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New extended model of hadrons*

A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf
Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
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We propose that a strongly interacting particle is a finite region of space to which fields are confined. The confinement is accomplished in a Lorentz-invariant way by endowing the finite region with a constant energy per unit volume , B. We call this finite region a "bag." The contained fields may be either fermions or bosons and may have any spin; they may or may not be coupled to one another. Equations of motion and boundary conditions are obtained from a variational principle. The confining region has no dynamical freedom but constrains the fields inside: There are no excitations of the coordinates determining the confining region. The model possesses many desirable features of hadron dynamics: (i) a parton interpretation and presumably Bjorken scaling; the confined fields are free or weakly interacting except close to the boundary; (ii) infinitely rising Regge trajectories as a consequence of the bag's finite extent; (iii) the Hagedorn degeneracy or limiting temperature; (iv) all physical hadrons are singlets under hadronic gauge symmetries. For example, in a theory of fractionally charged, "colored" quarks interacting with colored, massless gauge vector gluons, if both quark and gluon fields are confined to the bag, only color-singlet solutions exist. In addition to establishing these general properties, we present complete classical and quantum solutions for free scalars and also for free fermions inside a bag of one space and one time dimension. Both systems have linear mass-squared spectra. We demonstrate Poincaré invariance at the classical level in any dimension and at the quantum level for the above-mentioned explicit solutions in two dimensions. We discuss the behavior of specific solutions in one and three space dimensions. We also discuss in detail the problem of fermion boundary conditions, which follow only indirectly from the variational principle.

I. INTRODUCTION

In this paper we shall propose a new model for the structure of hadrons. It is a model which will be formulated in exact, quantitative language. However, it is conceptually simple and, consequently, we shall see immediately that it possesses many features which are in accord with the present understanding of hadron structure.

We assume that a region of space which is capable of containing hadronic fields has a constant, positive potential energy, B, per unit volume. B will be the only parameter of the theory, at least at the start. B will be of the order 1 GeV/(fm)^3, and the characteristic linear dimension of a hadron will be scaled by $(1/B)^{1/4}$. For short, we will call a region of space which contains hadron fields a "bag."

Because the action associated with B is proportional to the volume of the space-time hypertube swept out by it, the model is relativistically invariant. As an example, the simplest such system is described classically by the action

$$W = \int_{t_2}^{t_2} dt \int_R d^3r\left[\tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}(\vec{\nabla}\phi)^2 - B\right], \tag{1.1}$$

where the spatial region of integration extends over a closed, finite part of space (the bag). In (1.1), ϕ is the prototype of a hadronic field, that is, the field for partons or hadron constituents. To obtain the equations of motion and associated boundary conditions, we require W to be stationary with respect to variations $\delta\phi$ of ϕ which are arbitrary inside of and on the surface of the bag.

We also require W to be stationary with respect to independent variations of the position of the surface of the bag $[\delta\vec{R}(\alpha, t)]$. $\vec{R}$ will depend on two parameters α_1, α_2, which vary over the surface, and time. This feature is, of course, also essential for relativistic invariance. Since there are no kinetic terms in (1.1) which involve the bag surface, the equations which result from this requirement will be equations of constraint which (implicitly) define the geometrical variables $[\vec{R}(\alpha, t)]$ to be functions of the field degrees of freedom in the bag. Thus, this model of an extended relativistic object is distinct from earlier ones[1,2] in that in those models the geometrical variables were also, in part, dynamical. The field equations and boundary conditions which result from the variational principle are

$$-\Box\phi = 0 \text{ inside the bag},$$

$$\hat{n}\cdot\frac{\partial\vec{R}}{\partial t}\dot{\phi} + \hat{n}\cdot\vec{\nabla}\phi = 0 \text{ on the surface}, \tag{1.2}$$

$$\tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}(\vec{\nabla}\phi)^2 = B \text{ on the surface},$$

where $\hat{n}$ is the normal to the surface at any point. In Sec. III we discuss these in detail.

We emphasize that (1.1) is just a prototype of our model. The hadron constituent fields which are confined in the bag can carry any spin or quantum number. In this paper we generally shall assume that the fields confined in a bag are "massless," that is, we shall take the free Lagrangian for the fields to be the part which consists of the derivative terms. Thus, the only dimensional parameter will be B.

In this class of models the fields contained in a bag need have no interaction terms in the Lagrangian. Therefore, our model is capable of realizing in a covariant context the free-parton substructure for hadrons. Indeed, this feature was instrumental in suggesting the model. It is intuitively clear that a short-distance probe $(q^2 \gg \sqrt{B})$ when scattering from a constituent quantum in the bag will scatter from a free pointlike particle far from the walls, and hence Bjorken scaling of the scattering amplitude should result.

Up to this point we have only a model of a single hadron, albeit, in all of its possible states. In order to have a theory of strong interactions we must provide for a local coupling among hadrons. We may visualize this interaction to be one which allows a bag to fission or different bags to coalesce. This may be described classically in a local, causal way, if we couple two bags to form a single bag (or allow a single bag to become two) at points on their respective surfaces. We then integrate over the point. To obtain the quantum amplitude we would then (for example) further "sum over histories." Clearly, the constant B must be universal among bags for this interaction to make sense. It is needless to say that the calculation of this amplitude would be a formidable task. We make no apologies for this. The test of our model will result from the class of predictions of quantitative results which can be obtained from it. In particular, asymptotic calculations of many sorts can be made in a fairly simple way as we shall show in detail in this paper.

Because the bag model describes an extended hadron, it shares with other extended models such as the string[2] a leading Regge trajectory which is infinitely rising. Furthermore, the bag yields an asymptotic density of states of the form $\rho \sim e^{M/T_0}$ (see Sec. II) in common with the string model (and other extended models with different internal dimensions).

Although we have suggested that the fields in a bag should be free in first approximation, at the next level we shall propose that they be coupled weakly. We shall argue that such a weak coupling can account for the observed quantum numbers of the hadrons. The coupling should be weak enough so that the parton currents will show only small deviation from scaling by means of radiative corrections. We propose that the hadronic fields contained in the bag are "colored" quarks and gluons.[3]

The simplest model, which will have the correct quantum numbers for physical hadrons and approximately conserved color symmetry, is the Han-Nambu model[4] with the integer-charge rule for the colored quarks,

$$Q = I_3 + \tfrac{1}{2}Y + \tfrac{1}{3}C, \tag{1.3}$$

where I_3 and Y belong to the ordinary SU(3) and C belongs to the colored SU(3). The quarks have ordinary Fermi statistics. We construct the model so $C = 0$ for all physical hadrons. In this way all physical states will belong to zero-triality ordinary SU(3) representations. To achieve this we allow the current $\bar{q}\gamma^{\mu}Cq$ to be coupled to a massless Abelian gauge field (gluon) with a small coupling constant. This will weakly break color symmetry and give small radiative corrections to the parton structure. We assume that the gluon field is hadronic, that is, it is confined in the bag with the colored quark fields. This will prevent the appearance of hadron states with $C \neq 0$. In Sec. VI we show that solutions of the dynamical equations exist only if $C \equiv 0$. This is consistent with our fission interaction, for if a bag with $C = 0$

begins to fission (Fig. 1) by means of our assumed hadronic interaction into two bags, with C and $-C$, the bags will be connected with total flux lines C since the flux is confined in the bag. If we imagine, classically, that the two bags are connected by a neck of area A and length R, the gluon field energy in the neck will be proportional to $(C/A)^2AR=(C^2R/A)$. This neck energy diverges if the bags try to recede from each other ($R\to\infty$) or if the bags fission ($A\to 0$), that is, the flux lines cannot be broken by the strong interaction. The gluon coupling constant need not be large to achieve this. Thus, the massless gluon field confined in the bag allows for an intuitively simple, classical way of understanding why $C=0$ for all physical hadrons.[5]

An alternative scheme with color an exact symmetry (and, therefore, with many fewer states than in the Han-Nambu model) can be based on a massless non-Abelian colored-gluon field confined in the bag. Here, the masslessness is also necessary to ensure renormalizability.[6] In this model color symmetry would be exact, and a trivial generalization of the above argument to the non-Abelian case shows that it is hidden. However, in order that the non-Abelian gauge theory be consistent with electromagnetic interactions, it would be necessary to replace (1.3) by

$$Q=I_3+\tfrac{1}{2}Y.$$

That is, in the theory with exact color symmetry the quarks would have fractional charge.

The remainder of the paper is organized as follows. In Sec. II we examine the highly excited states of the bag from a semiclassical point of view. We are able to calculate very simply how energy is shared between the bag and the fields inside. We derive the level density, and discuss the large-quantum-number behavior of the Regge trajectories. In Sec. III we treat in detail both the classical and the quantum mechanics of a one-space, one-time dimensional bag containing scalar fields. In Sec. IV we offer some illustrative solutions of the equations of motion, in both one and three space dimensions. We turn in Sec. V to a discussion of the proper boundary conditions for Fermi fields, and we solve the quantum-mechanical fermion bag in two dimensions, analogous to the scalar solution of Sec. III.

The problem of interactions within the bag is briefly treated in Sec. VI, where we derive the boundary conditions for a system of colored quarks interacting with non-Abelian gauge fields, and we prove from these boundary conditions that colored hadrons cannot exist. In Sec. VII we outline the problems and challenges which lie ahead.

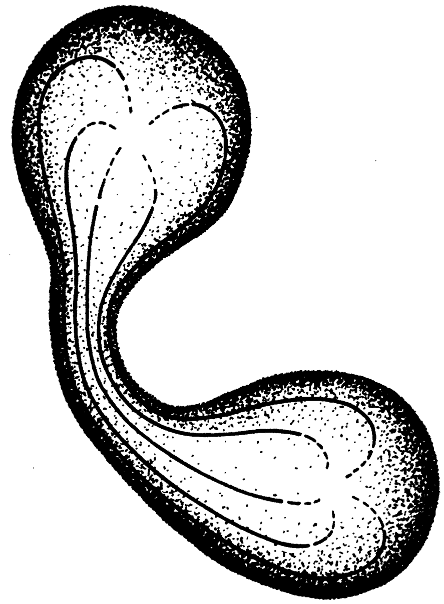

FIG. 1. A color-singlet bag attempting to fission into two bags which are not color singlets. The flux lines of the colored gluon field are shown explicitly.

II. SEMICLASSICAL DESCRIPTION OF A HADRON AT HIGH EXCITATION

Because of the simplicity of our model it is possible to draw a number of general, semiquantitative conclusions regarding the properties of the hadron. Some of these will be in the form of "virial theorems," which relate the time averages of dynamical quantities. These are rigorous on the classical level and probably remain so in the quantum theory, at least in the semiclassical limit when interpreted as relations between expectation values.

Many of our results are based upon a statistical treatment of the model at high excitation. In this "thermodynamic" limit we approximate the bag by a gas of free, massless particles—the quanta of the ϕ field which we shall call "partons"—enclosed in a region R and subject to an external pressure B. The extent to which this approximation is valid will be discussed below. For the moment we note that relations derived as time averages from virial theorems are reproduced as ensemble averages in our thermodynamics.

Our conclusions may be summarized as follows (the derivations will follow):

(a) The field in the bag behaves on the average like a perfect relativistic gas; that is, the trace of the energy-momentum tensor associated with the field, when averaged over space and time, is zero:

$$\left\langle \int_R d^3x(\Theta^\mu_\mu)_{\text{field}} \right\rangle = 0.$$

(b) The time-averaged volume of a bag is proportional to its energy:

$$E = 4B\langle V \rangle .$$

(c) The ground state and lowest excited states of the bag contain a few partons of average momentum of order $B^{1/4}$ enclosed in a volume of order $B^{-3/4}$. [B has the dimension (length)$^{-4}$ with $\hbar = c = 1$, and energies are expressed as reciprocal lengths.]

(d) In the thermodynamic limit the bag has a fixed temperature, T_0, independent of its energy. T_0 is of order $B^{-1/4}$. This is equivalent to the following statements:

(d_1) The average kinetic energy of the partons is of order T_0 independent of the bag's energy E provided the latter is larger than T_0: $E \gg T_0$.

(d_2) The asymptotic level density $\zeta(E)$ of the system is an exponential function of E:

$$\zeta \sim e^{E/T_0}.$$

(d_3) The number, N, of partons plus antipartons present in the hadron is proportional to its energy:

$$N \propto E/T_0 .$$

(e) If the classical dynamics is such that there is a maximum angular momentum of the hadron at a given total energy E, that maximum must be

$$J_{\max} = kB^{-1/3}E^{4/3},$$

where k is a dimensionless constant determined by the detailed dynamics. If the classical limit ($\hbar \to 0$) exists, quantum corrections to this formula would be down by powers of E. If there is no classical leading trajectory, a plausibility argument suggests that the leading trajectory might be (for large E)

$$J_{\max} = k' B^{-1/2}E^2 \quad (\hbar = 1) .$$

(f) The most likely angular momentum for large E is given by

$$\bar{J} \propto (B^{-1/4}E)^{5/6} .$$

Several of these results are familiar phenomenological attributes of hadrons. In particular if $B^{1/4}$ is of the order of $\frac{1}{3}$ GeV, (c) is familar from quark models and (d) is characteristic of statistical models. As we shall see, points (d) summarize the thermodynamics of a radiation field confined under constant pressure.

Point (b) is as yet untested since no experimental indication of the size of highly excited hadrons is available. According to (e) the highest Regge trajectory rises proportional to $(M^2)^{n/2}$ for large M, which, for $n=2$ is the usually assumed linear trajectory and for $n=\frac{4}{3}$, it is still compatible with present experimental evidence. (f) indicates that the angular momentum of the most frequent states increases with a smaller power of M, namely, $M^{5/6}$. The Veneziano model comes to a similar result with $\bar{J} \propto M$.[7]

To begin deriving these results we turn to simple thermodynamic and statistical arguments. Points (a) and (b) which are virial theorems will be discussed subsequently. According to our model the hadron is described by a field (or fields) and confined to a volume V. The boundary conditions on the fields ensure that they vanish outside V. Since we assume the fields confined in the bag to be quasifree and massless, it is natural to approximate their properties by those of a confined relativistic gas of massless particles. The quantum excitations of the fields, the "partons," correspond to the particles of the gas. As the basis of (1.1) the total energy of the gas is given by

$$E = E_r + BV, \tag{2.1}$$

where E_r is the internal energy of the gas ("radiation energy") and B is the constant defined in (1.1). The gas interacts at the boundary. Because the boundary conditions are nonlinear, this interaction allows an exchange of energy between the radiation (E_r) and the bag (BV), and also allows for the transformation of parton energy into new partons (if bosons) or parton pairs (if fermions). This is demonstrated explicitly in Sec. III for classical solutions in two dimensions. Thus the bag's surface serves as a means of establishing a thermal equilibrium in the gas. Since the field interacts at the boundary, we would expect the relativistic gas approximation to be valid only when the wavelength of the partons is much shorter than $V^{1/3}$.

For the lowest excitation states we do not apply thermodynamics. For these the number of partons is low and their wavelengths will be of the order $V^{1/3}$. Hence, $E_r \approx NV^{-1/3}$ where N is a small integer. Minimization of (2.1) gives immediately $V \approx B^{-3/4}$ and, as a consequence, the estimates quoted in point (c). These results follow from dimensional analysis ($\hbar = c = 1$) provided B is the only important dimensional parameter. In particular, the zero-point energy of the fields in the bag has been left out. In Sec. III we show that (at least for the case of one spatial dimension) this zero-point energy is not fixed by the theory and decouples from the dynamics.

