Modified expansion for virtual Compton amplitude

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(Received 13 October 1972; revised manuscript received 25 September 1973)

From considerations of analyticity and convergence it is shown that the virtual Compton amplitude can be expanded as $T_{\mu\nu}(q_0, \ldots) \sim \sum_n w^n C_{\mu\nu}^{(n)}(\cdots)$, where the variable w is obtained by conformally mapping the cut plane of analyticity in q_0 into the inside of a semi-infinite strip. This expansion reduces to the usual Bjorken expansion in the limit of large q_0.

I. INTRODUCTION

The hadronic contribution to the inelastic scattering of electrons from an unpolarized target is described by a tensor

$$W_{\mu\nu} = \tfrac{1}{2}\sum_n \langle p|j_\mu(0)|n\rangle\langle n|j_\nu(0)|p\rangle \times (2\pi)^3\delta^{(4)}(p+q-p_n), \tag{1}$$

where q is the four-momentum of the virtual photon, p is that of the target, and j_μ is the electromagnetic current operator. This tensor is also the absorptive part of the forward virtual Compton amplitude,

$$T_{\mu\nu} = \int d^4x\, e^{iq\cdot x}\langle p|T(j_\mu(x)j_\nu(0))|p\rangle. \tag{2}$$

From inelastic electron-proton scattering, information is available for negative q^2 and positive $\nu = p\cdot q/M$. Introducing a complete set of states $|n\rangle$, we can perform the x integration[1] in Eq. (2):

$$T_{\mu\nu} = -i\int_0^\infty \frac{dq_0'}{2\pi}\left[\frac{W_{\mu\nu}(q_0', \vec{q})}{q_0 - q_0'} - \frac{W_{\nu\mu}(q_0', -\vec{q})}{q_0 + q_0'}\right]. \tag{3}$$

The denominators $(q_0 \mp q_0')^{-1} = q_0^{-1}(1 \mp q_0'/q_0)^{-1}$ are expanded in a binomial series and it is easy to show that for large values of q_0, keeping $\vec{q}$, p_μ fixed, one can write

$$T_{\mu\nu} \underset{q_0 \to i\infty}{\sim} \sum_{n=1} C_{\mu\nu}^{(n)}(1/q_0)^n, \tag{4}$$

where

$$C_{\mu\nu}^{(n)} = \int d^3x\, e^{-i\vec{q}\cdot\vec{x}}\langle p|[j_\mu(\vec{x},0), \partial_0{}^{n-1}j_\nu(0)]|p\rangle. \tag{5}$$

This expansion is being widely used for the study of fundamental problems of particle physics like radiative corrections to β decay,[2] asymptotic sum rules,[3] asymptotic cross-section relations[4] for high-energy inelastic electron-neutrino scattering, electromagnetic mass differences,[5] and a large variety of problems of strongly interacting particles.

Johnson and Low[6] were first to indicate the nonuniqueness of the expansion by perturbation methods. Commenting on the work of Bjorken[1] and Callan and Gross,[4] Jackiw and Preparata[7] and independently Adler and Tung[8] have shown that the term $n=2$ of the expansion (4) is not correct and the coefficient of $C_{\mu\nu}^{(2)}$ behaves like $(a + b\ln q_0)/q_0{}^2$. Adler and Tung[8] considered a renormalizable theory of an SU(3) triplet of spin-$\frac{1}{2}$ particles interacting strongly with an SU(3)-singlet massive vector gluon. By direct computation they have demonstrated the existence of the $\ln q_0$ term. Therefore all results following from the expansion (4), $n \geq 2$, are suspect. Examining the deduction of this expansion, one realizes that this discrepancy could arise if the expansion is not convergent to the desired extent. As a function of q_0, the amplitude has certain assumed analytic properties. No single term of the Bjorken expansion has the desired analyticity, whereas a perturbative series obtained from model field-theoretic calculations exhibits correct analyticity in the desired order starting mostly from the box diagrams. It is from such diagrams that deviations from the Bjorken expansion have been deduced by Adler and Tung. The present work aims at providing a possible convergent expansion which is manifestly analytic in the complex q_0 plane.

II. CONVERGENT EXPANSION

The convergence properties of an analytic function depend on the cut structure in the complex plane of the variable for expansion. Consider a sequence $p_n(z)$ approximating a function $f(z)$ which is singular at points z_s. For each n, one wishes to minimize the error in approximation for a domain D of the z plane. It has been shown by Walsh[9] that the convergence properties are related to the electrostatic problem in which D forms an earthed conductor ($V=0$) with a negative charge. A family of equipotential curves $V(z)$ = constant is obtained and if V is any finite potential smaller than each of the potentials $V(z_s)$ at the singular points, then $p_n(z)$ converges uniformly to $f(z)$ inside the closed region that is bounded by the equipotential $V(z)=V$. Furthermore, the error in the nth approximation is bounded by $\exp(-nV)$. These ideas have been used to obtain optimally convergent expansion of scattering amplitudes, off-shell extrapolations, and model-independent analyses of experimental data.[10-12]

