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Quasicanonical quantum field theory*

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We call a quantum-field-theory model quasicanonical if it is defined by canonical equal-time field commutation relations (e.g., $[\dot{\phi}(x),\ \phi(0)]\delta(x^0) = -i\delta^4(x)$) and local field equations [e.g., $\Box\phi(x) = \lambda J(x)$], and is locally invariant to scale transformations [e.g., $\phi(x) \to \rho\phi(\rho x)$]. [These requirements are *not* consistent if the model is *purely* canonical, i.e., if $J(x)$ is the simple Wick product $:\phi^3(x):$.] Canonical Bjorken scaling is valid in such models provided that the field equations are also locally invariant to R transformations $[\phi(x) \to \phi(x) + r]$ and the physical currents are R-invariant. We discuss here further properties and consequences of these models. (a) We incorporate positivity and R-invariance restrictions on light-cone expansions and deduce the form of the consequent bilocal operators [e.g., $\int da\, \sigma(a) :\phi(ax)\phi(0):$]. (b) We exhibit a Hamiltonian formulation of the theory, both in the massless and massive cases. (c) We show that the theory is locally conformal- and inversion- $[\phi(x) \to (x^2)^{-1}\phi(-x/x^2)]$ invariant. These symmetries are spontaneously broken. (d) We discuss the implications of the model for deep-inelastic electron-positron annihilation. Exact scaling is obtained. (e) We study the possible low-energy consequences of the (spontaneously broken) R symmetry. These include the PCAC (partial conservation of axial-vector current) consistency conditions and the Gell-Mann charge algebra. (f) We consider the arguments for and consequences of a spontaneous breakdown of the dilatation symmetry.

I. INTRODUCTION

Canonical quantum field theory is based on canonical field equations such as[1]

$$\Box\phi(x) = \lambda :\phi^3(x):, \tag{1.1}$$

and canonical equal-time commutation relations such as

$$[\dot{\phi}(x), \phi(0)]\delta(x^0) = -i\delta^4(x). \tag{1.2}$$

This framework is unfortunately inconsistent except in the free-field case $\lambda = 0$.[2] This is because (1.1) and (1.2) imply the short-distance (SD) behavior $:\phi(x)\phi(0): \to \lambda(\ln x^2) :\phi^2(0):$, which precludes the existence of the simple Wick product $:\phi^3(x):$ in (1.1). The conventional approach to this problem is to give up (1.1) and (1.2) and to define the theory by the renormalized perturbation expansion. Then the source term in (1.1) is replaced by a complicated limit which subtracts out the singularities and (1.2) must be abandoned entirely.[3] Although consistent and explicit, this framework has been useless when strong interactions are involved. In particular, it seems impossible to understand (exact or approximate) canonical Bjorken scaling in this way.[4]

We have suggested an alternative approach to the problem in Ref. 2.[5] We sought to find a formalism sufficiently more singular than the canonical framework (1.1), (1.2) to avoid the above-mentioned inconsistencies, but sufficiently less singular than renormalized perturbation theory to obtain canonical scaling. We maintain (1.2), but replace (1.1) by

$$\Box\phi(x)=\lambda J(x)\,, \tag{1.3}$$

with the source $J(x)$ initially unspecified except for the normalization condition

$$[\dot{J}(x),\,\phi(0)]\delta(x^0)=j(0)\delta^4(x)\,, \tag{1.4}$$

which also defines $j(x)$, and the requirement that the theory is scale-invariant. These postulates require that $j(x)$ is a member of a two-dimensional indecomposable[6] (i.e., reducible but not completely reducible) representation of the dilatation group $\mathfrak{D}$. This provides explicit expressions for $j(x)$ and $J(x)$ in terms of the basic field ϕ.[2] Logarithmic singularities are involved, but do not ruin exact scale invariance because of the presence of indecomposable multiplets. We shall refer to theories such as (1.2)–(1.4), based on canonical commutators but noncanonical field equations, as *quasicanonical* theories.

In the above model, the $j(x)j(0)$ light-cone (LC) expansion will have the general form

$$j(x)j(0)\underset{x^2\to 0}{\sim}\Delta_+{}^2(x)I+\Delta_+(x)\mathcal{B}(x,0)+\Delta_+(x)(\ln x^2)\tilde{\mathcal{B}}(x,0)\,, \tag{1.5}$$

where the bilocal operators $\mathcal{B}$ and $\tilde{\mathcal{B}}$ are analytic. The amplitude[7]

$$W(q^2,\nu)\equiv\int dx\,e^{iqx}\langle p|[j(x),j(0)]|p\rangle\,,\quad \nu=q\cdot p\,, \tag{1.6}$$

will therefore behave as

$$\nu W(q^2,\nu)\to F(\omega)+(\ln q^2)\tilde{F}(\omega) \tag{1.7}$$

in the Bjorken limit $\nu\to\infty$ with $\omega\equiv -q^2/2\nu$ fixed. To obtain exact scaling $[\tilde{F}(\omega)\equiv 0]$, we can exploit the fact that the source $J(x)$ in (1.3) has not yet been fully specified. We showed in Ref. 2 that if J is chosen to be invariant under the one-parameter group of field-shift or R transformations,

$$R:\;\phi(x)\to\phi(x)+r\,, \tag{1.8}$$

then exact canonical scaling,

$$\nu W(q^2,\nu)\to F(\omega)\,, \tag{1.9}$$

is obtained. What happens is that, with (1.2)–(1.4) R-invariant, the operator-product expansions must be R-invariant (although the vacuum is *not* R-invariant) and this decouples the logarithms from relations among the (observable) R-invariant currents such as $j(x)$.[2]

To satisfy the above conditions, J must be a member of a three-dimensional indecomposable dilatation representation. The resultant quasicanonical theory has many interesting features: canonical commutation relations, simple field equations, scale invariance, R invariance, and Bjorken scaling. The same procedure can be applied to obtain a quasicanonical version of the gluon model.[2] To avoid unnecessary complications, in most of the paper the scalar theory will be dealt with. Our purpose will be to determine further properties and explore further consequences of the model.

We begin in Sec. II with a review of the quasicanonical R-invariant scalar theory and establish some notations. Light-cone expansions in the model are treated in Sec. III. In Sec. III A the restrictions of positivity are discussed, in Sec. III B the restrictions of R invariance implemented, and in Sec. III C simple forms for the bilocal operators, such as

$$\mathcal{B}(x,0)=\int da\,\sigma(a):\phi(ax)\phi(0):\,, \tag{1.10}$$

are deduced. It is shown in Sec. IV that the model can be derived from a Hamiltonian $\mathcal{H}$, which we construct. It is also shown there that if a mass term is added, it must be the R-variant term $m^2k(x)$ and not the R-invariant term $m^2j(x)$. In Sec. V we show that the model is conformal-invariant. The fields which mix under scale transformations are seen to also mix under inversion and special conformal transformations. The conformal symmetry must, however, be spontaneously broken. The implications of the model for e^+e^- annihilation are studied in Sec. VI. Using the methods developed in Ref. 8, we discuss scaling, the asymptotic behavior of the scaling functions, and the multiplicities of the produced hadrons. In Sec. VII we comment on the possible low-energy consequences of R invariance. These include all the usual consequences of partial conservation of axial-vector current (PCAC) and the Gell-Mann charge algebra. Finally, in Sec. VIII, arguments for and consequences of a spontaneous breakdown of scale invariance are discussed.

