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Generalized Euler-Pochhammer integral representation for single-loop Feynman amplitudes*

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Kershaw's power-series representation for the single-loop Feynman amplitudes is cast into a new integral representation which, as a generalization of the Euler-Pochhammer type, is closer in spirit to the Veneziano representation than the conventional Feynman-parameter representation. Landau singularities, which are obscure in the power-series expansion, are recovered in the new integral representation. Explicit calculation is carried out for the cases $N=3$ and $N=4$.

I. INTRODUCTION

Recently, Kershaw[1] has derived a new power-series expansion in several variables for the class of single-loop Feynman amplitudes, namely (after dropping an unessential numerical-constant multiplier),

$$F^{(N)}(u_{ij})=\left(\sum_i \frac{\partial}{\partial m_i}\right)^{N-4}\left(\prod_{k=1}^N \frac{1}{m_k}\right)\times\sum_{n_{ij}}\frac{\prod_{i=1}^N (1)_{n_i}}{(N-\frac{3}{2})_n}\prod_{i<j}\frac{u_{ij}{}^{n_{ij}}}{n_{ij}!}, \tag{1}$$

where

$$n_i=\sum_{j>i} n_{ij}+\sum_{j<i} n_{ji}, \qquad n=\sum_{i<j} n_{ij}, \tag{2}$$

and the scalar variables u_{ij} are defined as

$$u_{ij}=[z_{ij}-(m_i-m_j)^2]/4m_im_j, \qquad z_{ij}=(p_i+p_{i+1}+\cdots+p_{j-1})^2, \quad i<j. \tag{3}$$

We use the Pochhammer notation: $(a)_n=\Gamma(a+n)/\Gamma(a)$.

The normal threshold[2] behavior $u_{ij}=1$ is readily seen as the convergence requirement for the power series.[1] However, the existence of the anomalous thresholds[3] and all the other higher-order Landau singularities[4,5] are not at all obvious in an expression like (1). The question is then: Given a power-series expansion like (1), is there a simple way to see the Landau singularities besides the trivial ones $u_{ij}=1$? Of course, it is assumed that starting from Eq. (1) solely, we are not allowed to go back to the original Feynman-parameter representation.

The purpose of this note is to derive from (1) a new set of integral representations whereby the Landau singularities are recovered to each order. The new integral representation, being a generalization of the Euler[6]-Pochhammer[7] type, is closer in spirit to the Veneziano representation[8] than the conventional Feynman-parameter representation.

For the sake of clarity, we shall demonstrate explicitly the case $N=3$ (triangle graph) in Sec. II and the case $N=4$ (square graph) in Sec. III. Extension to higher-order single-loop graphs is obvious.

II. THE TRIANGLE GRAPH

For $N=3$, due to a slight technical complication as may be seen from (1), we find it more convenient to consider a slightly modified function, namely,

$$\tilde{F}^{(3)}\equiv\prod m_i\left(\sum_j \frac{\partial}{\partial m_j}\right)F^{(3)} = G^{(3)}(1,1,1;\tfrac{3}{2};u_{23},u_{13},u_{12}), \tag{4}$$

where

$$G^{(3)}(a_1,a_2,a_3;c;x_1,x_2,x_3)\equiv\sum_{k_i}\frac{(a_1)_{k_2+k_3}(a_2)_{k_3+k_1}(a_3)_{k_1+k_2}}{(c)_{k_1+k_2+k_3}}\prod_{i=1}^3\frac{x_i{}^{k_i}}{k_i!}. \tag{5}$$

With the aid of the identity $(a)_{r+s}=(a+r)_s(a)_r$, the right-hand side of (5) can be decomposed into

$$G^{(3)}=\sum_{k_1}\frac{(a_3)_{k_1}(a_2)_{k_1}}{(c)_{k_1}}\frac{x_1{}^{k_1}}{k_1!}\times\sum_{k_2,k_3}\frac{(a_1)_{k_2+k_3}(a_3+k_1)_{k_2}(a_2+k_1)_{k_3}}{(c+k_1)_{k_2+k_3}}\frac{x_2{}^{k_2}x_3{}^{k_3}}{k_2!\,k_3!}. \tag{6}$$