In the present problem the tensorial components of $T_{\mu\nu}$ are to be approximated by a series involving q_0, which is experimentally determinable from $i\infty$ to $-i\infty$. So the region D is the imaginary q_0 axis. Besides q_0, $T_{\mu\nu}$ is also a function of $\vec{q}$ and p_μ. The analytic properties of $T_{\mu\nu}$ as a function of q_0, q^2 are quite complicated.[13] For fixed $\vec{q}$ and p_μ, we assume that all the components of $T_{\mu\nu}$ are analytic in the complex q_0 plane except for the cuts extending from $-\infty$ to $-q_c$ and q_c to $+\infty$ on the real axis as shown in Fig. 1(a).

The equipotentials of a line charge are cylinders. So in the q_0 plane, they run parallel to the imaginary axis, and the region of convergence of the Bjorken expansion (4) is limited to the strip bounded by $q_0=\pm q_c$, whereas the region of analyticity is the entire q_0 plane minus the cuts.

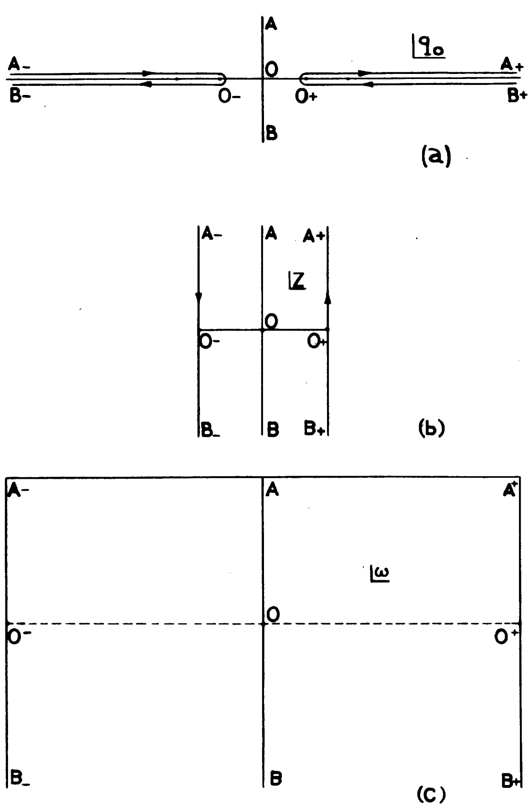

FIG. 1. (a) The cut plane of analyticity of the virtual Compton amplitude in q_0; (b) the strip mapping $z=\sin^{-1}q_0$; and (c) the semi-infinite strip mapping the point $i\infty$ to the origin. The mapping of the lines $A_+O_+B_+$, AOB, and $A_-O_-B_-$ in the conformally mapped planes is shown.

Following Cutkosky and Deo[10] and Ciulli,[11] a very good convergent expansion for the given problem can be obtained by conformally mapping the entire plane of analyticity into a strip.

III. STRIP MAPPING

This mapping is given by the simple relation[14] $q_0=q_c\sin(z/z_c)$ and is shown in Fig. 1(b). In the z plane, the imaginary q_0 axis has been mapped onto the imaginary z axis and the cuts have been mapped to form the boundaries $z=\pm z_c$ of the strip of convergence enclosing the whole of q_0 plane. z_c is a parameter whose value can be fixed by the following consideration: If the position of the cut q_c is made to go to infinity, q_0 could be the best variable for expansion, for then the q_0 plane will not have any cut structure. In this limit $q_0\to z$. This can be achieved if $z_c=q_c$. The correct mapping is then

$$q_0=\sin z\,, \tag{6}$$

where q_0 and z are measured in units of $q_c(\vec{q}^{\,2})$.

Any expansion of the functions $T_{\mu\nu}$ in terms of the variable

$$z=\sin^{-1}q_0\,, \tag{7}$$

will be analytic in the prescribed region. We can now try to expand

$$\begin{aligned} f(z) &= (1+q_0'/q_0)^{-1} \\ &= (1+q_0'/\sin z)^{-1} \end{aligned} \tag{8}$$

for large and purely imaginary values of z. One notices that $\sinh z$ has an essential singularity and, for a formal Taylor expansion in $1/z$, is not unambiguous. Assuming q_0' to be small and using a formula of the type

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{z^n}\frac{1}{n!}\left(-z^2\frac{\partial}{\partial z}\right)^n f\Big|_{z\to i\infty}\,, \tag{9}$$

one can isolate a set of terms:

$$f(z)\sim\sum_{n=0}^{\infty}(izq_0'/q_0)^n/n\,, \tag{10}$$

which gives an asymptotic expansion of the type

$$T_{\mu\nu}(q_0,\ldots)_{q_0\to i\infty}\sim\sum_{n=1}^{\infty}C_{\mu\nu}^{(n)}(1/q_0)[(\ln q_0)/q_0]^{n-1}\,. \tag{11}$$

However, the differentiation of f at the limit $z\to i\infty$ may not be permissible and the above result is nonrigorous and ambiguous.