II. REVIEW OF QUASICANONICAL THEORIES

We briefly review the short-distance behavior of quasicanonical theories. For simplicity we first discuss the case where there is only a single scalar elementary field $\phi(x)$ interacting through $\lambda\phi^4$. The usual quantization of the theory leads to the canonical equal-time commutation relation

$$[\dot{\phi}(x), \phi(0)]\delta(x^0) = -i\delta^4(x). \quad (2.1)$$

Canonical scaling in nature implies canonical behavior for products of currents near the LC, and, in particular, at SD. It was thought that a simple way to achieve this behavior for $j(x) \sim :\phi^2(x):$ is to demand that (2.1) be satisfied in nature. This naive framework fails however except for free ($\lambda = 0$) theories. Using the naive field equation

$$\Box\phi(x) = \lambda :\phi^3(x):, \quad (2.2)$$

the equal-time commutator $[\partial_0{}^3\phi(x), \phi(0)]\delta(x^0)$ is easily computed to be

$$[\partial_0{}^3\phi(x), \phi(0)]\delta(x^0) = -3i\lambda :\phi^2(0): \delta^4(x), \quad (2.3)$$

which in turn implies the SD operator-product expansion (OPE)

$$\phi(x)\phi(0) \underset{x\to 0}{\sim} \cdots - 3i\lambda \frac{\ln x^2}{16\pi^2} :\phi^2(0): + \cdots. \quad (2.4)$$

Thus the quantity

$$\lim_{x\to 0}[\phi(x)\phi(0) - \langle 0|\phi(x)\phi(0)|0\rangle] \quad (2.5)$$

does not exist (problem A), and the theory is moreover not irreducibly scale-invariant even in the absence of masses (problem B). A solution to problem B was offered by Dell'Antonio,[6] who observed that scale invariance can be implemented equally well in the presence of logarithms,[9] provided that one assumes that composite operators transform reducibly under the dilatation group $\mathfrak{D}$. We refer to this generalized concept as reducible scale invariance (RSI). With one power of the logarithm, a two-dimensional reducible representation is sufficient to restore RSI. We call the partners in the representation $j(x)$ and $k(x)$, and the transformation property is

$$U_\rho \begin{bmatrix} j(x) \\ k(x) \end{bmatrix} U_\rho{}^{-1} = \rho^2 \begin{bmatrix} j(\rho x) \\ \ln\rho\, j(\rho x) + k(\rho x) \end{bmatrix}, \quad (2.6)$$

where U_ρ effects the scale transformations.[10] The restrictions of RSI produce a $\phi\phi$ SD OPE of the form

$$\phi(x)\phi(0) \underset{x\to 0}{\sim} (\lambda_1 \ln x^2 + \lambda_2) j(0) + \lambda_1 k(0). \quad (2.7)$$

It is now possible from (2.7) to define j and k in a consistent manner:

$$j(0) = \lim_{x\to 0} \frac{:\phi(x)\phi(0):}{\lambda_1 \ln x^2 + \lambda_2}, \quad (2.8)$$

$$k(0) = \lim_{x\to 0} \lambda_1{}^{-1}[:\phi(x)\phi(0): - (\lambda_1 \ln x^2 + \lambda_2) j(0)], \quad (2.9)$$

so that problem A is also resolved. Equation (2.7) now no longer follows from the naive field equation (2.2). Assuming the validity of (2.7), we can produce a composite operator $J(x)$, intuitively like $:\phi^3(x):$, from the $j(x)\phi(0)$ expansion. A *finite* field equation[2,3] can now be written down:

$$\Box\phi(x) = \lambda J(x). \quad (2.10)$$

Application of RSI also enables us to deduce the singularity structure of OPE's like $j(x)j(0)$, $J(x)\phi(0)$, etc. For example,

$$\begin{bmatrix} j(x)j(0) \\ j(x)k(0) \\ k(x)k(0) \end{bmatrix} \underset{x\to 0}{\sim} \frac{1}{x^2} \begin{bmatrix} -b_1 \ln x^2 + a_1 \\ -b_1 \ln^2 x^2 + (a_1 - b_2)\ln x^2 + a_2 \\ -b_1 \ln^3 x^2 + (a_1 - 2b_2)\ln^2 x^2 + (2a_2 - b_3)\ln x^2 + a_3 \end{bmatrix} j(0) + \frac{1}{x^2} \begin{bmatrix} b_1 \\ b_1 \ln x^2 + b_2 \\ b_1 \ln^2 x^2 + 2b_2 \ln x^2 + b_3 \end{bmatrix} k(0). \quad (2.11)$$

If $b_1 \neq 0$, canonical scaling is violated by one power of logarithm; conversely, exact scaling requires $b_1 = 0$.

The condition is automatically satisfied if the symmetry of field-shift invariance (R invariance) governs SD OPE's. We define

$$R:\ \phi(x) \to \phi(x) + r, \quad (2.12)$$

where r is a constant. It is easy to deduce from (2.8) and (2.9) the transformation properties of j and k under R:

$$\delta_R j(0) = 0, \quad (2.13)$$

$$\delta_R k(0) = \lambda_1{}^{-1}[2r\phi(0) + r^2]. \quad (2.14)$$

Application of R invariance to (2.11) immediately gives

$$b_1 = b_2 = 0, \quad (2.15)$$

which decouples the $\ln x^2$ term in $j(0)$, giving the jj expansion a canonical structure

$$j(x)j(0) \underset{x\to 0}{\sim} \frac{a_1}{x^2} j(0). \tag{2.16}$$

The field equation (2.10) is R-invariant provided

$$\delta_R J(x) = 0. \tag{2.17}$$

To implement R invariance, it is actually necessary to have $J(x)$ belong to a three-dimensional reducible representation of $\mathfrak{D}$: (J, K, L). $J(x)$ will *not* be R-invariant with a two-dimensional representation. With this choice of $J(x)$, (2.10) is now R-invariant. Specifically, the R-invariant expression for $J(x)$ is

$$J(x) = \lim_{y\to x} \frac{j(x)\phi(y)}{-\alpha_2 \ln(x-y)^2 + \alpha_4}, \tag{2.18}$$

and the field equation, together with the R-invariant constraint,

$$[J, \dot\phi]_{\rm ET} \propto j, \tag{2.19}$$

can now be used to *derive* the expansion (2.7) in a way completely consistent with RSI and with R invariance. The above scheme then constitutes a quasicanonical $\lambda\phi^4$ quantum field theory.

For later convenience we introduce here the notation $\{\phi_i^n(x) | i = 1, \ldots, m\}$ to denote the m-dimensional multiplet of composite fields which replace in our scheme the ill-defined object $:\phi^n(x):$. Thus, $j = \phi_1^2$, $k = \phi_2^2$; $J = \phi_1^3$, $K = \phi_2^3$, $L = \phi_3^3$.

Mass terms are always present in the real world, and they always break R invariance (see Sec. IV). It is, however, well known that symmetries broken by mass terms are restored in SD OPE's.[3,11]

Quasicanonical theories are just as easily constructed for fields with spacetime and/or internal indices. In particular, the quark-gluon model was so treated in Ref. 2.

III. LIGHT-CONE EXPANSIONS

A. Positivity restrictions

In our scalar model, the structure function ($\kappa \equiv q^2$)

$$W(\omega, \kappa) \equiv \int dx\, e^{iqx} \langle p | [j(x), j(0)] | p \rangle \tag{3.1}$$

will satisfy Bjorken (canonical) scaling,

$$F(\omega, \kappa) \equiv \nu W(\omega, \kappa) \underset{\kappa\to\infty}{\longrightarrow} F(\omega), \tag{3.2}$$

provided the LC OPE has the canonical form[11]

$$j(x)j(0) \underset{x^2\to 0}{\sim} \frac{1}{x^2} \sum_{n=0}^{\infty} x^{\alpha_1}\cdots x^{\alpha_n} \mathcal{O}_{\alpha_1\cdots\alpha_n}(0). \tag{3.3}$$

We showed in Ref. 2, and reviewed in Sec. II, that the SD limit of the LC expansion has the canonical form

$$j(x)j(0) \underset{x\to 0}{\sim} \frac{1}{x^2} \mathcal{O}(0). \tag{3.4}$$

In this subsection, we will show how, in our model, (3.4) and the positivity of (3.1) lead essentially to (3.3).