The double sum in k_2, k_3 in (6) is recognized as the Appell F_1 function, which has the following integral representation[9,10]:

$$\frac{\Gamma(c+k_1)}{\Gamma(a_1)\Gamma(c-a_1+k_1)}\int_0^1 dt_1\, t_1^{a_1-1}(1-t_1)^{c-a_1-1+k_1}(1-x_2t_1)^{-a_3-k_1}(1-x_3t_1)^{-a_2-k_1} \tag{7}$$

The k_1 sum gives then simply a Gauss ${}_2F_1$ function, in the variable $x_1(1-t_1)/(1-x_2t_1)(1-x_3t_1)$, which in turn is cast into an Euler integral representation. The answer is

$$G^{(3)}(a_1,a_2,a_3;c;x_1,x_2,x_3)=\frac{\Gamma(c)}{\Gamma(a_1)\Gamma(a_2)\Gamma(c-a_1-a_2)}$$
$$\times\int_0^1 dt_1 t_1^{a_1-1}(1-t_1)^{c-a_1-1}(1-x_3t_1)^{-a_2}\int_0^1 dt_2\, t_2^{a_2-1}(1-t_2)^{c-a_1-a_2-1}$$
$$\times[1-x_2t_1-x_1t_2(1-t_1)/(1-x_3t_1)]^{-a_3}. \tag{8}$$

Thus, the modified triangle-graph amplitude has from (4) the following integral representation:

$$\tilde{F}^{(3)}(u_{ij})=-\tfrac{1}{4}\int\int_0^1 dt_1dt_2(1-t_1)^{-1/2}(1-t_2)^{-3/2}[(1-u_{12}t_1)(1-u_{13}t_1)-u_{23}t_2(1-t_1)]^{-1}. \tag{9}$$

The well-known anomalous thresholds come from the vanishing of the denominator function, subject to the standard pinching argument,[4] namely, at the extremum in t_1 and at the end point of $t_2=1$. We have the following identity:

$$D\equiv(1-u_{12}t_1)(1-u_{13}t_1)-u_{23}t_2(1-t_1)\big|_{t_1=\text{extremum};\,t_2=1}=(16u_{12}u_{13})^{-1}\begin{vmatrix}1 & 1-2u_{12} & 1-2u_{13}\\ 1-2u_{12} & 1 & 1-2u_{23}\\ 1-2u_{13} & 1-2u_{23} & 1\end{vmatrix}. \tag{10}$$

The vanishing of this determinant in (10) is immediately recognized as the source of the anomalous thresholds.[2] [For a technical point regarding the convergence of (8), see the Appendix.]

III. THE SQUARE GRAPH

For $N=4$, we have in an obvious recursive manner, (dropping an unessential Πm_i factor)

$$F^{(4)}(u_{ij})=\sum_{n_{ij}}\frac{(1)_{n_{12}+n_{13}+n_{14}}(1)_{n_{12}+n_{23}+n_{24}}(1)_{n_{13}+n_{23}+n_{34}}(1)_{n_{14}+n_{24}+n_{34}}}{(\frac{5}{2})_{n_{12}+n_{13}+n_{14}+n_{23}+n_{24}+n_{34}}}\prod_{i<j}\frac{u_{ij}^{n_{ij}}}{n_{ij}!} \tag{11}$$

$$=\sum_{n_{12},n_{13},n_{14}}\frac{(1)_{n_{12}+n_{13}+n_{14}}(1)_{n_{12}}(1)_{n_{13}}(1)_{n_{14}}}{(\frac{5}{2})_{n_{12}+n_{13}+n_{14}}}\frac{u_{12}^{n_{12}}u_{13}^{n_{13}}u_{14}^{n_{14}}}{n_{12}!\,n_{13}!\,n_{14}!}$$
$$\times G^{(3)}(1+n_{14},1+n_{13},1+n_{12};\tfrac{5}{2}+n_{12}+n_{13}+n_{14};u_{23},u_{24},u_{34}). \tag{12}$$