Returning to the relativistic-gas approximation (and of necessity to states of relatively high excitation), we note that the term BV in (2.1) may be interpreted as the energy associated with an external pressure B. The system corresponds therefore to a bubble of ideal,[8] relativistic gas within an ideal liquid under constant pressure, B. Equilibrium will obtain only when the radiation pressure of the gas balances the pressure exerted by the liquid. In a gas of particles of negligible rest mass the pressure is $p=\frac{1}{3}E_r/V$ where E_r is the energy of the gas. Equilibrium then requires

$$p=\tfrac{1}{3}E_r/V=B, \tag{2.2}$$

$$E=E_r+BV=4BV. \tag{2.3}$$

The assumptions made here are compatible with the virial theorems discussed at the end of this section, which state that

$$\left\langle \int d^3x(\partial\phi)^2\right\rangle_{\text{time average}}=0$$

and

$$E=4B\langle V\rangle_{\text{time average}}\ .$$

Since the trace of the energy-momentum tensor associated with the field is proportional to $(\partial\phi)^2$ the first is equivalent to $p=\frac{1}{3}E_r/V$ and the second is just the time-average analog of our equilibrium (ensemble average) result (2.3).

Continuing our approximation we estimate the entropy of the bag by calculating that of a free massless gas enclosed in a container of volume V and with total energy $E_r=3BV$. There are no degrees of freedom associated with the walls of the bag (the bag coordinates are determined from the motion of the field), so there will be no added contribution from the walls. It is true that the boundary conditions place constraints on the fields, but these should have negligible effect at high excitations (i.e., many short-wavelength partons). Also if the confined field theory were interacting our approximations would be valid only for small values of the coupling constants. We compute the entropy of a free massless gas in thermal equilibrium using the second law of thermodynamics

$$dS=\frac{dE_r}{T}+\frac{p}{T}dV$$

and the familiar black-body law

$$E_r=\alpha T^4V\ , \tag{2.4}$$

where $\alpha=(g\pi^2/30)$ for bosons and $g(7\pi^2/240)$ for fermions, and g is the number of internal degrees of freedom of the particles. [Note that (2.3) and (2.4) combine to give $T=(3B/\alpha)^{1/4}$]. Eliminating p, T, and V we obtain

$$\frac{dS}{dE_r}=\left(\frac{\alpha}{3B}\right)^{1/4}(1+\tfrac{1}{3})$$

or

$$\begin{aligned}S(E_r)&=\frac{4}{3}\left(\frac{\alpha}{3B}\right)^{1/4}E_r+S_0\\&=E/T_0+S_0\ ,\end{aligned} \tag{2.5}$$

where

$$T_0=\left(\frac{3B}{\alpha}\right)^{1/4}. \tag{2.6}$$

An alternative derivation of these results (not assuming $p=B$) would be to first compute the entropy of a free massless gas occupying a fixed volume and having a fixed energy:

$$\begin{aligned}S(E_r,V)&=\tfrac{4}{3}E_r^{\ 3/4}(\alpha V)^{1/4}\\&=\tfrac{4}{3}(E-BV)^{3/4}(\alpha V)^{1/4}+S_0\ .\end{aligned} \tag{2.7}$$

One then assumes that the walls have the effect of allowing interchange of energy between the gas (E_r) and the volume (BV). The equilibrium state of the bag at a given energy E is that which maximizes the entropy. Maximizing (2.7) with respect to V keeping E fixed we immediately obtain (2.3), (2.5), and (2.6).

The bag is characterized by a total energy proportional to its volume and by a fixed temperature $T=T_0$. It must be remembered that these results are valid only at high excitation; clearly the temperature will be less in low-lying states. Thus T_0 must be considered as the maximum temperature of the gas in the bag.

To pursue our thermodynamic analogy to its limit: Our system is a liquid which is boiling under constant pressure B. A bubble corresponds to a bag and is filled with a gas which is ideal and relativistic. As energy is delivered to this two-phase system it does not heat up the gas or the liquid; rather it is used to convert liquid to gas, creating new partons and increasing the size of the bag. As in the classical, nonrelativistic analog (boiling water) the temperature is fixed during the phase transition and the volume of the gas phase increases linearly with the heat delivered to the system.

The temperature T_0 is equivalent to the average kinetic energy of the partons. We therefore can introduce an average number N of partons (particles plus antiparticles in the Fermi case):

$$N=\frac{E_r}{T_0}=3\frac{B}{T_0}V=V/V_0\ , \tag{2.8}$$

with

$$V_0=(3B)^{-3/4}\alpha^{-1/4}$$

as the volume containing one quantum or particle.

The level density $\zeta(E)$ of the system can be deduced directly from the entropy: $\zeta=\zeta_0e^S$, where ζ_0 is a constant. From (2.5) we get (for high energies, at which the statistical method is applicable)

$$\zeta = \zeta_0 e^{(E/T_0)}$$

being an exponential function of E which, of course, is directly connected with the fact that the temperature reaches a constant value T_0. The exact two-dimensional statistical calculation for the scalar bag leads to the result[9]

$$\zeta(E) = \frac{1}{\sqrt{2}}\left(\frac{g}{24}\right)^{(g+1)/24} \frac{1}{(\pi B)^{1/2}} \left(\frac{E}{(4\pi B)^{1/2}}\right)^{(1-g)/4} \times \exp\left[\left(\frac{g\pi}{6B}\right)^{1/2} E\right],$$

where g is the number of components of the scalar field. Notice that the thermodynamical calculation fails to obtain the power of E multiplying the exponent. This corresponds to a logarithmic term in the entropy which has been ignored at high excitations.

We now turn to the angular momentum of the hadron states. The momentum resides only in the gas, since the bag's surface is not a dynamical variable. If averaged over many nearby states with energy E, the angular momentum J certainly is zero. However, the average J^2 does not vanish and can roughly be estimated as follows. The average momentum p of each parton is of the order T_0; the most probable shape of the bag will be spherical, so that the contribution j of each parton to the angular momentum will be of the order $j \sim T_0 V^{1/3}$. The directions of these individual momenta are at random. Since there are N partons, the resulting average total angular momentum $\bar{J}$ will be

$$\bar{J} \sim N^{1/2} T_0 V^{1/3} \sim (B^{-1/4}E)^{5/6}. \tag{2.9}$$

We used (2.1), (2.6), and (2.8) in this relation. Here and in the following relations the sign ~ means "order of magnitude." We will not attempt here to determine these quantities more exactly.

To estimate the largest possible angular momentum for a fixed energy E requires a detailed dynamical calculation. However, if the corresponding classical theory has a maximum angular momentum $J_{\max}(E)$ for a given energy, its functional form can be determined from pure dimensional analysis to be ($c = 1$)

$$J_{\max}(E) \propto \frac{E^{4/3}}{B^{1/3}}, \tag{2.10}$$

where we have used the fact that $J_{\max}(E)$ must be independent of initial conditions and must therefore depend only on B. If this situation is true, i.e., the leading Regge trajectory exists classically, we can use the correspondence principle to tell us that formula (2.10) should be valid for large angular momentum and energy. It is conceivable that there is no upper limit for J at given E in the classical theory. If this is so, the following plausibility consideration can be made, which is based upon the quantum theory of black-body radiation. The largest total angular momentum $J_{\max}$ at a given energy E will occur when the bag containing the radiation assumes a form in which one linear dimension is maximized. This is the case when it assumes a "cigar"-shaped spindle form. The thickness of that shape is limited by the condition that it must be larger than the wave length associated with T_0. Hence $V \sim L/T_0^2$, where L is of the order of the length of the spindle. The maximum angular momentum is obtained if the spindle rotates about an axis perpendicular to its longest extension, such that the speed of the ends is of order c. Then we obtain as a crude estimate

$$J_{\max} \sim NT_0 L \sim E\left(\frac{V}{T_0^2}\right) \sim (B^{-1/4}E)^2,$$

which is a straight-line Regge trajectory.

Finally we derive the virial theorems to which we have referred throughout this section. We shall derive points (a) and (b) which are the time average analogs of results already found in the thermodynamic analysis. The advantage of the virial approach is that the theorems emerge as exact, though classical, results. We may consider first the quantity

$$\Omega = \int_R d^3x\, \phi(x)\dot{\phi}(x).$$

Then

$$\frac{d\Omega}{dt} = \int_S ds\, \hat{n}\cdot\frac{\partial \vec{R}}{\partial t}\,\dot{\phi}\phi + \int_R d^3x(\ddot{\phi}\phi + \dot{\phi}^2).$$

If we apply the Dirichlet boundary[10] condition the first term vanishes and by means of the equation of motion

$$\frac{d\Omega}{dt} = \int_R d^3x\,(\dot{\phi}^2 + \phi\vec{\nabla}^2\phi)$$

$$= \int_R d^3x\,[\dot{\phi}^2 + \vec{\nabla}\cdot(\phi\vec{\nabla}\phi) - (\vec{\nabla}\phi)^2].$$

A second application of the boundary condition yields

$$\frac{d\Omega}{dt} = \int_R d^3x\,[\dot{\phi}^2 - (\vec{\nabla}\phi)^2].$$

Then for all motions where Ω remains bounded,

$$0=\left\langle\int_R d^3x\,\dot{\phi}^2\right\rangle-\left\langle\int_R d^3x\,(\vec{\nabla}\phi)^2\right\rangle$$
$$=\left\langle\int_R d^3x\,(\partial\phi)^2\right\rangle\,, \tag{2.11}$$

where $\langle\ \rangle$ stands for a time average. $(\partial\phi)^2$ is proportional to the trace of the energy-momentum tensor of the field [see (3.9)]. We next consider

$$\bar{\Omega}=\int_R d^3x\,\phi(x)\vec{\mathbf{r}}\cdot\vec{\nabla}\dot{\phi}(x)\,.$$

Again,

$$\frac{d\bar{\Omega}}{dt}=\int_S ds\,\hat{n}\cdot\frac{\partial\vec{\mathbf{R}}}{\partial t}\phi\vec{\mathbf{r}}\cdot\vec{\nabla}\phi+\int_R d^3x\,(\dot{\phi}\vec{\mathbf{r}}\cdot\vec{\nabla}\dot{\phi}+\vec{\mathbf{r}}\cdot\vec{\nabla}\ddot{\phi})\,.$$

With the Dirichlet boundary condition and the equation of motion,

$$\frac{d\bar{\Omega}}{dt}=\int_R d^3x\,[\vec{\mathbf{r}}\cdot\vec{\nabla}(\tfrac{1}{2}\dot{\phi}^2)+\phi\vec{\mathbf{r}}\cdot\vec{\nabla}\,\nabla^2\phi]\,.$$

The integrand of the second term may be rewritten in the form

$$\nabla_k(\phi\vec{\mathbf{r}}\cdot\vec{\nabla}\,\nabla_k\phi)-\vec{\mathbf{r}}\cdot\vec{\nabla}\,[\tfrac{1}{2}(\vec{\nabla}\phi)^2]-\vec{\nabla}\cdot(\phi\vec{\nabla}\phi)+(\vec{\nabla}\phi)^2\,,$$

so on integration and use of the boundary condition,

$$\frac{d\bar{\Omega}}{dt}=\int_R d^3x\,\{\vec{\mathbf{r}}\cdot\vec{\nabla}\,[\tfrac{1}{2}\dot{\phi}^2-\tfrac{1}{2}(\vec{\nabla}\phi)^2]+(\vec{\nabla}\phi)^2\}$$
$$=\int_R d^3x\,(\vec{\nabla}\cdot\{\vec{\mathbf{r}}[\tfrac{1}{2}\dot{\phi}^2-\tfrac{1}{2}(\vec{\nabla}\phi)^2]\}$$
$$-3[\tfrac{1}{2}\dot{\phi}^2-\tfrac{1}{2}(\vec{\nabla}\phi)^2]+(\vec{\nabla}\phi)^2)\,.$$

The first term can be integrated to the surface where by use of the second boundary condition, $\frac{1}{2}\dot{\phi}^2-\frac{1}{2}(\vec{\nabla}\phi)^2$ may be replaced by $(-B)$, where we again may replace the surface integral by the volume integral so,

$$\frac{d\bar{\Omega}}{dt}=\int_R d^3x\,[-B\vec{\nabla}\cdot\vec{\mathbf{r}}-\tfrac{3}{2}\dot{\phi}^2+\tfrac{5}{2}(\vec{\nabla}\phi)^2]\,.$$

We finally get

$$\frac{d\bar{\Omega}}{dt}=\int_R d^3x\,[-3B-\tfrac{3}{2}\dot{\phi}^2+\tfrac{5}{2}(\vec{\nabla}\phi)^2]\,.$$

Thus, if $\bar{\Omega}$ is bounded then, on the average,

$$0=-3B\langle V\rangle-\tfrac{3}{2}\left\langle\int_R d^3x\,\dot{\phi}^2\right\rangle+\tfrac{5}{2}\left\langle\int_R d^3x(\vec{\nabla}\phi)^2\right\rangle\,.$$

If we combine this with the other time average (2.11), we get

$$3B\langle V\rangle=\left\langle\int_R d^3x\,(\vec{\nabla}\phi)^2\right\rangle\,. \tag{2.12}$$

The total energy of the bag is given by [see (3.13a) though this can be read off from (1.1)]

$$E=\int_R d^3x\,[\tfrac{1}{2}\dot{\phi}^2+\tfrac{1}{2}(\vec{\nabla}\phi)^2+B]\,.$$

Combining this with (2.11) and (2.12) we obtain

$$\tfrac{1}{4}E=B\langle V\rangle\,.$$

The virials will be bounded only when the particle is at rest (i.e., when $\vec{\mathbf{p}}=0$); hence E refers to the rest mass.

III. SCALAR FIELDS

In this section we begin the quantitative study of the properties of field theories confined to a bag with the case of a single scalar field. We treat this model in some detail since many of the techniques we shall develop can be carried over to subsequent discussions of fermion, vector, and interacting fields. The charged scalar field is a trivial generalization and will not be discussed here.

We shall formulate the classical problem (boundary conditions and equations of motion) covariantly in an arbitrary number of dimensions of space-time. The Poincaré invariance of these equations will be demonstrated explicitly. We have not attempted a complete solution of the classical or quantum mechanics in arbitrary dimension (some features of the classical and quantum solutions in three spatial dimensions are discussed in Secs. II and IV). In two dimensions (one space, one time) both the classical and quantum problems are soluble. We shall present the solutions, verify the Poincaré covariance of the classical and quantum theory, and discuss some properties of the quantum theory.

A. Formulation of the classical problem

We begin with the Lagrangian which was discussed in the Introduction (generalized to $n-1$ spatial dimensions):

$$L=\int_R d^{n-1}x\,(-\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-B)$$
$$\equiv\int_R d^{n-1}x\,\mathcal{L} \tag{3.1}$$

(our metric is $-g^{00}=g^{ii}=1$), where B is the bag constant, that is, the energy density associated with the volume R to which the fields are confined. The boundary of the region R sweeps out a surface S in space-time. The coordinates X^μ of S are labeled by $n-1$ parameters α_j,

$$X^\mu=X^\mu(\{\alpha\})\,. \tag{3.2}$$

The unit normal (n_μ) to this surface is defined to be the unit vector orthogonal to the $n-1$ tangent vectors T^μ_j:

$$T_j^\mu \equiv \frac{d}{d\alpha_j} X^\mu(\{\alpha\}) . \tag{3.3}$$

It is useful to express n_μ in terms of the normal (m_μ) to the surface at constant time $(t \equiv x^0)$.[11] To do this we choose the parameter $\alpha_0 = t$ and rewrite (3.2) and (3.3):

$$X^\mu = (t, X^i(\{\alpha\}, t)), \quad i = 1, \ldots, n-1$$
$$T_j^\mu = \begin{cases} (1, \dot{X}^i(\{\alpha\}, t)), & j = 0 \\ \left(0, \dfrac{d}{d\alpha_j} X^i(\{\alpha\}, t)\right), & j = 1, \ldots, n-2 . \end{cases} \tag{3.4}$$

m_μ is then the purely spatial $[m_\mu = (0, m_i)]$ unit vector orthogonal to the $n-2$ tangent vectors $T_j^\mu (j = 1, \ldots, n-2)$:

$$m_\mu T_j^\mu = 0, \quad j = 1, \ldots, n-2$$
$$m_\mu m^\mu = 1.$$

Then define

$$n_\mu = \frac{-(m_\lambda \dot{X}^\lambda)\eta_\mu + m_\mu}{[1 - (m_\lambda \dot{X}^\lambda)^2]^{1/2}} , \tag{3.5}$$

where η_μ is the unit timelike vector:

$$\eta_\mu \equiv (1, 0, \ldots, 0)$$

and $\dot{X}^\lambda \equiv T_0^\lambda$. It is easy to verify that $n_\mu T_j^\mu = 0$ $(j = 0, \ldots, n-2)$ and $n_\mu n^\mu = 1$. To establish a convention we choose m_μ to be the interior normal to the spatial surface.