The most general possible LC OPE for the T product has the form[12]

$$T[j(x)j(0)] \underset{x^2\to 0}{\sim} \frac{1}{x^2} \sum_n G_n(x^2) x^{\alpha_1}\cdots x^{\alpha_n} \mathcal{O}_{\alpha_1\cdots\alpha_n}(0). \tag{3.5}$$

In our model, we must have

$$G_n(x^2) = g_n (\ln x^2)^{a_n}, \tag{3.6}$$

but we shall work with arbitrary G_n's for generality. It follows from (3.5) that the amplitude

$$T(\omega, \kappa) \equiv i \int dx\, e^{iqx} \langle p | T[j(x)j(0)] | p \rangle \tag{3.7}$$

satisfies

$$\nu T(\omega, \kappa) \underset{\kappa\to\infty}{\sim} i \sum_n c_n (1/\omega)^{n+1} \tilde G_n(\kappa) \tag{3.8}$$

in the region $\omega > 1$ of convergence, where

$$\langle p | \mathcal{O}_{\alpha_1\cdots\alpha_n}(0) | p \rangle = c_n p_{\alpha_1}\cdots p_{\alpha_n} + \cdots, \tag{3.9}$$

and

$$\tilde G_n(\kappa) \equiv -\frac{\kappa}{2} \left(i\kappa \frac{\partial}{\partial\kappa} \right)^n \int dx\, e^{iqx} \frac{1}{x^2} G_n(x^2). \tag{3.10}$$

Therefore,

$$\int_0^1 d\omega\, \omega^n F(\omega, \kappa) = 2\pi c_n \tilde G_n(\kappa). \tag{3.11}$$

The positivity of $F(\omega, \kappa)$ now gives the inequalities

$$c_0 \tilde G_0(\kappa) \geq c_1 \tilde G_1(\kappa) \geq \cdots \geq c_n \tilde G_n(\kappa) \geq c_{n+1} \tilde G_{n+1}(\kappa) \geq \cdots. \tag{3.12}$$

Therefore, if one moment of $F(\omega, \kappa)$ scales, so do all the higher moments. Now, (3.4) gives $G_0(x^2) = \text{const}$, and so $\tilde G_0(\kappa) = \text{const}$. Equation (3.12) then requires that each $c_n \tilde G_n(\kappa) \to \text{const}$, and $G_n(x^2) \to \text{const}$ for each n such that $c_n \neq 0$. We can conclude that (3.3) is valid at least between the states of interest. This means that each a_n in (3.6) effectively vanishes. Thus, in the present context, (3.2) follows just from (3.4).

More generally, ignoring the existence of the LC OPE (3.5), the positivity of $F(\omega, \kappa)$ still implies that the moments (3.11) satisfy the inequalities (3.12) and so, if (3.4) is valid so that

$$c_0 \tilde G_0(\kappa) \underset{\kappa\to\infty}{\longrightarrow} \text{const},$$

each $c_n \tilde{G}_n(\kappa)$ must have a finite limit for $\kappa \to \infty$; i.e., each moment (3.11) must scale. Although this is clearly *physically* equivalent to the scaling (3.2) of $F(\omega, \kappa)$, it is not mathematically equivalent to it. It is easy to construct positive functions $F(\omega, \kappa)$, each of whose moments scale but which do not themselves scale. Such functions are not, of course, of physical interest and do not correspond to OPE's like (3.5).

Let us next consider the LC behavior of the field product $\phi(x)\phi(0)$. The general LC OPE,

$$\phi(x)\phi(0) \underset{x^2\to 0}{\sim} \sum_n H_n(x^2) x^{\alpha_1}\cdots x^{\alpha_n} \mathcal{P}_{\alpha_1\cdots\alpha_n}(0) + \cdots, \tag{3.13}$$

$$H_n(x^2) = h_n (\ln x^2)^{b_n}, \tag{3.14}$$

is required by the SD behavior,

$$\phi(x)\phi(0) \underset{x\to 0}{\sim} (\ln x^2)\mathcal{P}(0) + \cdots, \tag{3.15}$$

and positivity to have the form

$$\phi(x)\phi(0) \underset{x^2\to 0}{\sim} (\ln x^2)\sum_n x^{\alpha_1}\cdots x^{\alpha_n} \mathcal{P}_{\alpha_1\cdots\alpha_n}(0) + \cdots. \tag{3.16}$$

B. *R* invariance

Since j is R-invariant, each $\mathcal{O}_{\alpha_1\cdots\alpha_n}$ occurring in (3.3) must also be R-invariant. In this subsection, we shall show how this can be achieved in the simplest possible way. The construction will also guarantee that the canonical structure (3.3) occurs.

We have noticed that for the multiplet $\{\phi_i{}^n(x) | i = 1, \ldots, m\}$, it is necessary to have $m \geq n$ in order that *at least one* of the operators in the set (defined to be $i = 1$) is R-invariant: $\delta_R \phi_1{}^n(x) = 0$. The most esthetically pleasing situation would prevail if $m = n$, in which case there would be *exactly one* R-invariant operator in the multiplet.

It turns out that if an analogous condition holds for all the operators $\mathcal{O}_{\mu_1\cdots\mu_n}(x)$ which occur in the $j(x)j(0)$ LC expansion, then the entire expansion is free from logarithms provided the SD limit (3.4) is. We explain: The $\mathcal{O}$'s carry two more indices: l (the level) $\equiv$ dimension $-$ spin,[13] and i, which labels the operators occurring in a representation of $\mathfrak{D}$ of dimension m. The condition is that for a given level l, exactly one element of the set $\{\mathcal{O}^{[l]}_{\mu_1\cdots\mu_n,i} | i = 1, \ldots, m\}$, defined to be $\mathcal{O}^{[l]}_{\mu_1\cdots\mu_n,1}(x)$, be R-invariant. We refer to the above situation as the *leader condition*, and the unique $i = 1$ operator as the *leader* of the multiplet. In the expansion of two R-invariant operators $j(x)j(0)$, the set $\mathcal{O}^{[l]}_{\mu_1\cdots\mu_n,i}(x)$ must occur in a way prescribed by invariance under $\mathfrak{D}$. For example, for $m = 3$,

$$j(x)j(0) \underset{x^2\to 0}{\sim} E(x)\{[\tfrac{1}{2}c_1(\ln x^2)^2 - c_2 \ln x^2 + c_3]\mathcal{O}_1(0) + (-c_1 \ln x^2 + c_2)\mathcal{O}_2(0) + c_1\mathcal{O}_3(0)\}, \tag{3.17}$$

where Lorentz indices on the power-behaved singular function $E(x)$ and $\mathcal{O}_i$ have been suppressed. Thus, $\delta_R\mathcal{O}_1 = 0$, but $\delta_R\mathcal{O}_2 \neq 0$, $\delta_R\mathcal{O}_3 \neq 0$. Application of R invariance on (3.17) gives at once

$$c_1 = c_2 = 0. \tag{3.18}$$

Then (3.17) becomes

$$j(x)j(0) \underset{x^2\to 0}{\sim} E(x)c_3\mathcal{O}_1(0), \tag{3.19}$$

and is purely power-behaved. Since $\delta_R\mathcal{O}_1 = 0$, (3.19) is R- invariant. On the other hand, if there were another R-invariant operator, say, $\delta_R\mathcal{O}_1 = \delta_R\mathcal{O}_2 = 0$, then $\mathcal{O}_2$ might occur with $c_2 \neq 0$, and the expansion would be

$$j(x)j(0) \underset{x^2\to 0}{\sim} E(x)[(-c_2 \ln x^2 + c_3)\mathcal{O}_1(0) + c_2\mathcal{O}_2(0)], \tag{3.20}$$

and scaling is violated. The feature in (3.17) which enables this argument to proceed is not changed for any m, provided that the leader condition holds. We have nothing to offer on the necessity of the leader condition for canonical scaling. In the absence of the condition, certain numerical coefficients would have to vanish by accident to achieve freedom from logarithms.

In summary, the existence of OPE's on the whole LC is a consequence of the leader condition for R-invariant theories. It is the simplest generalization of the "$m = n$ rule" previously proposed.[2] It will serve as another constraint in constructing solutions to $\mathfrak{D}$- and R-invariant dynamical systems.