There are certain advantages in using the strip-mapping variable. The convergence for a larger region is ensured. From all perturbative methods

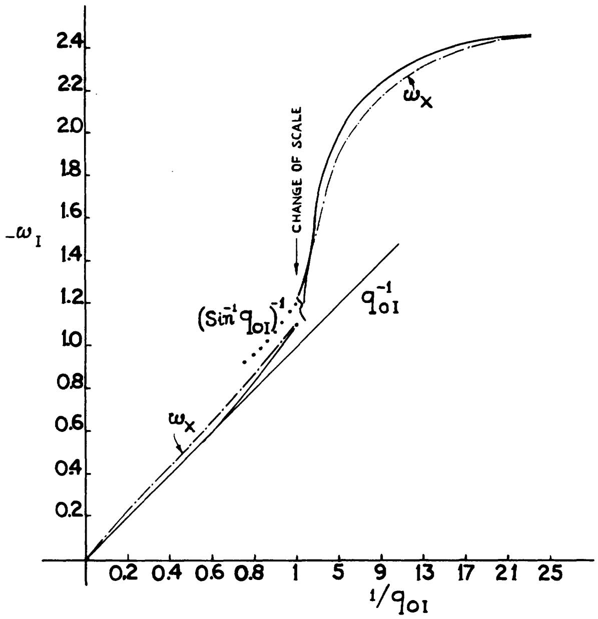

FIG. 2. The solid curve gives the values of w_I for different q_{0I}^{-1}. Note the change of scale at $q_{0I} = 1$. The straight line q_{0I}^{-1} matches the exact curve $\infty > q_{0I} > 1.2$ and $(\sin^{-1} q_{0I})^{-1}$ matches for $0 < q_{0I} < 5$. An assumed approximate formula w_x [see text Eq. (16)] is shown as the dot-dashed curve.

using field-theoretic models and Feynman diagrams, one obtains terms containing $z = \sin^{-1} q_0$ and its integral powers for the amplitude. The presence of this term ensures that the required cut structure of the amplitude is present in the expression. Such terms will occur in the expansion of $(\sin z + q_0')^{-1}$ in powers of z.

IV. MAPPING INTO A SEMI-INFINITE STRIP

A Taylor series expansion about $z \to i\infty$ will be possible if A of Fig. 1(b) representing this point, is conformally mapped to form the origin. So, for our purpose, the infinite strip should be mapped into a semi-infinite one. This is achieved by the mapping

$$z = -i \ln \eta \tag{12}$$

or

$$q_0 = (1/2i)[\eta - \eta^{-1}], \tag{13}$$

where

$$\eta = e^{iw/8} - e^{-3iw/8}, \tag{14}$$

which is shown in Fig. 1(c). This factor $\frac{1}{8}$ in the exponent for the expression of η is taken such that in the limit $q_0 \to i\infty$, $w \to 1/q_0$. The "physical" region is again the q_0-imaginary axis AOB with the strips at $w = \pm 4\pi$ forming the boundaries. The origin of the q_0 plane is at $w_0 = -i2.58$. With the help of this variable, one can now expand $[q_0(w) + q_0']^{-1}$ in powers of w. Thus our expansion for the Compton amplitude is

$$T_{\mu\nu} \sim \sum_{n=1} w^n C_{\mu\nu}^{(n)} . \tag{15}$$

This agrees with the Bjorken expansion for large q_0 and analytic properties are explicit in each term.

On the imaginary axis, the variation of $w = -iw_I$ with q_I^{-1} is shown in Fig. 2. One notices that for small w, it is proportional to $1/q_0$ for $q_0 > 1.2$ and for $q_0 < 0.8$, w is nearly proportional to $(\sin^{-1} q_0)^{-1}$.

These features suggest that "w" can be written in terms of q_0 and $\sin^{-1} q_0$. In keeping with the results of perturbation calculations of Ref. 8, we tried a simple formula:

$$-w \simeq +w_x = \frac{1}{q_{0I}} \left(1 + \frac{-1 + b \sinh^{-1} q_{0I}}{c + q_{0I}}\right) . \tag{16}$$

The factors b and c are chosen to coincide with ω at $q_I = 0$, ∞, and 1, and they are $b = 0.93$ and $c = 1.65$. The fit with the exact formula (12) is very good indeed. For large q_{0I}, $w_I \to q_{0I}^{-1} + (\ln q_0)/q_{0I}^2$. This suggests that perturbation results containing $\sin^{-1} q_0$ terms are probably valid for $q_{0I} < 1$ and cannot be extrapolated to large values of q_{0I}. We recommend the use of the w variable wherever a Bjorken type of expansion is used.

ACKNOWLEDGMENTS

I thank the Theory group of the Physics Department and Dr. S. Patnaik of the Mathematics Department of Utkal University for some useful discussions.

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