With this geometric preliminary in mind we derive the equations of motion of the system by requiring the action $W \equiv \int_{t_0}^{t_1} dt L$ to be stationary under variations of the field ϕ and of the boundary S which vanish at t_0 and t_1. Stability under variation in the boundary requires that the Lagrange density vanish on S:

$$\partial_\mu \phi \partial^\mu \phi = -2B \text{ on } S. \tag{3.6}$$

Variation of the field generates the Klein-Gordon equation inside the bag:

$$\partial_\mu \partial^\mu \phi = 0 \text{ in } R \tag{3.7}$$

and another boundary condition:

$$n_\mu \partial^\mu \phi = 0 \text{ on } S. \tag{3.8}$$

This final condition arises from surface terms in the partial integrations which are performed to free the variation $\delta\phi$ from the derivative ∂_μ. The resulting terms may be combined, using (3.5) to give the quoted result. According to the two boundary conditions, the vector field $\partial_\mu \phi$ is a tangent field of constant magnitude over the entire surface S.

B. Poincaré invariance of the classical problem

The equations of motion (3.6)–(3.8) are manifestly Poincaré-invariant. Corresponding to this invariance we have a set of momenta P_μ and Lorentz-rotation generators $M_{\mu\nu}$ which should be time-independent. These may be constructed via Noether's theorem from the Lagrangian (3.1). Since the derivatives of R do not appear in L, R has no conjugate momentum and will not appear in P_μ or $M_{\mu\nu}$ except to delimit the spatial integrals. The locally conserved currents are identical to those of the free Klein-Gordon field except for terms involving the energy density B:

$$T_{\mu\nu} \equiv g_{\mu\nu}\mathcal{L} + \partial_\mu \phi \partial_\nu \phi , \tag{3.9}$$

$$M_{\mu\nu\lambda} = x_\mu T_{\nu\lambda} - x_\nu T_{\mu\lambda} , \tag{3.10}$$

with

$$\partial^\nu T_{\mu\nu} = \partial^\lambda M_{\mu\nu\lambda} = 0 .$$

To show the constancy of the corresponding charges consider the integral of the divergence of a conserved current over the "world hypertube" of the bag:

$$0 = \int_V d^n x \partial_\mu \mathcal{J}^\mu \quad (\text{where } \partial_\mu \mathcal{J}^\mu = 0) . \tag{3.11}$$

V is the space-time volume swept out by the bag and is bounded by two spacelike or lightlike hypersurfaces R_1 and R_2 which may be taken as surfaces of constant time or may be kept more general, and also by the section of the boundary surface S contained between R_1 and R_2. Let n_μ be the normal to the boundary of V. Figure 2 illustrates the geometry. Integrating (3.11) we obtain[12]

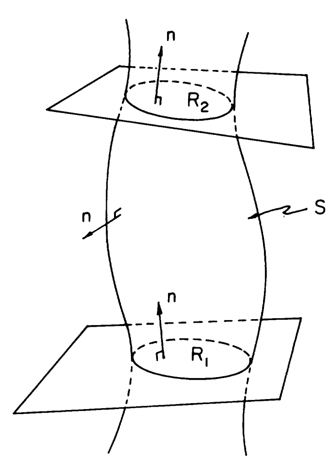

FIG. 2. The volume V and surfaces R_1, R_2, and S for a bag with two space dimensions. Note the normals to R_1 and R_2 are oppositely oriented.

$$Q \equiv \int_{R_1} ds\, n_\mu \mathcal{J}^\mu = \int_{R_2} ds\, n_\mu \mathcal{J}^\mu - \int_S ds\, n_\mu \mathcal{J}^\mu \,, \tag{3.12}$$

where ds is the surface element on the $(n-1)$-dimensional surfaces R_1, R_2, and S. For the conserved currents of (3.9) and (3.10) it is easily shown that $n_\mu \mathcal{J}^\mu = 0$ on S with the aid of the boundary condition (3.8). Therefore the integral of $n_\mu \mathcal{J}^\mu$ over a spacelike slice through the world hypertube of the bag is independent of the slice chosen. In less covariant language, this is simply the time independence of the conventional charges. For completeness, we quote the expressions for P_μ and $M_{\mu\nu}$ defined on surfaces of constant time:

$$P_\mu \equiv \int_R d^{n-1}x\, T_\mu{}^0 \,, \tag{3.13a}$$

$$M_{\mu\nu} \equiv \int_R d^{n-1}x (x_\mu T_\nu^0 - x_\nu T_\mu^0) \,. \tag{3.13b}$$

Since the primary function of the boundary conditions is to guarantee the conservation of the Poincaré generators, we may ask whether there exists an alternative set of boundary conditions, other than (3.6) and (3.8), which will achieve this goal. We observe that since $M_{\mu\nu\lambda}$ is constructed out of $T_{\mu\nu}$, it suffices to find a set of boundary conditions for which

$$n_\mu T^{\mu\nu} = 0 \text{ on } S \,.$$

We have, from (3.9),

$$n_\mu T^{\mu\nu} = n^\nu \mathcal{L} + (n_\mu \partial^\mu \phi)\partial^\nu \phi \,.$$

There are only two ways for this to vanish. Either the coefficients of n^ν and $\partial^\nu\phi$ must separately vanish, which is precisely the content of (3.6) and (3.8), or else we must have

$$\partial^\nu \phi = \beta n^\nu \,.$$

Inserting this and using the explicit form of $\mathcal{L}$, and the condition $n_\mu n^\mu = 1$, we find $n_\mu T^{\mu\nu} = 0$ if and only if $\beta^2 = 2B$.

Thus

$$\partial^\nu \phi \partial_\nu \phi = 2B \text{ on } S \tag{3.14a}$$

[note the difference in sign from (3.6)]. Furthermore, since the gradient of ϕ is normal to S, we deduce that ϕ is constant on S; for convenience, we choose this constant to be zero:

$$\phi = 0 \text{ on } S \,. \tag{3.14b}$$

In Sec. V we shall discuss an alternative derivation of these boundary conditions, in which the choice $\phi = 0$ will emerge naturally.

Because of the similarity with electrodynamics, we shall refer to the set (3.6)–(3.8) as Neumann boundary conditions, and (3.14) as Dirichlet boundary conditions.[13]

We note that the covariant definition of the charge (3.12) together with its independence of the spacelike slice chosen establishes the Lorentz transformation properties of Q. In particular it guarantees that P_μ is a four-vector and $M_{\mu\nu}$ is a second-rank Lorentz tensor.

C. Classical mechanics in two dimensions

In one space and one time dimension the equations of motion of the scalar field confined to a bag simplify considerably. In this section we will solve the classical equations, discuss some characteristics of the simplest solutions and set up the Poisson bracket (PB) formalism which will allow us to quantize the theory. Although we must sacrifice the generality of the previous sections, we expect to be amply compensated by being able to exhibit explicit solutions which realize many of the properties discussed in the Introduction.

We choose to work with the Dirichlet boundary conditions for two reasons: They are slightly simpler, and the Neumann conditions allow the fields to acquire zero modes which can have the effect of making the mass spectrum of the bag continuous.

Since the field inside the bag is massless, it is convenient to work with light-cone variables:

$$x^+ \equiv \tau \equiv \frac{1}{\sqrt{2}}(t+z) \,,$$

$$x^- \equiv x \equiv \frac{1}{\sqrt{2}}(t-z) \,.$$

Using light-cone variables, the metric tensor is off-diagonal $g^{+-} = g^{-+} = -1$, $g^{++} = g^{--} = 0$. We shall denote derivatives with respect to τ by dots:

$$\partial_+ \phi(x,\tau) = \frac{\partial}{\partial\tau}\phi(x,\tau) = \dot\phi(x,\tau)$$

and derivatives with respect to x by primes:

$$\partial_- \phi(x,\tau) = \frac{\partial}{\partial x}\phi(x,\tau) = \phi'(x,\tau) \,.$$

1. *Solution to the classical problem*

In two dimensions and in light-cone coordinates the equation of motion and boundary conditions (3.7) and (3.14) reduce to

$$\frac{\partial^2}{\partial x \partial\tau}\phi(x,\tau) = 0, \text{ in } R \tag{3.15a}$$

$$\dot\phi(x_i(\tau),\tau)\,\phi'(x_i(\tau),\tau) = -B, \quad i = 0, 1 \tag{3.15b}$$

$$\phi(x_i(\tau),\tau) = 0, \tag{3.15c}$$

where $x_i(\tau)$ $(i = 0, 1)$ are the two points which bound the bag. (3.15a) is satisfied by any function of τ

or x, corresponding to left- or right-moving waves in the bag:

$$\phi(x, \tau)=f(\tau)+g(x) \ . \qquad (3.16)$$

The boundary conditions may be rewritten in terms of $f(\tau)$ and $g(x)$:

$$\dot{f}(\tau)g'(x_i(\tau))=-B, \quad i=0,1 \qquad (3.17a)$$

$$\dot{f}(\tau)+\dot{x}_i(\tau)g'(x_i(\tau))=0 \ , \qquad (3.17b)$$

where we have differentiated (3.15c) to obtain (3.17b). The constants of the motion are given by (3.13):

$$P^- \equiv H=B(x_1(\tau)-x_0(\tau)) \ , \qquad (3.18a)$$

$$P^+ \equiv P=\int_{x_0(\tau)}^{x_1(\tau)} dx[g'(x)]^2 \ , \qquad (3.18b)$$

$$M^{+-} \equiv M=H\tau-\int_{x_0(\tau)}^{x_1(\tau)} dx\, x[g'(x)]^2 \ . \qquad (3.18c)$$

The time independence of H, P, and M may be verified with the help of the boundary conditions (3.17). For example, a suitable combination of Eqs. (3.17a) and (3.17b) yields

$$\dot{x}_i(\tau)=\frac{[\dot{f}(\tau)]^2}{B} \ ,$$

so $\dot{x}_i(\tau)$ is independent of i and $\dot{H}=0$. The extent of the bag in x (which is not conventional length, but rather length along the light cone) measures its energy.

To proceed we will find it convenient to linearize the boundary conditions. This may be done by defining a new space parameter $\sigma=\sigma(x)$ according to the differential equation

$$\frac{d\sigma}{dx}=\frac{1}{p}[g'(x)]^2 \qquad (3.19)$$

and initial condition $\sigma(x_0(0))=0$, where p is a constant to be specified later.

We define a new field $\tilde{g}(\sigma)$ in terms of $g(x)$ by this change of independent variables,

$$\tilde{g}(\sigma)\equiv g(x(\sigma)). \qquad (3.20)$$

$x(\sigma)$ will be determined from the inverse of (3.19)

$$\frac{dx}{d\sigma}=\frac{1}{p}[\tilde{g}'(\sigma)]^2 \ , \qquad (3.21)$$

where $\tilde{g}'(\sigma)\equiv(d/d\sigma)\tilde{g}(\sigma)$. The boundaries of the bag are $\sigma_i(\tau)\equiv\sigma(x_i(\tau))$. When described in terms of σ, the boundary motion will be quite simple. When we transform to σ as independent variable (3.17a) and (3.17b) become

$$\dot{f}(\tau)=-\frac{B}{p}\tilde{g}'(\sigma_i(\tau)) \ , \qquad (3.22a)$$

$$\dot{f}(\tau)+\dot{\sigma}_i(\tau)\tilde{g}'(\sigma_i(\tau))=0 \ , \qquad (3.22b)$$

so $\dot{\sigma}_i(\tau)=B/p$. Using the initial condition $\sigma(x_0(0))=\sigma_0(0)=0$, $\sigma_0(\tau)=B\tau/p$, $\sigma_1(\tau)=(B\tau/p)+\sigma_1$, where σ_1 is a constant of integration. To specify σ_1 and p consider the momentum (3.18b) in conjunction with (3.19):

$$P=p(\sigma_1(\tau)-\sigma_0(\tau))=p\sigma_1 \ .$$

Consequently, if we choose for convenience p to be the constant P, then $\sigma_1=1$ and

$$\sigma_1(\tau)=\frac{B\tau}{P}+1 \ , \qquad (3.23a)$$

$$\sigma_0(\tau)=\frac{B\tau}{P} \ . \qquad (3.23b)$$

The solution is now immediate. From (3.22a) it is apparent that $\tilde{g}(\sigma)$ is periodic in the interval $[\sigma_0, \sigma_1]$, so a general expression for $\tilde{g}(\sigma)$ is

$$\tilde{g}(\sigma)=\frac{i}{(4\pi)^{1/2}}\sum_{n\neq 0}\frac{a_n}{n}e^{-2\pi i n\sigma}+g_0 \ . \qquad (3.24)$$

$a_n=a_{-n}^\dagger$ ensures that $\tilde{g}(\sigma)$ is real. If we write (3.15c) for either end we conclude that

$$f(\tau)=-\tilde{g}\left(\frac{B\tau}{P}\right) \ ,$$

so that

$$f(\tau)=\frac{-i}{(4\pi)^{1/2}}\sum_{n\neq 0}\frac{a_n}{n}e^{-2\pi i n(B\tau/P)}-g_0 \ . \qquad (3.25)$$

Notice that g_0 cancels in the field ϕ, so we may set it equal to zero. Finally (3.21) may be integrated to obtain

$$x(\sigma)=\bar{x}_0+\frac{i}{P}\sum_{n\neq 0}\frac{L_n}{n}e^{-2\pi i n\sigma}+\frac{2\pi}{P}L_0(\sigma-\tfrac{1}{2}) \ . \qquad (3.26)$$

The constants L_n (which are the generators of the conformal group in two dimensions) are defined by

$$L_n=\frac{1}{2}\sum_{p=-\infty}^{\infty} a_p a_{n-p} \ . \qquad (3.27)$$

In this and in similar expressions below, $a_0=0$. The reason we take the constant of integration to be $\bar{x}_0-(\pi L_0/P)$ in (3.26) is to give $\bar{x}_0$ the interpretation of the average of $x(\sigma)$ over the bag at $\tau=0$:

$$\bar{x}(\tau)=\int_{B\tau/P}^{1+(B\tau/P)} d\sigma\, x(\sigma)=\bar{x}_0+\frac{H}{P}\tau \ .$$

The constants of motion are related to the explicit solution (3.24)–(3.26):

$$H=\frac{2\pi B}{P}L_0 \ , \qquad (3.28a)$$

$$M=\tau H-P\bar{x}(\tau) \ ,$$

$$M=\tau H-P\int_{B\tau/P}^{1+(B\tau/P)} d\sigma\, x(\sigma)=-\bar{x}_0 P \ , \qquad (3.28b)$$

where P is the momentum. (3.28a) determines the $(\text{mass})^2$ of the classical bag to be

$$m^2 = 2HP = 4\pi B L_0 . \tag{3.29}$$

2. Poisson-bracket formalism

To prepare the ground for quantizing the scalar bag we shall formulate the classical solution in terms of Poisson brackets.[14] Rather than return to the Lagrangian in order to determine the dynamical variables,[15] we shall simply guess what they are and verify that they satisfy Hamilton's equations in PB form. We shall then show that H, M, and P satisfy the correct Poincaré algebra in terms of PB and finally verify that they transform the fields $f(\tau)$, $\tilde{g}(\sigma)$, and $x(\sigma)$ according to naive expectations. It will then be trivial to quantize our system by the correspondence-principle substitution PB $\to 1/i$ commutator.