C. Bilocal operators

It follows from the leader condition that

$$j(x)j(0) \underset{x^2\to 0}{\sim} \frac{1}{x^2}\sum_{n=0}^{\infty} x^{\mu_1}\cdots x^{\mu_n}\mathcal{O}^{[2]}_{\mu_1\cdots\mu_n,1}(0) \equiv \frac{1}{x^2}\mathcal{C}(x;0), \tag{3.21}$$

where $\mathcal{O}^{[2]}_{\mu_1\cdots\mu_n,1}$ are all leaders with level two. In the old naive canonical approach,

$$j(x)j(0) \underset{x^2\to 0}{\sim} \Delta_+(x) : \phi(x)\phi(0) :, \tag{3.22}$$

where $:\phi(x)\phi(0):$ is a bilocal operator. In quasicanonical theories, (3.22) certainly cannot hold. Already at SD

$$:\phi(x)\phi(0): \underset{x\to 0}{\sim} (\alpha_0 \ln x^2 + \beta_0)\phi_1{}^2(0) + \alpha_0\phi_2{}^2(0), \tag{3.23}$$

and so, as we saw in Sec. III A, by positivity $:\phi(x)\phi(0):$ has a logarithmic singularity on the whole LC. In deriving the LC OPE, one needs the SD OPE's of the derivatives of $\phi(x)\phi(0)$, and they must take the form dictated by RSI:

$$:\partial_{\mu_1}\phi(x)\phi(0): \underset{x\to 0}{\sim} \partial_{\mu_1}[(\alpha_0 \ln x^2+\beta_0)\phi_1{}^2(0)+\alpha_0\phi_2{}^2(0)] + [(\alpha_1 \ln x^2+\beta_1)\mathcal{O}_{\mu_1,1}(0)+\alpha_1\mathcal{O}_{\mu_1,2}(0)], \tag{3.24}$$

$$:\partial_{\mu_1}\partial_{\mu_2}\phi(x)\phi(0): \underset{x\to 0}{\sim} \partial_{\mu_1}\partial_{\mu_2}[(\alpha_0 \ln x^2+\beta_0)\phi_1{}^2(0)+\alpha_0\phi_2{}^2(0)] + \partial_{\mu_2}[(\alpha_1 \ln x^2+\beta_1)\mathcal{O}_{\mu_1,1}(0)+\alpha_1\mathcal{O}_{\mu_1,2}(0)]$$
$$+[(\alpha_2 \ln x^2+\beta_2)\mathcal{O}_{\mu_1\mu_2,1}(0)+\alpha_2\mathcal{O}_{\mu_1\mu_2,2}(0)], \tag{3.25}$$

and so on. From the general form

$$:\phi(x)\phi(0): \underset{x^2\to 0}{\sim} (\ln x^2)\mathfrak{F}(x,0)+\mathcal{G}(x,0), \tag{3.26}$$

we deduce

$$\lim_{y\to 0}\lim_{x^2\to 0}\frac{:\phi(x+y)\phi(0):}{\ln(x+y)^2} = \mathfrak{F}(x,0); \tag{3.27}$$

on the other hand,

$$\lim_{y\to 0}\lim_{x^2\to 0}\frac{:\phi(x+y)\phi(0):}{\ln(x+y)^2} = \lim_{y\to 0}\lim_{x^2\to 0}\sum_{n=0}^{\infty}\frac{1}{n!}[(y\cdot\partial)^n\phi(x)]\phi(0)\frac{1}{\ln(x+y)^2}. \tag{3.28}$$

Substituting (3.23), (3.24), (3.25), etc. for the behavior of $\partial_{\mu_1}\cdots\partial_{\mu_n}\phi(x)\phi(0)$, we obtain

$$\mathfrak{F}(x,0)=2\sum_{n=0}^{\infty}\frac{1}{n!}y^{\mu_1}\cdots y^{\mu_n}\alpha_n\mathcal{O}_{\mu_1\cdots\mu_n,1}(0). \tag{3.29}$$

The bilocal $\mathfrak{F}(x,0)$ as defined in (3.26) is thus composed solely of sums of leaders, and is thus R-invariant.

However, the bilocal $\mathcal{C}(x,0)$ occurring in the $j(x)j(0)$ expansion (3.21) is in general not identical with $\mathfrak{F}(x,0)$. R invariance only requires that it be a sum of leader operators $\mathcal{O}_{\mu_1\cdots\mu_n,1}(0)$. It is therefore useful to introduce a general form of the bilocal operator adequate even for such situations.

By a simple change of normalizations, we can write the LC OPE as

$$:\phi(x)\phi(0): \underset{x^2\to 0}{\sim} (\ln x^2)\sum_{n=0}^{\infty}x^{\alpha_1}\cdots x^{\alpha_n}\mathcal{O}^{[2]}_{\alpha_1\cdots\alpha_n,1}(0) + \sum_{n=0}^{\infty}x^{\alpha_1}\cdots x^{\alpha_n}\mathcal{O}^{[2]}_{\alpha_1\cdots\alpha_n,2}(0). \tag{3.30}$$

The irreducibility of $\phi(x)$ requires that the local operators occurring in the $j(x)j(0)$ LC OPE are taken from the set of R-invariant operators which can occur in (3.30). Thus,

$$:j(x)j(0): \underset{x^2\to 0}{\sim} \Delta_+(x)\sum_{n=0}^{\infty}c_n x^{\alpha_1}\cdots x^{\alpha_n}\mathcal{O}^{[2]}_{\alpha_1\cdots\alpha_n,1}(0), \tag{3.31}$$

for some constants c_n, $n=0,2,4,\ldots$.

To write (3.30) in another way, we introduce a function $\tau(a)$ which satisfies the conditions

$$\int da\,\tau(a)a^n=0,\quad n=0,2,4,\ldots, \tag{3.32a}$$

$$\int da\,\tau(a)(\ln a^2)a^n=1,\quad n=0,2,4,\ldots . \tag{3.32b}$$

Then we have

$$:\phi(x)\phi(0): \underset{x^2\to 0}{\sim} (\ln x^2)\int da\,\tau(a):\phi(ax)\phi(0):. \tag{3.33}$$

Similarly, introduction of a function $\sigma(a)$ which satisfies the conditions

$$\int da\,\sigma(a)a^n=0,\quad n=0,2,4,\ldots, \tag{3.34a}$$

$$\int da\,\sigma(a)(\ln a^2)a^n=c_n,\quad n=0,2,4,\ldots, \tag{3.34b}$$

enables us to write (3.31) as

$$:j(x)j(0): \underset{x^2\to 0}{\sim} \Delta_+(x)\int da\,\sigma(a):\phi(ax)\phi(0):. \tag{3.35}$$

We shall not discuss the existence or uniqueness of $\sigma(a)$ in detail. We have had no difficulty in explicitly constructing it in all cases of interest. A few general remarks are, however, in order. We define the function

$$A(n)=\int da\,\sigma(a)a^n \tag{3.36}$$

for all complex n for which the integral exists or by analytic continuation. By (3.34a), we have

$$A(n)=0,\quad n=0,2,4,\ldots, \tag{3.37a}$$

and by (3.34b), we have

$$A'(n)=\tfrac{1}{2}c_n\,,\quad n=0,2,4,\ldots\,. \tag{3.37b}$$

Now suppose that $\sigma(a)$ has support in $[0,\infty)$ and that $A(n)$ is sufficiently well behaved so that the inverse Mellin transform,

$$\sigma(a)=\frac{1}{2\pi i}\int_{N-i\infty}^{N+i\infty}dn\,A(n)a^{-n}\,, \tag{3.38}$$

exists. This can then be used to determine $\sigma(a)$. In practice, one is given c_n for complex n. Then, if $(\partial/\partial n)c_n$ is regular at the integers, we can take $A(n)=B(n)c_n$, where $B(n)$ is zero for integer n and $B'(n)$ is unity for integer n, like $(2\pi)^{-1}\sin 2\pi n$. $B(n)$ must also be such that the integral in (3.38) exists.

Another method for constructing $\sigma(a)$ is to take a sequence $\sigma_k(a)$, $k=1,2,3,\ldots$, of functions, each of which satisfies (3.34a), and take

$$\sigma(a)=\sum_{k=1}^{\infty}d_k\sigma_k(a)\,, \tag{3.39}$$

with the coefficients d_k chosen to satisfy (3.34b). An example is

$$\sigma_k(a)=\exp[-a^{1/k}\cos(\pi/k)]\sin[a^{1/k}\sin(\pi/k)]\,,\quad k>2\,. \tag{3.40}$$

For illustration, we consider the class of examples

$$\begin{aligned}A^{(b,m)}(n)&=\int_0^{\infty}da\,a^n\sigma^{(b,m)}(a)\\&=2\Gamma(2n+8m)(\sqrt{2}\,b)^{-2(n+4m)}(\sin\tfrac{1}{2}n\pi)\,,\\ \sigma^{(b,m)}(a)&=a^{4m-1}e^{-b\sqrt{a}}\sin b(\sqrt{a}\,)\,.\end{aligned} \tag{3.41}$$

These satisfy

$$A^{(b,m)}(n)=0\quad\text{for } n=-4m,-4m+2,\ldots,-2,0,2,4,\ldots\,, \tag{3.42}$$

and

$$\frac{\partial}{\partial n}A^{(b,m)}(n)=\pi\Gamma(2n+8m)(\sqrt{2}\,b)^{-2(n+4m)}\quad\text{for } n=-4m,-4m+2,\ldots,-2,0,2,4,\ldots\,. \tag{3.43}$$

Related examples can be constructed by, e.g., differentiating with respect to b.

