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Remarks on the physical degrees of freedom in two-dimensional electrodynamics*

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A simple argument is advanced for how it happens that two-dimensional electrodynamics is a theory of massive spinless bosons.

It is well known that, once the computational dust has settled, two-dimensional quantum electrodynamics (TDQED) collapses to a theory of a massive spinless noninteracting Bose field. Sophisticated arguments for why this occurs have been presented by Lowenstein and Swieca.[1] The purpose of this note is to supply a simple way of seeing why it is so.

The first observation required is that in the gauge $A_1(x,t)=0$, the interaction reduces to a self-interaction of the charge density via the (two-dimensional) Coulomb potential,[2]

$$\mathcal{L}=i\bar{\psi}\partial\!\!\!/\psi+\tfrac{1}{4}e^2 j^0(x,t)\int dy\,|x-y|\,j^0(y,t)\,. \quad (1)$$

Next, as is known from work on the Thirring model,[3] the free *massless* Dirac theory in two dimensions may equivalently be discussed in terms of the associated vector current

$$j^\mu(x,t)=:\bar{\psi}\gamma^\mu\psi:\,, \quad (2)$$

and the symmetric, traceless, conserved tensor operator which may be constructed from the current,

$$T^{\mu\nu}(x,t)=\tfrac{1}{2}\pi(\{j^\mu,j^\nu\}-g^{\mu\nu}j_\lambda j^\lambda)\,. \quad (3)$$

That is, $H=\int dx\,T^{00}$ generates the free-field equation of motion for ψ, given the anticommutator $\{\psi(x,t),\psi^\dagger(y,t)\}=\delta(x-y)$, and the definitions Eqs. (2) and (3). The result is not obvious, however, and depends for its demonstration on the operator equation[3]

$$\partial_x\psi(x,t)=\tfrac{1}{2}i\pi\{j^1+\gamma^5 j^0,\psi\}\,, \quad (4)$$

which is true in two dimensions.

It is possible to proceed by postulating a set of commutation relations satisfied by j^μ with itself and with ψ.[4,5] For the free theory, this is unnecessary. The required "current algebra" can be derived, and is

$$[j_0(x,t),j_0(y,t)]=0\,, \quad (5a)$$

$$[j_1(x,t),j_1(y,t)]=0\,, \quad (5b)$$

$$[j_0(x,t),j_1(y,t)]=\frac{i}{\pi}\partial_x\delta(x-y)\,. \quad (5c)$$

This current algebra is solved by setting[6]

$$\sqrt{\pi}\,j_\mu(x,t)=\epsilon_{\mu\nu}\partial^\nu\tilde{\Phi}(x,t)\,, \quad (6)$$

where $\tilde{\Phi}$ is a canonical (pseudo) scalar field. Then $T^{\mu\nu}$ is the canonical energy-momentum tensor for this massless field. A consistent choice for an associated Lagrangian density is

$$\mathcal{L}^{(0)}=\tfrac{1}{2}(\partial_\mu\tilde{\Phi})(\partial^\mu\tilde{\Phi})\,. \quad (7)$$

[Note $\sqrt{\pi}j_\mu=\partial_\mu\Phi$ is also a possible choice. However, current conservation demands Φ is a massless free field, but places no constraints on $\tilde{\Phi}$.]

The result essential to our argument is that, using Eqs. (4) and (6), one can show that

$$\begin{aligned}H_0&=-i\bar{\psi}_0\gamma^1\partial_1\psi_1\\&=\tfrac{1}{2}[(\partial_0\tilde{\phi})^2+(\partial_1\tilde{\phi})^2]\,,\end{aligned} \quad (8)$$

where ψ_0 is the free Dirac field in two dimensions. This amazing relation is expected to be true only in two dimensions, and is a result of the fact that Eq. (4) reduces trilinears in ψ to a single ψ. Using this equation, *in the interaction representation* Eq. (1) becomes

$$\begin{aligned}\mathcal{L}&=\tfrac{1}{2}(\partial_\mu\tilde{\Phi})(\partial^\mu\tilde{\Phi})+\frac{e^2}{4\pi}\int dy\,|x-y|\,\partial_x\tilde{\Phi}(x,t)\partial_y\tilde{\Phi}(y,t)\\&\to\tfrac{1}{2}(\partial_\mu\tilde{\Phi})(\partial^\mu\tilde{\Phi})-\frac{e^2}{2\pi}\tilde{\Phi}^2\,.\end{aligned} \quad (9)$$

But this just describes a massive (pseudo) scalar field of mass $\mu^2=e^2/\pi$ and nothing else; this was the desired result.

Admittedly, we have seemed cavalier in obtaining this result, paying little attention to defining the various operators we have introduced with any rigor, and substituting free-theory equations into the interacting theory. In fact, however, a rigorous momentum-space analysis can be carried out. This analysis verifies the conclusions stated above.

As an example, consider TDQED in the finite spatial interval $[0,\pi]$. The use of a finite interval may be viewed as an alternative to Klaiber's procedure for regularizing the bad infrared behavior of the theory.[1,7,8] The particular interval chosen is a matter of convenience. Any interval of arbitrary length L may be adopted. Taking the limit $L\to\infty$ at the end of the calculation turns momentum sums into momentum integrals.