As coordinates we choose

$$q_n(\tau) \equiv \frac{i}{2\sqrt{\pi}\, n}\left(a_n e^{-2\pi i n(B\tau/P)} - a_{-n} e^{2\pi i n(B\tau/P)}\right),\ n>0 \tag{3.30}$$

and as momenta we have

$$p_n(\tau) \equiv \sqrt{\pi}\left(a_n e^{-2\pi i n(B\tau/P)} + a_{-n} e^{2\pi i n(B\tau/P)}\right),\ n>0 . \tag{3.31}$$

In terms of the $q_n(\tau)$ and $p_n(\tau)$,

$$H = \frac{\pi B}{P}\sum_{n>0}\left[\frac{1}{2\pi}p_n{}^2(\tau) + 2\pi n^2 q_n{}^2(\tau)\right] . \tag{3.32}$$

From the definition of the Poisson bracket

$$\{q_n, p_m\}_{\text{PB}} = \delta_{m,n}, \tag{3.33}$$

$$\{q_n, q_m\}_{\text{PB}} = \{p_n, p_m\}_{\text{PB}} = 0, \tag{3.34}$$

it is easy to verify Hamilton's equations,

$$\begin{aligned} \dot{p}_n(\tau) &= \{p_n(\tau), H\}_{\text{PB}} ,\\ \dot{q}_n(\tau) &= \{q_n(\tau), H\}_{\text{PB}} . \end{aligned} \tag{3.35}$$

The fields $\tilde{g}(\sigma)$ and $x(\sigma)$ depend explicitly on τ as well as on the dynamical variables $q_n(\tau)$ and $p_n(\tau)$. For example,

$$\tilde{g}(\sigma) = \frac{1}{2}\sum_{n>0}\left[\frac{1}{\pi}\frac{p_n(\tau)}{n}\sin 2\pi n\left(\sigma - \frac{B\tau}{P}\right) + q_n(\tau)\cos 2\pi n\left(\sigma - \frac{B\tau}{P}\right)\right] . \tag{3.36}$$

The Poisson bracket of $\tilde{g}(\sigma)$ with H corresponds to the time derivative of $\tilde{g}$ keeping $\hat{\sigma} = \sigma - (B\tau/P)$ fixed. In order to compute the time derivative keeping σ fixed one must, of course, take account of the explicit time dependence:

$$\left(\frac{\partial \tilde{g}}{\partial \tau}\right)_{\text{fixed}\,\sigma} = \{\tilde{g}(\sigma), H\}_{\text{PB}} + \frac{\partial \tilde{g}}{\partial \tau} = 0 .$$

The parameter $\hat{\sigma}$ is clearly the appropriate one to keep fixed during transformations involving τ since the range of σ is τ-dependent, whereas $\hat{\sigma}$ always ranges between 0 and 1. We therefore define a third set of fields

$$\hat{g}(\hat{\sigma}, \tau) \equiv \tilde{g}\left(\hat{\sigma} + \frac{B\tau}{P}\right) , \tag{3.37a}$$

$$\hat{x}(\hat{\sigma}, \tau) \equiv x\left(\hat{\sigma} + \frac{B\tau}{P}\right) . \tag{3.37b}$$

The PB of $\hat{g}$ and $\hat{x}$ with H are then simply time derivatives at fixed $\hat{\sigma}$.

Up to this point we have said nothing about PB involving $\bar{x}_0$. The Poincaré algebra requires

$$\{M, H\}_{\text{PB}} = -H , \tag{3.38a}$$

$$\{M, P\}_{\text{PB}} = +P , \tag{3.38b}$$

$$\{H, P\}_{\text{PB}} = 0 . \tag{3.38c}$$

From the definition $M = -\bar{x}_0 P$, it is clear that the algebra requires $\{\bar{x}_0, H\}_{\text{PB}} = +H/P$. Since $\bar{x}_0$ is constant, it is not a dynamical variable. In fact, $\bar{x}_0$ is the initial value of the variable

$$\bar{x}(\tau) = \int_{B\tau/P}^{1+(B\tau/P)} x(\sigma) = \bar{x}_0 + \frac{H}{P}\tau , \tag{3.39}$$

which we have already introduced and which is a dynamical variable:

$$\{\bar{x}(\tau), H\}_{\text{PB}} = \frac{d\bar{x}}{d\tau} .$$

From (3.38b) we expect $\bar{x}(\tau)$ to be canonically conjugate to P:

$$\begin{aligned} &\{\bar{x}(\tau), P\}_{\text{PB}} = -1 ,\\ &\{\bar{x}(\tau), q_n(\tau)\}_{\text{PB}} = \{\bar{x}(\tau), p_n(\tau)\}_{\text{PB}} = 0 ,\\ &\{P, q_n(\tau)\}_{\text{PB}} = \{P, p_n(\tau)\}_{\text{PB}} = 0 . \end{aligned} \tag{3.40}$$

(3.39) and (3.40) ensure the validity of the Poincaré algebra (3.38a)–(3.38c).

We finally consider how the fields transform under translations and boosts. For an infinitesimal translation we have[16]

$$\delta u = \epsilon\{u, -P\}_{\text{PB}}$$

for any dynamical variable u. Direct calculation yields

$$\delta\hat{g}(\hat{\sigma}, \tau) = 0 ,$$

$$\delta f(\tau) = 0 ,$$

$$\delta\hat{x}(\hat{\sigma}, \tau) = \epsilon .$$

It is not difficult to verify that these changes imply that the original variables change as expected:

$$\delta\phi(x,\tau)=-\epsilon\frac{\partial\phi}{\partial x},$$

$$\delta x_i(\tau)=\epsilon.$$

For an infinitesimal boost we have

$$\delta u=\epsilon\{u,M\}_{\rm PB}$$

from which we compute

$$\delta f(\tau)=\epsilon\{f(\tau),\tau H-P\overline{x}(\tau)\}_{\rm PB}=\epsilon\tau\dot{f}(\tau),$$

$$\delta\hat{g}(\hat{\sigma},\tau)=\epsilon\{\hat{g}(\hat{\sigma},\tau),\tau H\}_{\rm PB}=\epsilon\tau\frac{\partial\hat{g}}{\partial\tau},$$

$$\delta\hat{x}(\hat{\sigma},\tau)=\epsilon\tau\frac{\partial\hat{x}}{\partial\tau}+\epsilon\hat{x}(\hat{\sigma},\tau).$$

Again it is a straightforward exercise to show that these transformations imply the expected transformations of the original variables

$$\phi(x,\tau)\to\phi((1-\epsilon)x,(1+\epsilon)\tau),$$

$$x_i(\tau)\to(1+\epsilon)x_i((1+\epsilon)\tau).$$

3. *Quantum mechanics in two dimensions*

We quantize the scalar bag via the correspondence principle

$$i\{A,B\}_{\rm PB}\to[A,B],$$

which yields the commutation relations

$$[\overline{x}(\tau),P]=-i, \tag{3.41a}$$

$$[a_m(\tau),a_n(\tau)]=m\delta_{m,-n}, \tag{3.41b}$$

$$[\overline{x}(\tau),a_n(\tau)]=[P,a_n(\tau)]=0, \tag{3.41c}$$

where

$$a_n(\tau)=a_n e^{-2\pi i n(B\tau/P)}. \tag{3.41d}$$

The generators of translations and Lorentz transformations are carried over from the classical theory—the ordering ambiguity (which occurs only for M) is resolved by the requirement that M be Hermitian:

$$P\equiv P, \tag{3.42a}$$

$$H=\frac{\pi B}{P}\sum_n a_n a_{-n}, \tag{3.42b}$$

$$M\equiv-\tfrac{1}{2}[\overline{x}(\tau)P+P\overline{x}(\tau)]+H\tau. \tag{3.42c}$$

From (3.41) it is trivial to verify the Poincaré algebra:

$$[M,H]=-iH, \tag{3.43a}$$

$$[M,P]=iP, \tag{3.43b}$$

$$[H,P]=0, \tag{3.43c}$$

which guarantees that the quantum theory is Lorentz- and translation-invariant.

The mass-squared operator has a simple form:

$$m^2=2\pi B\sum_n a_n a_{-n}. \tag{3.44}$$

Actually all of these operators, except P, must be normal-ordered to make them finite. To determine a normal-ordering prescription, we must define the vacuum or empty bag. Unlike the vacuum in conventional field theory, the empty bag is a particle state. Indeed since $[P,a_n]=0$ the momentum of any state created by operating with a_n on the empty bag is the same as the momentum of the empty bag. We define the empty bag of momentum p by the requirements

$$P|\Omega_p\rangle=p|\Omega_p\rangle, \tag{3.45a}$$

$$a_n|\Omega_p\rangle=0 \text{ for } n>0. \tag{3.45b}$$

In general the empty bag can have mass as well as momentum. When the m^2 operator of Eq. (3.44) is normal-ordered an infinite constant will be generated, which we absorb into m_0^2, the mass of the empty bag,

$$m^2=4\pi BL_0+m_0^2. \tag{3.46}$$

Normal ordering is defined in the usual way, with all a_n, $n>0$ lying to the right of all a_n, $n<0$. The various vacua of different momenta are related by the Lorentz boost:

$$|\Omega_{e^\beta p}\rangle\equiv e^{i\beta M}|\Omega_p\rangle.$$

The operators L_n which occur in $x(\sigma)$ obey the conformal algebra:

$$[L_n,L_m]=(n-m)L_{n+m}+\tfrac{1}{12}\delta_{n,-m}(n^3-n), \tag{3.47a}$$

$$[L_n,a_m]=-ma_{n+m}, \tag{3.47b}$$

where

$$L_n=\frac{1}{2}\sum_{m=-\infty}^{\infty}:a_{-m}a_{m+n}:,$$

so that L_0 in (3.26) is replaced by $L_0+(m_0^2/4\pi B)$.

When normal-ordered, the symmetry generators of Eq. (3.42) become

$$P=P, \tag{3.48a}$$

$$H=\frac{2\pi B}{P}L_0+\frac{m_0^2}{2P}, \tag{3.48b}$$

$$M=-\tfrac{1}{2}[\overline{x}(\tau)P+P\overline{x}(\tau)]+H\tau. \tag{3.48c}$$

The algebra of (3.43) is unaffected by normal ordering since the normal-ordered operators differ from the original ones only by c numbers.

The Hamiltonian correctly generates the time dependence of dynamical fields,

$$[H, f(\tau)] = \frac{1}{i}\dot{f}(\tau)\,, \tag{3.49a}$$

$$\left[H, \tilde{g}\left(\sigma + \frac{B\tau}{P}\right)\right] = \frac{1}{i}\frac{d}{d\tau}\tilde{g}\left(\sigma + \frac{B\tau}{P}\right), \tag{3.49b}$$

$$\left[H, x\left(\sigma + \frac{B\tau}{P}\right)\right] = \frac{1}{i}\frac{d}{d\tau}x\left(\sigma + \frac{B\tau}{P}\right). \tag{3.49c}$$

The Hamiltonian does not commute with $\tilde{g}(\sigma)$ nor should it despite the fact that $(d/d\tau)\tilde{g}(\sigma) = 0$. The reason was explained in the Poisson-bracket formalism: When expressed in terms of dynamical variables $\tilde{g}(\sigma)$ has explicit time dependence which must be included in a calculation of $(d/d\tau)\tilde{g}(\sigma)$.

One may also determine the Lorentz transformation properties of the dynamical fields

$$[M, f(\tau)] = -i\tau\frac{df}{d\tau}\,, \tag{3.50a}$$

$$\left[M, \tilde{g}\left(\sigma + \frac{B\tau}{P}\right)\right] = -i\tau\frac{d}{d\tau}\tilde{g}\left(\sigma + \frac{B\tau}{P}\right), \tag{3.50b}$$

$$\left[M, x\left(\sigma + \frac{B\tau}{P}\right)\right] = -i\tau\frac{d}{d\tau}x\left(\sigma + \frac{B\tau}{P}\right). \tag{3.50c}$$

In addition to these symmetries, a generator of translations in σ may be defined. From the commutators of Eq. (3.41) the following may be shown:

$$\left[\frac{m^2}{2B}, \tilde{g}(\sigma)\right] = \frac{1}{i}\frac{d\tilde{g}}{d\sigma}(\sigma)\,, \tag{3.51a}$$

$$\left[\frac{m^2}{2B}, x(\sigma)\right] = \frac{1}{i}\frac{d}{d\sigma}x(\sigma) - \frac{H}{B}\,, \tag{3.51b}$$

where m^2 is the mass-squared operator of (3.46).

Finally we may construct the commutators of the following fields:

$$[f(\tau_1), \dot{f}(\tau_2)] = \frac{i}{2}\sum_n \delta(\sigma_1 - \sigma_2 + nP/B) - \tfrac{1}{2}iB/P\,, \tag{3.52a}$$

$$[\tilde{g}(\sigma_1), \tilde{g}'(\sigma_2)] = \frac{i}{2}\sum_n \delta(\sigma_1 - \sigma_2 + n) - \tfrac{1}{2}i\,, \tag{3.52b}$$

$$[\tilde{g}(\sigma), \dot{f}(\tau)] = \frac{i}{2}\sum_n \delta\left(\sigma - \frac{B\tau}{P} + n\right) + \tfrac{1}{2}iB/P\,. \tag{3.52c}$$

Note that all of these are periodic in the intervals which define the bag. In other words, the quantum mechanics we have generated is appropriate to a series of linked bags rather than a single bag. Generally we shall truncate the theory to consider a single bag.

Observe that the nonlocality in the commutators (3.52), which arises because of the absence of zero modes in the expansion of f and $\tilde{g}$ separately, disappears if we compute the quantity of physical interest:

$$[\tilde{\phi}(\tau_1, \sigma_1), \tilde{\phi}'(\tau_2, \sigma_2)] = \frac{i}{2}\left[\delta(\sigma_1 - \sigma_2) - \delta\left(\frac{B\tau_1}{P} - \sigma_2\right)\right],$$

which can be integrated to yield

$$[\tilde{\phi}(\tau_1, \sigma_1),\ \tilde{\phi}(\tau_2, \sigma_2)] = \frac{i}{4}\left[\epsilon(\sigma_2 - \sigma_1) + \epsilon\left(\frac{B\tau_2}{P} - \frac{B\tau_1}{P}\right) - \epsilon\left(\sigma_2 - \frac{B\tau_1}{P}\right) - \epsilon\left(\frac{B\tau_2}{P} - \sigma_1\right)\right].$$

IV. SIMPLE CLASSICAL MOTION

Since the bag is not a familiar classical system, it is worthwhile to pause and see what sort of motion occurs in some simple cases. We begin with some examples of motion in one space dimension. Later we will find approximate solutions to the spherically symmetric three-dimensional bag in the limit of large energy and small oscillations. From (3.18a) it is clear that the interval $\Delta x(\tau) \equiv x_1(\tau) - x_0(\tau)$ is fixed and equal to H/B. It is equally easy to show from (3.18b) that $\Delta\tau(x) \equiv \tau_1(x) - \tau_0(x)$ is fixed and equal to P/B. These requirements fix the motion of the end points to be almost periodic (i.e., periodic plus a linear function) with period H/B in x and P/B in τ. We may give a simple geometric interpretation to the boundary motion. Any monotonically increasing curve $x(\tau)$ is a suitable boundary for a bag provided

$$x\left(\tau + \frac{P}{B}\right) = x(\tau) + \frac{H}{B} \tag{4.1}$$

for some (positive) numbers P and H. In the x, τ plane we draw a horizontal vector of length P/B and a vertical vector of length H/B originating at the same point A and terminating at $x(\tau)$. (4.1) requires that the vectors remain, respectively, horizontal and vertical as the point A is moved. The point A then traces out the bag's other boundary. An example is shown in Fig. 3. It should be noted that the boundary motion just described is a necessary but not always sufficient condition for solution to the equation of motion. For example, the neutral scalar field subject to Dirichlet boundary conditions must not have any linear term (a_0). This is not ensured by our geometrical construction.