IV. HAMILTONIAN FORMALISM

To define a complete quantum field theory, it would be necessary to construct the appropriate energy-momentum tensor. In particular, the equation of motion of the quantized theory can be obtained by commuting with Θ^{00}, the Hamiltonian density, in accordance with the Heisenberg equation

$$[\mathcal{O}(x),H]=i\partial_0\mathcal{O}(x)\,. \tag{4.1}$$

We define the kinetic-energy term from the OPE

$$\begin{aligned}\partial^{\mu_1}\phi(x)\partial^{\mu_2}\phi(0)\underset{x\to0}{\sim}&(a\ln x^2+b)K_1^{\mu_1\mu_2}(0)\\&+aK_2^{\mu_1\mu_2}(0)\,,\end{aligned} \tag{4.2}$$

where $(K_1^{\mu_1\mu_2},K_2^{\mu_1\mu_2})$ as usual is a two-dimensional multiplet under $\mathfrak{D}$. The normalization of $K^{\mu_1\mu_2}$ is fixed by considering the OPE

$$\begin{bmatrix}K_1^{\mu_1\mu_2}(x)\phi(0)\\K_2^{\mu_1\mu_2}(x)\phi(0)\end{bmatrix}\underset{x\to0}{\sim}\left(\partial^{\mu_1}\frac{1}{x^2}\right)\begin{bmatrix}c_1\\c_1\ln x^2+c_2\end{bmatrix}\partial^{\mu_2}\phi(0)+\cdots\,, \tag{4.3}$$

which gives the equal-time commutator

$$[K_{1\mu}^{\mu}(x),\phi(0)]\delta(x^0)=4\pi^2c_1i\delta^4(x)\dot{\phi}(0)\,. \tag{4.4}$$

Thus, the kinetic term ($=\tfrac{1}{2}$"$\partial_\mu\phi\partial^\mu\phi$") is just given by the R-invariant composite operator

$$E_K=\frac{1}{2}\left(\frac{-1}{2\pi^2c_1}\right)K_{1\mu}^{\mu}(x)\,. \tag{4.5}$$

For the example of a ϕ^4 theory, the interaction term is given by the composite operator $(\lambda/4!)\times\phi_1^4(x)$. Using the Heisenberg equation and the fact that

$$[\phi_1^4(x),\dot{\phi}(0)]\delta(x^0)\sim\delta^4(x)\phi_1^3(0)\,, \tag{4.6}$$

one easily obtains the R-invariant equation of motion

$$\Box\phi(x)=-\frac{\lambda}{3!}\phi_1^3(x)\,. \tag{4.7}$$

The Hamiltonian that generates this equation is

$$\mathcal{H}=\frac{1}{2}\left(\frac{-1}{2\pi^2c_1}\right)K_{1\mu}^{\mu}(x)+\frac{\lambda}{4!}\phi_1^4(x)\,, \tag{4.8}$$

and is itself R-invariant. This is a desirable state of affairs, since the Hamiltonian is an observable quantity.

In more realistic theories, mass terms are always present, and they break scale invariance. A consistent incorporation of mass terms in quasicanonical theories turns out to *require* that they also break R invariance. A mass term for scalar theories would correspond to either ϕ_1^2 (R-invariant) or ϕ_2^2 (R-variant). Suppose the mass term is R-invariant; then we have the OPE

$$\begin{bmatrix}\phi_1^2(x)\phi(0)\\\phi_2^2(x)\phi(0)\end{bmatrix}\underset{x\to0}{\sim}\frac{1}{x^2}\begin{bmatrix}\beta_1\\\beta_1\ln x^2+\beta_2\end{bmatrix}\phi(0)\,. \tag{4.9}$$

R invariance for (4.9) implies that

$$\beta_1 = 0. \tag{4.10}$$

Thus,

$$[\phi_1{}^2(x), \dot{\phi}(0)]\delta(x^0) = 0. \tag{4.11}$$

This means that an R-invariant mass term $\sim\phi_1{}^2(x)$ will *not* yield a mass term in the field equation generated by the Hamiltonian. On the other hand,

$$[\phi_2{}^2(x), \dot{\phi}(0)]\delta(x^0) = 4\pi^2\beta_2 i\delta^4(x)\phi(0). \tag{4.12}$$

We can thus construct the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\left(\frac{-1}{2\pi^2 c_1}\right) K^\mu_{1\mu}(x) - \frac{m^2}{2}\phi_2{}^2(x) + \frac{\lambda}{4!}\phi_1{}^4(x), \tag{4.13}$$

with the mass term explicitly breaking scale and R invariance, and it generates the field equation

$$(\Box + m^2)\phi(x) = -\frac{\lambda}{3!}\phi_1{}^3(x). \tag{4.14}$$

The presence of the mass term in (4.13) thus implies the breaking of R invariance in the field equation.

V. CONFORMAL INVARIANCE

A large class of models which are formally scale-invariant are also formally conformal-invariant.[14] In particular, both symmetries are present in the Thirring model[15] and in the constructive bootstrap models.[16] More generally, conformal invariance will be present whenever scale invariance is present as a consequence of a Gell-Mann–Low eigenvalue.[17] It is therefore natural to ask if our quasicanonical scale-invariant model is also conformal-invariant. In this section, we will answer this question in the affirmative, but see that the conformal symmetry must be spontaneously broken.

The generators of the conformal group are the generators $M_{\mu\nu}$ of the orthochronous Lorentz group, the generators P_μ of translations, the generator D of dilatations, and the generators K_μ of special conformal transformations.[18] The special conformal transformations act on spacetime according to

$$x_\mu \to x'_\mu = \frac{x_\mu - c_\mu x^2}{1 - 2c\cdot x + c^2 x^2} \equiv K(c)x_\mu. \tag{5.1}$$

The special conformal generators and translation generators are related by[19]

$$NP_\mu N = K_\mu, \tag{5.2}$$

where N is the inversion operator

$$N:\ x_\mu \to -\frac{x_\mu}{x^2} \equiv \tilde{x}_\mu \equiv Nx_\mu. \tag{5.3}$$

It follows that any theory which is invariant under N and translations is invariant under the special conformal transformations. We note also the commutation relations[18,19]

$$[K_\mu, P_\nu] = 2i(g_{\mu\nu}D - M_{\mu\nu}). \tag{5.4}$$

We take the field $\phi(x)$ to transform under N as

$$N:\ \phi(x) \to \frac{1}{x^2}\phi(Nx) = \frac{1}{x^2}\phi(-x/x^2) \equiv \tilde{x}^2\phi(\tilde{x}), \tag{5.5}$$

where

$$\tilde{x} = Nx = -x/x^2. \tag{5.6}$$

Then

$$N:\ \Box_x\phi(x) \to (\tilde{x}^2)^3\Box_{\tilde{x}}\phi(\tilde{x}), \tag{5.7}$$

and so the classical field equation $\Box_x\phi(x) = \lambda\phi^3(x)$ is N-invariant. To establish the N invariance of our quantum field equation (2.10), the limit involved in the definition (2.18) of the source $J(x)$ must be taken into account.

We must determine first the behavior of the dimension-two scalar currents $j(x)$ and $k(x)$ under N. We rewrite Eqs. (2.8) and (2.9) as[20]

$$j(x) = \lim_{y\to x}\frac{\phi(y)\phi(x) - \Delta_+(x-y)}{g\ln(x-y)^2}, \tag{5.8}$$

$$k(x) = \lim_{y\to x}\{\phi(y)\phi(x) - \Delta_+(x-y) - [g\ln(x-y)^2 + 1]j(x)\}. \tag{5.9}$$

Using

$$(\tilde{x} - \tilde{y})^2 = \frac{(x-y)^2}{x^2 y^2}, \tag{5.10}$$

we find from (5.5) that

$$N:\ j(x) \to (\tilde{x}^2)^2 j(\tilde{x}), \tag{5.11}$$

$$N:\ k(x) \to (\tilde{x}^2)^2[k(\tilde{x}) + g\ln(\tilde{x}^2)^2 j(\tilde{x})]. \tag{5.12}$$

We see that j and k mix under N transformations just as they do under the scale transformations (3.6). The fields j and k form a two-dimensional reducible, but not completely reducible, representation of the discrete group $\{N, I\}$ $(N^2 = I)$.

