. \quad \text{(A9b)}$$

The subtraction terms are necessary in order to avoid the nonintegrable singularities when $t_i=0$. As in (A5), these terms can be obtained using the Zimmermann forest formula[13] for the integrand renormalized in the α-parameter space and then performing the scaling. As an example, look at expression (A5).

Again we use the approximations illustrated in (A6)–(A8), which for the explicit term shown in (A9b) correspond to $E(t)\to 1$ and taking the external masses equal to zero. With these approximations the general structure of the relevant integrand will be

$$1-\frac{\sum p_iF^\Gamma_{ij}(t,t_\beta,t_\gamma)p_j}{m^2E(t)}\to 1-\frac{q^2}{m^2}[P(t,t_\beta,t_\gamma)+M(t_\beta,t_\gamma)]. \quad \text{(A10)}$$

In (A10) $M(t_\beta,t_\gamma)$ is a monomial in the variables t_β,t_{γ_i} (does not involve the variables $t_1,t_2,\ldots,t_n$) (see Fig. 5) of degree 2 and $P(t,t_\beta,t_\gamma)$ is a polynomial in the variables t, t_β,t_{γ_i} of degree >2. These conclusions come from an analysis of the structure of two-trees and trees, in the language of Speer.[12] The relevant region of integration is the region where all variables of integrations are very small $[O(1/q^2)]$, so roughly speaking we can drop the polynomial as compared to the monomial. Furthermore, for the variables $t_1,t_2,\ldots,t_n$, owing to the good behavior at the origin, introduced by the renormalization, we can integrate only in the relevant region $[\lambda/q^2,1]$. These conclusions are illustrated in (A8), where the polynomial is $t_1^2t_2t_3+t_1\{t_2+t_3[t_u+t_2(t_4+t_5)]\}$ and the monomial is t_4t_5. The asymptotic behavior does not change by dropping the polynomial when taking the divergent term and regularizing with the cutoff λ/q^2.

So, for leading contributions in $2n$th order of perturbation theory we have

$$\Gamma^{(2n)}(q^2,m^2)\cong\int_{\lambda/(-q^2/m^2)}^1\frac{dt_1}{t_1}\int_{\lambda/(-q^2/m^2)}^1\frac{dt_2}{t_2}\cdots\int_{\lambda/(-q^2/m^2)}^1\frac{dt_{n-1}}{t_{n-1}}\int_0^1 dt_{\beta_1}\cdots dt_{\beta_n}\int_0^1 dt_{\gamma_1}\cdots dt_{\gamma_n}\ln\left(1-\frac{q^2}{m^2}t_{\beta_n}t_{\gamma_n}\right)\underset{q^2\to-\infty}{\sim}\left[\ln\left(\frac{-q^2}{m^2}\right)\right]^n,$$

$$\Delta\Gamma^{(2n)}(q^2,m^2)\cong\int_{\lambda/(-q^2/m^2)}^1\frac{dt_1}{t_1}\int_{\lambda/(-q^2/m^2)}^1\frac{dt_2}{t_2}\cdots\int_{\lambda/(-q^2/m^2)}^1\frac{dt_{n-1}}{t_{n-1}}\int_0^1 dt_{\beta_1}\cdots dt_{\beta_n}\int_0^1 dt_{\gamma_1}\cdots dt_{\gamma_n}\frac{1}{1-(q^2/m^2)t_{\beta_n}t_{\gamma_n}}\underset{q^2\to-\infty}{\sim}\frac{m^2}{q^2}\left[\ln\left(\frac{-q^2}{m^2}\right)\right]^{n+1}.$$

Now we will consider the nonleading contributions for fourth-order graphs. We consider again only some scalings. The situation does not change for other scalings. The approximations are the same as considered before.