To make the discussion more concrete consider the motion with the longest period,

$$a_1 = a_{-1} \equiv a\,.$$

From (3.24)–(3.26) we find

$$\hat{x}(\hat{\sigma}) = \hat{\sigma} + \sin\hat{\sigma}\,, \tag{4.2a}$$

$$\tilde{g}(\hat{\sigma}) = \frac{a}{\sqrt{\pi}}\sin\hat{\sigma}/2\,, \tag{4.2b}$$

$$f(\hat{\tau}) = \frac{-a}{\sqrt{\pi}}\sin\hat{\tau}/2\,, \tag{4.2c}$$

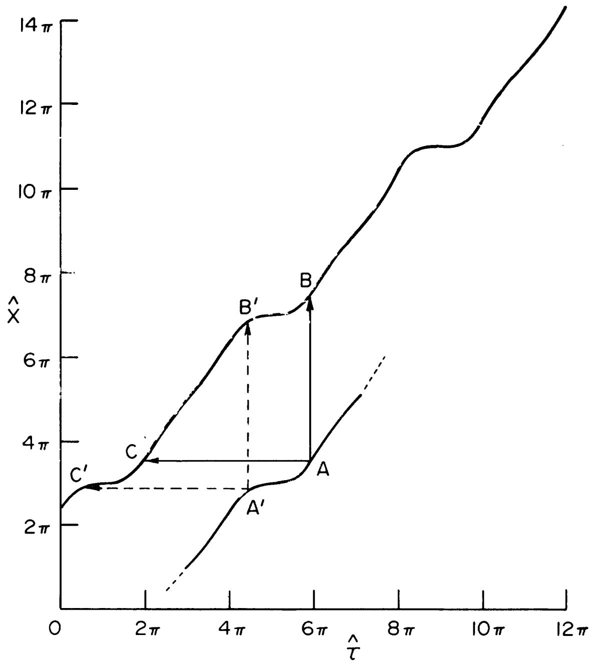

FIG. 3. Construction of the boundary motion of a one-space-dimensional bag by parallel transport of lightlike momentum and energy vectors.

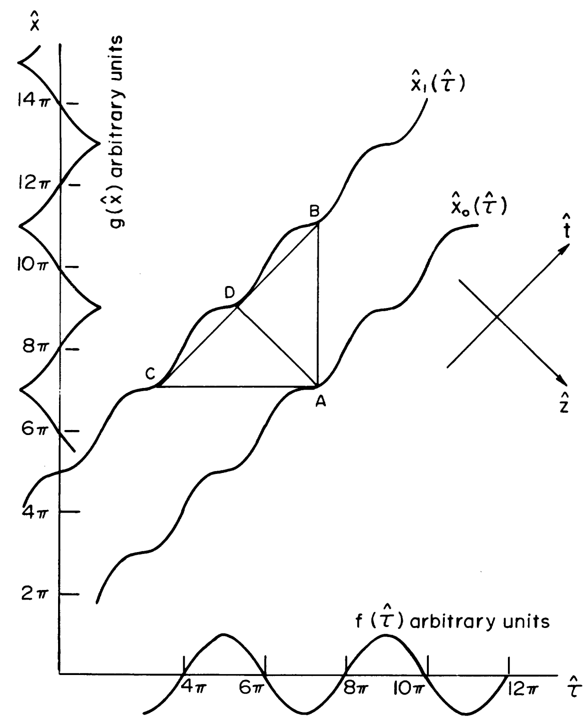

FIG. 4. Explicit solution for one-space dimensional rattling bag.

where we have defined scaled variables

$$\hat{x}=\frac{4\pi B}{H}x,\quad \hat{\tau}=\frac{4\pi B}{P}\tau,\quad \hat{\sigma}=4\pi\sigma. \tag{4.3}$$

The mass of this excitation is determined by (3.28a) to be

$$m^2=4\pi Ba^2\ .$$

The trajectories of the end points are $x(\sigma_0(\tau)) = x(B\tau/P)$, $x(\sigma_1(\tau))=x(B\tau/P+1)$, or correspondingly,

$$\hat{x}_0(\hat{\tau})=\hat{\tau}+\sin\hat{\tau}\ , \tag{4.4a}$$

$$\hat{x}_1(\hat{\tau})=\hat{\tau}+\sin\hat{\tau}+4\pi\ . \tag{4.4b}$$

To display the physical field $\phi(\hat{x},\hat{\tau})$ we require $g(\hat{x})$. This may be constructed graphically from (4.2a) and (4.2b). The resulting solution $f(\hat{\tau})$, $g(\hat{x})$ and the trajectories $\hat{x}_i(\hat{\tau})$ are shown in Fig. 4. The boundary conditions (3.17a) and (3.17b) may be verified by inspection. Several features of this solution should be noted:

(a) $\partial\hat{x}/\partial\hat{\tau}=0$ whenever $\dot{f}(\hat{\tau})=0$, i.e., when $\hat{\tau}=2\pi n$. This is required by (3.17a) and (3.17b). $\partial\hat{x}/\partial\hat{\tau}=0$ corresponds to lightlike motion of the end points.

(b) $g'(\hat{x})=\infty$ when $\dot{f}(\hat{\tau})=0$, also required by (3.17a). $g'(\hat{x})=\infty$ corresponds to an infinite momentum density at the particular $\hat{x}$. The total momentum of course remains finite and conserved.

(c) In the bag's rest frame ($H=P$), its conventional length Δz is fixed. The conventional coordinates of the rest frame [scaled as in (4.2)] are inclined at 45° to the $\hat{x}$ and $\hat{\tau}$ axes. As noted above the lengths $\Delta\hat{x}$ and $\Delta\hat{\tau}$, labeled by the line segments AB and AC in Fig. 4, are constant. Since $\hat{x}(\hat{\tau})$ has twice the frequency of $f(\hat{\tau})$, the midpoint (D) of BC is also on the trajectory $\hat{x}_1(\hat{\tau})$ and the line seqment AD is also constant. However, AD is the (scaled) length of the bag in conventional coordinates.

(d) In a moving frame Δz is not constant (θ, the angle between light cone and conventional coordinates, is no longer 45°) as required by Lorentz invariance.

(e) In the rest frame potential energy $B\Delta z$ is separately conserved.

(f) $g(\hat{x})$ has high-frequency components. Even rather simple excitations of the bag have considerable structure.

In short, we have constructed an excitation which rattles. A wave, carrying momentum, is confined to the bag. The ends of the bag oscillate as momentum is transferred from right- to left-moving waves as dictated by the boundary conditions.

Not all of the excitations of the bag rattle (have constant length in the bag's rest frame) (see, for example, Fig. 2). This is rather important since the thermodynamics developed in Sec. II assumes a mechanism for the exchange of energy between the trapped field and the potential energy (rest-system size) of the bag. Clearly, any solution in which the midpoint of the line BC is on the trajectory $\hat{x}_1(\hat{\tau})$ will rattle. This happens in general if the frequencies occurring in $\hat{x}(\hat{\sigma})$ are even integer multiples of the fundamental frequency which, in our scaled units, is $\frac{1}{2}$. In terms of the a_n, the bag will rattle if $a_n \neq 0$ only for odd or only for even values of n. When both odd and even modes are excited the rest length of the bag Δz changes with time.

So far we have studied only the simple, one-dimensional theory which is exactly soluble. In order to get some insight into the real problem in three-dimensional space, we shall discuss the situation closest to the one-dimensional case, namely, the S waves in three dimensions. We have no exact solution of the classical equations in this case, but we can solve them approximately when the energy is large enough so that the radius of the bag is large in comparison with its time-dependent fluctuations. We do this by finding solutions with fixed boundaries ("static solutions"). We may then study small oscillations about these by expanding the boundary conditions to first order about the static solution. As our example we will study the charged scalar field subject to Dirichlet boundary conditions.

For the charged scalar bag with spherical symmetry (in conventional coordinates) the field equation and boundary conditions become

$$\phi = 0 \quad \left(\dot{\phi} + \dot{R}\frac{\partial \phi}{\partial r}\bigg|_{r=R} = 0\right), \tag{4.5a}$$

$$\left|\frac{\partial \phi}{\partial r}\right|^2 - |\dot{\phi}|^2 = B, \tag{4.5b}$$

$$\Box \phi = 0 . \tag{4.5c}$$

The general solution of (4.5c) may be written

$$\phi(r,t) = \frac{1}{r}[g(t+r) - g(t-r)] , \tag{4.6}$$

where we have chosen the solution regular at the origin.

We shall solve these first for the case of static walls ($\dot{R}=0$).

With $\dot{R}=0$, $R=R_0$ (4.5) and (4.6) reduce to

$$g(t-R_0) = g(t+R_0), \tag{4.7}$$

$$|g'(t-R_0)|^2 = BR_0^{\,2}/4 . \tag{4.8}$$

The general solution to (4.7) and (4.8) is

$$g'(u) = \left(\frac{BR_0^{\,2}}{4}\right)^{1/2} e^{i\psi(u)}, \tag{4.9}$$

where $\psi(u)$ satisfies[17]

$$\psi(u+2R_0) = \psi(u) + 2\pi N \tag{4.10a}$$

and

$$\int_0^{2R_0} e^{i\psi(u)} du = 0 . \tag{4.10b}$$

As one simple example of such a solution take

$$\psi(u) = ku, \text{ where } k = \pi N/R_0 .$$

Then

$$\phi(r,t) = \frac{(BR_0^{\,2})^{1/2}}{\pi N} \frac{\sin kr}{r} e^{ikt} . \tag{4.11}$$

Note that all static solutions have energy

$$E(R_0) = 4B\langle V\rangle = \tfrac{16}{3}\pi R_0^{\,3}B ,$$

which depends only upon the radius and not on ψ in virtue of the virial theorem of Sec. II. In particular the energy is independent of k for solutions of the form of (4.11). The spherical static bag is indeed a curious classical system with energy independent of the motion of its contents.

We will now solve for small-amplitude spherically symmetric (breathing) oscillations about some static bag characterized by $\psi(u)$. We will show that only real frequencies independent of $\psi(u)$ occur in these solutions and therefore that the solutions given above are stable with respect to small spherical perturbations. We put $R(t) = R_0 + R_1(t)$, and assume $R_1 \ll R_0$. In this case we let

$$\phi(r,t) = \phi_0(r,t) + \phi_1(r,t)$$

and assume

$$|\phi_1| \ll |\phi_0| ,$$

where ϕ_0 is a solution of (4.9) and (4.10). We then linearize the equations in ϕ_1 and R_1. We first write (4.5) in the form

$$\phi = 0, \tag{4.12a}$$

$$\left|\frac{\partial \phi}{\partial r}\right|^2 (1-\dot{R}^2) = B . \tag{4.12b}$$

To first order $\dot{R}^2$ is negligible. Hence, the linearized equations are

$$R_1(t)\frac{\partial}{\partial r}\phi_0(r,t) + \phi_1(r,t) = 0 \text{ at } r = R_0 , \tag{4.13a}$$

$$\mathrm{Re}\left[\frac{\partial}{\partial r}\phi_0^*(r,t)\left(R_1(t)\frac{\partial^2}{\partial r^2}\phi_0(r,t) + \frac{\partial}{\partial r}\phi_1(r,t)\right)\right] = 0 \text{ at } r = R_0 . \tag{4.13b}$$

Using the properties of ϕ_0 [(4.9) and (4.10)] we find

$$R_1(t)=\frac{1}{\sqrt{B}}\,\phi_1(R_0,t)e^{-i\psi(t+R_0)} \tag{4.14a}$$

$$=\frac{R_0}{2\sqrt{B}}\,\mathrm{Re}\left[\frac{\partial}{\partial r}\,\phi_1(r,t)\bigg|_{r=R_0} e^{-i\psi(t+R_0)}\right], \tag{4.14b}$$

where $R_1(t)$ must be real. Here $\phi_1(r,t)$ satisfies the wave equation

$$\phi_1(r,t)=\frac{1}{r}[f(t+r)-f(t-r)]. \tag{4.15}$$

The constraints on f, imposed by (4.14), are most easily formulated in terms of

$$F(u)\equiv e^{-i\psi(u)}f(u)\equiv G(u)+H(u)\ , \tag{4.16}$$

which is required to satisfy

$$H(u)=H(u+2R_0)\ , \tag{4.17a}$$

$$G'(u)+G'(u+2R_0)-\frac{1}{R_0}[G(u)-G(u+2R_0)] =\psi'(u)M(u)\ . \tag{4.17b}$$

The most general solution of (4.17b) is

$$G(u)=\sum_{j\neq 0} a_j\frac{(|x_j|)^{1/2}}{\sin\pi x_j}\,\frac{1}{8\sqrt{\pi}\,B}\,e^{i\pi x_j(u/R_0)} +\tfrac{1}{2}\int_0^u \psi'(u')H(u')\,du' -\frac{u}{8R_0}\int_0^{2R_0}\psi'(u')H(u')\,du'+a_0\ . \tag{4.18}$$

The frequencies of small oscillation are $\pi x_j/R_0$, where

$$\tan\pi x_j=-\pi x_j\ ; \tag{4.19}$$

we note that (4.19) has the same form for all ψ. In (4.18), $a_j^*=a_{-j}$ [where $\pm j$ label the $\pm x_j$ solutions of (4.19)], in order to make $G(u)$ real. The normalization of a_j is chosen for later convenience. The motion of the bag's surface may be constructed from (4.14):

$$R_1(t)=\frac{1}{4(\pi B)^{1/2}R_0}\sum_j i\,a_j\{(\pi|x_j|)^{1/2}\}\,e^{i\pi x_j(t/R_0)} +\frac{\sqrt{B}}{4R_0}\int_{-R_0}^{R_0}\psi'(u')H(u')du'\ . \tag{4.20}$$

The eigenvalue condition (4.19) has only real solutions which means that the small spherical oscillations are stable. The second term in (4.20) corresponds to an over-all dilation of the sphere and may be reabsorbed into the definition of R_0. Henceforth we choose $H(u)$ so that

$$\int_{-R_0}^{R_0}\psi(u')H(u')du'=0\ . \tag{4.21}$$

The energies of small oscillations are easily found from the virial theorem:

$$E=\frac{16\pi B}{3}\langle[R_0+R_1(t)]^3\rangle =E_0+16\pi BR_0\langle R_1(t)^2\rangle\ , \tag{4.22}$$

where $E_0=16\pi BR_0^3/3$ is the energy of the static sphere. From (4.20)–(4.22) we obtain

$$E=E_0+\sum_j a_j\,a_{-j}\,x_j\,\frac{\pi}{R_0}\ . \tag{4.23}$$

The small oscillations may be quantized in the same manner that the two-dimensional theory was quantized in Sec. III. The resulting quantum mechanics is specified by the Hamiltonian

$$H=\sum_j\frac{|x_j|\pi}{R_0}\,a_j\,a_{-j} \tag{4.24a}$$

and commutators

$$[a_j,a_k]=\delta_{j,-k}\ . \tag{4.24b}$$

The reader may verify that (4.24) generate the correct time dependence for $R_1(t)$ in the language of Poisson brackets.

This analysis has been extended to nonspherically symmetric small oscillations about a zeroth-order solution $\psi(u)=\pi Nu/R_0$ [see (4.10a)].[18] The resulting eigenvalue condition

$$\frac{j_l'(Z_{nl})}{j_l(Z_{nl})}=-2/Z_{nl}$$

again admits only real frequencies, so the spherical solutions are stable with respect to arbitrary small amplitude deformations. These oscillations have nonzero angular momentum. They may be quantized in direct analogy to the above.[18] Finally we note that the static fermion bag may be treated in exactly the same manner as we have treated the scalar bag.