The behavior of the three-dimensional scale multiplet $\{J, K, L\}$ under N transformations can be similarly determined. We find

$$N:\ J(x) \to (\tilde{x}^2)^3 J(\tilde{x}), \tag{5.13}$$

$$N:\ K(x) \to (\tilde{x}^2)^3[K(\tilde{x}) - \ln x^2 J(\tilde{x})], \tag{5.14}$$

$$N:\ L(x) \to (\tilde{x}^2)^3[L(\tilde{x}) - \ln x^2 K(\tilde{x})]. \tag{5.15}$$

We see that $\{J, K, L\}$ define a three-dimensional indecomposable representation of the discrete group $\{N, I\}$.

The N invariance of our field equation (2.10) now follows from (5.7) and (5.13). The N invariance of the theory follows from this and the N invariance of the equal-time commutation relations (1.2) and (1.4). The conformal invariance of the theory finally follows from (5.2). We note that, if we use K or L instead of J as the source of the field equation, both scale and conformal invariance would be lost. The conformal invariance is not, however, connected with the R invariance. If an irreducibly scale-invariant source of scale dimension three is used, the R invariance is lost but the conformal invariance remains.

It is straightforward to determine the transformation properties of the fields we have considered under the special conformal transformations (formally generated by $e^{ic\cdot K}$), either directly or using (5.2). The results are

$$\phi(x)\to\sigma^{-1}(x,c)\phi(K(c)x), \tag{5.16a}$$

$$j(x)\to\sigma^{-2}(x,c)j(K(c)x), \tag{5.16b}$$

$$k(x)\to\sigma^{-2}(x,c)[k(K(c)x)-\ln\sigma^2(x,c)j(K(c)x)], \tag{5.16c}$$

$$J(x)\to\sigma^{-3}(x,c)J(K(c)x), \tag{5.16d}$$

$$K(x)\to\sigma^{-3}(x,c)[K(K(c)x)+\ln\sigma^2(x,c)J(K(c)x)], \tag{5.16e}$$

$$L(x)\to\sigma^{-3}(x,c)[\tfrac{1}{2}\ln^2\sigma^2(x)J(K(c)x) +\ln\sigma^2(x)K(K(c)x)+L(K(c)x)], \tag{5.16f}$$

where

$$\sigma(x,c)\equiv 1-2c\cdot x+c^2x^2. \tag{5.17}$$

Suppose that the N symmetry is of the ordinary sort so that the vacuum is invariant. Then the Wightman function of j is[12]

$$\mathcal{W}_{11}((x-y)^2)\equiv\langle 0|j(x)j(y)|0\rangle=\frac{c_{11}}{(x-y)^4}, \tag{5.18}$$

a result which also follows from scale invariance. Consider next

$$\mathcal{W}_{12}((x-y)^2)\equiv\langle 0|j(x)k(y)|0\rangle. \tag{5.19}$$

Applying N transformations gives the relation

$$\mathcal{W}_{12}((x-y)^2)=\frac{-c_{11}\ln y^4}{(x-y)^4}+\frac{1}{x^4y^4}\mathcal{W}_{12}\left(\frac{(x-y)^2}{x^2y^2}\right), \tag{5.20}$$

which is clearly only consistent if $c_{11}=0$. The vanishing of (5.18), however, violates positivity, and so one presumably must conclude that the N symmetry is spontaneously broken so that the vacuum is not invariant.[21] It follows that the special conformal symmetry is spontaneously broken. The same conclusion follows directly from the special conformal transformation properties (5.16), without the use of N. It also follows just from the infinitesimal transformations.[22] The general conclusion is that conformal invariance must be spontaneously broken if indecomposable dilatation multiplets are present.[21,22]

There have been other suggestions, independent of the presence of indecomposable dilatation multiplets, that conformal invariance is spontaneously broken in theories with canonical scaling. Otherwise such theories must possess an infinite number of conserved local tensors,[21,23] and this might be unacceptable if the theory is not free. Further discussion of this will be given in Sec. VIII.

The consequences of the (spontaneously broken) conformal symmetry of our model are of two sorts. The symmetry places interesting restrictions on the LC OPE's, which, unfortunately, only become significant when nonforward matrix elements are involved.[24] The symmetry also leads to Ward identities and low-energy theorems in the Goldstone manner. Because of the apparently large symmetry-breaking effects in nature, this program has not yet proved to be particularly fruitful. We will return to these matters in connection with spontaneously broken scale invariance in Sec. VIII.

VI. DEEP-INELASTIC ANNIHILATION

To inquire further into the possibility that R invariance is a (broken) symmetry in nature, it is important to determine the consequences of R invariance in experimentally accessible processes other than electroproduction. In this section we will study the important electron-positron annihilation (via single-photon exchange) process,

$$\gamma(q)\to H(p)+\text{anything}, \tag{6.1}$$

in the context of our R-invariant theories. Here $H(p)$ represents any (elementary or composite) hadron. We will use the methods developed in Ref. 8 for the analysis of processes such as (6.1) in (asymptotically) scale-invariant theories. The nonperturbative character of these methods renders them ideally suited for our purposes. We will see that our quasicanonical theories imply canonical scaling laws for (6.1), whereas purely canonical theories in general violate these laws.

In the scalar theory with scalar photons, the amplitude for (6.1) is given by

$$\overline{W}(\kappa,\nu)=\int dx\, e^{iqx}\langle 0|\mathfrak{A}(x,0;p)|0\rangle, \tag{6.2}$$

with

$$\mathcal{A}(x, 0; p) = \int dy\, dz\, e^{ip(y-z)} R[j(x)S(y)] \times R[j(0)S^\dagger(z)], \tag{6.3}$$

where R denotes the retarded commutator, S is a source operator for the hadron H,[25] and the variables are $\kappa = q^2 > 0$, $\nu = q \cdot p$. In the purely canonical theory, one has

$$\mathcal{A}(x, 0; p) \underset{x^2 \to 0}{\sim} \Delta_+(x)\mathcal{B}(x, 0; p), \tag{6.4}$$

where

$$\mathcal{B}(x, 0; p) = \int dy\, dz\, e^{ip(y-z)} R[\phi(x)S(y)] R[\phi(0)S^\dagger(y)], \tag{6.5}$$

whereas in the quasicanonical theory, we have

$$\mathcal{A}(x, 0; p) \underset{x^2 \to 0}{\sim} \Delta_+(x)\mathcal{C}(x, 0; p), \tag{6.6}$$

where the alternative forms (3.27) and (3.33) for the bilocal operator give the alternative expressions

$$\mathcal{C}(x, 0; p) = \lim_{\xi \to 0} \lim_{x^2 \to 0} \frac{\mathcal{B}(x + \xi, 0; p)}{\ln(x + \xi)^2} \tag{6.7}$$

and

$$\mathcal{C}(x, 0; p) = \int da\, \sigma(a)\mathcal{B}(ax, 0; p) \tag{6.8}$$

for $\mathcal{C}$.

Denote the minimal dimension of the field operators in (6.5) by

$$\Delta = \dim\phi = 1, \quad D = \dim S. \tag{6.9}$$

It is shown in Ref. 8 that

$$\langle 0|\mathcal{B}(x, 0; p)|0\rangle \underset{\substack{x^2 \to 0 \\ x \cdot p \to 0}}{\sim} (x \cdot p)^{-\Sigma}(a + b \ln x^2 + c \ln x \cdot p) + O(x^2) + O(1/x^2), \tag{6.10}$$

where Σ, the *slant* of (6.5), is given by

$$\Sigma = D + \Delta - 3 = D - 2. \tag{6.11}$$

The constants a, b, and c are unknown but are nonvanishing in general. The $O(1/x^2)$ terms in (6.10) occur if $\Sigma > 0$ and have the form $b_N(x^2)^{-\Sigma+N}$, $N = 0, 1, 2, \ldots, -1$, with the b_N unknown. The SD singularities in (6.10) do not contribute to the electroproduction matrix element $\langle p|j(x)j(0)|p\rangle$.

