PHYSICAL REVIEW D VOLUME 9, NUMBER 12 15 JUNE 1974

Asymptotic behavior of spin-dependent Feynman integrals: Fourth-order spinor-vector scattering*

Samuel Prum
Department of Physics and Astronomy, Tel-Aviv University, Ramat-Aviv, Israel†
and Department of Physics, University of California at Los Angeles, Los Angeles, California 90024
(Received 3 July 1973)

Several methods for obtaining the leading and subleading asymptotic dependence of the fourth-order spinor-vector scattering amplitude are considered. Under the assumption of equal masses, $s=0$, and large t, the nonpolynomial asymptotic expansion for the helicity-dependent amplitude is evaluated and shown to consist of terms proportional to $t^{-n}\ln t + t^{-m}(\ln t)^2$, where $n \geq 0$ and $m \geq 1$. The origin of terms quadratic in $\ln t$ is discussed as an example of the singularity-enhancing properties of spin, and it is shown that a possible mechanism for the enhancement effects is through the introduction of subtractions to the dispersion relations of the theory.

I. INTRODUCTION

Perturbation theory has enjoyed wide popularity among high-energy physicists because of the success it has achieved in conjunction with quantum electrodynamics. This achievement derives for the most part from predictions obtained in second- and fourth-order calculations; exact higher-order computations are seldom possible due to the complex form that the Feynman integrals of the theory assume. Spin, in particular, compounds the problem by injecting into the integrals numerators which depend on the integration variables and which are themselves complex algebraic expressions.

Many authors have hoped that a study of spinless integrals, while meritorious in its own, may also pave the way to the spin-dependent case. *Prima facie* this idea seems plausible, and indeed the study of scalar diagrams has helped to identify the types of singular behavior associated with each integral.[1-6] However, the notion that a spin-derived numerator does not change the singular behavior of the scalar integral is incorrect. Azimov,[7] for example, has remarked on the translational effects that spin induces, while Polkinghorne[8] has pointed out that the presence of internal-momentum factors in the numerator can also result in changes to the class of singular configurations associated with each diagram. The understanding of these effects, however, has been very limited and has been concerned only with the leading asymptotic term in the amplitude, so that very little is known about next-to leading-order terms.

We wish to examine in this paper the asymptotic properties of a simple model, that of a spin-$\frac{1}{2}$ particle interacting with a neutral vector meson through a conserved current in fourth-order perturbation theory. For this model, we discuss several techniques useful in studying the leading and subleading terms in the asymptotic expansion, and apply them to the evaluation of the helicity-dependent amplitude for the equal-mass process at large t (or u) and $s=0$. Our results point out several differences in singular behavior between the scalar and spin counterparts. In particular, we find that while the singular behavior of the leading term exhibits only translational effects (i.e., it is of the form $\ln t$ for the sense-sense amplitudes), the succeeding terms develop an enhanced singular behavior of the form $t^{-n}(\ln t)^2$ (where n is a positive integer). These consequences are of special interest in the study of Regge properties of perturbative models, such as the one originally proposed by Gell-Mann, Goldberger, and collaborators,[9,10] since the subleading terms should be directly connected with the daughters of the theory.[11]

Our main approach to the evaluation of the amplitude for this process follows a suggestion of Keller,[12] and consists of taking advantage of the Cutkosky rules[13] to obtain the imaginary part of the amplitude, followed by the use of dispersion relations to recover the real part. At $s=0$, which is the point we consider in our calculation due to the existence of interesting kinematic constraints,[14] the amplitude in the s channel becomes nonphysical; therefore, we circumvent the difficulties of an analytic continuation by evaluating the amplitudes in the cross channels, and make use of crossing relations[15-17] to recover the s-channel amplitude. This method is discussed in Secs. II through IV, and it has the advantage of yielding directly the large-t and the large-u limits for the asymptotic amplitude.

In Sec. II we consider some preliminaries regarding the diagrams involved in the calculation, and the general approach to the problem. The evaluation of the discontinuities due to the t- and u-channel cuts and the application of dispersion relations to the recovery of the real parts are discussed in Sec. III. These results are incorporated with the reduced spin numerators in Sec. IV, where the t- and u-channel contributions are obtained. From these, the s-channel asymptotic amplitude is regained.

An alternate method for the evaluation of the amplitude through direct integration techniques is discussed in Sec. V. This method, which is illustrated by means of the spin-dependent box-diagram, allows one to carry out exact calculations and the results obtained by this approach find complete agreement with the answer previously derived. We conclude our discussion in Sec. VI by using the unitarity principle as an aid in understanding the effects that spin numerators can have on the asymptotic form of the amplitude. In this way we show how the presence of these numerators can lead to enhancements in the singular behavior as a result of cancellations between numerator and denominator factors in the integrals in question; and how some of the enhancement effects are tied to the existence of subtractions in the Mandelstam representation.

II. SYSTEMATICS OF THE CALCULATION

Spinor-vector scattering is an amenable model to study since the spins involved can yield nontrivial and interesting information on the properties of spin-dependent Feynman integrals while at the same time the theory has the advantage of having a renormalizable perturbation expansion. Even this simple model, however, presents great calculational difficulties for diagrams beyond second order, especially if one is interested in the spin (or helicity) dependence for other than the dominant term. For this reason, it is necessary to restrict one's study to the somewhat simpler case of equal masses with the expectation that many of the features that one finds are shared with the unequal-mass problem.

The diagrams that contribute to the fourth-order process are shown in Fig. 1. These diagrams have an asymptotic behavior for large t which consists of terms of the form t^{-n} $(\ln t)^m$ for nonnegative integral n and m. If one neglects the terms for which $m=0$, i.e., the polynomial terms, one is able to set aside the renormalization questions while keeping the singular features of the amplitude; thus, in our discussion, we ignore all polynomial terms arising in the calculation. This

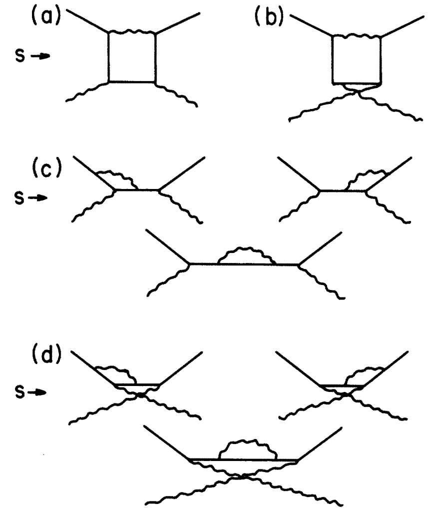

FIG. 1. Fourth-order diagrams for spinor-vector scattering: (a) planar box, (b) crossed box, and (c) and (d) corrections to the second-order graphs. Wavy lines are vector mesons, solid lines represent the spinor.

enables us, for example, to ignore any subtractions to the dispersion relations that we write.

Our calculations make use of the helicity formalism of Jacob and Wick,[18] with the conventions as summarized in Appendix A. For $s=0$, the planar box [Fig. 1(a)] has a normal threshold cut in t for $t \geq 4m^2$; the crossed box [Fig. 1(b)] has normal threshold cuts in t for $t \geq 4m^2$ and in u for $u \geq 4m^2$; while the vertex corrections and self-energy correction to the second-order crossed-graph [Fig. 1(d)] possess normal threshold discontinuities in u for $u \geq 4m^2$. The diagrams in Fig. 1(c) do not possess any dynamical cuts in t or u and need not be considered.

Rules for obtaining the discontinuities associated with branch cuts due to normal thresholds in Feynman integrals have been given by Cutkosky.[13] Although these rules are given for scalar particles, they can be applied to integrals having numerator factors provided one is careful not to introduce any new discontinuities in the numerator. In addition, caution must be taken in their use for unphysical values of the intermediate momenta q_j, since at these points the Dirac δ functions $\delta^+(q_j^2 - m_j^2)$ that replace the propagators take on complex arguments, whose mathematical significance can only be ascertained by reverting back to the much more intricate analysis of singularities pinching contours in the q_j^2 hyperspace. At $s=0$, the scattering amplitude in the s channel is at an unphysical value, so that in a direct application of these rules one would be besieged by increased complexity. We can avoid these difficulties by taking advantage of crossing symmetry; thus we compute the t discontinuities in the t channel and similarly, the u discontinuities in the u channel, which has the advantage of yielding the s-channel amplitude for large positive as well as large negative values of z_s, the cosine of the scattering angle.

The t- and u-channel diagrams giving nonzero contributions to the s-channel amplitude are shown in Figs. 2 and 3, respectively. In each channel, we work in the appropriate center-of-mass system, with the convention that particles a and c travel along the $-z$ direction with positive momentum for the t channel, and particles a and b for the u channel. Associated with each diagram we have an integral of the form

$$T_{\{\lambda\}}(s=0,t)=\frac{2mg^4}{(2\pi)^4}\int d^4l\,\frac{N_{\{\lambda\}}(l,t)}{D(l,t)}, \tag{2.1}$$

where $g^2\equiv e^2/4\pi$ and the helicity labels $\{\lambda\}$ are given by $\{\lambda\}=\lambda_c,\lambda_a;\lambda_d,\lambda_b$ for the t channel, and $\{\lambda\}=\lambda_a,\lambda_d;\lambda_c,\lambda_b$ for the u channel. The numerator and denominator functions, $N_{\{\lambda\}}(l,t)$ and $D(l,t)$, in the case of the planar box diagram are given by

$$N_{\{\lambda\}}(l,t)=\bar{v}(\bar{p}_2,\lambda_d)N_{\mu\nu}(l,t)u(p_1,\lambda_b)\times\epsilon^{\mu*}(k_2,\lambda_c)\epsilon^{\nu*}(\bar{k}_1,\lambda_a), \tag{2.2}$$

with

$$N_{\mu\nu}(l,t)=\gamma^\sigma(\not{l}-\not{\bar{p}}_2+m)\gamma_\mu(\not{l}+m)\gamma_\nu(\not{l}+\not{p}_1+m)\gamma_\sigma,$$

and

$$D(l,t)=[(l-\bar{p}_2)^2-m^2](l^2-m^2)^2[(l+p_1)^2-m^2]. \tag{2.3}$$

Similar expressions hold for the other diagrams.

We proceed now by rewriting the numerator expression as a polynomial in the integration variable l; each term of the polynomial being a tensor in the components of l. Thus we have

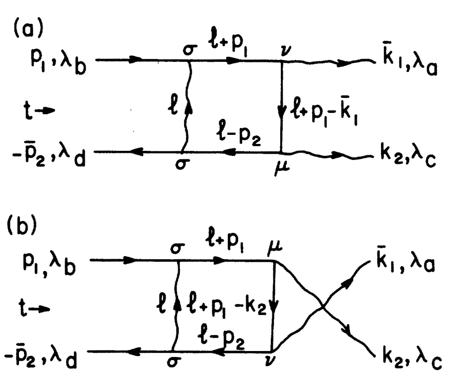

FIG. 2. t-channel diagrams having a nonzero discontinuity in t: (a) planar box, (b) crossed box.