We can then ask, what quantum conditions characterized the static solutions? That is, how is R_0 quantized? A classical formalism may be set up which describes the general time development of the field in a bag of radius R_0 that is the motion associated with the phase ψ. If we pass to the quantum description through the Poisson-bracket formalism this gives quantum conditions which govern R_0, or equivalently E_0 since

$$E_0=4\times\frac{4\pi}{3}\,R_0^3B$$

for these solutions. Here E is the generator of the time development of the field in the interior of a bag of radius R_0. We than may write (4.23) in the form

$$\frac{\delta E}{\delta n_j}=\left(\frac{16\pi B}{3E}\right)^{1/3}|x_j|\pi$$

or

$$E^{4/3}=E_0{}^{4/3}+\frac{4}{3}\left(\frac{16\pi B}{3}\right)^{1/3}\sum_{j>0}a_{-j}a_j\,|x_j|\pi \qquad (4.25)$$

and then fix the constant E_0 by quantization of the motion of the phase ψ. Thus we obtain

$$E^{4/3}=\frac{4\pi}{3}\left(\frac{16\pi B}{3}\right)^{1/3}\left(\sum_n A_n^\dagger A_n+\sum_j a_{-j}a_j\,|x_j|+M_0\right), \qquad (4.26)$$

where $\{A_n\}$ are the quantum variables associated with the field ϕ_0, and $\{a_j\}$ are the variables associated with ϕ_1 and R_1. In (4.26), M_0 is an undetermined over-all constant associated with the ground state, that is, the empty bag. In (4.26) we see that the motion of the field inside a bag of a given radius, and the motion associated with fluctuations in the radius, decouple in this approximation.[19]

V. FERMION FIELDS

A. Statement of the boundary conditions

An advantage of the bag over other extended models of hadrons is the conceptual ease with which a wide variety of systems may be studied. Since confinement to a bag does not alter the short-distance behavior of a field theory, it is reasonable to restrict our attention to renormalizable theories, i.e., to spins 0, $\frac{1}{2}$, and 1, where the vector fields should be gauge fields apart from masses. Only the Dirac field requires the development of formalism beyond the previous section. We turn now to a study of this problem and leave to the following section the relatively straightforward treatment of gauge fields and other interacting theories.

Suppose we consider a single Dirac field in the bag described by the action

$$W_1=\int_V d^4x[\tfrac{1}{2}i(\bar\psi\overleftrightarrow{\partial\!\!\!/}\psi)-m\bar\psi\psi-B]. \qquad (5.1)$$

Varying this action in the usual way we obtain

$$i\partial\!\!\!/\psi=m\psi \text{ inside the bag, } R \qquad (5.2a)$$

and

$$\tfrac{1}{2}i(\bar\psi\overleftrightarrow{\partial\!\!\!/}\psi)-m\bar\psi\psi-B=0\,, \qquad (5.2b)$$

$$\tfrac{1}{2}in_\mu\gamma^\mu\psi=0 \text{ on the boundary, } S, \qquad (5.2c)$$

where n_μ is the normal to S. Unfortunately (5.2a) and (5.2b) are incompatible unless $B=0$. Furthermore, if B were maintained $\neq 0$ by the introduction of extra boson fields, (5.2c) would require that even though the motion of the surface would be completely determined by the boson fields, the surface must always move with the speed of light, which would clearly not be true of all solutions.

These problems are associated with the fact that only terms linear in the derivatives of ψ appear in the Lagrangian, and are well known in the context of the problem of a Dirac particle confined to a static box. Our method of handling these difficulties will be similar to the way in which one can handle the Dirac field confined between static walls. We shall obtain boundary conditions by allowing the Dirac field to permeate all of space-time and then proceed to a limit in which the field is confined inside the bag. This approach will be exhibited in detail below. First, however, let us state the boundary conditions which we get by this limiting method, and verify that they imply that all the required conservation laws are obtained.

To replace (5.2b) and (5.2c) we postulate the boundary conditions

$$in_\mu\gamma^\mu\psi=\psi \qquad (5.3a)$$

and

$$\left(n_\mu\frac{\partial}{\partial x_\mu}\right)\bar\psi\psi=2B\,. \qquad (5.3b)$$

Here n_μ is the inward-directed four-normal defined in Sec. III. Note that the eigenvalue equation (5.3a) implies the normalization condition $n_\mu n^\mu=1$, as well as the requirement that $\bar\psi\psi=0$ on the boundary of the bag.

Recall from Sec. III that a locally conserved current density

$$\partial_\mu\mathcal{J}^\mu=0 \qquad (5.4)$$

will yield a constant of the motion

$$Q=\int_R d^3x\,\mathcal{J}^0(x,t)\,, \qquad (5.5)$$

with the correct transformation properties, provided the current also satisfies

$$n_\mu\mathcal{J}^\mu=0 \qquad (5.6)$$

on the surface of the bag. The Dirac equation allows for a number of locally conserved quantities: The electromagnetic current

$$j^\mu=\bar\psi\gamma^\mu\psi; \qquad (5.7)$$

the energy-momentum tensor

$$T^{\mu\nu}=-\frac{i}{2}\left[\bar\psi\gamma^\mu\frac{\partial}{\partial x_\nu}\psi-\left(\frac{\partial}{\partial x_\nu}\bar\psi\right)\gamma^\mu\psi\right]-Bg^{\mu\nu}; \qquad (5.8)$$

and the angular momentum tensor

$$M^{\mu\nu\lambda}=x^\mu T^{\nu\lambda}-x^\nu T^{\mu\lambda} - \tfrac{1}{4}(\bar\psi\gamma^\lambda\sigma^{\mu\nu}\psi+\bar\psi\sigma^{\mu\nu}\gamma^\lambda\psi)\,, \tag{5.9}$$

where $\sigma^{\mu\nu}=\frac{1}{2}i[\gamma^\mu,\gamma^\nu]$. We have immediately

$$n_\mu j^\mu=\bar\psi\not n\psi=i\bar\psi\psi=0$$

on the surface. Also,

$$n_\mu T^{\mu\nu}=\frac{1}{2}\frac{\partial}{\partial x_\nu}(\bar\psi\psi)-Bn^\nu$$

from (5.3a). Since $\bar\psi\psi\equiv 0$ on the surface, its gradient defines a normal. Using (5.3b) we have

$$\frac{\partial}{\partial x_\nu}(\bar\psi\psi)=n^\nu\left(n_\lambda\frac{\partial}{\partial x_\lambda}\right)\bar\psi\psi=2Bn^\nu\ .$$

Therefore

$$n_\mu T^{\mu\nu}=0\ . \tag{5.10}$$

Finally, it is immediate from (5.10) and (5.3a) that

$$n_\lambda M^{\mu\nu\lambda}=0$$

on the surface as well.

Note that in the massless case, the axial-vector current $\bar\psi\gamma^\mu\gamma_5\psi$ is locally conserved. However, we have $n_\mu\bar\psi\gamma^\mu\gamma_5\psi=i\bar\psi\gamma_5\psi$ on the surface which does not necessarily vanish. The symmetry generated by the axial charge will be broken by the boundary conditions. This is related to the fact that in stating the boundary conditions (5.3a) and (5.3b) we have established a convention regarding the intrinsic parity of the fermion field. The field $\psi'\equiv\gamma_5\psi$, which has opposite intrinsic parity to ψ, would satisfy the boundary conditions with the opposite sign:

$$in_\mu\gamma^\mu\psi'=-\psi'\,, \tag{5.11}$$

$$\left(n_\mu\frac{\partial}{\partial x_\mu}\right)\bar\psi'\psi'=-2B\ . \tag{5.12}$$

We shall see below that in deriving the boundary conditions the sign of a mass parameter M outside the bag will be responsible for the conventional choice of parity implied by (5.3).

B. Derivation of the boundary conditions

To derive the boundary conditions, we begin with a field ψ defined over all space-time, but with different masses inside and outside the bag[20]:

$$i\gamma^\mu\partial_\mu\psi=m\psi \text{ inside the bag,} \tag{5.13a}$$

$$i\gamma^\mu\partial_\mu\Psi=M\Psi \text{ outside the bag,} \tag{5.13b}$$

where we have denoted the outside field by capital Ψ. The action for this system is then

$$W=\int_V d^4x(\tfrac{1}{2}i\bar\psi\overleftrightarrow{\not\partial}\psi-m\bar\psi\psi-B) + \int_{\bar V} d^4x(\tfrac{1}{2}i\bar\Psi\overleftrightarrow{\not\partial}\Psi-M\bar\Psi\Psi)\,,$$

with V the region of space-time swept out by the interior of the bag R, and $\bar V$ the region swept out by its complement, $\bar R$. In addition to (5.13), we have

$$\psi=\Psi \tag{5.14a}$$

and

$$M\bar\Psi\Psi=B+m\bar\psi\psi \tag{5.14b}$$

on the surface of the bag. The strategy for confining the field to the bag is now to let $M\to\infty$, in which limit we expect Ψ to vanish outside the bag, leaving only the ψ field nonzero within and on the surface of the bag

A few remarks are in order before we begin to implement this idea. First, observe that before taking M to ∞, we must limit ourselves to solutions with energies which are small compared with M. Otherwise the particle will not be confined to the bag as the limit is taken. Second, note that it is not necessary to require $\Psi=0$ on the surface, even as $M\to\infty$. We shall see in fact, that Ψ is normalized in an M-independent manner on the surface, and falls exponentially in M outside the surface. Third, (5.14b) clearly requires that

$$\lim_{M\to\infty}\bar\Psi\Psi=0$$

on the surface. What we require, and what will be the major objective of the calculation which follows is the term of order $1/M$ on the left-hand side of (5.14b). This term will provide us with a nontrivial boundary condition involving B to replace the naive result (5.2).

We begin with the following ansatz for the exterior solution:

$$\Psi_\alpha=\Phi_\alpha e^{Mj_\alpha}\ . \tag{5.15}$$

Here α labels the Dirac components. The specification of Ψ in terms of Φ and j is made precise by the further requirement that j be independent of M. The form of (5.15) is motivated by the observation that each component of Ψ satisfies the Klein-Gordon equation:

$$\Box\Psi=+M^2\Psi\ ,$$

so that an exponential behavior such as (5.15) is suggested. Only after this exponential is explicitly taken into account do we expect the remaining factor Φ to possess an orderly expansion in increasing powers of $(1/M)$. The real part of j_α must, of course, be negative outside the bag in order that

Ψ vanish as $M\to\infty$.

We first observe that j_α must be independent of α. To see this, use (5.13b):

$$\sum_\beta i(\gamma^\mu)_{\alpha\beta}\left(\frac{\partial\Phi_\beta}{\partial x^\mu}+M\Phi_\beta\frac{\partial j_\beta}{\partial x^\mu}\right)e^{Mj_\beta}=M\Phi_\alpha e^{Mj_\alpha}. \tag{5.16}$$

Having assumed that j is independent of M and not positive and that Φ is expandable in powers of $1/M$,

$$\Phi=\Phi^{(0)}+\frac{1}{M}\Phi^{(1)}+\cdots, \tag{5.17}$$

we can isolate the leading M dependence in (5.16) as $M\to\infty$,

$$\sum_\beta\left[i(\gamma^\mu)_{\alpha\beta}\frac{\partial j_\beta}{\partial x^\mu}\right]\Phi_\beta^{(0)}e^{Mj_\beta}=\Phi_\alpha^{(0)}e^{Mj_\alpha}. \tag{5.18}$$

This is a matrix eigenvalue equation for $\Phi_\alpha^{(0)}e^{Mj_\alpha}$, in which the matrix is independent of M. Therefore the eigenvector must likewise be independent of M except for a possible over-all normalization factor. Hence $j_\alpha\equiv j$ is independent of the Dirac index α.

We can now rewrite (5.18) as

$$(i\partial\!\!\!/ j-1)\Phi^{(0)}=0. \tag{5.19}$$

We observe that (5.19) requires $\partial_\mu j\partial^\mu j=1$. For later use, we compute the results of applying $(\partial_\mu j\,\partial/\partial x_\mu)$ to (5.19):

$$\begin{aligned}(i\partial\!\!\!/ j-1)\left(\partial_\mu j\frac{\partial}{\partial x_\mu}\right)\Phi^{(0)}&=-i[\partial^\mu j\partial_\mu\partial_\nu j]\gamma^\nu\Phi^{(0)}\\&=0.\end{aligned} \tag{5.20}$$

The last equality follows from differentiating $\partial_\mu j\partial^\mu j=1$. We rewrite (5.14b) using (5.13b):

$$m\bar\psi\psi+B=\tfrac{1}{2}i(\bar\Psi\overrightarrow{\partial\!\!\!/}\Psi-\bar\Psi\overleftarrow{\partial\!\!\!/}\Psi) \tag{5.21}$$

and insert (5.15):

$$m\bar\psi\psi+B=\tfrac{1}{2}i[\bar\Phi(\overrightarrow{\partial\!\!\!/}\Phi+M\partial\!\!\!/ j\Phi)-(\bar\Phi\overleftarrow{\partial\!\!\!/}+M\bar\Phi\partial\!\!\!/ j^a)\Phi]e^{M(j+j^*)} \tag{5.22}$$

where

$$\partial\!\!\!/ j^a=\gamma^0\partial\!\!\!/ j^+\gamma^0=\partial_\mu j^*\gamma^\mu.$$

Two important points emerge from (5.22). First, we see that the terms of order M will cancel only if $\partial_\mu j$ is real. This is a restatement of the necessity of eliminating energies comparable to M from the solution: By choosing j, and hence $\partial_\mu j$ real we forbid oscillatory behavior in the exponential e^{Mj}. The second condition we deduce from (5.22) is that, in order for the limit as $M\to\infty$ to exist on the right-hand side, j must vanish on the surface S. The gradient of j is therefore normal to S. Since j is less than zero outside of S, $\partial_\mu j$ is in fact the inward-directed normal on S as defined in Sec. III and required by (5.3):

$$\partial_\mu j\equiv n_\mu \text{ on } S.$$

Had we chosen $M<0$, Eqs. (5.11) and (5.12) would have been obtained. As $M\to\infty$, (5.22) becomes

$$m\bar\psi\psi+B=\tfrac{1}{2}i[\bar\Phi^{(0)}\overrightarrow{\partial\!\!\!/}\Phi^{(0)}-\bar\Phi^{(0)}\overleftarrow{\partial\!\!\!/}\Phi^{(0)}]. \tag{5.23}$$

On S we can write

$$\frac{\partial\Phi^{(0)}}{\partial x^\mu}=n_\mu\left[n^\nu\frac{\partial\Phi^{(0)}}{\partial x^\nu}\right]+\left[\frac{\partial\Phi^{(0)}}{\partial x^\mu}-n_\mu\left(n^\nu\frac{\partial\Phi^{(0)}}{\partial x^\nu}\right)\right]. \tag{5.24}$$

The second term on the right-hand side of (5.24) is orthogonal to n_μ. Thus it contains only derivatives of $\Phi^{(0)}$ which lie wholly on the surface. From the continuity equation (5.4b), which now reads

$$\psi=\Phi^{(0)} \text{ on } S, \tag{5.25}$$

we see that $[\partial\Phi^{(0)}/\partial x^\nu]$ may be replaced by $(\partial\psi/\partial x^\nu)$ in this term. The first term on the right-hand side of (5.23) is now easily evaluated:

$$\begin{aligned}\tfrac{1}{2}i\bar\Phi^{(0)}\overrightarrow{\partial\!\!\!/}\Phi^0&=\frac{-1}{4}\bar\Phi^{(0)}(in\!\!\!/-1)n^\nu\frac{\partial\Phi^{(0)}}{\partial x^\nu}\\&\quad+\tfrac{1}{2}i\bar\psi\overrightarrow{\partial\!\!\!/}\psi+\tfrac{1}{2}\bar\psi n\cdot\overrightarrow{\partial}\psi,\end{aligned} \tag{5.26}$$

where the eigenvalue conditions

$$-i\bar\Phi^{(0)}n\!\!\!/=\bar\Phi^{(0)};\quad -i\bar\psi n\!\!\!/=\bar\psi$$

have been used. However, the first term on the right-hand side of (5.26) vanishes by (5.20), so that we need only add (5.26) to its Hermitian conjugate to obtain

$$(n\cdot\partial)(\bar\psi\psi)=2B. \tag{5.3b}$$

Notice that the mass term, $m\bar\psi\psi$, has canceled out of the boundary condition.