It follows from (6.2)–(6.5) and (6.10) that the scaling limit of $\nu\overline{W}(\kappa, \nu)$ will in general be divergent in the purely canonical theory. This divergence is related to the inconsistency of this theory. The canonical field equations require the simple vacuum-subtracted product $\phi(x)\phi(0) - \Delta_+(x)$ to be singular on the LC if the theory is not free. These LC singularities do not, however, reveal themselves simply in considering the matrix element $\langle p|\phi(x)\phi(0)|p\rangle_{\text{conn}}$ relevant in electroproduction, which could, at this level, be analytic as in the free-field case. The singularities do, on the other hand, reveal themselves simply from consideration of the matrix element $\langle 0|\mathcal{B}(x, 0; p)|0\rangle$ relevant in annihilation. Free-field analyticity is manifestly excluded in (6.10) since in the absence of interaction ($S = 0$), one has $\Sigma < 0$ and the absence of singularities in (6.10), whereas in the presence of interactions ($\dim S \geq 3$), one has $\Sigma > 0$ and the presence of singularities. Annihilation is thus seen to probe deeper into the underlying dynamics than does electroproduction.

In the quasicanonical theory, the LC singularity in (6.10) is removed in general by the limiting procedure in (6.7) or by the line integral in (6.8) and therefore does *not* give rise to divergences in the scaling limit of $\nu\overline{W}(\kappa, \nu)$. The quasicanonical theory, which was designed to give Bjorken scaling for electroproduction, is thus seen to give scaling for annihilation. Although not surprising, this result is not trivial. One could imagine ignoring field equations and assuming that the simple bilocal $\langle p|\phi(x)\phi(0)|p\rangle$ is nonsingular so that scaling occurs in electroproduction. Scaling would then, however, not occur in general in annihilation. Our field-equation approach yields the more complicated forms (6.7) or (6.8) for the bilocals, and these forms automatically imply scaling in both electroproduction and annihilation.

The scaling law in the quasicanonical theory is

$$\nu\overline{W}(\kappa, \nu) \to \overline{F}(\omega), \quad 1 \leq \omega \leq \infty \tag{6.12}$$

where

$$\overline{F}(\omega) = \pi \int d\lambda\, e^{i\lambda\omega}\bar{f}(\lambda), \tag{6.13}$$

with

$$\bar{f}(x \cdot p) = \lim_{x^2 \to 0} \langle 0|\mathcal{C}(x, 0; p)|0\rangle. \tag{6.14}$$

The result (6.10) now gives the asymptotic behavior

$$\overline{F}(\omega) \underset{\omega \to \infty}{\sim} \text{const} \times \omega^{\Sigma - 1}. \tag{6.15}$$

The generalization of the above considerations to the more interesting quasicanonical gluon model is immediate. There the scaling law is

$$\nu\overline{W}_2(\kappa, \nu) \to \overline{F}_2(\omega), \quad 1 \leq \omega \leq \infty \tag{6.16}$$

in an obvious notation, whereas this scaling would be violated in general in the purely canonical gluon model. Experimental confirmation of (6.16) will therefore constitute further support for the quasicanonical approach. It further follows from the analysis of Ref. 8 that in our theory

$$\bar{F}_2(\omega) \underset{\omega\to\infty}{\sim} \omega^\sigma, \tag{6.17}$$

where

$$\sigma = D - 1 \text{ or } \sigma = d - \tfrac{1}{2} \tag{6.18}$$

for a scalar hadron of dimension D (e.g., a pion source) or a spinor source of dimension d (e.g., a nucleon source), respectively. Also, the multiplicity of the produced hadron is asymptotically

$$N(\kappa) \to \begin{cases} (\sqrt{\kappa})^{\sigma-3}, & 1 > \sigma - 3 > 0 \\ \ln\kappa, & \sigma - 3 = 0 \\ \text{const}, & \sigma - 3 < 0. \end{cases} \tag{6.19}$$

As discussed in Ref. 8, these results can be used to determine D and d experimentally and thus provide information on how the observed composite hadrons are constructed out of their constituents. Comparison of these determinations of the source dimensions with other such determinations[26] will provide an important test of the consistency and relevance of the whole approach.

VII. LOW-ENERGY CONSEQUENCES

In the preceding sections, we have discussed a number of high-energy consequences of R invariance—primarily Bjorken scaling. To further indicate the relevance of R invariance in nature, it is important to seek further consequences and compare them with experiment. It might be expected that R symmetry, being spontaneously broken, would have associated low-energy theorems which are approximately valid in nature. There have, in fact, been several attempts to deduce Adler consistency conditions from an assumed invariance of the S operator under R transformations of the pion field.[27] In this section we shall study such low-energy consequences of R invariance in our quasicanonical theories.

In the previous approaches,[27] the applicability of R invariance was severely limited by the fact that R transformations could only be applied to neutral fields because of the noninvariance of the electric current $:\phi^\dagger \overleftrightarrow{\partial}_\mu \phi:$ or $:\bar{\psi}\gamma_\mu\psi:$. As we have already stressed, this problem is elegantly circumvented in our RSI theories in which the electric current and all other observables *are* R-invariant.

Consider first the scalar quasicanonical R-invariant theory. The R invariance implies that the n-to-m particle scattering amplitude $T_{nm}(k_1, \ldots, k_n; k_1', \ldots, k_m')$ vanishes when any four-momentum is set to zero.[27] If a mass term is included in the model, then not only does the zero-momentum point become unphysical ($k^2 = m^2 > 0$), but the low-energy theorem itself becomes invalid because the R invariance is broken. If m^2 is sufficiently small, it might be hoped that the low-energy theorem is not too badly violated.

In a more realistic model containing an elementary pion field π_a ($a = 1, 2, 3$) and other elementary fields χ_i ($i = 1, \ldots, N$), if one has invariance under shifts of the pion fields in the absence of a pion-mass term, the scattering amplitudes will vanish whenever a pion four-momentum is set to zero. The experimentally well-satisfied Adler consistency conditions will therefore be satisfied in such a model, and this follows without invoking PCAC in any form. Furthermore, this is sufficient to give the Gell-Mann charge algebra[28] and all of its experimentally well-satisfied consequences such as the Adler-Weisberger relation. Thus, much of the progress in particle physics of the past decade (Gell-Mann charge algebra, the Adler consistency condition, Bjorken scaling) may be understood in a unified way as consequences of R invariance. In the presence of the pion-mass term, the low-energy theorems become invalid and unphysical, but, because of the small pion mass, they should remain as good approximations.

The pion is, however, presumably not an elementary particle, and so it must be asked how the above conclusions are altered when this is taken into account. Suppose in the scalar R-invariant theory that the composite "pion" is a bound state of two of the elementary scalar particles. If $j(x)$ [Eq. (2.8)] is a good interpolating field for the pion, then there will not necessarily be an interesting low-energy theorem, because j is R-invariant. If $k(x)$ is a good interpolating field, interesting low-energy theorems will be obtained because of the behavior [Eq. (2.14)] of k under R transformations. The theorems will, of course, also be valid if *both* j and k are good interpolating fields, because one is free to use k and the (on-shell) amplitudes are independent of the choice of fields if they are suitably normalized.

In the more realistic R-invariant quasicanonical gluon model, the low-energy theorems will be valid if the R-variant pseudoscalar field ($\sim\bar{\psi}\gamma_5\psi$) is a good interpolating field for the pion. The nucleon interpolating field ($\sim\psi\psi\psi$) may be either R-invariant or R-variant, since the nucleon mass is presumably too large for the nucleon low-energy theorems to be physically relevant. Under these circumstances, the R-invariance of the model exemplifies the above-mentioned common origin for the consistency conditions, charge algebra, and scaling in what is probably the most physical model yet studied.

The remarks above are meant only to illustrate some possibilities for the low-energy significance of R invariance. Much work remains to be done to see if these ideas can be implemented in a consistent and predictive way. If such a program is successful, the consequent common origin for many of the highlights of the recent era of particle physics will be most remarkable. Such a unification of pion-pole dominance and Bjorken scaling, in the framework of a consistent equal-time and light-cone current algebra, seems to us to be very appealing.

VIII. SPONTANEOUS BREAKDOWN OF DILATATION SYMMETRY

The R symmetry in our models must be spontaneously broken, and the resulting low-energy effects were discussed in Sec. VII. If the dilatation symmetry is also spontaneously broken, there will be further low-energy theorems which may be of interest. In this section we shall briefly look into these matters.