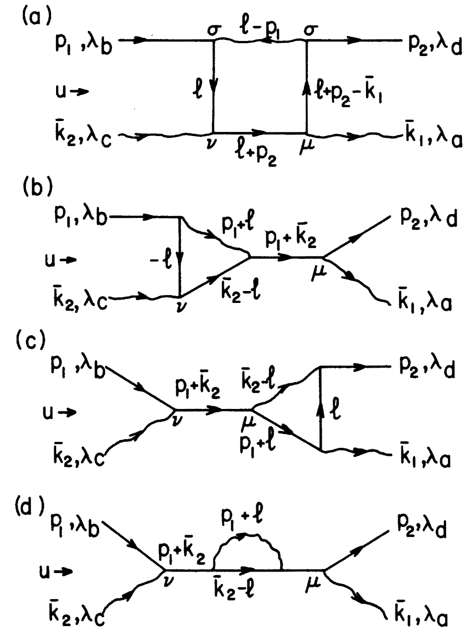

FIG. 3. u-channel diagrams having a nonzero discontinuity in u: (a) planar box, (b) and (c) vertex corrections; and (d) self-energy correction.

$$T_{\{\lambda\}}(t)=A(\{\lambda\},t)I+B_\alpha(\{\lambda\},t)I^\alpha+C_{\alpha\beta}(\{\lambda\},t)I^{\alpha\beta}+D_{\alpha\beta\gamma}(\{\lambda\},t)I^{\alpha\beta\gamma}, \tag{2.4}$$

where the coefficients A, B_α, $C_{\alpha\beta}$, and $D_{\alpha\beta\gamma}$ are obtained by an algebraic reduction of the numerator and depend only on the external variables and helicities; the terms I, I^α, $I^{\alpha\beta}$, and $I^{\alpha\beta\gamma}$ corresponds to integrals with numerators of the form 1, l^α, $l^\alpha l^\beta$, and $l^\alpha l^\beta l^\gamma$, respectively. Thus for example, $I^{\alpha\beta\gamma}$ is defined as

$$I^{\alpha\beta\gamma}\equiv(2\pi)^{-4}\int d^4l\,\frac{l^\alpha l^\beta l^\gamma}{D(l,t)} \tag{2.5}$$

and similar definitions are used for the other tensors. Care must be taken when using Eq. (2.4) to adhere to a standard set of labels for the graphs, since the values of A, B_α, $C_{\alpha\beta}$, and $D_{\alpha\beta\gamma}$ depend on the choice of labels.

The procedure for determining the amplitude for each channel will consist of evaluating the tensors I, I^α, etc., from the discontinuities ΔI, ΔI^α, etc., by means of dispersion relations; whereas the coefficients A, B_α, $C_{\alpha\beta}$, and $D_{\alpha\beta\gamma}$ are determined by an algebraic reduction of the numerator. This procedure introduces several subtractions in the various dispersion integrals; however, these subtractions will at best modify the behavior of the polynomial in t^{-1}. The evaluation of the discontinuities and the integration of the dispersion relations are discussed in the next section.

III. EVALUATION OF THE TENSOR INTEGRALS

In order to see how the discontinuities are evaluated, we begin by considering the tensor integral I for the t-channel planar box. Its discontinuity $\Delta_t I$ is given by

$$\Delta_t I = -(2\pi)^{-2}\int d^4l\, \delta^+((l+p_1)^2-m^2) \times \delta^+((l-\bar{p}_2)^2-m^2)\,(l^2-m^2)^{-2}\,. \tag{3.1}$$

The two Dirac δ functions can be conveniently simplified by going to the t center of mass. Defining

$$C(l,t)=\delta^+((l+p_1)^2-m^2)\,\delta^+((l-\bar{p}_2)^2-m^2)$$

we have the following relation:

$$\begin{aligned}C(l,t) &= \delta^+(l^2+2l\cdot p_1)\delta^+(l^2-2l\cdot\bar{p}_2)\\ &= \delta^+(l^2-2l\cdot\bar{p}_2)\delta^+(2l\cdot(p_1+\bar{p}_2))\\ &= (1/\sqrt{t}\,)\delta(l^0)\delta(-\vec{l}^2+2l_i\bar{p}_{2i})\\ &= [2\sqrt{t}\,(t-4m^2)^{1/2}\cos\theta]^{-1}\delta(l^0)\\ &\quad\times\delta(|\vec{l}|-(t-4m^2)^{1/2}\cos\theta)\,,\end{aligned} \tag{3.2}$$

where $\cos\theta\equiv\hat{l}\cdot\hat{p}_2$. Using this relation, the discontinuity $\Delta_t I$ can be simply evaluated giving

$$\Delta_t I = -(8\pi m^2)^{-1}\frac{(1-4m^2/t)^{1/2}}{(t-3m^2)}\Theta(t-4m^2)\,, \tag{3.3}$$

where Θ is the Heaviside function.

One can determine the applicability of the Cutkosky rules to the more complicated tensors I^α, $I^{\alpha\beta}$, and $I^{\alpha\beta\gamma}$ by analyzing the numerators present in these integrals. Consider I^α as an example; the extension to the other tensors will be self-evident. The singularities (if any) introduced by the numerator l^α, must be viewed in terms of the line momenta q_j for which the Cutkosky rules are derived. Note that the denominator of the integral is symmetric in l, and the range of integration is spherically symmetric; the tensors I^1 and I^2 must therefore vanish, since they possess an odd azimuthal symmetry. Defining the line momenta for the integral by: $q_1{}^2=q_3{}^2=l^2$, $q_2{}^2=(l+p_1)^2$, and $q_4{}^2=(l-\bar{p}_2)^2$, we can express the components of l in terms of the $q_j{}^2$. We find

$$l^0=(\sqrt{t}\,)^{-1}(q_2{}^2-q_4{}^2),$$

and

$$l^3=[2(t-4m^2)^{1/2}]^{-1}(q_2{}^2+q_4{}^2-2q_1{}^2-2m^2)\,.$$

As functions of the $q_j{}^2$, l^0, and l^3 are analytic and cannot affect the argument used in deriving the Cutkosky rules. However, we see that the discontinuities obtained will be those corresponding to $\Delta_t(\sqrt{t}\,I^0)$ and $\Delta_t[(t-4m^2)^{1/2}I^3]$. This has little effect on our program because we can write dispersion relations for $\Delta_t(\sqrt{t}\,I^0)/\sqrt{t}$ and $\Delta_t[(t-4m^2)^{1/2}I^3]/(t-4m^2)^{1/2}$ which would differ from the expressions that come from $\Delta_t I^0$ and $\Delta_t I^3$ merely by a polynomial in t^{-1}. Thus, up to a polynomial arbitrariness in the real part of the amplitude, the Cutkosky rules are applicable to the remaining tensor integrals.

With these modifications in mind, we can write expressions for the discontinuities of I^α. These are given by

$$\Delta_t(\sqrt{t}\,I^0)=-\frac{\sqrt{t}}{(2\pi)^2}\int d^4l\,\frac{l^0C(l,t)}{(l^2-m^2)^2}\,, \tag{3.4}$$

$$\Delta_t[(t-4m^2)^{1/2}I^3]=-\frac{(t-4m^2)^{1/2}}{(2\pi)^2}\int d^4l\,\frac{l^3C(l,t)}{(l^2-m^2)^2}\,. \tag{3.5}$$

By our choice of labels in the diagram, we see that $C(l,t)$, as given in Eq. (3.2), contains as a factor $\delta(l^0)$; so that the integral in Eq. (3.4) gives a vanishing contribution. In fact, any tensor $I^{\alpha\beta\gamma}$ having one or more of its indices equal to zero can be discarded, greatly reducing the number of calculations required. In addition, any tensor having an odd number of indices equal to 1 or 2 will also vanish exactly because of azimuthal asymmetry.

The imaginary part of $(t-4m^2)^{1/2}I^3$ for the planar box can now be evaluated using simple integration techniques, with the result that

$$\begin{aligned}\Delta_t[(t-4m^2)^{1/2}I^3] = \frac{[(t-4m^2)/t]^{1/2}}{8\pi}&\\ \times\{(t-3m^2)^{-1}&\\ -(t-4m^2)^{-1}\ln[(t-3m^2)/m^2]\}&\\ \times\Theta(t-4m^2)\,.&\end{aligned} \tag{3.6}$$

This procedure can be applied without any amendments to the evaluation of the discontinuities of all graphs shown in Figs. 2 and 3. The results are presented in Tables I and II.

An examination of these results points out that the various contributions to the imaginary part of the scattering amplitude have the form $t^{-n}+t^{-m}\times(\ln t)$ for large t, with n and m non-negative integers. We thus expect the asymptotic (real) amplitude to consist of various terms behaving like $t^{-n}(\ln t)+t^{-m}(\ln t)^2$. The leading term in this expansion has been previously obtained[10] and is proportional to $\ln t$ in the case of sense-sense

TABLE I. Discontinuities of the t-channel graphs ($t \geq 4m^2$, $\bar{\Delta} \equiv 8\pi[t/(t-4m^2)]^{1/2}\Delta_t$).

(a) Planar box [Fig. 2(a)]

$$\bar{\Delta} I = -[m^2(t-3m^2)]^{-1}$$

$$\bar{\Delta}[(t-4m^2)^{1/2}I^3] = (t-3m^2)^{-1} - (t-4m^2)^{-1}\ln[(t-3m^2)/m^2]$$

$$\sum_{i=1}^{3}\bar{\Delta}[(t-4m^2)I^{ii}] = (t-4m^2)\bar{\Delta}[(t-4m^2)^{1/2}I^3]$$

$$\bar{\Delta}[(t-4m^2)I^{33}] = -\frac{t-2m^2}{t-3m^2} + \frac{2m^2}{t-4m^2}\ln\left(\frac{t-3m^2}{m^2}\right)$$

$$\sum_{i=1}^{3}\bar{\Delta}[(t-4m^2)^{3/2}I^{ii3}] = (t-4m^2)\bar{\Delta}[(t-4m^2)I^{33}],$$

$$\bar{\Delta}[(t-4m^2)^{3/2}I^{333}] = -\frac{\frac{1}{2}(t^2-11m^2t+22m^4)}{t-3m^2} - \frac{3m^4}{t-4m^2}\ln\left(\frac{t-3m^2}{m^2}\right)$$

(b) Crossed box [Fig. 2(b)]

$$\bar{\Delta} I = 2[(t-4m^2)(t-2m^2)]^{-1}\ln[(t-3m^2)/m^2]$$

$$\bar{\Delta}[(t-4m^2)^{1/2}I^3] = (t-2m^2)^{-1}\ln[(t-3m^2)/m^2]$$

$$\sum_{i=1}^{3}\bar{\Delta}[(t-4m^2)I^{ii}] = (t-4m^2)\bar{\Delta}[(t-4m^2)^{1/2}I^3]$$

$$\bar{\Delta}[(t-4m^2)I^{33}] = -1 + \frac{t^2-6m^2t+10m^4}{(t-2m^2)(t-4m^2)}\ln\left(\frac{t-3m^2}{m^2}\right)$$

$$\sum_{i=1}^{3}\bar{\Delta}[(t-4m^2)^{3/2}I^{ii3}] = (t-4m^2)\bar{\Delta}[(t-4m^2)I^{33}],$$

$$\bar{\Delta}[(t-4m^2)^{3/2}I^{333}] = -\tfrac{3}{2}(t-4m^2) + \frac{t^2-5m^2t+7m^4}{t-2m^2}\ln\left(\frac{t-3m^2}{m^2}\right)$$

TABLE II. Discontinuities of the u-channel graphs ($u \geq 4m^2$, $\hat{\Delta} \equiv 8\pi[u/(u-4m^2)]^{1/2}\Delta_u$).