Together with (5.3a), which follows from (5.19) and (5.25), these are the boundary conditions with which we replace (5.2b) and (5.2c).

The Dirichlet boundary conditions for the scalar field, Eq. (3.14), can be derived in a similar manner. Denoting the fields inside and outside the bag by ϕ and Φ, respectively, we have

$$\Box\phi=0 \text{ inside } R,$$

$$\Box\Phi=M^2\Phi \text{ outside } R,$$

$$\tfrac{1}{2}M^2\Phi^2=B \text{ on } S,$$

and

$$\phi=\Phi,\ \partial_\mu\phi=\partial_\mu\Phi \text{ on } S.$$

Note that we demand continuity of the first derivative as well, since the equation of motion is second order. By writing $\Phi = \chi e^{Mj}$, $\chi = \chi^{(0)} + (1/M)\chi^{(1)} + \cdots$ and performing manipulations analogous to the spinor case, we arrive at the boundary conditions (3.14), with ϕ specified to be zero on S.

C. Quantization in two dimensions

Having formulated the classical fermion problem in arbitrary dimension, we turn to a detailed study of the quantization of a massless Dirac field in two space-time dimensions. Just as in the case of the scalar field, the quantization is particularly simple on the light cone. We therefore parametrize the motion by τ and x introduced in Sec. III . We choose as our Dirac matrices

$$\beta = \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\alpha = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{5.27}$$

Writing the Dirac spinor $\psi = \binom{ig}{f}$, the Dirac equation inside the bag becomes

$$\left(\gamma^+ \frac{\partial}{\partial \tau} + \gamma^- \frac{\partial}{\partial x}\right)\begin{pmatrix} ig \\ f \end{pmatrix} = 0 , \tag{5.28}$$

which implies that g depends only on x, while f depends only on τ, hence

$$\psi = \begin{pmatrix} ig(x) \\ f(\tau) \end{pmatrix} \tag{5.29}$$

is the most general solution inside.

If we denote the ends of the bag by $x_0(\tau)$ and $x_1(\tau)$, we easily obtain the following equations for the components of $\partial_\mu j\,(\equiv n_\mu)$:

$$\frac{\partial j}{\partial \tau}\frac{\partial j}{\partial x} = -\frac{1}{2}\,, \quad \text{for } x = x_i(\tau)$$

$$\frac{\partial j}{\partial \tau} + \frac{\partial x}{\partial \tau}\frac{\partial j}{\partial x} = 0, \quad \text{for } x = x_i(\tau)$$

the latter equation being obtained by differentiating the condition $j(x_i(\tau), \tau) = 0$ with respect to τ. Then, if we choose the solution for which $\partial_\mu j$ is the inward-directed normal, we obtain

$$\left.\frac{\partial j}{\partial x}\right|_{x=x_i(\tau)} = (-)^i \frac{1}{(2\dot{x}_i)^{1/2}}\,, \tag{5.30a}$$

$$\left.\frac{\partial j}{\partial \tau}\right|_{x=x_i(\tau)} = -(-)^i \left(\frac{\dot{x}_i}{2}\right)^{1/2} . \tag{5.30b}$$

The boundary conditions on ψ reduce to

$$\sqrt{2}\, i \beta \left[\frac{\partial j}{\partial \tau}\begin{pmatrix} ig \\ 0 \end{pmatrix} + \frac{\partial j}{\partial x}\begin{pmatrix} 0 \\ f \end{pmatrix}\right] = \begin{pmatrix} ig \\ f \end{pmatrix} \quad \text{for } x = x_i(\tau), \tag{5.31}$$

from which follows

$$f(\tau) = -\sqrt{2}\, g(x_i(\tau)) \frac{\partial j}{\partial \tau}(x_i(\tau), \tau)$$
$$= (-)^i \left(\frac{dx^i}{d\tau}\right)^{1/2} g(x^i(\tau)), \tag{5.32}$$

together with

$$\left(\frac{\partial j}{\partial \tau}\frac{\partial}{\partial x} + \frac{\partial j}{\partial x}\frac{\partial}{\partial \tau}\right) i\,(f^* g - g^* f) = -2B \tag{5.33a}$$

or

$$\left.\operatorname{Im}\left(f^* g' \frac{\partial j}{\partial \tau} + \dot{f}^* g \frac{\partial j}{\partial x}\right)\right|_{x=x_i(\tau)} = B\,. \tag{5.33b}$$

For convenience, we also cast this formula in two alternate forms by using the relation between f and g and between $dx_i/d\tau$ and $\partial j/\partial \tau$ on the boundary and dropping all purely real terms inside the imaginary part (recall that j is real):

$$\operatorname{Im} \dot{f}^*(\tau) f(\tau) = \frac{B}{\sqrt{2}}\frac{dx_i}{d\tau} \tag{5.33c}$$

or

$$\frac{dx_i}{d\tau}\operatorname{Im} g'^*(x_i(\tau)) g(x_i(\tau)) = \frac{B}{\sqrt{2}}\,. \tag{5.33d}$$

The components of the energy-momentum tensor and electromagnetic current are summarized by the following:

$$T^{++} = \sqrt{2}\, \operatorname{Im} g'^*(x) g(x), \tag{5.34a}$$

$$T^{-+} = B, \tag{5.34b}$$

$$T^{--} = \sqrt{2}\, \operatorname{Im} \dot{f}^*(\tau) f(\tau), \tag{5.34c}$$

$$j^+ = \sqrt{2}\, g^*(x) g(x), \tag{5.34d}$$

$$j^- = \sqrt{2}\, f^*(\tau) f(\tau), \tag{5.34e}$$

$$M^{+-+} = \tau B - x\sqrt{2}\, \operatorname{Im} g'^*(x) g(x), \tag{5.34f}$$

$$M^{+--} = \tau\sqrt{2}\, \operatorname{Im} \dot{f}^*(\tau) f(\tau) - xB\,. \tag{5.34g}$$

So we have for the boost generator, energy-momentum, and charge

$$P^- = B(x_1(\tau) - x_0(\tau)) \equiv H\,, \tag{5.35}$$

$$P^+ = \sqrt{2} \int_{x_0(\tau)}^{x_1(\tau)} dx \operatorname{Im} g'^*(x) g(x) \equiv P\,, \tag{5.36}$$

$$Q = \sqrt{2} \int_{x_0(\tau)}^{x_1(\tau)} dx\, g^*(x) g(x)\,, \tag{5.37}$$

$$M = M^{+-} = \tau H^0 - \sqrt{2} \int_{x_0(\tau)}^{x_1(\tau)} dx\, x \operatorname{Im} g'^*(x) g(x)\,, \tag{5.38}$$

and one can check directly that they are all conserved, though this has been proved in general in the first part of this section.

In order to quantize the fermion system we resort to a redefinition of the x coordinate similar to the scalar case. We define a function $\sigma(x)$ by the equation

$$\frac{d\sigma}{dx}=\frac{\sqrt{2}}{P}\operatorname{Im} g^{*\prime}(x)g(x)\ ,\qquad \sigma(x_0(0))=0\ , \tag{5.39}$$

so that

$$\sigma_0(\tau)=\frac{B\tau}{P}\ ,\quad \sigma_1(\tau)=\frac{B\tau}{P}+\sigma_0\ .$$

Notice that $\operatorname{Im} g^{*\prime}(x)g(x)$ is not an intrinsically positive-definite quantity. It is, in fact, proportional to T^{++}, the P^{+} density of the bag. This lack of positive definiteness of T^{++} is a familiar feature of the classical Dirac theory. We expect that this difficulty will be removed in the usual way by quantizing with anticommutation relations. For the purposes of setting up our canonical formalism we shall assume that $T^{++}/P \propto d\sigma/dx>0$ throughout the bag and so $\sigma(x)$ can be inverted. We will cease to worry whether we have made an error after verifying the self-consistency of the quantized theory.

We shall take the field $g(x)$ to be a conformal spinor, i.e., we define a new $\tilde{g}(\sigma)$ by

$$\tilde{g}(\sigma)=\left(\frac{dx}{d\sigma}\right)^{1/2} g(x(\sigma))\ . \tag{5.40}$$

This is in contrast with the scalar theory where g transformed as a conformal scalar. If we substitute (5.40) in the equation for $d\sigma/dx$ we find

$$\frac{dx}{d\sigma}=\frac{\sqrt{2}}{P}\operatorname{Im}(\tilde{g}'(\sigma)^*\tilde{g}(\sigma))\ . \tag{5.41}$$

We may express P, H, M, and Q in terms of $\tilde{g}(\sigma)$:

$$Q=\sqrt{2}\int_{\sigma_0(\tau)}^{\sigma_1(\tau)} d\sigma\, \tilde{g}^*(\sigma)\tilde{g}(\sigma)\ , \tag{5.42a}$$

$$H=\frac{B\sqrt{2}}{P}\int_{\sigma_0(\tau)}^{\sigma_1(\tau)} d\sigma \operatorname{Im}(\tilde{g}'^*(\sigma)\tilde{g}(\sigma))\ , \tag{5.42b}$$

$$P=P(\sigma_1(\tau)-\sigma_0(\tau))\rightarrow\sigma_0=1\ , \tag{5.42c}$$

so that

$$M=\tau H-P\bar{x}(\tau)\ , \tag{5.43}$$

where we have defined the average $\bar{x}(\tau)$ by

$$\bar{x}(\tau)=\int_{\sigma_0(\tau)}^{\sigma_1(\tau)} d\sigma\, x(\sigma)\ , \tag{5.44}$$

from which

$$\frac{d\bar{x}}{d\tau}=\frac{B}{P}(x_1(\tau)-x_0(\tau))=\frac{H}{P}\ ,$$
$$\bar{x}(\tau)=\bar{x}_0+\frac{H}{P}\tau\ ,$$

so that

$$M=-P\bar{x}_0\ . \tag{5.43a}$$

From (5.30), the connection between f and $\tilde{g}$ is just

$$f(\tau)=(-)^i\left(\frac{B}{P}\right)^{1/2}\tilde{g}(\sigma_i(\tau))\ , \tag{5.45}$$

from which the odd periodic condition

$$\tilde{g}\left(\frac{B\tau}{P}\right)=-\tilde{g}\left(\frac{B\tau}{P}+1\right) \tag{5.46}$$

follows. Thus we can expand $\tilde{g}$ in half-integral modes:

$$\tilde{g}(\sigma)=\frac{1}{(\frac{1}{2})^{1/4}}\sum_{m=-\infty}^{\infty}{}' \, b_m e^{-2\pi i m\sigma}\ , \tag{5.47}$$

where the prime on $\sum$ indicates that m is summed over half-odd integers. From (5.45) and (5.47) we have

$$f(\tau)=\frac{1}{(\frac{1}{2})^{1/4}}\left(\frac{B}{P}\right)^{1/2}\sum_{m=-\infty}^{\infty}{}' \, b_m e^{-2\pi i m(B\tau/P)} \tag{5.48}$$

and from (5.42b)

$$H=\frac{2\pi B}{P}\sum_{m=-\infty}^{\infty}{}' \, m b_m^* b_m\ . \tag{5.49}$$

We take $b_m(\tau)\equiv b_m e^{-2\pi i\, m(B\tau/P)}$ to be dynamical variables and therefore impose the anticommutation relations

$$\{b_m(\tau), b_n(\tau)\}=0\ ,\qquad \{b_m(\tau), b_n^\dagger(\tau)\}=\{b_m^\dagger(\tau), b_n(\tau)\}=\delta_{m,n}\ , \tag{5.50}$$

which ensures Heisenberg's equations

$$\dot{b}_m(\tau)=\frac{1}{i}[b_m(\tau), H]$$

and, as in the scalar case, the operators $x(\sigma)$, $\tilde{g}(\sigma)$ will have explicit time dependence.

To achieve a positive-energy spectrum we must interpret the b_m's as follows ($m>0$):

b_m annihilates a fermion,

$b_{-m}\equiv d_m^\dagger$ creates an antifermion,

$b_m{}^\dagger$ creates a fermion,

$b_{-m}{}^\dagger\equiv d_m$ annihilates an antifermion.

Then we can write the Hamiltonian (5.48) as

$$H=\frac{2\pi B}{P}\left[\,:\sum_{m=1/2}^{\infty}{}'(m b_m^\dagger b_m - m b_{-m}^\dagger b_{-m}):+\frac{m_0{}^2}{4\pi B}\right]$$
$$=\frac{2\pi B}{P}\left[\sum_{m=1/2}^{\infty}{}' \, m(b_m^\dagger b_m+d_m^\dagger d_m)+\frac{m_0{}^2}{4\pi B}\right]\ , \tag{5.51a}$$

$$H \equiv \frac{2\pi B}{P}\mathcal{L}_0 + \frac{m_0{}^2}{4\pi B} \quad , \tag{5.51b}$$

where we have introduced m_0, the mass of the empty bag, which is undetermined due to ordering ambiguities. We list the expressions for Q and M which follow from (5.42a) and 5.43):

$$Q = \sum_{m=1/2}^{\infty}{}' (b_m^\dagger b_m - d_m^\dagger d_m) + Q_0 \,, \tag{5.52a}$$

$$M = -\tfrac{1}{2}(\bar{x}_0 P + P\bar{x}_0)\,, \tag{5.52b}$$

where Q_0 is the charge of the empty bag; we shall assume $Q_0 = 0$ in our model. It is essentially trivial to show that P, H, and M obey the Lorentz algebra if we make the usual requirement

$$[\bar{x}(\tau), P] = -i\,,$$
$$[\bar{x}(\tau), b_n(\tau)] = [\bar{x}(\tau), b_n^\dagger(\tau)] = 0\,.$$

We shall not burden the reader with the evaluation of the operator $x(\sigma)$. It is straightforward to show that

$$x(\sigma) = \bar{x}_0 + \frac{2\pi}{P}\left(\mathcal{L}_0 + \frac{m_0{}^2}{4\pi B}\right)(\sigma - \tfrac{1}{2}) + \frac{i}{P}\sum_{n\neq 0}\frac{\mathcal{L}_n}{n}e^{-2\pi i n\sigma}\,, \tag{5.52}$$

where $\mathcal{L}_n$ are the fermion conformal generators:

$$\mathcal{L}_n = :\sum_{m=-\infty}^{\infty}{}' (m + \tfrac{1}{2}n) b_m^\dagger b_{n+m} :\,, \tag{5.54}$$

which obey the algebra

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m} + \tfrac{1}{12}\delta_{n,-m}(n^3 - n)\,, \tag{5.55}$$
$$[\mathcal{L}_n, b_m] = -(m + \tfrac{1}{2}n) b_{m+n}\,.$$

We observe the striking similarity in the forms of $x(\sigma)$ for the scalar and fermion cases.

