see S. Fubini and G. Veneziano, Nuovo Cimento 64A, 811 (1969); K. Huang and S. Weinberg, Phys. Rev. Lett. 25, 895 (1970).

[10] See Sec. III, especially (3.14) for the definition of these boundary conditions which may be used in place of (1.2) for a bag containing scalar fields. Identical virial theorems are obtained with Neumann boundary conditions [see (11.1)] and in the case of Dirac particles.

[11] Time may be placed by x^+ in light-cone variables, or indeed by any other time or lightlike coordinate.

[12] The normals are chosen with opposite orientation on R_1 and R_2.

[13] The Dirichlet boundary conditions have been discovered independently by T. T. Wu, B. M. McCoy, and H. Cheng, following paper, Phys. Rev. D 9, 3495 (1974).

[14] Our quantization procedure is modeled after that discussed in P. Goddard, J. Goldstone, C. Rebbi, and C. B. Thorn, Nucl. Phys. B56, 109 (1973).

[15] This is not trivial since there is no momentum conjugate to the surface variable $\vec{R}$.

[16] The generator of translations in x is $P_- = -P^+ \equiv -P$.

[17] This is the reason that we study the complex scalar field. For the real scalar field ψ must be a periodic step function which takes on the values 0, π, so the only solutions with a fixed radius are rather formal. We can see that this is a characteristic of static solutions of the real field problem regardless of shape as follows. Consider the integral

$$\int_R d^3x\, \nabla^2\phi = \int_S ds\hat{n}\cdot\vec{\nabla}\phi, \qquad \text{(a)}$$

where we assume the surface is static. The boundary conditions (3.14) require $\vec{\nabla}\phi = \pm(2B)^{1/2}\hat{n}$ on the surface. If we demand smoothness, i.e., only one sign of the square root then (a) reduces to

$$(2B)^{1/2}\int_S ds = (2B)^{1/2}\,\text{Area} = \left(\int_R d^3x\, \frac{\partial^2}{\partial t^2}\phi\right). \qquad \text{(b)}$$

Since R is static,

$$2B\int_S ds = \frac{\partial}{\partial t}\left(\int_R d^3x\, \frac{\partial\phi}{\partial t}\right). \qquad \text{(c)}$$

However, because the energy is bounded and depends upon the integral of ϕ^2, $\dot{\phi}$ is bounded in the bag. Hence, if we average (c) over the time we find Area = 0, which is clearly not possible. Hence, there can only be solutions with static walls of the bag if $\vec{\nabla}\phi$ is discontinuous on the surface, as a function of time (or position).

[18] P. Hayes (unpublished).

[19] We have been rather sketchy in this analysis. Details will be published elsewhere.

[20] Putting a very large mass outside is equivalent to confining particles by a scalar potential. The fact that such a scalar potential confines both particles and antiparticles has been mentioned by N. N. Bogoliubov *et al.*, Dubna Report Nos. D-1968, 1965 (unpublished) and D-2569, 1966 (unpublished); H. J. Lipkin and A. Tavkhelidze, Phys. Lett. 17, 331 (1965).

[21] Hadronic interactions were introduced into the string model by just such a fissioning mechanism. See S. Mandelstam, Nucl. Phys. B64, 205 (1973).

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Theory of hadron "bags" with scattering

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We derive the boundary conditions satisfied by a boson field in the theory of hadron "bags." The scattering problem, the fission problem, and the fusion problem in this theory of one spatial dimension are discussed.

I. INTRODUCTION

In the preceding paper, Chodos, Jaffe, Johnson, Thorn, and Weisskopf[1] (CJJTW) proposed a very interesting model for the structure of hadrons. They assume that hadron fields are contained inside a "bag" which has a constant, positive potential energy density B. By requiring that the action of this Lagrangian be an extremum, they obtain the field equations inside the bag and the conditions satisfied by the wave functions at the boundary. These equations also determine the location of the boundary.

Their boundary conditions do not require the field to vanish at the boundary. This seems to lead to difficulties when two hadron bags scatter from each other. In CJJTW troubles with boundary conditions are already encountered in the fermion case, and are solved by introducing an outside field with large mass. In this paper we propose to apply the same treatment to the boson case.