(a) Planar box [Fig. 3(a)]

$$\hat{\Delta} I = -2[(u-2m^2)(u-4m^2)]^{-1}\ln[(u-3m^2)/m^2]$$

$$\hat{\Delta}[(u-4m^2)^{1/2}I^3] = (u-2m^2)^{-1}\ln[(u-3m^2)/m^2]$$

$$\sum_{i=1}^{3}\hat{\Delta}[(u-4m^2)I^{ii}] = -(u-4m^2)\hat{\Delta}[(u-4m^2)^{1/2}I^3]$$

$$\hat{\Delta}[(u-4m^2)I^{33}] = 1 - \frac{u^2-6m^2u+10m^4}{(u-2m^2)(u-4m^2)}\ln\left(\frac{u-3m^2}{m^2}\right)$$

$$\sum_{i=1}^{3}\hat{\Delta}[(u-4m^2)^{3/2}I^{ii3}] = -(u-4m^2)\hat{\Delta}[(u-4m^2)I^{33}]$$

$$\hat{\Delta}[(u-4m^2)^{3/2}I^{333}] = -\tfrac{3}{2}(u-4m^2) + \frac{u^2-5m^2u+7m^4}{(u-2m^2)}\ln\left(\frac{u-3m^2}{m^2}\right)$$

(b) Vertex-correction graphs [Figs. 3(b) and 3(c)]

$$\hat{\Delta} I = (u-4m^2)^{-1}\ln[(u-3m^2)/m^2]$$

$$\hat{\Delta}[(u-4m^2)^{1/2}I^3] = 1 - \frac{m^2}{u-4m^2}\ln\left(\frac{u-3m^2}{m^2}\right)$$

$$\sum_{i=1}^{3}\hat{\Delta}[(u-4m^2)I^{ii}] = (u-4m^2)\hat{\Delta}[(u-4m^2)^{1/2}I^3]$$

$$\hat{\Delta}[(u-4m^2)I^{33}] = \tfrac{1}{2}(u-6m^2) + \frac{m^4}{u-4m^2}\ln\left(\frac{u-3m^2}{m^2}\right)$$

(c) Self-energy graph [Fig. 3(d)]

$$\hat{\Delta} I = -1$$

$$\hat{\Delta}[(u-4m^2)^{1/2}I^3] = -\tfrac{1}{2}(u-4m^2)$$

amplitudes; therefore, when all contributions are added together one should expect cancellations amongst the $(\ln t)^2$ terms. As we shall see, these terms do eventually cancel, but the contributions of terms of the form $t^{-m}(\ln t)^2$ for $m \geq 1$ remain.

Once the discontinuity across the cut is known for each tensor integral, the real part can be obtained by means of a dispersion relation. By ignoring polynomial contributions to the amplitude, we are able to neglect any difficulties introduced by the need for subtractions. Thus, for example, the dispersing function $\Delta_t[(t-4m^2)I^{33}]/(t-4m^2)$ is, within our sense of equality up to a polynomial, equivalent to $\Delta_t I^{33}$. A similar argument holds for all other cases.

The technical aspects involved in the evaluation of the dispersion relations are discussed in Appendix B. There it is shown that the following relations hold:

$$\int_{4m^2}^{\infty} dx\,\frac{Q(x)}{x-t} \sim Q(t)\ln\left(\frac{4m^2}{t}\right) + f(t^{-1}), \tag{3.7}$$

$$\int_{4m^2}^{\infty} dx\,R(x)\,\frac{\ln[(x-3m^2)/m^2]}{x-t} \sim -\tfrac{1}{2}R(t)\{\ln[(t-3m^2)/m^2]\}^2 + g(t^{-1}), \tag{3.8}$$

where Q and R are rational functions of x, vanishing for $x\to\infty$ and possessing a polynomial expansion in x^{-1} for m^2x^{-1} near zero, and f and g are polynomials in t^{-1}. These two expressions allow the evaluation of all the dispersion integrals, and show that the dispersion integrals have the effect of increasing by one power the logarithmic dependence of the real part of the amplitude over that of the imaginary part.

IV. ASYMPTOTIC FORM OF THE AMPLITUDE

Having obtained the asymptotic form of the tensor integrals, we turn to the evaluation of the spin-dependent coefficients A, B_α, $C_{\alpha\beta}$, $D_{\alpha\beta\gamma}$, which is mainly a tedious exercise in the algebra of Dirac matrices. Some simplification is obtained as a result of the discussion in Sec. III, where we have seen that there are only four independent tensor integrals, namely I, I^3, I^{33}, and I^{333}; all other tensors being either zero or linear combinations of these four (for example, the relation

$$I^3+\sum_{i=1}^{3} I^{ii}=0$$

holds). Thus there are in effect only four scalar coefficients F_0, F_1, F_2, and F_3, to be determined. One can rewrite the expansion given in Eq. (2.4) in terms of these coefficients obtaining

$$T_{\{\lambda\}}(t)=F_0(\{\lambda\},t)I+F_1(\{\lambda\},t)I^3 +F_2(\{\lambda\},t)I^{33}+F_3(\{\lambda\},t)I^{333}. \qquad (4.1)$$

An additional simplification is gained as a result of working in the cross channels, with $s=0$; since this last requirement, combined with angular momentum conservation, reduces the number of independent helicity states to four for each channel. These are chosen, for the t-channel amplitudes which are denoted by $T^t_{\lambda_c,\lambda_a;\lambda_d,\lambda_b}$, as the following:

$$T^t_{1,1;1/2,1/2},\quad T^t_{-1,-1;1/2,1/2},\quad T^t_{0,0;1/2,1/2},\quad T^t_{1,0;1/2,-1/2};$$

for the u-channel amplitudes, which we denote by $T^u_{\lambda_a,\lambda_d;\lambda_c,\lambda_b}$, we choose

$$T^u_{1,1/2;-1,-1/2},\quad T^u_{1,-1/2;-1,1/2},\quad T^u_{0,1/2;0,-1/2},\quad T^u_{1,1/2;0,1/2}.$$

All other nonvanishing amplitudes are related to these four by the requirements of parity, time reversal, and charge conservation.[18]

The procedure of evaluating the numerator coefficients consists of algebraic manipulations of the Dirac γ matrices. This is a tedious task which is somewhat alleviated by the judicious use of the Dirac equation: $(\not p-m)u(\vec{p},\lambda)=0$, $(\not p+m)v(\vec{p},\lambda)=0$; and identities between γ matrices such as $\gamma^\sigma\gamma^{\alpha_1}\cdots\gamma^{\alpha_n}\gamma_\sigma=-2\gamma^{\alpha_n}\cdots\gamma^{\alpha_1}$ for odd integral n. The values of the spinors and polarization vectors required in the calculation are given in Appendix A. After a long calculation, the numerators are reduced to simple rational functions of t, the results of which are shown in Tables III and IV where the values of the coefficients of Eq. (4.1) are given.

By combining these coefficients with the values of the tensor integrals obtained previously, the asymptotic contributions due to each diagram are found. These are given by the following expressions:

(i) For the t channel we define

$$\zeta\equiv m^2/t,$$
$$L_t\equiv\ln(4m^2/t),$$
$$M_t\equiv\{\ln[(t-3m^2)/m^2]\}^2,$$

and write the amplitude in the form

$$T^t_{\{\lambda\}}(s=0,t)=-mg^4(4\pi^2\sqrt{t})^{-1}(1-4m^2/t)^{-1}\overline{T}_{\{\lambda\}}. \qquad (4.2)$$

We then obtain the following contributions for the planar box [Fig. 2(a)]:

TABLE III. t-channel spin-dependent numerator factors.

Coefficients \ Helicities $\lambda_c,\lambda_a;\lambda_d,\lambda_b$	$1,1;\frac12,\frac12$	$-1,-1;\frac12,\frac12$	$0,0;\frac12,\frac12$	$1,0;\frac12,-\frac12$
(a) Planar box				
$m[2g^4(t-4m^2)^{1/2}]^{-1}F_0$	$2m^2(t-2m^2)$	0	$m^2(t-2m^2)$	$\sqrt2\, m^2(t-2m^2)$
$m(2g^4)^{-1}F_1$	$2m^2(t-4m^2)$	$2m^2(t-4m^2)$	$t(t-2m^2)$	$\sqrt2\,(t-2m^2)^2$
$m[2g^4(t-4m^2)^{1/2}]^{-1}F_2$	0	0	$-2m^2$	$\sqrt2\,(2t-m^2)$
$m(2g^4)^{-1}F_3$	$-4m^2$	$-4m^2$	$-2t$	$\sqrt2\, t$
(b) Crossed box				
$m[2g^4(t-4m^2)^{1/2}]^{-1}F_0$	0	$-2m^2(t-2m^2)$	$-(t+m^2)(t-2m^2)$	$-\sqrt2\,(t-m^2)(t-2m^2)$
$m(2g^4)^{-1}F_1$	$2m^2(t-4m^2)$	$2m^2(t-4m^2)$	$t(t-2m^2)$	$\sqrt2\,(3t^2-12m^2t+4m^4)$
$m[2g^4(t-4m^2)^{1/2}]^{-1}F_2$	$4m^2$	$4m^2$	$2(t+m^2)$	$-\sqrt2\,(3t-m^2)$
$m(2g^4)^{-1}F_3$	$-4m^2$	$-4m^2$	$-2t$	$\sqrt2\, t$

TABLE IV. u-channel spin-dependent numerator factors.

Coefficients \ Helicities $\lambda_a, \lambda_d; \lambda_c, \lambda_b$	$1, \frac{1}{2}; -1, -\frac{1}{2}$	$1, -\frac{1}{2}; -1, \frac{1}{2}$	$0, \frac{1}{2}; 0, -\frac{1}{2}$	$1, \frac{1}{2}; 0, \frac{1}{2}$
(a) Planar box				
$(2mg^4\sqrt{u})^{-1}F_0$	$-2m^2$	$-2(u-2m^2)$	$u-7m^2$	$-\sqrt{2}m^2$
$\{2mg^4[u(u-4m^2)]^{1/2}\}^{-1}F_1$	-2	-6	$(u+4m^2)/m^2$	0
$(2mg^4\sqrt{u})^{-1}F_2$	-2	-6	$(u+4m^2)/m^2$	$-\sqrt{2}(u-5m^2)/m^2$
$\{2mg^4[u(u-4m^2)]^{1/2}\}^{-1}F_3$	0	0	0	$-\sqrt{2}/m^2$
(b) Vertex corrections (sum)				
$(u-m^2)(2mg^4\sqrt{u})^{-1}F_0$	$-4m^2$	0	$2(u-5m^2)$	$2\sqrt{2}(u-3m^2)$
$(u-m^2)\{2mg^4[u(u-4m^2)]^{1/2}\}^{-1}F_1$	0	0	$-2(u+5m^2)/m^2$	$-3\sqrt{2}$
$(u-m^2)(2mg^4\sqrt{u})^{-1}F_2$	-4	0	4	$-\sqrt{2}$
(c) Self-energy graph				
$(u-m^2)^2(2mg^4\sqrt{u})^{-1}F_0$	$-2(2u-5m^2)$	0	$-(u^2-13m^4)/m^2$	$-\sqrt{2}(u-7m^2)$
$\{2mg^4[u(u-4m^2)]^{1/2}\}^{-1}F_1$	0	0	0	$\sqrt{2}/[(u-m^2)m^2]$

$$\bar{T}_{1,1;1/2,1/2} = -(1-8\zeta+10\zeta^2)(1-4\zeta)^{-1}M_t + 2\zeta^{-1}(1-12\zeta+51\zeta^2-70\zeta^3)(1-3\zeta)^{-1}L_t\,, \tag{4.3a}$$

$$\bar{T}_{-1,-1;1/2,1/2} = -(1-8\zeta+10\zeta^2)(1-4\zeta)^{-1}M_t - 2(2-19\zeta+38\zeta^2)(1-3\zeta)^{-1}L_t \tag{4.3b}$$

$$\bar{T}_{0,0;1/2,1/2} = -\tfrac{1}{2}\zeta^{-1}(1-6\zeta+6\zeta^2-16\zeta^3)(1-4\zeta)^{-1}M_t - \zeta^{-1}(1-5\zeta-14\zeta^2+48\zeta^3)(1-3\zeta)^{-1}L_t\,, \tag{4.3c}$$

$$\sqrt{2}\,\bar{T}_{1,0;1/2,-1/2} = -\zeta^{-1}(1-4\zeta+5\zeta^2-8\zeta^3)(1-4\zeta)^{-1}M_t - \zeta^{-1}(3-11\zeta-2\zeta^2+16\zeta^3)(1-3\zeta)^{-1}L_t\,. \tag{4.3d}$$