For the graph of Fig. 6(a) and the E family of Fig. 6(b)

$$\Gamma^{(4)} \cong \int_0^1 dt_1 dt_2 \cdots dt_5 \ln\left(1 - \frac{q^2}{m^2} t_1[t_4 + t_5 + t_4(t_3 + t_1 t_2) + t_2 t_4(1 + t_1 t_5)]\right)$$

$$\underset{q^2 \to -\infty}{\sim} \ln\left(\frac{-q^2}{m^2}\right),$$

$$\Delta\Gamma^{(4)} \cong \int_0^1 dt_1 dt_2 \cdots dt_5 \left(1 - \frac{q^2}{m^2} t_1[t_4 + t_5 + t_4(t_3 + t_1 t_2) + t_2 t_4(1 + t_1 t_5)]\right)^{-1}$$

$$\underset{q^2 \to -\infty}{\sim} \frac{m^2}{q^2}\left[\ln\left(\frac{-q^2}{m^2}\right)\right]^2 .$$

The vertex correction of Fig. 7(a) with the scaling corresponding to the E family of Fig. 7(b) leads us to

$$\Gamma^{(4)}(q^2, m^2) \cong \int_0^1 \frac{dt_1}{t_1} dt_2 \cdots dt_5 \left[\ln\left(1 - \frac{q^2}{m^2}[t_4(1 + t_2 + t_3) + t_1 t_2]\right) - \ln\left(1 - \frac{q^2}{m^2} t_4\right)\right]$$

$$\underset{q^2 \to -\infty}{\sim} \ln\left(\frac{-q^2}{m^2}\right),$$

$$\Delta\Gamma^{(4)}(q^2, m^2) \cong \int_0^1 \frac{dt_1}{t_1} dt_2 \cdots dt_5 \left\{\frac{1}{1 - (q^2/m^2)[t_4(1 + t_2 + t_3) + t_1 t_2]} - \frac{1}{1 - (q^2/m^2)t_4}\right\}$$

$$\underset{q^2 \to -\infty}{\sim} \frac{m^2}{q^2}\left[\ln\left(\frac{-q^2}{m^2}\right)\right]^2,$$

while for the self-energy insertion of Fig. 8,

$$\Gamma^{(4)}(q, p, p', m^2) = \int_0^1 \frac{dt_1}{t_1^2} \int_0^1 dt_2 \cdots dt_5 \left\{\frac{1}{E^3} \ln\left(\frac{M}{E}\right) - \left[\frac{1}{E^3} \ln\left(\frac{M}{E}\right)\right]_{t_1=0} - t_1 \frac{d}{dt_1}\left[\frac{1}{E^3} \ln\left(\frac{M}{E}\right)\right]\bigg|_{t_1=0}\right\}$$

$$\underset{q^2 \to -\infty}{\sim} \ln\left(\frac{-q^2}{m^2}\right),$$

$$\Delta\Gamma^{(4)}(q, p, p', m^2) = m^2 \int_0^1 \frac{dt_1}{t_1^2} \int_0^1 dt_2 \cdots dt_5 \left[\frac{\sum \beta_l}{E^2 M} - \left(\frac{\sum \beta_l}{E^2 M}\right)_{t_1=0} - t_1 \frac{d}{dt_1}\left(\frac{\sum \beta_l}{E^2 M}\right)\bigg|_{t_1=0}\right]$$

$$\underset{q^2 \to -\infty}{\sim} \frac{1}{q^2} \ln\left(\frac{-q^2}{m^2}\right),$$

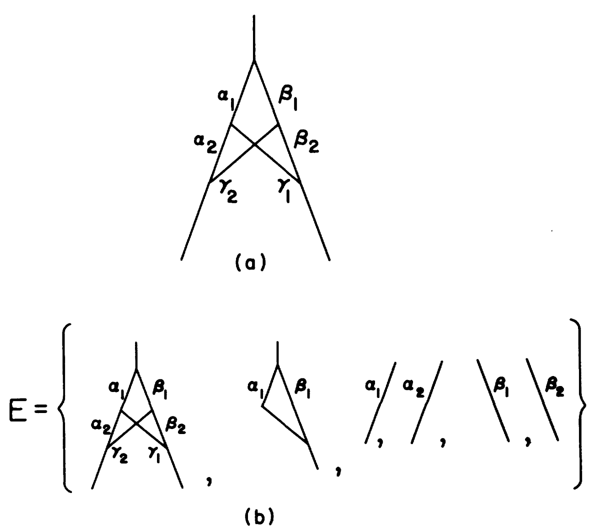

FIG. 6. (a) The crossed-ladder-graph contribution in fourth order. (b) One particular E family.