It is found that this procedure leads to a different set of boundary conditions from those given by CJJTW. In particular, the boson field is indeed required to be zero at the boundary. In Sec. II we derive these boundary conditions, and in the remaining sections we discuss solutions of these equations in two dimensions, with special emphasis on the scattering of two bags.

II. BOUNDARY CONDITIONS

Following CJJTW, we consider the action

$$W=\int_V d^4x[\tfrac{1}{2}(\partial_\mu\phi)^2-B]+\int_{\overline{V}}d^4x[\tfrac{1}{2}(\partial_\mu\Phi)^2-\tfrac{1}{2}M^2\Phi^2], \tag{1}$$

where V is the space-time region occupied by the hadron bag, and $\overline{V}$ is the region outside the bag. We are interested in the limit $M\to\infty$. For any M, the variation of W by changing ϕ and Φ and keeping V fixed leads to

$$\partial_\mu\partial_\mu\phi=0 \tag{2}$$

in V and

$$(\partial_\mu\partial_\mu+M^2)\Phi=0 \tag{3}$$

in $\overline{V}$, together with the boundary condition

$$n_\mu\partial_\mu\phi=n_\mu\partial_\mu\Phi, \tag{4}$$

where n_μ is the unit vector in the direction of the normal to the boundary. Equation (4), together with the continuity condition on the boundary

$$\phi=\Phi, \tag{5}$$

implies that,[2] on the boundary,

$$\partial_\mu\phi=\partial_\mu\Phi. \tag{6}$$

We next vary V and obtain

$$\tfrac{1}{2}(\partial_\mu\phi)^2-B=\tfrac{1}{2}(\partial_\mu\Phi)^2-\tfrac{1}{2}M^2\Phi^2. \tag{7}$$

From (6) and (7) we get

$$\phi=\Phi=(2B)^{1/2}/M \tag{8}$$

on the boundary.

When M is large, (3) may be solved[1] by the WKB method in the form

$$\Phi=e^{Mj}, \tag{9}$$

where j satisfies approximately

$$(\partial_\mu j)^2+1\sim 0. \tag{10}$$

Thus

$$(\partial_\mu\Phi)^2\sim -M^2\Phi^2. \tag{11}$$

The boundary conditions in the limit $M\to\infty$ are obtained by substituting (6) and (8) into (11). The results are

$$\phi=0 \tag{12}$$

and

$$(\partial_\mu\phi)^2=-2B. \tag{13}$$

III. CLASSICAL SOLUTIONS IN TWO DIMENSIONS

We shall next discuss the solutions of the field equation (2) with the boundary conditions (12) and (13) in the simple case of one spatial dimension. Following CJJTW, we denote

$$\tau=\frac{1}{\sqrt{2}}(t+z),$$

$$x=\frac{1}{\sqrt{2}}(t-z),$$

where z is the spatial variable. Then the solution of (2) is

$$\phi(x,\tau)=f(\tau)+g(x). \tag{14}$$

The boundary conditions (12) and (13) give

$$f(\tau(x))+g(x)=0 \tag{15}$$

and

$$f'(\tau(x))g'(x)=-B, \tag{16}$$

where $\tau(x)$ describes an end point of the bag. From (15) and (16) we get

$$\frac{d\tau(x)}{dx}=[g'(x)]^2/B. \tag{17}$$

Thus, if $\tau_1(x)$ and $\tau_2(x)$ are the two end points of the bag, then

$$\tau_2(x)=\tau_1(x)+a, \tag{18}$$

where a is independent of x. Substituting (18) into (15), we find that $f(\tau)$ is a periodic function of period a. Similarly, if we denote the end points of the bag as $x_1(\tau)$ and $x_2(\tau)$, we find that

$$x_2(\tau)=x_1(\tau)+b, \tag{19}$$

where b is independent of τ, and that $g(x)$ is a periodic function of period b.