For the crossed-box graph of Fig. 2b we find:

$$\bar{T}_{1,1;1/2,1/2} = -(1-10\zeta+22\zeta^2)(1-2\zeta)^{-1}M_t + 2(1-4\zeta)L_t\,, \tag{4.4a}$$

$$\bar{T}_{-1,-1;1/2,1/2} = (1-2\zeta-6\zeta^2)(1-2\zeta)^{-1}M_t + 2(1-4\zeta)L_t\,, \tag{4.4b}$$

$$\bar{T}_{0,0;1/2,1/2} = \tfrac{1}{2}\zeta^{-1}(1-4\zeta+2\zeta^2-4\zeta^3)(1-2\zeta)^{-1}M_t + \zeta^{-1}(1-2\zeta)(1-4\zeta)L_t\,, \tag{4.4c}$$

$$\sqrt{2}\,\bar{T}_{1,0;1/2,-1/2} = \zeta^{-1}(1-4\zeta+5\zeta^2-10\zeta^3)(1-2\zeta)^{-1}M_t + \zeta^{-1}(3-14\zeta+8\zeta^2)L_t\,. \tag{4.4d}$$

(ii) For the u channel we define $\xi = m^2/u$ with $u = 4m^2 - t$, $L_u \equiv \ln(4m^2/u)$, $M_u \equiv \{\ln[(u-3m^2)/m^2]\}^2$, and write the amplitude in the form

$$T^{\mathrm{u}}_{\{\lambda\}}(s=0,t) = -mg^4(4\pi^2\sqrt{u})^{-1}(1-4m^2/u)^{-3/2}\,\hat{T}_{\{\lambda\}}\,. \tag{4.5}$$

In terms of these definitions, the contribution of the u-channel box [Fig. 3(a)] is found to be

$$\hat{T}_{1,1/2;-1,-1/2} = 2(1-2\xi)^{-1}[(2\xi-7\xi^2)M_u + (1-4\xi)L_u]\,, \tag{4.6a}$$

$$\hat{T}_{1,-1/2;-1,1/2} = 2(1-2\xi)^{-1}[(1-3\xi-\xi^2)M_u + 3(1-4\xi)L_u]\,, \tag{4.6b}$$

$$\hat{T}_{0,1/2;0,-1/2} = -(1-2\xi)^{-1}[2(1-5\xi+8\xi^2)M_u + \xi^{-1}(1-16\xi^2)L_u]\,, \tag{4.6c}$$

$$\sqrt{2}\,\hat{T}_{1,1/2;0,1/2} = -(1-2\xi)^{-1}[(2-15\xi+30\xi^2)M_u + \xi^{-1}(1-2\xi)(1-4\xi)L_u]\,; \tag{4.6d}$$

the sum of the vertex correction diagrams [Figs. 3(b) and 3(c)] give a contribution of the form

$$\hat{T}_{1,1/2;-1,-1/2} = 2(1-\xi)^{-1}[-\xi(1-3\xi)M_u + (1-6\xi)(1-4\xi)L_u]\,, \tag{4.7a}$$

$$\hat{T}_{1,-1/2;-1,1/2} = 0\,, \tag{4.7b}$$

$$\hat{T}_{0,1/2;0,-1/2} = 2(1-\xi)^{-1}[(1-4\xi+\xi^2)M_u + \xi^{-1}(1-4\xi)(1-14\xi^2)L_u]\,, \tag{4.7c}$$

$$\sqrt{2}\,\hat{T}_{1,1/2;0,1/2} = (1-\xi)^{-1}[(2-11\xi+11\xi^2)M_u + (7-30\xi)(1-4\xi)L_u]\,; \tag{4.7d}$$

the self-energy graph of Fig. 3(d) gives

$$\hat{T}_{1,1/2;-1,-1/2} = -2(1-4\xi)^2(2-5\xi)(1-\xi)^{-2}L_u\,, \tag{4.8a}$$

$$\hat{T}_{1,-1/2;-1,1/2} = 0\,, \tag{4.8b}$$

$$\hat{T}_{0,1/2;0,-1/2} = -(1-4\xi)^2(1-13\xi^2)\xi^{-1}(1-\xi)^{-2}L_u\,, \tag{4.8c}$$

$$\sqrt{2}\,\hat{T}_{1,1/2;0,1/2} = (1-4\xi)^2(1-7\xi+18\xi^2) \times \xi^{-1}(1-\xi)^{-2}L_u\,. \tag{4.8d}$$

Once all diagrams for each channel are added together, the s-channel amplitude is recovered by means of crossing matrix relations; these relations are listed in Appendix C. In this way, one obtains the asymptotic s-channel amplitude corresponding to the limits of large t and of large u. The amplitude coming from the right-hand cut in z_s ($t\sim\infty$ and $s=0$) is given by

$$T^s_{\lambda_c,\lambda_d;\lambda_a,\lambda_b} = ig^4\sqrt{t}\,[8m\pi^2(1-4\zeta)]^{-1} \times \bar{T}^s_{\lambda_c,\lambda_d;\lambda_a,\lambda_b}\,, \tag{4.9}$$

with $\bar{T}^s_{\{\lambda\}}$ having the following values:

$$\bar{T}^s_{1,1/2;1,1/2} = (-3\zeta+22\zeta^2-60\zeta^3+92\zeta^4)\bar{M}_t + (1-28\zeta+148\zeta^2-196\zeta^3)\bar{L}_t\,, \tag{4.10a}$$

$$\bar{T}^s_{1,1/2;0,-1/2} = \sqrt{2}\,(-2\zeta+14\zeta^2-38\zeta^3+44\zeta^4)\bar{M}_t + \sqrt{2}\,(1-16\zeta+58\zeta^2-52\zeta^3)\bar{L}_t\,, \tag{4.10b}$$

$$\bar{T}^s_{0,1/2;0,1/2} = (-2\zeta+28\zeta^2-112\zeta^3+152\zeta^4)\bar{M}_t + (2-24\zeta+112\zeta^2-168\zeta^3)\bar{L}_t\,, \tag{4.10c}$$

$$\bar{T}^s_{1,-1/2;0,1/2} = \sqrt{2}\,(6\zeta^2-26\zeta^3+20\zeta^4)\bar{M}_t + \sqrt{2}\,(1-4\zeta+6\zeta^2-12\zeta^3)\bar{L}_t\,, \tag{4.10d}$$

$$\bar{T}^s_{1,1/2;-1,1/2} = (-\zeta+14\zeta^2-64\zeta^3+84\zeta^4)\bar{M}_t + (1-8\zeta+16\zeta^2-12\zeta^3)\bar{L}_t\,, \tag{4.10e}$$

$$\bar{T}^s_{-1,1/2;-1,1/2} = (\zeta+6\zeta^2-36\zeta^3+44\zeta^4)\bar{M}_t + (1-4\zeta+44\zeta^2-116\zeta^3)\bar{L}_t\,, \tag{4.10f}$$

where

$$\bar{L}_t \equiv (1-3\zeta)^{-1}L_t$$

and

$$\bar{M}_t \equiv [(1-2\zeta)(1-4\zeta)]^{-1}M_t\,.$$

For the left-hand-cut contribution to the amplitude ($u\sim\infty$, $s=0$), we obtain

$$T^s_{\lambda_c,\lambda_d;\lambda_a,\lambda_b} = -g^4\sqrt{u}\,[8m\pi^2(1-4\xi)^{3/2}]^{-1} \times \hat{T}^s_{\lambda_c,\lambda_d;\lambda_a,\lambda_b}\,, \tag{4.11}$$

where $\hat{T}^s_{\{\lambda\}}$ is given by

$$\hat{T}^s_{1,1/2;1,1/2} = (\xi-\xi^2-22\xi^3+22\xi^4)\hat{M}_u + (5\xi-31\xi^2-25\xi^3+132\xi^4)\hat{L}_u\,, \tag{4.12a}$$

$$\hat{T}^s_{1,1/2;0,-1/2} = \sqrt{2}\,(-\xi+4\xi^2)\hat{M}_u + \sqrt{2}\,(4\xi-25\xi^2+54\xi^3-42\xi^4)\hat{L}_u\,, \tag{4.12b}$$

$$\hat{T}^s_{0,1/2;0,1/2} = (2\xi-6\xi^2-4\xi^3+4\xi^4)\hat{M}_u + (18\xi-62\xi^2+62\xi^3)\hat{L}_u\,, \tag{4.12c}$$

$$\hat{T}^s_{1,-1/2;0,1/2} = \sqrt{2}\,(-\xi+6\xi^2-12\xi^3+4\xi^4)\hat{M}_u + \sqrt{2}\,(2\xi-13\xi^2-4\xi^3+42\xi^4)\hat{L}_u\,, \tag{4.12d}$$

$$\hat{T}^s_{1,1/2;-1,1/2} = (\xi-3\xi^2+6\xi^3-10\xi^4)\hat{M}_u + (11\xi-19\xi^2+27\xi^3-48\xi^4)\hat{L}_u\,, \tag{4.12e}$$

$$\hat{T}^s_{-1,1/2;-1,1/2} = (\xi-5\xi^2+2\xi^3+6\xi^4)\hat{M}_u + (9\xi-55\xi^2+91\xi^3-36\xi^4)\hat{L}_u\,, \tag{4.12f}$$

where we have defined

$$\hat{L}_u \equiv (1-4\xi)\,[(1-2\xi)(1-\xi)^2]^{-1}L_u$$

and

$$\hat{M}_u \equiv [(1-2\xi)(1-\xi)]^{-1}M_u\,.$$

A look at the asymptotic expressions that are obtained show that, aside from the factor proportional to $\sqrt{t}$ which comes from the kinematic singularity in the amplitude, the asymptotic dependence is of the form $t^{-n}(\ln t)+t^{-m}(\ln t)^2$ for n,m non-negative integers. Moreover, the leading $(\ln t)^2$ dependence has been canceled by the process of summing up all contributions, thus leaving the expected $\ln t$ leading behavior.[10] This cancellation, however, does not take place for the next leading order, an effect which can be attributed to the singularity-enhancing properties of spin numerators. The basis for this mechanism will be considered in Sec. VI, where the unitarity condition is discussed.