We summarize briefly the results of our quantization procedure. There are only positive-energy fermions and antifermions with a (mass)^2 spectrum of half-odd-integral multiples of $4\pi B$, in contrast with the boson and even fermion spectrum of integral multiples of $4\pi B$. The possible states of the bag are represented by polynomials of $b_m^\dagger$ and $d_m^\dagger$ acting on the empty bag state, $|\Omega_p\rangle$, which is defined by

$$d_m|\Omega_p\rangle = b_m|\Omega_p\rangle = 0\,,$$
$$P|\Omega_p\rangle = p|\Omega_p\rangle\,.$$

VI. THE MIXED BAG

As remarked in Sec. V, the confinement of gauge vector and other interacting fields to a bag poses no new problems once the scalar and spinor theories have been understood. To exemplify the treatment of interacting fields, we consider the colored quark-gluon theory proposed in the Introduction as a realistic model for hadrons. Specifically, we will study a model consisting of three triplets of massless quarks, interacting via an SU(3) vector gauge group (which operates only on the color indices of the quarks) with eight massless self-coupled, gauge vector bosons. The quarks could be chosen to be massive and be treated identically to the massive fermions of Sec. V. We leave them massless for simplicity, although it may be desirable to add a mass term to break the hadronic (as opposed to color) SU(3) symmetry. As remarked in the Introduction, we expect that the infrared divergences of the massless Yang-Mills theory will be absent since the bag cuts off long wavelengths.

We begin by formulating the massless Yang-Mills theory. The quarks will be included later. Define

$$F_i^{\mu\nu} = \partial^\mu A_i^\nu - \partial^\nu A_i^\mu + g f_{ijk} A_j^\mu A_k^\nu\,. \tag{6.1}$$

$F_i^{\mu\nu}$ transforms under the regular representation of the gauge group when A_i^μ undergoes the infinitesimal gauge transformation

$$A_i^\mu \to A_i^\mu + g f_{ijk} A_j^\mu \alpha_k(x) + \partial^\mu \alpha_i(x)\,, \tag{6.2}$$

where f_{ijk} are the structure constants of the group. We then consider the gauge-invariant action for A_i^μ confined to the bag:

$$W = \int_V d^4x(-\tfrac{1}{4}F_i^{\mu\nu}F_{i\mu\nu} - B)\,. \tag{6.3}$$

Stability of the action under variation of the fields A_i^μ implies the equation of motion

$$D_{ij}^\mu F_{j\mu\nu}(x) = 0 \tag{6.4a}$$

and the boundary conditions

$$n_\mu F_i^{\mu\nu} = 0\,, \tag{6.4b}$$

$$-\tfrac{1}{4}F_{i\mu\nu}F_i^{\mu\nu} - B = 0 \tag{6.4c}$$

on the surface of the bag. In (6.4) we have introduced the gauge-covariant derivative

$$D_{ij}^\mu = \delta_{ij}\partial^\mu - g f_{ijk} A_k^\mu\,. \tag{6.5}$$

It is straightforward to show that the usual locally conserved $T^{\mu\nu}$ and $M^{\mu\nu\lambda}$ satisfy

$$n_\nu T^{\mu\nu} = n_\lambda M^{\mu\nu\lambda} = 0$$

on the surface, so the Poincaré generators are conserved and Lorentz-covariant as they must be.

The methods described in Sec. V can be easily extended to massless quarks interacting with the gauge boson through a conserved current. We denote the quark field by $g_{i\alpha}$ where the first index refers to the SU(3) of color and the second refers to the hadronic SU(3). Terms without indices are understood to be summed over omitted indices. The naive action is

$$W=\int_V d^4x\left(-\tfrac{1}{4}F_{i\mu\nu}F_i^{\mu\nu}+\tfrac{1}{2}i\bar{q}\overleftrightarrow{\partial\!\!\!/}q\right.$$
$$\left.+g\bar{q}\underline{A\!\!\!/}\cdot\underline{G}q-B\right), \qquad (6.6)$$

where $\underline{G}$ is a suitable representation of the gauge group. Following the limiting procedure described in Sec. V, it is not difficult to obtain the following set of equations for F and q:

$$D^\mu_{ij}F_{j\mu\nu}=-g\bar{q}G_i\gamma_\nu q\,, \qquad (6.7a)$$

$$i\partial\!\!\!/q_{i\alpha}+g(\underline{A\!\!\!/}\cdot\underline{G})_{ij}\,q_{j\alpha}=0 \qquad (6.7b)$$

inside the bag, and

$$n_\mu F_j^{\mu\nu}=0\,, \qquad (6.7c)$$

$$i n\!\!\!/q_{i\alpha}=q_{i\alpha}\,, \qquad (6.7d)$$

$$-\tfrac{1}{4}F_{i\mu\nu}F_i^{\mu\nu}+\tfrac{1}{2}n\cdot\partial\,\bar{q}q-B=0 \qquad (6.7e)$$

on the surface of the bag. Notice that these boundary conditions are manifestly gauge-invariant. In particular the coupling between the fermion and boson fields drops out. Equations (6.7) ensure the conservation of the Poincaré generators in much the same way as in Sec. V.

Using the boundary conditions of Eq. (6.7), we can make explicit the argument in the Introduction that no bag with nonzero color can exist. The color generators are

$$Q_i=\int_R d^3x\,j_i^0(x)\,, \qquad (6.8)$$

where the current $j_i^\mu(x)$ is given by

$$j_i^\mu(x)=g\left(\bar{q}G_i\gamma^\mu q+f_{ijk}F_j^{\nu\mu}A_{k\nu}\right)$$
$$=\partial_\nu F_i^{\nu\mu}\,. \qquad (6.9)$$

The two terms in j_i^μ are the contributions to the color current from the quarks and the gluons, respectively. From (6.9) we have

$$j_i^0=\partial_l F_i^{l0}\,,$$

where the sum runs over only spatial indices. Therefore

$$Q_i=\int_S dS\,n_l F_i^{l0}\,,$$

where the integral is now over the surface of the bag. However, $n_l F_i^{l0}\propto n_\mu F_i^{\mu0}=0$ on the surface by Eq. (6.7c). Hence, the bag necessarily transforms as a singlet.

We have confined our detailed discussion to the non-Abelian case where color symmetry is exact. Similar considerations can be applied to the Abelian Han-Nambu model mentioned in the Introduction. Of course, these considerations do not apply to the electromagnetic charge since the photon is not confined to the bag.

VII. SPECULATIONS AND CONCLUSIONS

The foregoing sections have dealt entirely with the properties of single hadron states. No interactions either with an external system or with other hadrons have been introduced. It is clearly of primary importance to incorporate such interactions into the theory. At the present stage of our understanding of the theory, we have only qualitative ideas about how they can be included. For example, we expect the weak and electromagnetic interactions of a single hadron to be governed by the appropriate currents. These currents are in turn coupled to the photon (or intermediate vector bosons) to which the bag is, of course, transparent. At the classical level we know what these currents are; the electromagnetic current, for example, has been written down in previous sections for the various sorts of fields confined to the bag. The quantum treatment of the currents is complicated by the fact that the classical current is zero outside the bag. This can be expressed formally by multiplying the locally conserved current by step functions: For example, in the two-dimensional case,

$$j^\mu(x)=\tfrac{1}{2}\left[\epsilon(x_1(\tau)-x)+\epsilon(x-x_0(\tau))\right]j^\mu_{\text{local}}(x)\,.$$

Of course, in the quantum theory $x_1(\tau)$, $x_0(\tau)$ are operator functions of the fields inside. They therefore do not commute with the fields which go into the construction of the locally conserved current. Consequently, there is an ambiguity in how to order the quantum operators. This ambiguity is presumably to be resolved by the requirement that the currents be conserved and local. The proper treatment of locality will involve the creation of bag-antibag pairs, as is evident from the need for both positive and negative frequencies to construct the step functions. In the context of our single-bag theory locality could only be achieved by introducing negative-energy states. We expect the creation of pairs to be related to the form factor by a kind of duality, as indicated in Fig. 5.

We have not yet gone beyond this qualitative description. Clearly, a detailed understanding of the relationship between this kind of duality and locality is a prerequisite for understanding how to resolve the ordering ambiguity. Only then will we be able to make rigorous statements about the elastic form factors and deep-inelastic structure functions of the low-lying quantum states. Nonetheless, we feel that the quasifree parton structure of our model will lead to approximate scaling of the deep-inelastic structure functions, because a short-wavelength ($\lambda < B^{-1/4}$) virtual photon will see the free-field short-distance structure of the contained fields, but will not be sensitive to the boundaries which are characterized by the length $B^{-1/4}$. We might even venture to guess that the elastic form factors fall like powers of the momentum transfer by virtue of the sharp walls of the bag.

The qualitative picture for purely hadronic interactions, as we have mentioned, is based on the possibility of a bag fissioning into two bags at a point.[21] This interaction is clearly consistent with causality and Lorentz invariance at the classical level. Hopefully this would also be true of the quantum theory. This idea is most naturally incorporated in the "sum over histories" formalism. Again we have yet to go beyond this simple qualitative description. The detailed understanding of the hadronic interaction will involve the problems associated with the final states in inelastic hadron-hadron scattering. Finally we stress once again the dramatic qualitative difference between weak and electromagnetic interactions, on the one hand, and purely hadronic interactions, on the other.

We conclude our paper with a summary of the principal features of our model for hadrons. It is a relativistically invariant model which confines free or nearly free parton constituents. The confinement leads naturally to a hadron spectrum with infinitely rising Regge trajectories and to a density of states characterized by a maximum temperature for the constituents. When the con-

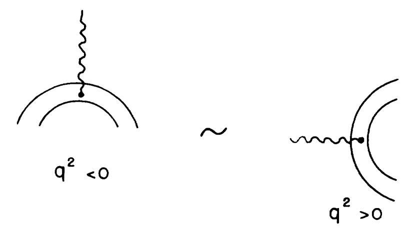

FIG. 5. Diagram for the form factor of a bag for spacelike and timelike q^2 (involving creation of a bag-antibag pair).

stituent fields are allowed to interact through a massless gauge field which is also confined, we find a natural understanding of why the physical hadrons have only the observed quantum numbers even though the constituents carry quark quantum numbers.

We have shown that the classical field equations can be quantized by explicitly accomplishing this in one space dimension for both Bose and Fermi fields. In the real case of three dimensions, in this paper we have only developed a quantization for large quantum numbers. Clearly, much work will be required to get a detailed understanding of the low-lying quantum states—the observed particles.

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[1] P. A. M. Dirac, Proc. R. Soc. A268, 57 (1962).

[2] Y. Nambu, Lectures at the Copenhagen Summer Symposium, 1970 (unpublished).

[3] For convenience, we shall refer to any model based on SU(3) × SU(3)′ as a color model, where SU(3)′ involves the color quantum numbers. For a discussion of such models see H. Fritzsch and M. Gell-Mann, in *Proceedings of the International Conference on Duality and Symmetry in Hadron Physics*, edited by E. Gotsman (Weizmann Science Press, Jerusalem, 1971).

[4] M. Y. Han and Y. Nambu, Phys. Rev. 139, B1006 (1965).

[5] Other mechanisms for preventing the appearance of colored hadrons have been discussed by A. Casher, J. Kogut, and L. Susskind [Phys. Rev. D (to be published)], and by C. Carlson, L.–N. Chang, F. Mansouri, and J. F. Willemsen, Phys. Lett. 49B, 377 (1974).

[6] A spontaneously broken non-Abelian gauge theory would be renormalizable, but color symmetry would then not be exact and colored (i.e., nonzero triality) states would necessarily appear. Thus, exact color symmetry requires massless non-Abelian gluons. In our model we avoid the infrared difficulty in such theories because the finite size of the bag cuts off long wavelengths. Another interesting feature of the non-Abelian gauge theories is asymptotic freedom [H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973); D. J. Gross and F. Wilczek, *ibid.* 30, 1343 (1973)], but we stress that we do not have to invoke asymptotic freedom to obtain Bjorken scaling, which in our model is an intuitive consequence of the almost-free character of the constituents inside the bag.

[7] G. Veneziano, MIT report, 1968 (unpublished).

[8] In the Fermi case this would be exactly true only if number of particles minus antiparticles is zero. For high excitations this will be a good approximation.

[9] This is exactly the result in dual resonance models;

see S. Fubini and G. Veneziano, Nuovo Cimento 64A, 811 (1969); K. Huang and S. Weinberg, Phys. Rev. Lett. 25, 895 (1970).

[10]See Sec. III, especially (3.14) for the definition of these boundary conditions which may be used in place of (1.2) for a bag containing scalar fields. Identical virial theorems are obtained with Neumann boundary conditions [see (11.1)] and in the case of Dirac particles.

[11]Time may be placed by x^+ in light-cone variables, or indeed by any other time or lightlike coordinate.

[12]The normals are chosen with opposite orientation on R_1 and R_2.

[13]The Dirichlet boundary conditions have been discovered independently by T. T. Wu, B. M. McCoy, and H. Cheng, following paper, Phys. Rev. D 9, 3495 (1974).

[14]Our quantization procedure is modeled after that discussed in P. Goddard, J. Goldstone, C. Rebbi, and C. B. Thorn, Nucl. Phys. B56, 109 (1973).

[15]This is not trivial since there is no momentum conjugate to the surface variable $\vec{R}$.

[16]The generator of translations in x is $P_- = -P^+ \equiv -P$.

[17]This is the reason that we study the complex scalar field. For the real scalar field ψ must be a periodic step function which takes on the values 0, π, so the only solutions with a fixed radius are rather formal. We can see that this is a characteristic of static solutions of the real field problem regardless of shape as follows. Consider the integral

$$\int_R d^3x\, \nabla^2\phi = \int_S ds\,\hat{n}\cdot\vec{\nabla}\phi, \qquad \text{(a)}$$

where we assume the surface is static. The boundary conditions (3.14) require $\vec{\nabla}\phi = \pm(2B)^{1/2}\hat{n}$ on the surface. If we demand smoothness, i.e., only one sign of the square root then (a) reduces to

$$(2B)^{1/2}\int_S ds = (2B)^{1/2}\,\text{Area} = \left(\int_R d^3x\,\frac{\partial^2}{\partial t^2}\phi\right). \qquad \text{(b)}$$

Since R is static,

$$2B\int_S ds = \frac{\partial}{\partial t}\left(\int_R d^3x\,\frac{\partial\phi}{\partial t}\right). \qquad \text{(c)}$$

However, because the energy is bounded and depends upon the integral of ϕ^2, $\dot{\phi}$ is bounded in the bag. Hence, if we average (c) over the time we find Area = 0, which is clearly not possible. Hence, there can only be solutions with static walls of the bag if $\vec{\nabla}\phi$ is discontinuous on the surface, as a function of time (or position).

[18]P. Hayes (unpublished).

[19]We have been rather sketchy in this analysis. Details will be published elsewhere.

[20]Putting a very large mass outside is equivalent to confining particles by a scalar potential. The fact that such a scalar potential confines both particles and antiparticles has been mentioned by N. N. Bogoliubov *et al.*, Dubna Report Nos. D-1968, 1965 (unpublished) and D-2569, 1966 (unpublished); H. J. Lipkin and A. Tavkhelidze, Phys. Lett. 17, 331 (1965).

[21]Hadronic interactions were introduced into the string model by just such a fissioning mechanism. See S. Mandelstam, Nucl. Phys. B64, 205 (1973).

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Theory of hadron "bags" with scattering

Tai Tsun Wu* and Barry M. McCoy†
Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138

Hung Cheng‡
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
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We derive the boundary conditions satisfied by a boson field in the theory of hadron "bags." The scattering problem, the fission problem, and the fusion problem in this theory of one spatial dimension are discussed.

I. INTRODUCTION

In the preceding paper, Chodos, Jaffe, Johnson, Thorn, and Weisskopf[1] (CJJTW) proposed a very interesting model for the structure of hadrons. They assume that hadron fields are contained inside a "bag" which has a constant, positive potential energy density B. By requiring that the action of this Lagrangian be an extremum, they obtain the field equations inside the bag and the conditions satisfied by the wave functions at the boundary. These equations also determine the location of the boundary.

Their boundary conditions do not require the field to vanish at the boundary. This seems to lead to difficulties when two hadron bags scatter from each other. In CJJTW troubles with boundary conditions are already encountered in the fermion case, and are solved by introducing an outside field with large mass. In this paper we propose to apply the same treatment to the boson case.

It is found that this procedure leads to a different