We observe that we may restrict ourselves to the solutions in which a and b are equal, as the other solutions may be obtained from these solutions by Lorentz transformations. Indeed, the latter solutions are the stationary ones which may oscillate in position but have no mean velocity over a long period of time. For such solutions, we have by (17) and (15) that

$$f\left(B^{-1}\int_0^x[g'(y)]^2dy\right)+g(x)=0 \tag{20}$$

and

$$\int_0^a [g'(x)]^2 dx = aB, \tag{21}$$

where $\tau_1(0)$ is chosen to be zero.

IV. SOLUTIONS OF A SINGLE BAG

We have found that the solution of a stationary bag is completely specified if the periodic function $g(x)$ satisfying the normalization (21), is given. A periodic function must have extrema. At an extremum of $g(x)$, there are two possibilities: (i) $g'(x)$ is equal to zero; (ii) $g'(x)$ is discontinuous.

If $g'(x)$ is equal to zero at an extremum, then by (16) $f'(\tau(x))$ is infinite. Indeed, let

$$g(x) \approx -x^2 \tag{22}$$

near $x=0$; then (20) shows that

$$f(\tau) \approx \left(\frac{3B\tau}{4}\right)^{2/3} \tag{23}$$

near $\tau=0$. Thus $f'(\tau)$ blows up like $|\tau|^{-1/3}$ at the extremum $\tau=0$.

There also exist solutions for which $g'(x)$ is discontinuous at an extremum. To see that such solutions also satisfy the variational principle, we apply the Weierstrass-Erdmann conditions[3] for solutions with discontinuous derivatives. Let $g(x)$ have discontinuous derivatives at x_0; then these conditions are that $f'(\tau)$ and $[f'(\tau)g'(x) - L]$ are both continuous. The second condition is trivially satisfied as $[f'(\tau)g'(x) - L]$ is equal to the constant B, while the first condition is implied by the continuity of $\phi(x, \tau)$ at x_0. Thus $g(x)$ is allowed to have discontinuous derivatives.

Two explicit solutions of a single bag are illustrated in Figs. 1 and 2. The first example is a bag with fixed shape traveling with a uniform velocity, while the second example is a bag at rest with pulsating shape. Both of these solutions have discontinuities. These discontinuities travel with the velocity of light, reversing direction after they hit the boundary.[4]

V. SCATTERING, FUSION, AND FISSION

We shall next consider the scattering of two bags. Let the incident bags be described by $f(\tau) + g(x)$ and $F(\tau) + G(x)$, respectively. We shall show that there exists a solution to (2), (12), and (13) such that the two bags emerge elastically with the wave functions acquiring only phase shifts. Specifically, if bag 1 is behind bag 2 at the distant past and catches up with bag 2 at some finite time, then the scattered bags are respectively described by

$$f(\tau - a') + g(x) \tag{24}$$

and

$$F(\tau) + G(x - b), \tag{25}$$

where a and b [a' and b'] are the periods of $f(\tau)$ and $g(x)$ [$F(\tau)$ and $G(x)$], respectively. Furthermore, the boundaries of the scattered bags are respectively given by