V. AN ALTERNATE METHOD

The new features in our calculation can be corroborated by means of an alternate scheme. For this purpose, we consider in this section the t-channel planar box diagram as a representative example and discuss an alternative to the Cutkosky rules approach. This scheme takes advantage of some symmetry properties of the integrals and of the properties of dilogarithms, and enables one to directly evaluate the integrals in question. While the method is somewhat more involved than the one previously used, it has the advantage of giving exact solutions and, at the same time, it provides useful insight into the *modus operandi* of other asymptotic methods, such as the techniques developed by Federbush and Grisaru.[2]

Consider the integral representing the planar-box diagram of Fig. 2(a). In terms of the Feynman α parameters it has the form

$$T(s=0,t)=\frac{3!}{(2\pi)^4}\int_0^1\prod_{j=1}^{4}(d\alpha_j)\,\delta\left(\sum_{i=1}^{4}\alpha_i-1\right)\times\int d^4l\,\frac{\tilde{N}(l,t)}{[\psi(l,\{\alpha_i\},t)]^4}\qquad(5.1)$$

with

$$\psi(l,\{\alpha_i\},t)\equiv(\alpha_1+\alpha_2)\,(l^2-m^2)+\alpha_3[(l+p_1)^2-m^2]+\alpha_4[(l-\bar{p}_2)^2-m^2]\,,$$

where

$$\tilde{N}(l,t)\equiv 2mg^4N(l,t)\,.$$

$N(l,t)$ is as defined by Eq. (2.2), and we have omitted the helicity labels $\{\lambda\}$. The four-dimensional integration on l is performed by making the change of variables $k=l+\alpha_3p_1-\alpha_4p_2$ and applying standard methods. Since any numerator having an odd dependence in k vanishes as a result of this integration, and since the numerator is at most a cubic polynomial in k, the spin numerator can be written as

$$\tilde{N}(l,t)=\tilde{N}(l(k,\alpha_3,\alpha_4,t),t)=A(\alpha_3,\alpha_4,t)+B(\alpha_3,\alpha_4,t)k^2\,,\qquad(5.2)$$

plus vanishing terms, where A and B are helicity-dependent functions. The integration of k thus gives

$$T(s=0,t)=\left(-\frac{1}{16\pi^2}\right)[I_A(t)+I_B(t)]\,,$$

where

$$I_A(t)=\int_0^1\prod_{j=1}^{4}(d\alpha_j)\,\delta\left(\sum_{i=1}^{4}\alpha_i-1\right)\frac{A(\alpha_3,\alpha_4,t)}{[\phi(\alpha_3,\alpha_4,t)]^2}\,,\qquad(5.3)$$

$$I_B(t)=2\int_0^1\prod_{j=1}^{4}(d\alpha_j)\,\delta\left(\sum_{i=1}^{4}\alpha_i-1\right)\frac{B(\alpha_3,\alpha_4,t)}{\phi(\alpha_3,\alpha_4,t)}\,,\qquad(5.4)$$

and

$$\phi(\alpha_3,\alpha_4,t)\equiv\alpha_3\alpha_4t-m^2[1-(\alpha_3+\alpha_4)+(\alpha_3+\alpha_4)^2]+i\epsilon\,.$$

At this point it is possible to extract the leading behavior of t by noting that for large t the dominant term must come from making the denominator, $\phi(\alpha_3,\alpha_4,t)$ as small as possible; thus one restricts α_3 and/or α_4 to values near zero. This is the approach suggested by Federbush and Grisaru,[2] who showed that the correct leading asymptotic value is obtained by neglecting terms which are small whenever the α's that multiply the asymptotic variable in the denominator are restricted to values near zero. This result, they showed, was independent of the size of the range $[0,\epsilon']$ to which each α was restricted. In our case for example, the integral I_A would be approximated by taking

$$\phi(\alpha_3,\alpha_4,t)\approx\alpha_3\alpha_4t-m^2+i\epsilon\,,$$

and restricting the limits of integration in α_3 and α_4 to the ranges $[0,\epsilon_3']$ and $[0,\epsilon_4']$, respectively. In the case of the scalar numerator [i.e., $A(\alpha_3,\alpha_4,t)=1$ and $B(\alpha_3,\alpha_4,t)=0$] one finds exact agreement with the value of the leading term that is obtained from integrating the dispersion relation for ΔI in Table I. When one attempts to extend this method to integrals with α dependent numerators, one finds that the results of the integrations in α_3 and α_4 are not always independent of ϵ_3' and ϵ_4'. Furthermore, this technique does not yield any information on subleading asymptotic terms.

Returning to the integration of Eq. (5.1), we proceed by introducing a new set of variables to replace the α's; thus we let

$$x=\tfrac{1}{2}(\alpha_3+\alpha_4),\quad y=\tfrac{1}{2}(\alpha_3-\alpha_4)\,,\qquad(5.5)$$

which is essentially a rotation of the axes by 45 degrees. This, however, displays adscititious symmetries of the integrand. Applying this transformation and carrying out the α_1 and α_2 integrations one finds

$$I_A=\frac{2}{t^2}\int_0^{1/2}dx\int_{-x}^{x}dy\,\frac{(1-2x)\tilde{A}(x,y)}{[y^2-\delta^2(x)]^2}\,,\qquad(5.6)$$

$$I_B=\frac{-4}{t}\int_0^{1/2}dx\int_{-x}^{x}dy\,\frac{(1-2x)\tilde{B}(x,y)}{y^2-\delta^2(x)}\,,\qquad(5.7)$$

where $\delta^2(x) \equiv (1-4\zeta)x^2 + 2\zeta x - \zeta$ with $\zeta \equiv m^2/t$. The functions $\tilde{A}(x,y)$ and $B(x,y)$ are obtained from $A(\alpha_3, \alpha_4)$ and $B(\alpha_3, \alpha_4)$ by applying the above transformation, and the small imaginary part $i\epsilon$ in the denominator has been omitted with the understanding that these expressions actually refer to the Cauchy principal value of the integral.

With the aid of Fig. 4, one can see the advantage gained by changing to the x-y variables. The path of singularities of the integrals in Eqs. (5.6) and (5.7), which corresponds to zeros in the denominator, i.e., $y^2 = \delta^2(x)$, has the form of an equilateral hyperbola when viewed in the $\alpha_3 - \alpha_4$ frame of reference. In the x-y frame this path becomes symmetric in the y variable, thus making the y integrand symmetric.

In order to carry out the y integration, we separate the range into two regions: one for which $\delta^2(x)$ is non-negative (called region I), where the integration obtains hyperbolic functions; and the second region where $\delta^2(x)$ is negative, thereby resulting in trigonometric functions after integration. The dividing line for these two regions is given by $x = x_0 - \rho$ for ρ a small positive number, where x_0 is the positive root of $\delta^2(x)$ and is given by

$$x_0 = \{-\zeta + [\zeta(1-3\zeta)]^{1/2}\}(1-4\zeta)^{-1}\ ; \tag{5.8}$$

thus, for large t (ζ small) x_0 has the form

$$x_0 \sim 0 + \zeta^{1/2} - \zeta + \tfrac{5}{2}\zeta^{3/2} + \cdots .$$

Region I includes the hyperbolic singularity path $y^2 = \delta^2(x)$, so that in order to carry out the y integration for this region, one must check to see that the principal value exists. Although we shall omit the proof, one finds by carefully defining ϵ-neighborhoods around the hyperbolic singularity path, and splitting the integration range into appropriate limits in ϵ, that all ϵ dependence is canceled at the end of the calculations; and that the contributions coming from ϵ neighborhoods of $y^2 = \delta^2(x)$ vanish as $\epsilon \to 0$.

It is interesting to note the interpretation that our picture gives to the limit $t \to \infty$. In this limit the value of x_0 goes to zero while the hyperbola $y^2 = \delta^2(x)$ tends to its asymptotes. The contributions coming from region II and from the section of region I to the left of $y^2 = \delta^2(x)$ will therefore be negligible in the limit of large t. Since after the y integration is performed, the hyperbola coalesces into the point $x = x_0$, which is tending to zero, the asymptotic behavior of the integral must be determined by the properties of the integrand in the neighborhood of x_0. In the language of Federbush and Grisaru, this is just the statement that the α's should be small, excepting that our refinement indicates that the leading behavior is actually connected with letting $x = \frac{1}{2}(\alpha_3 + \alpha_4)$ tend to $x_0 \sim 0 + \zeta^{1/2} + \cdots$ from the right. Furthermore, the general nonpolynomial asymptotic behavior is given only by the behavior of the integrand for $y^2 \geq \delta^2(x)$.

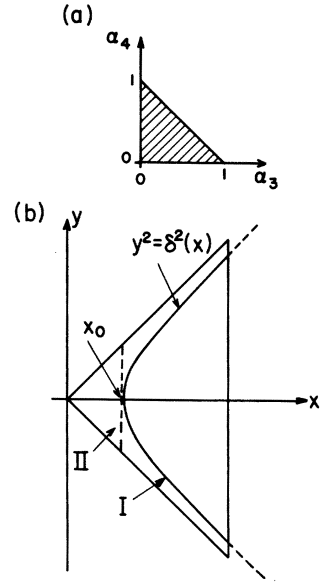

FIG. 4. Integration boundaries for the t-channel planar box diagram: (a) α space, and (b) x-y space.

The y integration is simple to perform, and only terms which are even in y contribute. Thus one obtains for the contribution of region I the following expression:

$$-16\pi^2 T(0,t) = \int_{x_0}^{1/2} dx \frac{xF(x,t)}{\{\delta^2(x)[\delta^2(x)-x^2]\}} + \int_{x_0}^{1/2} dx\, \tfrac{1}{2}F(x,t)\,[\delta^2(x)]^{-3/2}\ln\{[x+\delta(x)]/[x-\delta(x)]\}$$
$$+ \int_{x_0}^{1/2} dx\, G(x,t)\,[\delta(x)]^{-1}\ln\{[x+\delta(x)]/[x-\delta(x)]\}\ , \tag{5.9}$$

where $F(x,t)$ and $G(x,t)$ are defined in terms of the

$$\tilde{A}(x,y) \equiv A_1(x) + tA_2(x)y^2$$

and

$$\tilde{B}(x,y) \equiv B_1(x)$$

by the relations

$$F(x,t)\equiv 2[A_1(x)+t\,\delta^2(x)A_2(x)]/t^2\,, \qquad (5.10)$$

$$G(x,t)\equiv 2[2B_1(x)-A_2(x)]/t\,. \qquad (5.11)$$

The contribution coming from region II is similar to the above expression, except that the integration limits are $[0,x_0]$ and the logarithm function is replaced by $2\tan^{-1}\{x/[-\delta^2(x)]^{1/2}\}$.

Two general types of integrals need to be considered in order to evaluate the amplitude in Eq. (5.9). These are

$$C^n=\int_{x_0}^{1/2}dx\,x^n/\{\delta^2(x)\,[\delta^2(x)-x^2]\}\,, \qquad (5.12)$$

$$D^{n,m}=\int_{x_0}^{1/2}dx\,x^n[\delta^2(x)]^{-m/2}\ln\left(\frac{x+\delta(x)}{x-\delta(x)}\right). \qquad (5.13)$$

Type C^n integrals are evaluated by simply separating the function into a sum of fractions; $D^{n,m}$ on the other hand, requires an elaborate procedure which consists first of an integration by parts in which one defines

$$u=\ln\{[x+\delta(x)]/[x-\delta(x)]\}$$

and

$$dv=x^n[\delta^2(x)]^{-m/2}dx\,.$$

This reduces Eq. (5.13) to integrals of the form C^n plus the following integral:

$$E=\int_{x_0}^{1/2}dx(x-1)\,\{\delta^2(x)\,[\delta^2(x)-x^2]\}^{-1}\times\ln[z+(z^2-a^2)^{1/2}]\,, \qquad (5.14)$$

where $z=x+\zeta/(1-4\zeta)$ and $a\equiv\frac{1}{2}\Delta/(1-4\zeta)$, with Δ being the discriminant of $\delta^2(x)=0$, i.e., $\Delta^2=4\zeta^2+4\zeta(1-4\zeta)$. This last integral is extremely involved. One way to evaluate it is to introduce the change of variables $w=z+(z^2-a^2)^{1/2}$; the denominator then assumes the form of a quartic polynomial in w which can be factored into two real quadratic polynomials. By separating the factor multiplying $\ln w$ into a sum of two fractions, one is then able to cast this integral in dilogarithmic form. This calculation then obtains

$$\zeta E=\tfrac{1}{4}\ln^2\zeta-\tfrac{1}{2}\ln\zeta\ln\left(\frac{\frac{1}{2}\sigma^3[1+(1-4\zeta)^{1/2}]}{(1-4\zeta)}\right)+\ln\tfrac{1}{2}[1+(1-4\zeta)^{1/2}]\ln\left(\frac{\sigma^3}{1-4\zeta}\right)+\mathrm{Li}_2\{2\sigma[1-(1-4\zeta)^{1/2}],\theta\}$$
$$-\mathrm{Li}_2\{2\sigma[1+(1-4\zeta)^{1/2}],\theta\}+\mathrm{Li}_2\{2\zeta[1-(1-4\zeta)^{1/2}],\theta\}-\mathrm{Li}_2\{2\zeta[1+(1-4\zeta)^{1/2}],\theta\}\,, \qquad (5.15)$$

where $\sigma\equiv(1-3\zeta)^{1/2}$ and $\cos\theta=\frac{1}{2}\sigma$.