In (12), we insert the representation (8) for $G^{(3)}$. The summations over n_{13} and n_{14} become trivial. The n_{12} sum gives another ${}_2F_1$ function. Thus, we have altogether a threefold integral representation:

$$F^{(4)}(u_{ij})=-\tfrac{3}{8}\int\int\int_0^1 dt_1dt_2dt_3(1-t_1)^{1/2}$$
$$\times(1-t_2)^{-1/2}(1-t_3)^{-3/2}$$
$$\times(1-u_{34}t_1)D_4^{-1}, \tag{13}$$

where the denominator function is

$$D_4\equiv[(1-u_{24}t_1)(1-u_{34}t_1)-u_{23}t_2(1-t_1)]$$
$$\times[(1-u_{14}t_1)(1-u_{34}t_1)-u_{13}t_2(1-t_1)]$$
$$-u_{12}t_3(1-t_2)(1-t_1)(1-u_{34}t_1)^2. \tag{14}$$

The pinching conditions[4] now for both t_1 and t_2, together with the end-point condition $t_3=1$, result in the following identity:

$$D_4|_{t_1,t_2=\text{extrema};\,t_3=1}=u_{12}\lambda^{-1}\{1-\lambda^{-1}u_{34}[(u_{12}-u_{13}-u_{23})(u_{12}u_{34}-u_{13}u_{24}-u_{14}u_{23})+2u_{13}u_{23}(u_{12}-u_{14}-u_{24})]^2\}$$
$$\times\begin{vmatrix}1 & 1-2u_{12} & 1-2u_{13} & 1-2u_{14}\\ 1-2u_{12} & 1 & 1-2u_{23} & 1-2u_{24}\\ 1-2u_{13} & 1-2u_{23} & 1 & 1-2u_{34}\\ 1-2u_{14} & 1-2u_{24} & 1-2u_{34} & 1\end{vmatrix}, \qquad (15)$$

where

$$\lambda\equiv\lambda(u_{12}u_{34},u_{13}u_{24},u_{14}u_{23}), \qquad (16)$$

with

$$\lambda(a,b,c)\equiv a^2+b^2+c^2-2ab-2bc-2ca.$$

Thus the vanishing of the denominator in (13), subject to the pinching argument, implies the vanishing of the determinant (15), and the Landau singularities for the square graph are immediately recognized.[3]

IV. CONCLUSION

We have recovered the Landau singularities in a new integral representation of the Kershaw power-series expansion for the single-loop Feynman amplitudes, in particular for the cases $N=3$ and $N=4$. The representations (9) and (13), which are different from the conventional Feynman-parameter representation, are more natural from the point of view of generalized hypergeometric functions.

Extension of the present consideration to higher N-point *single-loop* Feynman amplitudes is straightforward. On the other hand, whether Kershaw's analysis can be generalized for graphs with several loops remains to be seen.

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APPENDIX

In going from Eqs. (6) to (8) and also from Eqs. (12) to (13), the Euler integral for ${}_2F_1$, namely,

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\times\int_0^1 dt\,t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}, \qquad (A1)$$

is used which, strictly speaking, holds for[11,12]

$$\mathrm{Re}c>\mathrm{Re}b>0, \qquad (A2)$$

$$|\arg(1-z)|<\pi. \qquad (A3)$$

Restriction to the single-loop Feynman amplitudes calls for special values of the coefficients such that conditions such as (A2) are not fulfilled. The implication of this is that the t_2 integral in (8) and the t_3 integral in (13) may be ill defined at the upper limit of integration ($t=1$). To cure this, two approaches may be invoked:

(a) One decrees that the function $G^{(3)}$ in (8) be *formally* defined for $\mathrm{Re}(c-a_1-a_2)>0$ and its identification with $\bar{F}$ in Eq. (4) be done by analytic continuation afterward at $c=\frac{3}{2}$. The discussion of the algebraic structure of the Landau singularity made in the text remains valid. It should be noted that changing the values of the coefficients a_i and c will *not* affect the algebraic structure of the singularity manifold in the u_{ij} variables. (It may of course alter the Riemann sheet structure of the functions involved.)