It has been shown recently[9] that

$$H=\int_0^{\pi} dx[\psi^\dagger\partial_0\psi-\mathcal{L}]$$
$$=\sum_{n=1}^{\infty}(n-\tfrac{1}{2})(b_n^\dagger b_n+c_n^\dagger c_n)$$
$$+(\mu^2/4)\sum_{p=1}^{\infty}p^{-1}(\rho^\dagger\rho^\dagger+\rho\rho+2\rho^\dagger\rho)_p\,, \qquad (10)$$

where $\mathcal{L}$ is given by Eq. (1), and

$$\rho(p)=p^{-1/2}\int_0^{\pi}dx[j^0(x,0)\cos px-ij^1(x,0)\sin px] \qquad (11)$$

satisfy Bose commutation relations $[\rho(p),\rho^\dagger(q)]=\delta_{p,q}$. This Hamiltonian may be diagonalized by means of a Bogoliubov transformation,

$$\tilde{H}=e^{iS}He^{-iS}$$
$$=(H_0-T)+\sum_{p=1}^{\infty}E(p)\rho^\dagger(p)\rho(p)+E_0\,, \qquad (12)$$

where H_0 is the free Dirac Hamiltonian, and

$$S=-\frac{i}{2}\sum_{p=1}^{\infty}\left[\tanh^{-1}\left(\frac{\mu^2}{\mu^2+p^2}\right)\right][\rho^\dagger\rho^\dagger-\rho\rho]_p\,, \qquad (13a)$$

$$T=\sum_{p=1}^{\infty}p\rho^\dagger(p)\rho(p)\,, \qquad (13b)$$

$$E(p)=(\mu^2+p^2)^{1/2}\,, \qquad (13c)$$

$$E_0=\frac{1}{2}\sum_{p=1}^{\infty}\left[E(p)-p-\left(\frac{\mu^2}{2p}\right)\right]\,. \qquad (13d)$$

The analog of Eq. (8) in this case is[10]

$$H_0=\tfrac{1}{2}Q^2+T\,, \qquad (14)$$

where the conserved charge $Q=\int dx\,j^0$. This equation may be verified by explicit, though tedious, calculation, using identities which follow from Eq. (4). Fourier-analyzing Eq. (4), one obtains

$$Qb_k+\sum_{p=1}^{\infty}p\rho_p^\dagger b_{k+p}+\sum_{p=1}^{\infty}(p+k-1)^{1/2}c_p^\dagger\rho_{p+k-1}$$
$$+\sum_{p=1}^{k-1}\sqrt{p}\,\rho_p b_{k-p}=0\,, \qquad (15)$$

and an accompanying equation for c obtained by interchanging $b \leftrightarrow c$, where they appear explicitly in Eq. (15).

The validity of Eq. (15) as an operator equation may be verified explicitly by brute force. As mentioned earlier, the key to why this relation is possible is that ρ and $\rho^\dagger$ are bilinear in fermion operators, hence many terms anticommute to zero under summation.[8]

Implicit in Eq. (10) was the consistency requirement for TDQED in the axial gauge,

$$Q|\Phi_{\rm phys}\rangle=0\,. \qquad (16)$$

This constraint follows, essentially, because the vector current is conserved; but the divergence of the axial-vector current contains an anomaly. However, these currents are related by $a^\mu=\epsilon^{\mu\nu}j_\nu$. Consistency demands (in this gauge) that Eq. (16) be true.[11]

Combining Eqs. (12), (14), and (16), we have

$$\tilde{H}=E_0+\sum_{p=1}^{\infty}E(p)\rho^\dagger(p)\rho(p)\,. \qquad (17)$$

All reference to the fermions has disappeared. This is a property of the solution independent of our choice of interval and of boundary conditions in the interval.[12] As the size of the system $L\to\infty$, E_0 diverges logarithmically. This is the only remnant of the infrared problem.

To summarize, one first casts TDQED as a theory of self-interacting fermions by choosing a convenient gauge. This done, one attempts to represent as much of the theory as possible in terms of the currents, based on the experience with the Thirring model that these are the only genuine observables.[3-5] Unlike the Thirring model, however, in which ψ preserves a role as an intertwining operator between inequivalent irreducible representations of the current algebra,[5] the vanishing of the charge in TDQED deprives ψ of even this role. Indeed, the presence of massive excitations follows trivially from $(\Box+\mu^2)j^\mu=0$. The point being made is that ψ can be eliminated entirely from the problem. In the language of Ref. 12, there are no "quasiparticles," *only* plasmons.

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[9]C. E. Carlson, L.-N. Chang, F. Mansouri, and J. F. Willemsen, Phys. Lett. B (to be published).

[10]This result is not unexpected. It is known that $[H_0 - T]$ commutes with the current's Fourier components, ρ and $\rho^\dagger$ (Ref. 12). It is then argued on general grounds ("irreducibility assumption") that $[H_0 - T]$ must be a function of Q only (Ref. 5). Dimensional arguments and a few low-order matrix elements can determine the precise form

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