The integrals corresponding to region II can be evaluated by means of similar integration techniques. However, it has been previously remarked in connection with the effect that asymptotic t has on the hyperbolic singularity path, that their contribution will be at most a polynomial in t^{-1}. That this is so can also be gleaned from the fact that for $0\le x\le x_0$ the integrand is bounded and has a uniformly convergent power-series expansion in t^{-1}; so that term by term integration of the power series is possible. For the purpose of studying nonpolynomial asymptotic behavior, this contribution can therefore be ignored.

Through the use of Eq. (5.15) and the integration techniques outlined above, one is able to evaluate the contribution to region I exactly. The final answer consists of a rather complex algebraic expression involving a number of dilogarithms. By a careful application of dilogarithmic relations such as those listed in Appendix B, one can then compute the asymptotic expansion of the result. One finds, not surprisingly so, exact agreement with the t-channel box calculation that was obtained by the Cutkosky rules approach.

An analogous approach to that discussed above can be used in evaluating all other graphs. If one considers, for example, the integral corresponding to the t-channel crossed box, one obtains expressions similar to Eqs. (5.3) and (5.4) with the denominator ϕ replaced by

$$\phi(\alpha_1,\alpha_2,\alpha_3,\alpha_4,t)=f(\alpha_1,\alpha_2,\alpha_3,\alpha_4)+(\alpha_1\alpha_3-\alpha_2\alpha_4)t$$

for some function $f(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, and subject to the δ-function constraint $\delta(\sum_{i=1}^4\alpha_i-1)$. In this case one introduces two rotations by defining

$$x_1=\tfrac{1}{2}(\alpha_1+\alpha_3),\quad y_1=\tfrac{1}{2}(\alpha_1-\alpha_3)\,,$$
$$x_2=\tfrac{1}{2}(\alpha_2+\alpha_4)\,,\quad y_2=\tfrac{1}{2}(\alpha_2-\alpha_4)\,,$$

which transform the δ function into $\delta(2x_1+2x_2-1)$. The picture associated with the singularities of the integral becomes in this case that of a hyperboloid in four dimensions intersecting a plane. The hyperbolic surface thus obtained tends to its triangular asymptotes as t becomes large, and the analysis can proceed along a similar but somewhat more complicated line as before. The dominant asymptotic behavior, in fact, becomes clearly determined by the values of the α's near the apex of the hyperboloid and the problem is again simplified by virtue of the geometrical picture associated with the integral.

VI. ELASTIC UNITARITY AND ENHANCED SINGULAR BEHAVIOR

In this section we study the origins of the increased singular behavior observed in our amplitudes in terms of the unitarity principle. Two-body elastic unitarity provides an alternative way of examining the fourth-order process, since it relates it to second-order amplitudes and allows one to view the scattering process in a complementary and useful way. For a t-channel process, in which large t corresponds to high energy, unitarity can be expressed in the form

$$D_t(s(z_t,t),t)=(8\pi\sqrt{t})^{-1}(t-4m^2)^{1/2}\times\int d\Omega_t A^{\dagger}_{in}(s',t_-)A_{nf}(s'',t_+)\,, \tag{6.1}$$

where $A(s,t)$ is an amplitude, $D_t(s,t)$ is the discontinuity across a t-channel cut and i, n, and f label the initial, intermediate, and final states, respectively; $t_\pm=t\pm i\epsilon$ for small positive ϵ; $s'=2q_t^{\,2}(1-z_t')$ and $s''=2q_t^{\,2}(1-z_t'')$ are related by the addition theorem for cosines:

$$z_t'=z_tz_t''+[1-(z_t'')^2]^{1/2}[1-(z_t)^2]^{1/2}\cos\phi$$

and Ω_t and ϕ refer to the intermediate-state phase space.

The amplitudes $A^{\dagger}_{in}(s',t)$ and $A_{nf}(s'',t)$ can be written by means of the Mandelstam representation as dispersion integrals in s, over the s and u discontinuities. By making use of this representation in Eq. (6.1), and interchanging orders of integration, one obtains

$$D_t(s,t)=(t-4m^2)^{1/2}(4\pi^2\sqrt{t})^{-1}\int_{s_0}^{\infty}ds_1\int_{s_0}^{\infty}ds_2\{[D_s^{in}(s_1,t_-)+D_u^{in}(s_1,t_-)]\times[D_s^{nf}(s_2,t_+)+D_u^{nf}(s_2,t_+)]\}\,U(s_1,s_2,s,t)\,, \tag{6.2}$$

where $D^{in}(s_1,t)$ and $D^{nf}(s_2,t)$ refer to the discontinuities of $A^{\dagger}_{in}$ and A_{nf}, respectively, and where the Mandelstam unitarity function[19] $U(s_1,s_2,s,t)$ is defined by

$$U(s_1,s_2,s,t)=\tfrac{1}{2}\pi\int_{-1}^{1}dz_t''\int_0^{2\pi}d\phi[s_1-s'(z_t',t)]^{-1}[s_2-s''(z_t'',t)]^{-1}. \tag{6.3}$$

In writing Eq. (6.2) we have assumed an unsubtracted form for the Mandelstam representation; later on we will point out what modifications subtractions introduce.

The phase-space integral given by $U(s_1,s_2,s,t)$ can be shown with some effort to give[19]

$$U(s_1,s_2,s,t)=K^{-1/2}\ln\left(\frac{s-s_1-s_2-2s_1s_2(t-4m^2)^{-1}+\sqrt{K}}{s-s_1-s_2-2s_1s_2(t-4m^2)^{-1}-\sqrt{K}}\right), \tag{6.4}$$

where

$$K\equiv s^2+s_1^{\,2}+s_2^{\,2}-2(ss_1+ss_2+s_1s_2)-4ss_1s_2(t-4m^2)^{-1}.$$

This result, together with Eq. (6.2), provides a way to compute the fourth-order discontinuity from the second-order amplitudes.

The relevant diagrams contributing to the unitarity equation up to fourth order are shown in Fig. 5. As can be easily seen, the discontinuities associated with these second-order processes are simply given by Dirac δ functions in the appropriate energy transfer variables. Thus one can write

$$D_s^{in}(s_1,t)=A_1(s_1,t)\,\delta(s_1-m^2)\,,$$

$$D_u^{in}(s_1,t)=0\,,$$

$$D_s^{nf}(s_2,t)=A_{21}(s_2,t)\delta(s_2-m^2)\,,$$

$$D_u^{nf}(s_2,t)=A_{22}(s_2,t)\delta(u_2-m^2)\,,$$

where A_1, A_{21}, and A_{22} are given spin-dependent functions, and $u_2=4m^2-t-s_2$.

Using these expressions for the second-order discontinuity, the integrations in Eq. (6.2) can be carried out. One then finds that the fourth-order

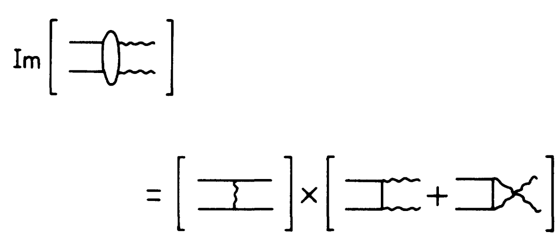

FIG. 5. The unitarity relation for the t-channel scattering up to fourth order.

discontinuity in t is given by

$$D_t(s,t)=A(t)U(s_1=m^2,s_2=m^2,s,t)+B(t)U(s_1=m^2,s_2=3m^2-t,s,t)\,, \tag{6.5}$$

where

$$A(t)=(t-4m^2)^{1/2}(4\pi^2\sqrt{t}\,)^{-1}[A_1(m^2,t)+A_{21}(m^2,t)]$$

and

$$B(t)=(t-4m^2)^{1/2}(4\pi^2\sqrt{t}\,)^{-1}A_{22}(3m^2-t,t)\,.$$

The analysis of $D_t(s,t)$ reduces thus to a consideration of the properties of $U(s_1,s_2,s,t)$.

Keeping in mind that a logarithmic dependence for D_t corresponds to a $(\ln t)^2$ behavior in the amplitude, we can study the form of the logarithm in $U(s_1,s_2,s=0,t)$. We find that in general U contributes a $\ln t$ behavior for large t, and only at the value $s=0$, $s_1=m^2=s_2$, a limiting process cancels the $\ln t$ contribution. For the values of U in Eq. (6.5) we obtain

$$U(s_1=m^2,s_2=m^2,s=0,t)=\frac{t-4m^2}{m^2(t-3m^2)}\,,$$

$$U(s_1=m^2,s_2=3m^2-t,s=0,t)=2(t-2m^2)^{-1}\times\ln\left(\frac{t-3m^2}{m^2}\right).$$

We therefore conclude that whenever the coefficient B is nonzero, which corresponds to the inclusion of the crossed-box contribution, one can expect a polynomial-times-$(\ln t)^2$ type of dependence in the amplitude. This is in fact the behavior observed in the case of scalar scattering, which corresponds to setting the coefficients A and B equal to constants; the answer obtained from Eq. (6.5) for this case is in full agreement with the values of the tensor integral I of Table I.

When one extends the analysis to spin-dependent amplitudes, a second mechanism for the generation of a $\ln t$ dependence in D_t arises; this is through the presence of subtractions in the dispersion relations used to replace A_{in} and A_{nf} in Eq. (6.1). For the scalar case, such subtractions were not necessary since the absorptive part of the amplitude vanished at large t. However, for the case with spin, where numerators affect the convergence, subtractions are almost always the rule.

Subtracted dispersion relations, when inserted in Eq. (6.1), modify the unitarity integral and give rise to additional terms with a single linear polynomial factor in the denominator. Thus, for example, one obtains additional contributions of the form

$$\int_{-1}^{1}dz''\int_0^{2\pi}d\phi[s_2-s''(z'',t)]^{-1}$$

to the discontinuity D_t. Such a term gives rise to a $\ln t$ dependence in D_t, which is present, contrary to the previously discussed case, also for the direct box diagram. Thus we have a clear example of how a diagram, which in the scalar case has an nonlogarithmic imaginary part, can be so affected by a spin numerator as to develop a more singular dependence in t — in this case $\ln t$.