$$\tau_i(x) + a', \quad i = 1, 2 \tag{26}$$

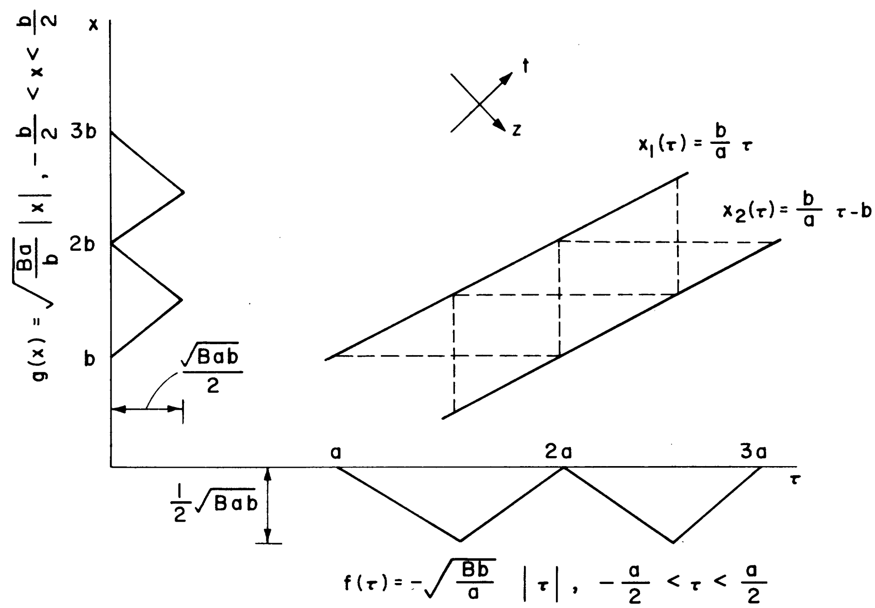

FIG. 1. An explicit solution for a bag with a fixed size $ab/(a^2+b^2)^{1/2}$. Both of the end points of the bag travel with the uniform velocity $(b-a)/(b+a)$. The dashed lines are the surfaces of discontinuous derivatives.

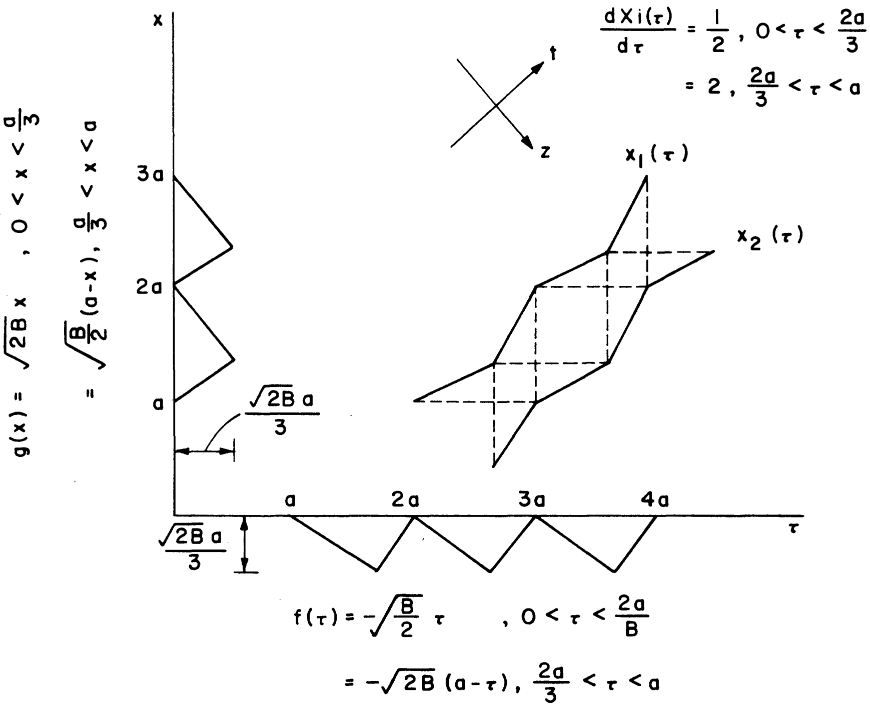

FIG. 2. An explicit solution for a stationary bag with a pulsating size. The dashed lines are the surfaces of discontinuous derivatives.