(b) Alternatively, one may deform the integration contour to avoid the point $t=1$. A modified Euler integral for ${}_2F_1$ reads[11,12]

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\frac{ie^{i\pi(b-c)}}{2\sin\pi(c-b)}\times\int_0^{(1+)} dt\,t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a},$$

$$\mathrm{Re}b>0,\ c-b\neq\text{integers} \qquad (A4)$$

where the closed contour encircles the point $t=1$ in the counterclockwise direction. In this approach, the t_2 integral in (8) and the t_3 integral in (13) would be thus modified. This entails a slight modification of the pinching argument for the denominator functions. The points $t_2=1$ for (10) and $t_3=1$ for (15) are no longer end points of integration, but serve as additional pinches for the contour. The algebraic structure of the singularity manifold remains the same.

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PHYSICAL REVIEW D VOLUME 9, NUMBER 2 15 JANUARY 1974

Quasicanonical quantum field theory*

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We call a quantum-field-theory model quasicanonical if it is defined by canonical equal-time field commutation relations (e.g., $[\dot\phi(x),\ \phi(0)]\delta(x^0) = -i\delta^4(x)$) and local field equations [e.g., $\Box\phi(x) = \lambda J(x)$], and is locally invariant to scale transformations [e.g., $\phi(x) \to \rho\phi(\rho x)$]. [These requirements are *not* consistent if the model is *purely* canonical, i.e., if $J(x)$ is the simple Wick product $:\phi^3(x):$.] Canonical Bjorken scaling is valid in such models provided that the field equations are also locally invariant to R transformations [$\phi(x) \to \phi(x) + r$] and the physical currents are R-invariant. We discuss here further properties and consequences of these models. (a) We incorporate positivity and R-invariance restrictions on light-cone expansions and deduce the form of the consequent bilocal operators [e.g., $\int da\, \sigma(a) :\phi(ax)\phi(0):$]. (b) We exhibit a Hamiltonian formulation of the theory, both in the massless and massive cases. (c) We show that the theory is locally conformal- and inversion- [$\phi(x) \to (x^2)^{-1}\phi(-x/x^2)$] invariant. These symmetries are spontaneously broken. (d) We discuss the implications of the model for deep-inelastic electron-positron annihilation. Exact scaling is obtained. (e) We study the possible low-energy consequences of the (spontaneously broken) R symmetry. These include the PCAC (partial conservation of axial-vector current) consistency conditions and the Gell-Mann charge algebra. (f) We consider the arguments for and consequences of a spontaneous breakdown of the dilatation symmetry.

I. INTRODUCTION

Canonical quantum field theory is based on canonical field equations such as[1]

$$\Box\phi(x) = \lambda : \phi^3(x) : , \qquad (1.1)$$

and canonical equal-time commutation relations such as

$$[\dot\phi(x), \phi(0)]\delta(x^0) = -i\delta^4(x) . \qquad (1.2)$$

This framework is unfortunately inconsistent except in the free-field case $\lambda = 0$.[2] This is because (1.1) and (1.2) imply the short-distance (SD) behavior $:\phi(x)\phi(0): \to \lambda(\ln x^2) : \phi^2(0):$, which precludes the existence of the simple Wick product $:\phi^3(x):$ in (1.1). The conventional approach to this problem is to give up (1.1) and (1.2) and to define the theory by the renormalized perturbation expansion. Then the source term in (1.1) is replaced by a complicated limit which subtracts out the singularities and (1.2) must be abandoned entirely.[3] Although consistent and explicit, this framework has been useless when strong interactions are involved. In particular, it seems impossible to understand (exact or approximate) canonical Bjorken scaling in this way.[4]