The singularity enhancement effects that spin produces are thus shown to be connected with the existence of subtractions in the Mandelstam representation. These subtractions are not, however, the only way in which one can view the singularity producing mechanism, for our discussion of Eq. (6.1) could have also proceeded by inserting the appropriate second-order expressions for A_{in} and A_{nf}. These consist of spin-dependent numerators divided by single propagator-type denominators. Proceeding, one would then obtain integral expressions of the form

$$\int_{-1}^{1}dz''\int_0^{2\pi}d\phi\,\frac{N(z'',t)}{[s'(z',t)-m^2]\,[s''(z'',t)-m^2]}\,,$$

where $N(z'',t)$ is some spin-derived function. The mechanism for producing a $\ln t$ dependence can then also be viewed in terms of the existence of cancellations between the numerator and denominator factors of the integral. These numerator factors are of course the same ones responsible for the introduction of subtractions to the Mandelstam representation.

ACKNOWLEDGMENTS

I wish to thank Professor Ernest Abers for his constant encouragement and interest during all stages of this work. For several interesting conversations, I extend my appreciation to Professor Vigdor Teplitz.

APPENDIX A: CONVENTIONS, SPINORS, AND POLARIZATION VECTORS

A summary of the conventions used in calculating the fourth-order amplitude is presented here. In our calculation, we consider the elastic scattering of a massive photon having initial four-momentum $k_1=(E,k\hat{n}_z)$ and helicity λ_a by an electron of equal mass with initial four-momentum $p_1=(E,-k\hat{n}_z)$ and helicity λ_b in the s-channel center of mass. The final four-momenta are labeled $k_2=(E,k\hat{n})$ and $p_2=(E,-k\hat{n})$, with helicities λ_c and λ_d, respectively, where the scattering angle is given by $\cos\theta_s=\hat{n}\cdot\hat{n}_z$. Helicity states $|\vec{p},\lambda\rangle$ are constructed following the phase conventions of Jacob and Wick,[18] and are normalized to give $\langle \vec{p},\lambda|\vec{p}\,',\lambda'\rangle = \delta^3(\vec{p}-\vec{p}\,')\delta'_{\lambda\lambda}$. The invariant scattering matrix is defined as

$$T_{\lambda_c,\lambda_d;\lambda_a,\lambda_b}(s,t)=\langle k_2,p_2;\lambda_c,\lambda_d\,|\,T\,|\,k_1,p_1;\lambda_a,\lambda_b\rangle\,,$$

with T being related to the S matrix by

$$\langle f\,|\,S\,|\,i\rangle = \langle f\,|\,i\rangle + (i/2E^2)\delta^4(p_1+k_1-p_2-k_2) \times\langle f\,|\,T\,|\,i\rangle\,,$$

where $|i\rangle$ and $|f\rangle$ denote the respective initial and final two-particle states. In terms of this definition, the differential cross section is given by $d\sigma/d\Omega = |\pi T/2E|^2$.

Spinor and polarization vectors for each scattering channel are constructed by using the Jacob and Wick prescriptions, with the following choices for the "number two" particles: In the s channel these are taken to be particles b and d (the two electrons); for the t and u channels, particles a and b are chosen. The spinor normalizations are taken as

$$\bar{u}(\vec{p},\lambda)u(\vec{p},\lambda')=\delta_{\lambda\lambda'}\,,$$
$$\bar{v}(\vec{p},\lambda)v(\vec{p},\lambda')=-\delta_{\lambda\lambda'}\,,$$

and the polarization vectors are normalized to unity. Working in the appropriate center-of-mass frame for each channel, the following parameters are obtained:

For the s channel, we define

$$E_s=\tfrac{1}{2}\sqrt{s}\,,$$
$$q_s=\tfrac{1}{2}(s-4m^2)^{1/2}$$

and take the x-z plane as the scattering plane. The polarization vectors are hence given by

$$\epsilon(k_1,\lambda_a)=\begin{cases}(-\sqrt{2}\,)^{-1}(0,\lambda_a,i,0)\,, & \lambda_a=\pm1\\ m^{-1}(q_s,0,0,E_s)\,, & \lambda_a=0\end{cases}$$

$$\epsilon^*(k_2,\lambda_c)=\begin{cases}(\sqrt{2})^{-1}(0,-\lambda_c\cos\theta_s,i,\lambda_c\sin\theta_s)\,, & \lambda_c=\pm1\\ m^{-1}(q_s,E_s\sin\theta_s,0,E_s\cos\theta_s)\,, & \lambda_c=0\end{cases}$$

and the spinors describing the initial and final nucleons are

$$u(p_1,\lambda_b)=[2m(E_s+m)]^{-1/2}\begin{pmatrix}E_s+m\\2\lambda_b q_s\end{pmatrix}\chi_{-\lambda_b}\,,$$

$$\bar{u}(p_2,\lambda_d)=[2m(E_s+m)]^{-1/2}\chi^{\dagger}_{-\lambda_d} \times(E_s+m,-2\lambda_d q_s)\exp(-i\theta_s\tfrac{1}{2}\sigma_y)\,.$$

For the t channel, we set $s=0$ and define $E_t=\frac{1}{2}\sqrt{t}$, $q_t=\frac{1}{2}(t-4m^2)^{1/2}$. The four-momenta for the process (see Fig. 2) are then $\bar{k}_1=p_1=(E_t,0,0,-q_t)$, and $k_2=\bar{p}_2=(E_t,0,0,q_t)$. The polarization vectors have the form

$$\epsilon^*(\bar{k}_1,\lambda_a)=(\sqrt{2}\,)^{-1}(0,\lambda_a,i,0)\,,\quad \lambda_a=\pm1$$
$$=m^{-1}(-q_t,0,0,E_t)\,,\quad \lambda_a=0$$
$$\epsilon(k_2,\lambda_c)=(\sqrt{2}\,)^{-1}(0,-\lambda_c,i,0)\,,\quad \lambda_c=\pm1$$
$$=m^{-1}(q_t,0,0,E_t)\,,\quad \lambda_c=0$$

and for the spinors one obtains

$$u(p_1,\lambda_b)=[2m(E_t+m)]^{-1/2}\begin{pmatrix}E_t+m\\2\lambda_b q_t\end{pmatrix}\chi_{-\lambda_b}\,,$$

$$\bar{v}(\bar{p}_2,\lambda_d)=(-1)^{\lambda_d+1/2}[2m(E_t+m)]^{-1/2} \times(2\lambda_d q_t,E_t+m)\chi^{\dagger}_{-\lambda_d}\,.$$

In the u channel (Fig. 3), with $s=0$, $E_u\equiv\frac{1}{2}\sqrt{u}$, and $q_u\equiv\frac{1}{2}(u-4m^2)^{1/2}$, the four-momenta are given by

$$\bar{k}_2=p_2=(E_u,0,0,q_u)$$

and

$$\bar{k}_1=p_1=(E_u,0,0,-q_u)\,.$$

The polarization vectors are

$$\epsilon(\bar{k}_2,\lambda_c)=(\sqrt{2}\,)^{-1}(0,-\lambda_c,-i,0)\,,\quad \lambda_c=\pm1$$
$$=m^{-1}(q_u,0,0,E_u)\,,\quad \lambda_c=0$$
$$\epsilon^*(\bar{k}_1,\lambda_a)=(\sqrt{2}\,)^{-1}(0,\lambda_a,i,0)\,,\quad \lambda_a=\pm1$$
$$=m^{-1}(-q_u,0,0,E_u)\,,\quad \lambda_a=0$$

and the spinors are given by

$$u(p_1,\lambda_b)=[2m(E_u+m)]^{-1/2}\begin{pmatrix}E_u+m\\2\lambda_b q_u\end{pmatrix}\chi_{-\lambda_b}\,,$$

$$\bar{u}(p_2,\lambda_d)=[2m(E_u+m)]^{-1/2}(E_u+m,-2\lambda_d q_u)\chi^{\dagger}_{\lambda_d}\,.$$

APPENDIX B: THE INTEGRATION OF DISPERSION RELATIONS

Our aim in this section is to describe in some detail the procedure for carrying out the dispersion integrals used in this paper. Consider first an unsubtracted dispersion relation

$$I(x)=\pi^{-1}\int_{4m^2}^{\infty}dt\,\frac{\Delta_t I(t)}{t-x} \tag{B1}$$

for an arbitrary function I with imaginary part $\Delta_t I$. Later on, we will comment on the modifications needed to account for subtractions. For the class of diagrams of interest here, $\Delta_t I$ can be taken to be of the form $\Delta_t I(t)=Q(t)+R(t)\ln t$, where Q and R are rational functions of t, vanishing for $t\to\infty$, and possessing a polynomial expansion in t^{-1} for m^2t^{-1} near zero. This leads to two types of integrals requiring consideration, i.e.,

$$I_1(x)=\int_{4m^2}^{\infty} dt\, Q(t)/(t-x)\,, \tag{B2}$$

$$I_2(x)=\int_{4m^2}^{\infty} dt\, \frac{R(t)\ln t}{t-x}\,. \tag{B3}$$

The integral given by I_1 is easily evaluated by means of conventional techniques. I_2, on the other hand, cannot in general be expressed in terms of elementary functions; its solution involves the dilogarithm function[20] $\mathrm{Li}_2(x)$, defined by

$$\mathrm{Li}_2(x)=-\int_0^x dt\, \frac{\ln(1-t)}{t} \tag{B4}$$

and the modified dilogarithm function $\mathrm{Li}_2(x,\theta)$ defined by

$$\mathrm{Li}_2(x,\theta)=-\tfrac{1}{2}\int_0^x dt\, \frac{\ln(1-2t\cos\theta+t^2)}{t}\,. \tag{B5}$$

These functions are extremely practical in the evaluation of asymptotic behavior, since they possess many interesting properties. In particular, $\mathrm{Li}_2(1)=\frac{1}{6}\pi^2$, $\mathrm{Li}_2(-1)=-\frac{1}{12}\pi^2$ and the following relations hold:

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(1/x)=\tfrac{1}{3}\pi^2-\tfrac{1}{2}(\ln x)^2-i\pi\ln x\,, \quad x>1 \tag{B6a}$$

$$\mathrm{Li}_2(-x)+\mathrm{Li}_2(-1/x)=-\tfrac{1}{6}\pi^2-\tfrac{1}{2}(\ln x)^2\,, \quad x>0 \tag{B6b}$$

$$\mathrm{Li}_2(x,0)=\mathrm{Li}_2(x)+i\pi\Theta(x-1)\ln x\,, \quad x\geq -1 \tag{B6c}$$

$$\mathrm{Li}_2(x,\theta)+\mathrm{Li}_2(1/x,\theta)=-\tfrac{1}{6}\pi^2-\tfrac{1}{2}(\ln x)^2+\tfrac{1}{2}(\pi-\theta)^2\,, \quad 0\leq\theta\leq 2\pi\,. \tag{B6d}$$

Noteworthy is the fact, as these relations show, that the dilogarithm has an asymptotic behavior of the form

$$\mathrm{Li}_2(x)\underset{x\to\infty}{\simeq}-\tfrac{1}{2}(\ln x)^2+\text{const}\,, \tag{B7}$$

with a similar expression applying for $\mathrm{Li}_2(x,\theta)$.