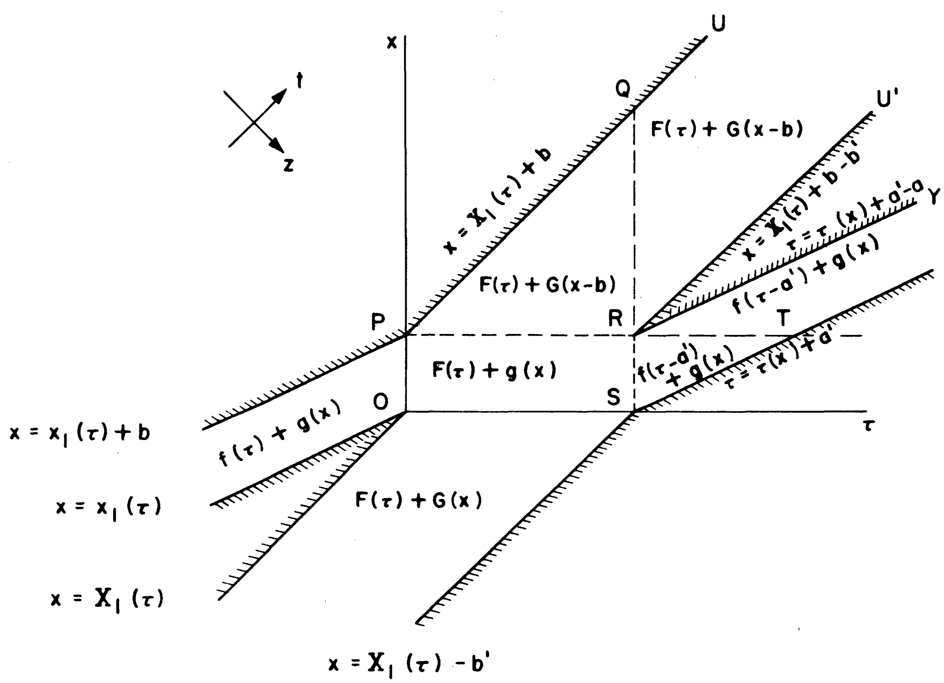

FIG. 3. An explicit example of the elastic scattering of two bags.

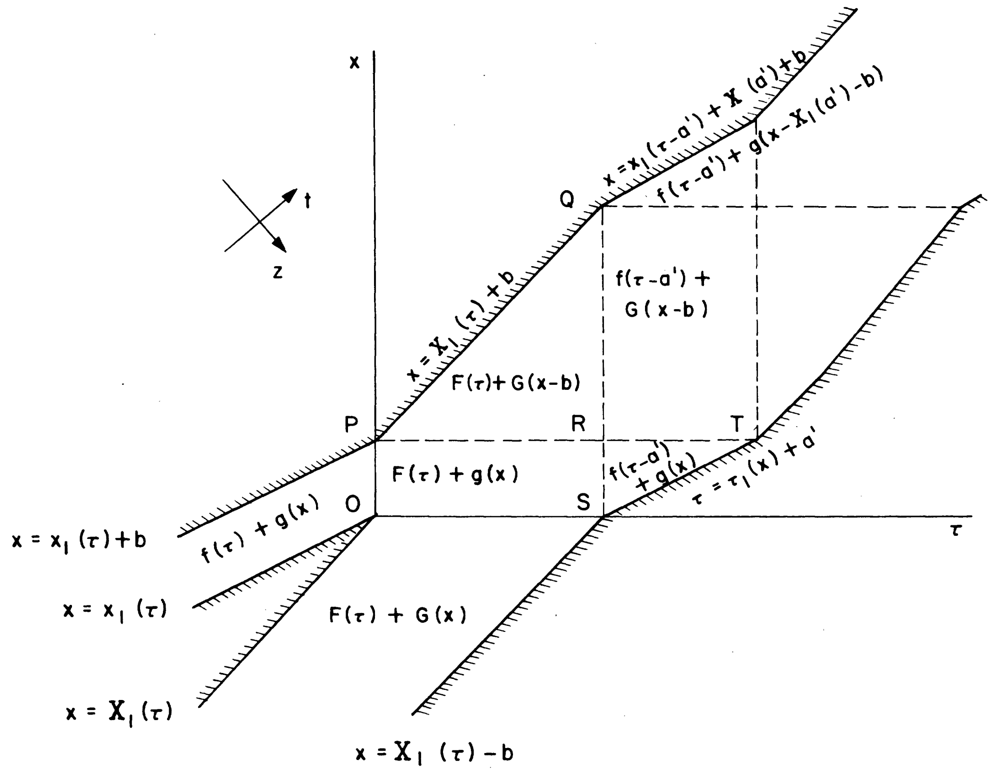

FIG. 4. An explicit example of a fusion process.

and

$$X_i(\tau)+b\,, \quad i=1,2\,, \tag{27}$$

where $\tau_i(x)$ and $X_i(\tau)$ denote the boundary of the incident bags.