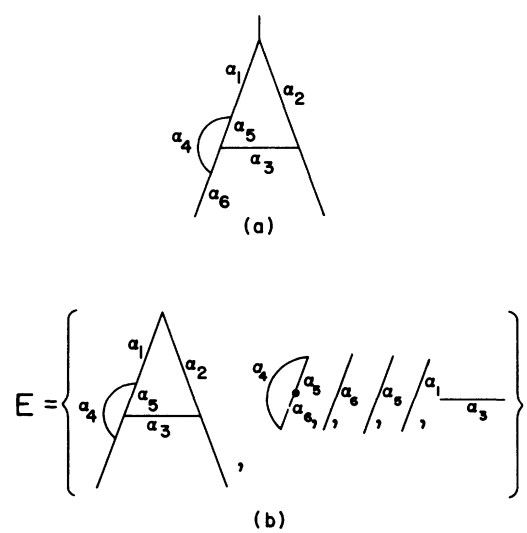

FIG. 7. (a) The fourth-order vertex correction to the vertex function. (b) One of the E families.

with

$$\sum\beta_i = 1+t_3+t_4+t_5+t_1(1+t_2),$$

$$E=(1+t_2)(1+t_3+t_4+t_5)+t_1t_2,$$

$$M=-m^2\left(\sum\beta_i\right)E$$
$$+p^2[t_3t_2(1+t_5)+t_3(1+t_5+t_1t_2)]+q^2t_3t_4(1+t_2)$$
$$+p'^2t_4[t_1t_2+(1+t_3)(1+t_2)].$$

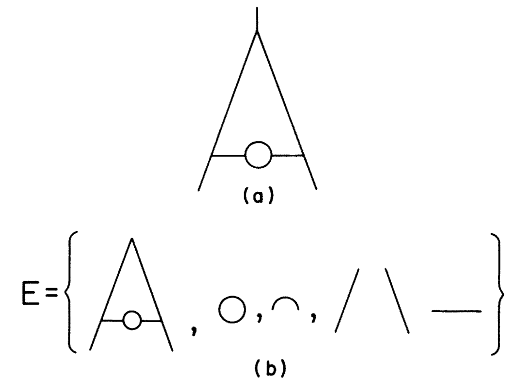

FIG. 8. (a) The fourth-order self-energy correction to the vertex function. (b) One E family is exhibited.

APPENDIX B

In this appendix we exhibit the expressions for the constants c_0 and $b_0-2c_0^{(1)}-c_0^{(2)}$, introduced in Sec. II. With respect to b_0 we refer to Ref. 16.

c_0 can be obtained directly from the complete Callan-Symanzik equation using the renormalization conditions. We get from the isospin pseudoscalar model [for the U(1) model the last terms, corresponding to the electromagnetic interaction of the pion, do not appear]

$$c_0\gamma^\mu=-\frac{i^3}{(2\pi)^4}\int d^4k\left(m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}\right)\left[\gamma_5\frac{1}{k\!\!\!/+m}\gamma_\mu\frac{1}{k\!\!\!/+m}\gamma_5\frac{1}{k^2-\mu^2}+2\gamma_5\frac{1}{k\!\!\!/-m}\gamma_5\frac{1}{k^2-\mu^2}(-2k_\mu)\frac{1}{k^2-\mu^2}\right]. \tag{B1}$$