In terms of the dilogarithm, one can then easily evaluate the asymptotic series expansion of I_2, since the integrand has been assumed to be uniformly convergent. Thus one expands $R(t)$ in a power series in t^{-1} and carries out the term by term integration by expressing each integral in dilogarithmic form. By using relations such as those given in Eq. (B6), one can obtain the asymptotic behavior and sum up the series to obtain closed expressions. For example, if we write

$$R(t)=\sum_n a_n t^{-n}$$

and consider

$$I_2^n(x)=\int_{4m^2}^{\infty} dt\, \frac{\ln(4m^2/t)}{t^n(t-x)}\,, \tag{B8}$$

a change of variable to $v=4m^2/t$ gives

$$I_2^n(x)=x^{-n}\beta^{-n+1}\int_0^1 dv\, v^{n-1}\frac{\ln v}{\beta-v}\,,$$

where $\beta\equiv 4m^2/x$. This integral is then cast in dilogarithmic form by rewriting

$$\frac{v^{n-1}}{\beta-v}=-(v^{n-2}+\beta v^{n-3}+\cdots+\beta^{n-2})+\frac{\beta^{n-1}}{\beta-v} \tag{B9}$$

and one gets

$$\begin{aligned} I_2^n(x)&\simeq x^{-n}\int_0^1 dv\,\frac{\ln v}{\beta-v}+f(x^{-1})\\ &\simeq x^{-n}\left[\ln^2\beta-\ln(1-\beta)\ln\beta+\mathrm{Li}_2\left(\frac{1-\beta}{\beta}\right)\right]\\ &\quad+g(x^{-1})\\ &\simeq \tfrac{1}{2}x^{-n}[\ln(4m^2/x)]^2+h(x^{-1})\,, \end{aligned} \tag{B10}$$

where f, g, and h are polynomial functions of x^{-1}. Thus, it follows that $I_2(x)$ has the following asymptotic form:

$$I_2(x)=\int_{4m^2}^{\infty} dt\,\frac{R(t)\ln t}{t-x}\underset{x\to\infty}{\simeq}-\tfrac{1}{2}R(x)\,(\ln x)^2\,, \tag{B11}$$

up to a polynomial in x^{-1}.

For integrals of the type of I_1, a similar result can be obtained by expanding $Q(t)$ in a power series in t^{-1}. Term by term integration is then carried out by using the substitution $v=4m^2/t$, and Eq. (B9); the resulting series is then summed up. As a result one obtains the following relation:

$$I_1(x)=\int_{4m^2}^{\infty} dt\,\frac{Q(t)}{t-x}\underset{x\to\infty}{\simeq}Q(x)\ln\left(\frac{4m^2}{x}\right), \tag{B12}$$

up to a polynomial in x^{-1}.

Although Eqs. (B11) and (B12) provide a general solution to the problem, it is instructive to evaluate a specific example without resorting to a series expansion in order to point out some additional techniques useful in handling dilogarithms. Consider the discontinuity function $\Delta_t[(t-4m^2)^{1/2}I^3]$ for the t-channel planar box obtained in Eq. (3.6), as an example. For this function, the integrals I_1 and I_2 are given by

$$I_1(x)=(8\pi)^{-1}\int_{4m^2}^{\infty} dt\,\frac{[1-4m^2/t]^{1/2}}{(t-x)\,(t-3m^2)}\,, \tag{B13}$$

$$I_2(x)=(8\pi)^{-1}\int_{4m^2}^{\infty} dt\,[t(t-x)\,(1-4m^2/t)^{1/2}]^{-1}\times\ln[(t-3m^2)/m^2]\,. \tag{B14}$$

I_1 is evaluated by changing to the variable $v=4m^2/t$

and making use of standard tricks of integration. One then obtains up to a polynomial in x^{-1}

$$I_1 = \frac{\beta}{m^2(4-3\beta)}\{(1-\beta)^{1/2}\ln[\beta(1+\beta)^{-2}]+(2\pi/3\sqrt{3})\}$$

$$\underset{x\to\infty}{\simeq} \frac{(1-4m^2/x)^{1/2}}{(x-3m^2)}\ln(4m^2/x)\,, \qquad \text{(B15)}$$

where $\beta \equiv 4m^2/x$.

For I_2, we take advantage of two useful integral relations which we list. These are

$$e\int_0^x dt\,\frac{\ln(a+bt)}{c+et} = \ln(\delta/e)\ln(\rho/c) - \mathrm{Li}_2(-b\rho/\delta) + \mathrm{Li}_2(-bc/\delta)\,, \qquad \text{(B16)}$$

where $\rho = c+ex$, $\delta = ae-bc$, and a, b, c, and e are arbitrary constants, and

$$b\int_0^x dt\,\frac{\ln(A+Bt+Ct^2)}{a+bt} = \ln(D/b^2)\ln[(a+bx)/a] - 2\mathrm{Li}_2[(a+bx)/(D/C)^{1/2},\theta] + 2\mathrm{Li}_2[a/(D/C)^{1/2},\theta] \qquad \text{(B17)}$$

with $\cos\theta = (aC - \frac{1}{2}bB)/\sqrt{CD}$ and $D = Ab^2 - abB + Ca^2$, for arbitrary constants a, b, A, B, and C.

The substitution $w = (1-4m^2/t)^{1/2}$, when inserted in I_2, reduces this integral to an expression of the form

$$I_2 = [x(1-\beta)^{1/2}]^{-1}\int_0^1 dw\,[w^2-(1-\beta)]^{-1}[\ln(1+3w^2) - \ln(1-w^2)]\,. \qquad \text{(B18)}$$

The expression involving the first logarithm in the integrand is of the form of Eq. (B17), while the second term can be expressed in the form of Eq. (B16) by writing $\ln(1-w^2)$ as $\ln(1-w)+\ln(1+w)$. In this way, the integral is evaluated with the result that

$$I_2 = \beta[4m^2(1-\beta)^{1/2}]^{-1}\{\ln(\alpha/\eta)\ln(\gamma/\alpha\eta) + \mathrm{Li}_2(\eta/\alpha) - \mathrm{Li}_2(\alpha/\eta) + 2\mathrm{Li}_2[\sqrt{3}\,\eta/\sqrt{\gamma}\,,\theta^+] - 2\mathrm{Li}_2[-\sqrt{3}\,\alpha/\sqrt{\gamma}\,,\theta^-]\}\,, \qquad \text{(B19)}$$

where $\alpha = (1-\beta)^{1/2}-1$, $\eta = (1-\beta)^{1/2}+1$, $\gamma = 4-3\beta$, and $\cos\theta^\pm = \pm[3(1-\beta)/\gamma]^{1/2}$. Hence I_2 has an asymptotic dependence of the form

$$I_2 \underset{x\to\infty}{\simeq} -\tfrac{1}{2}[x(1-4m^2/x)^{1/2}]^{-1}\{\ln[(x-3m^2)/m^2]\}^2 \qquad \text{(B20)}$$

up to a polynomial in x^{-1}, which is the same expression one obtains on the basis of Eq. (B11).

The existence of subtractions in the dispersion integrals does not cause any new difficulties. Rather, by modifying the functions $R(t)$ and $Q(t)$ to include the extra subtraction factor in the denominator, the convergence of the integrand is retained. Thus the line of reasoning leading up to Eqs. (B11) and (B12) continues to be valid and the results obtained remain unchanged.

APPENDIX C: CROSSING RELATIONS FOR HELICITY AMPLITUDES

Crossing relations connecting the scattering amplitude for the various channels have been obtained by several authors.[15-17] In our discussion, we make use of the results of Hara,[17] who has calculated explicitly the values of the phase constants associated with each choice of the "particle number two" assignments in the scattering process. For a two-body elastic scattering $a+b\to c+d$, the crossing matrix connecting the amplitude for any two channels has the form

$$H_{\{\lambda_i,\mu_i\}} = \epsilon(-1)^{\sigma(\lambda_i)}(-1)^{\eta(\mu_i)}\prod_{i=a,b,c,d} d^{s_i}_{\lambda_i\mu_i}(\psi_i)\,, \qquad \text{(C1)}$$

where ϵ is an arbitrary over-all phase, σ and η are channel and helicity-dependent phases, and $d^s_{\lambda u}$ is the Wigner function. The crossing angles ψ_i are channel-dependent functions of the kinematic variables; for the case of equal masses, the t to s crossing angles are given by the following expressions:

$$\cos\psi^t_a = -\cos\psi^t_b = -\cos\psi^t_c = \cos\psi^t_d = \left(\frac{st}{(s-4m^2)(t-4m^2)}\right)^{1/2} \qquad \text{(C2)}$$

and the u to s crossing angles are given by

$$\cos\psi^u_a = \cos\psi^u_b = -\cos\psi^u_c = \cos\psi^u_d = \left(\frac{su}{(s-4m^2)(u-4m^2)}\right)^{1/2}. \qquad \text{(C3)}$$

Thus, at $s=0$, all angles are equal to $\frac{1}{2}\pi$.

With the explicit "particle number two" assignments that were given in Appendix A, the phase factors σ and η have the following values:

$$\sigma^t = \eta^t = 0\,, \qquad \sigma^u = \lambda_b + \lambda_a - 2\lambda_c\,, \tag{C4a}$$

$$\eta^u = \mu_c + \mu_d - \mu_a - \mu_b\,. \tag{C4b}$$

The over-all phase factor ϵ can be determined for the spinor-vector scattering problem by computing the second-order amplitude in the various channels, and comparing the s-channel answer with the results obtained through the use of crossing relations. When this is done, one finds that $\epsilon^t = \epsilon^u = -i$.

As a result of applying the crossing matrix of Eq. (C1) to the t- and u-channel amplitudes, one obtains the following crossing relations:

$$T^s_{0,1/2;0,1/2}(0,t) = (-\tfrac{1}{2}i)\,(T^t_{1,1;1/2,1/2} + T^t_{-1,-1;1/2,1/2}) = \tfrac{1}{2}(T^u_{1,1/2;-1,-1/2} + T^u_{1,-1/2;-1,1/2})\,, \tag{C5a}$$

$$T^s_{\pm1,1/2;\pm1,1/2}(0,t) = (-\tfrac{1}{4}i)\,(T^t_{1,1;1/2,1/2} + T^t_{-1,-1;1/2,1/2} + 2T^t_{0,0;1/2,1/2} \pm 2\sqrt{2}\,T^t_{1,0;1/2,-1/2}) = \tfrac{1}{4}(T^u_{1,1/2;-1,-1/2} + T^u_{1,-1/2;-1,1/2} + 2T^u_{0,1/2;0,-1/2} \pm 2\sqrt{2}\,T^u_{1,1/2;0,1/2})\,, \tag{C5b}$$

$$T^s_{1,1/2;-1,1/2}(0,t) = (-\tfrac{1}{4}i)\,(T^t_{1,1;1/2,1/2} + T^t_{-1,-1;1/2,1/2} - 2T^t_{0,0;1/2,1/2}) = \tfrac{1}{4}(T^u_{1,1/2;-1,-1/2} + T^u_{1,-1/2;-1,1/2} - 2T^u_{0,1/2;0,-1/2})\,, \tag{C5c}$$

$$T^s_{1,\pm1/2;0,\mp1/2}(0,t) = (-i/2\sqrt{2})\,(T^t_{1,1;1/2,1/2} - T^t_{-1,-1;1/2,1/2} \pm \sqrt{2}\,T^t_{1,0;1/2,-1/2}) = (1/2\sqrt{2})\,(T^u_{1,1/2;-1,-1/2} - T^u_{1,-1/2;-1,1/2} \pm \sqrt{2}\,T^u_{1,1/2;0,1/2})\,. \tag{C5d}$$

*Work supported in part by the National Science Foundation. This work forms part of a dissertation submitted by the author to the University of California at Los Angeles in partial fulfillment of the Ph.D. degree.

†Present address: Aerojet Electro Systems Company, Azusa, California 91702.

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