It suffices to demonstrate the above by a simple example, as the extension to general cases is evident. Let us consider the scattering problem illustrated in Fig. 3, where the two bags first touch at $x=\tau=0$. Thus the solution is equal to the superposition of the wave functions of the incident bags everywhere except in the first quadrant. To find the solution in the first quadrant, we note that it is also in the form of a function of τ plus a function of x. In order to determine these functions of x and τ, we observe that the function of τ at any two points connected by a vertical line must have the same value, if this vertical line crosses no boundaries of the bags. Thus the function of τ in the region OPQRS of Fig. 3 is simply $F(\tau)$. Similarly, the function of x in the region OPRTS is simply $g(x)$. Thus by (17), the τ coordinate for the boundary curve ST obeys the same equation as that satisfied by $\tau_1(x)$. Therefore, the curve ST must be $\tau=[\tau_1(x)+a']$. Furthermore, the function of τ in the region SRT must be $f(\tau-a')$ so that the solution vanishes at the boundary curve ST. Similarly, the solution in the region PQR is $F(\tau)+G(x-b)$, and the boundary curve PQ is given by $x=[X_1(\tau)+b]$.

We have thus seen that the solution to the left of the curve SOP completely determines the solution to the left of the curve TRQ. We may therefore expect to be able to determine the solution farther to the right in a step-by-step way. It is somewhat surprising that there exist not one, but two solutions in the farther region.[5] One of the solutions is illustrated in Fig. 3, and is given by (24)–(27). This is the solution describing the elastic scattering of two bags. The other solution is illustrated in Fig. 4, and describes a fusion process. If we make a reflection about the origin, the solution in Fig. 4 becomes a fission process. Thus we have seen that there are scattering solutions, fusion solutions, and fission solutions for the wave equation (2) with the boundary conditions (12) and (13). Furthermore, the solution is not unique even if all initial conditions are specified.[6]

This nonuniqueness is related to the existence of points like R and O. At such a point, we have the following: (i) the wave function vanishes; (ii) two surfaces of discontinuity (in the partial derivatives) intersect; (iii) Eq. (13) is satisfied for both sets of $(\partial\phi/\partial t, \partial\phi/\partial x)$.

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[1]A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf, preceding paper, Phys. Rev. D 9, 3471 (1974). This paper will be referred to as CJJTW.

[2]Strictly speaking, this is true only if the boundary does not move with the volocity of light. To see this, let us consider for simplicity the case of two dimensions. In this case, (4) is $n_t(\phi_t-\Phi_t)-n_z(\phi_z-\Phi_z)=0$, while (5) gives $n_z(\phi_t-\Phi_t)-n_t(\phi_x-\Phi_x)=0$. These two equations imply (6) only if $n_t^{\,2}\neq n_z^{\,2}$.

[3]See, for example, O. Bolza, *Calculus of Variations* (Chelsea, New York, 1960), p. 36. This book only gives these conditions for one-dimensional integrals. However, generalization to two-dimensional integrals is immediate. Let x and y be the two independent variables and $\vec{n}=n_x\vec{e}_x+n_y\vec{e}_y$ be the normal to the surface of discontinuity; then the Weierstrass-Erdmann conditions are that

$$n_x\frac{\partial L}{\partial\phi_x}+n_y\frac{\partial L}{\partial\phi_y}$$

and

$$(n_x\phi_x+n_y\phi_y)n_x\frac{\partial L}{\partial\phi_x}+n_y\frac{\partial L}{\partial\phi_y}-L$$

are both continuous.

[4]Generally speaking, a discontinuity in the wave function gives dominant contributions to scattering processes with large momentum transfer. It is therefore interesting to speculate about the relation between the discontinuities discussed here and the pointlike structure inside a hadron as seen in *ep* scattering. Note that these discontinuities are also present in three-dimensional bags.

[5]Nonuniqueness of solutions also occurs in the string model, where a string may or may not break up at a point traveling with the velocity of light. See, for example, S. Mandelstam, Nucl. Phys. B64, 205 (1973). We thank Professor J. Mandula for informing us of this paper.

[6]In classical mechanics, this type of nonuniqueness is not unusual, such as a particle falling off a sphere, or an inclined ladder sliding off a frictionless wall. In these cases, the nonuniqueness can be resolved by the requirement that, for sufficiently small time intervals, the action is a minimum, not merely an extremal. So far as the authors are aware, such resolution is unavailable for problems of classical field theory.