Using (1.1), (1.2), integrating over the internal momentum k, and performing the scaling described in Appendix A it follows that

$$\begin{aligned}c_0=-\frac{i^3}{(2\pi)^4}\frac{\pi^2}{i}\Bigg[&m^2\int_0^1\frac{dt_2t_2}{(1+t_2)^2(m^2t_2+\mu^2)}+m^2\int_0^1\frac{dt_1}{(1+t_1)^2(m^2+\mu^2t_1)}-m^2\int_0^1\frac{dt_2}{(1+t_2)^3(m^2+\mu^2t_2)}\\&-m^2\int_0^1\frac{dt_1t_1^2}{(1+t_1)^3(\mu^2+m^2t_1)}+\mu^2\int_0^1\frac{dt_2t_2}{(1+t_2)^2(m^2+\mu^2t_2)}+\mu^2\int_0^1\frac{dt_1t_1}{(1+t_1)^2(\mu^2+m^2t_1)}\\&-\mu^2\int_0^1\frac{dt_2t_2}{(1+t_2)^3(m^2+\mu^2t_2)}-\mu^2\int_0^1\frac{dt_1t_1}{(1+t_1)^3(\mu^2+m^2t_1)}+m^2\int_0^1\frac{dt_2t_2}{(1+t_2)^3(\mu^2t_2+m^2)}\\&+m^2\int_0^1\frac{dt_1t_1}{(1+t_1)^3(\mu^2+m^2t_1)}+\mu^2\int_0^1\frac{dt_2}{(1+t_2)^3(\mu^2+m^2t_2)}+\mu^2\int_0^1\frac{dt_1t_1^2}{(1+t_1)^3(\mu^2t_1+m^2)}\Bigg].\end{aligned} \tag{B2}$$

From (B2) follows

$$c_0=\frac{1}{16\pi^2}\left(\tfrac{1}{2}+1\right), \tag{B3}$$

where the one-half comes from the first eight terms in (B2) or from the first term in square brackets in (B1) and the factor 1 comes from the electromagnetic interaction of the pion.

For SU(n) we have

$$(b_0-2c_0^{(1)}-c_0^{(2)})\mathrm{Tr}(\gamma^5\lambda^i\lambda^i\gamma^5)=\frac{i^3}{(2\pi)^4}\int d^4k\left(m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}\right)\mathrm{Tr}\left(\gamma^5\frac{1}{k\!\!\!/+m}\gamma^5\frac{1}{k\!\!\!/+m}\gamma^5\gamma^5\frac{1}{k^2-\mu^2}\lambda^j\lambda^i\lambda^j\lambda^i\right). \tag{B4}$$

In (B4) all repeated indices must be summed. After a trivial algebra (B4) can be written in the form

$$\begin{aligned}b_0-2c_0^{(1)}-c_0^{(2)}&=\left[\frac{-i^3}{(2\pi)^4}\int d^4k\left(m^2\frac{\partial}{\partial m^2}+\mu^2\frac{\partial}{\partial\mu^2}\right)\left(\frac{1}{(k^2-m^2)(k^2-\mu^2)}\right)\right]\frac{\mathrm{Tr}(\lambda^j\lambda^i\lambda^j\lambda^i)}{\mathrm{Tr}(\lambda^i\lambda^i)}\\&=\left(\frac{1}{16\pi^2}\right)S_1.\end{aligned} \tag{B5}$$

*Work supported in part by the National Science Foundation under Grant No. NSF 32418 and by Fundacaõ de Amparo a Pesquisa No Estado de São Paulo.

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$$\langle p|j_\mu(0)|p'\rangle = \overline{u}(p)\Lambda_\mu u(p') = \overline{u}(p)[\gamma_\mu F_1(q^2) + i\sigma_{\mu\nu}q^\nu F_2(q^2)]u(p').$$

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[19]Unless the Regge trajectory is restricted by the condition

$$-t\frac{\partial}{\partial t}\alpha(t/m^2) + \beta(g)\frac{\partial}{\partial g}\alpha(t/m^2) = 0,$$

in which case we need to analyze the nonleading contributions.

[20]There are two main shortcomings of this method for the analysis of the asymptotic behavior in perturbation theory: The number of "sectors" (E families) increases very fast with the order of the perturbation, and the method is not reliable when we are dealing with massless particles. This implied that we will have to face difficulties when dealing with higher-order graphs with exceptional momenta. The author thanks P. K. Mitter for this comment.