One argument that scale invariance is spontaneously broken in our models is based on the Pohlmeyer[29] theorem. Consider, for example, our scalar model. Since the scalar field ϕ has the canonical dimension one, if the vacuum were invariant to scale transformations, the two-point function would be a constant multiple of the free-field two-point function:

$$\langle 0|\phi(x)\phi(0)|0\rangle = \text{const}\times\frac{1}{x^2 - i\epsilon x^0}\,. \tag{8.1}$$

According to the Pohlmeyer theorem, ϕ must then be a free field and our theory would be a free one. This argument is not completely compelling because the presence of the R symmetry leads to the possibility that an indefinite-metric formalism may be necessary and this could vitiate the theorem. Because of our ignorance of such dynamical properties of the model, we are obviously in no position to reach a definite conclusion. There is, however, certainly a strong suggestion that the scale symmetry *is* spontaneously broken if our model is not a free-field one.

A second argument[22] that scale invariance is spontaneously broken in our models is based on the conformal invariance. We have seen that, in the absence of spontaneous breaking, non-completely-reducible representations of dilatations (with canonical *or* noncanonical dimensions) are incompatible with conformal invariance and positivity. The conclusion is, again, not compelling for the above reasons and because it requires a possibly overly strong form of conformal invariance.[21] Nevertheless, there is again a strong suggestion that the scale invariance of our models *is* spontaneously broken.

Let us now assume the spontaneous breakdown and explore the consequences. The usual low-energy theorems and Ward identities associated with scale invariance will then be valid and pole-dominance assumptions will lead to interesting and testable predictions.[30,31] These predictions can be deduced by either a dispersive approach[30] or an effective Lagrangian approach.[31] In the latter treatment, a scalar field $\sigma(x)$, the Goldstone boson, is introduced with the unusual transformation properties

$$i[D,\sigma(0)] = b^{-1}I\,, \tag{8.2}$$

$$e^{isD}\sigma(0)e^{-isD} = \sigma(0) + b^{-1}s\,, \tag{8.3}$$

under dilatations, where b is a constant with the dimension of length. The field $\sigma(x)$ can be thought of as the logarithm of an ordinary scalar field $\chi(x)$ with nonvanishing vacuum expectation value:

$$\sigma(x) = b^{-1}\ln[b\chi(x)]\,. \tag{8.4}$$

Any mass term in a Lagrangian, for example,

$$m^2\phi^2\,, \tag{8.5}$$

can then be made formally scale-invariant by multiplication by appropriate powers of $b\chi = e^{b\sigma}$; for example,

$$m^2\phi^2 \to m^2\phi^2 e^{2b\sigma} = m^2 b^2\phi^2\chi^2\,. \tag{8.6}$$

Then the Lagrangian becomes manifestly scale-invariant even though all particles (except σ) have nonvanishing masses (at least in the tree approximation, when $e^{2b\sigma}$ is expanded as $1 + 2b\sigma + \cdots$). Finally, explicit scale-symmetry breaking can be introduced to provide σ with a mass and the scale current with a nonzero (but smooth) divergence. Various broken-symmetry relations can then be deduced in the manner familiar to us from current algebra and PCAC. The results of such analyses, which are consistent with experiment, are reviewed in Refs. 30 and 31.

There is a possible relation between the above formalism and the reducible representations of the dilatation group. Consider a scalar field $A(x)$ of dimension d:

$$A(x) \xrightarrow[D]{} \lambda^d A(\lambda x)\,. \tag{8.7}$$

If another scalar field[9]

$$B(x) = \sigma(x)A(x) \tag{8.8}$$

is formally defined, it transforms under dilatations as

$$B(x) \xrightarrow[D]{} \lambda^d[B(\lambda x) + b^{-1}(\ln\lambda)A(\lambda x)]\,. \tag{8.9}$$

Thus A and B formally constitute a two-dimensional indecomposable representation of the dilatation group. Similarly, A, B, and $C \equiv \sigma^2 A$ constitute a three-dimensional representation.

The above considerations suggest that indecomposable representations have a natural origin in theories with spontaneously broken scale invariance. It is, however, difficult to assess the significance of the construction. It is certainly not clear, e.g., from (8.4), that $\sigma(x)$ is a local field operator.[32] Whatever $\sigma(x)$ is, the construction (8.8) is only of interest if $B(x)$ *is* a local field. If $\sigma(x)$ is a local field, the product in (8.8) must be precisely defined to make $B(x)$ local and well defined. If only the ordinary product is involved, the inversion $\sigma(x)=B(x)/A(x)$ again suggests that $\sigma(x)$ is not an ordinary field. A warning is provided by the example of free-field theory, where the absence[6] of indecomposable representations means that constructions such as (8.8) do not produce local fields.

It was shown in Sec. IV that the mass term to be appended to our scalar theory is $m^2k(x)$. The formally scale-invariant version is

$$m^2k(x)e^{2b\sigma(x)}\,. \tag{8.10}$$

Let us inquire into the R-transformation properties of (8.10). If $k(x)$ is identified with $\sigma(x)j(x)$, we have

$$\sigma(x)=\frac{k(x)}{j(x)}\underset{R}{\longrightarrow}\sigma(x)+\frac{2r\phi(x)+r^2}{j(x)}\,, \tag{8.11}$$

and so, since

$$\tau(x)\equiv\frac{\phi(x)}{j(x)}\underset{R}{\longrightarrow}\tau(x)+\frac{r}{j(x)} \tag{8.12}$$

and

$$\rho(x)\equiv\frac{1}{j(x)}\underset{R}{\longrightarrow}\rho(x)\,, \tag{8.13}$$

the operators σ, τ, and ρ formally define a three-dimensional indecomposable representation of the R group. If, on the other hand, $\sigma(x)$ is defined by (8.4) with $\chi(x)$ an ordinary scalar field which transforms as usual,

$$\chi(x)\underset{R}{\longrightarrow}\chi(x)+r\,, \tag{8.14}$$

then

$$\sigma(x)\underset{R}{\longrightarrow}b^{-1}\ln b[\chi(x)+r]\equiv\sigma_r(x)\,, \tag{8.15}$$

and so the continuous infinity of operators $\{\sigma_r(x)\,|\,-\infty<r<\infty\}$ define another representation of the R group. Finally, we note that the transformation property

$$\sigma\to\sigma+\frac{1}{2b}\ln\frac{k}{k+2r\phi+r^2} \tag{8.16}$$

is required in order that (8.10) be formally R-invariant as well as scale-invariant. Considerations such as these may be interesting, and we hope to take them up at greater length and depth in a future publication.

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Callan-Symanzik equation and asymptotic behavior in field theory: form factors*

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We show that even for some external lines on the mass shell, the procedure of dropping the mass-insertion term in the Callan-Symanzik equation is justified for the form factor at high squared momentum transfer in a certain class of models. This provides a very quick method of summing leading contributions in perturbation theory, as well as summing the next-to-leading terms.

INTRODUCTION

In the Lagrangian formulation of quantum field theory, because of the singular behavior of products of operators at short distance there are anomalies in the Ward identities, compared to naive ones. For example, the Callan-Symanzik equation[1] is the correct Ward identity for broken scale invariance in perturbation theory. Another aspect which has been emphasized, mainly by Symanzik,[2,3] is that this equation can be used to estimate the asymptotic behavior of Green's functions.

In general, the usefulness of this equation may be limited due to the following reasons:

(a) We are ignorant with respect to the mass-insertion term.

(b) The parameters which appear in this equation are unknown.

(c) Even if we know something about (a) and (b), we need to face the problem of the solution of this equation.

In spite of these restrictions, there are situations in which our knowledge of the asymptotic behavior of Green's functions can be improved or some results from the perturbation theory can easily be reproduced, using this equation.

Let us consider, for example, the asymptotic behavior in momentum space of Green's functions in such a configuration that no partial sum of external momenta can be zero (except for the overall energy-momentum conservation), or be on the light cone, i.e., the situation of so-called nonexceptional momenta. When all external variables are very far from the mass shell and Euclidean (all $p_i^2 \to -\infty$), from the usual arguments on power counting[4] the inhomogeneous term can be dropped,[1-3] and we are left with a homogeneous partial-differential equation of first order govern-