Nonperturbative approach to the electron-muon mass ratio in gauge theories*

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A pole approximation to the Dyson-Schwinger equations for the fermion self-energy parts is proposed to investigate the consequences of dynamical symmetry breaking on the pattern of lepton mass generation in gauge theories. This approach finesses known difficulties, encountered by Georgi and Glashow, in attempts to calculate perturbatively the electron-muon mass ratio in theories with elementary Higgs fields. Its self-consistency would imply, via an eigenvalue equation, the existence of ultraheavy gauge bosons.

Among the long-standing unresolved puzzles of particle physics, the role of the muon has proved as frustrating as it is, presumably, of central importance. Intimately connected to this question is the relationship between the muon and electron masses. Recently, Georgi and Glashow[1] emphasized the possibility[2] that a solution to both of these problems may be found within the context of a perturbative approach to renormalizable gauge theories of the weak interactions[3] and illustrated their ideas with "rational if implausible" models. The *raison d'etre* for the muon in these models is that it is part of the multiplet that provides the representation space for the gauge group. The electron mass is made calculable in terms of the muon mass by forcing the electron to be massless in zeroth order of the gauge coupling and by then letting it acquire a mass as a result of finite radiative corrections of order α involving a virtual muon. The suggestion that the electron mass comes entirely from its being a "part-time muon" due to an interaction of electromagnetic strength is aesthetically appealing and also supported by the numerological observation that the electron mass (m_e) is roughly α times the muon mass (m_μ).

A consistent and satisfactory implementation of these ideas turns out to be difficult to achieve because of the multiple conditions that need to be imposed on the choice of the gauge group, the lepton representation, and the zeroth-order lepton mass spectrum [generated through the Yukawa coupling by the vacuum expectation value (VEV) of the elementary canonical scalar (Higgs) fields present in the Lagrangian]. For instance, the zeroth-order spectrum, which differentiates between electron and muon by giving the muon a mass while keeping the electron massless, must obviously correspond to a configuration of VEV which represents a minimum of the potential, stable under small variations of any of the parameters in the Lagrangian. Such a condition is hard to meet in simple models because its specific symmetry-breaking character[4] tends to conflict with the symmetrical structure of the theory as embodied in the gauge interaction.

This conflict is clearly exemplified by considering Weinberg's chiral $SU(3)\times SU(3)$ model. The observed leptons are arranged in a Konopinski-Mahmoud triplet (μ^+, ν, e^-) with left-handed and right-handed components transforming under the gauge group as a $(1, \overline{3})$ and $(3, 1)$, respectively. The only meson representation which couples to the leptons by the gauge-invariant Yukawa coupling $(f\overline{\psi}_R\phi\psi_L + \text{H.c.})$ is a complex 3-by-3 elementary spinless matrix field ϕ transforming as a $(3, 3)$. Since the VEV of the latter is responsible for the zeroth-order lepton masses, to obtain the desired zeroth-order spectrum we must insist that the meson field in the array which is coupled to the electron (ϕ_e) have identically vanishing VEV at the physical minimum of the action. A necessary condition for such a minimum to exist is that there be no (destabilizing) terms in the Lagrangian linear in ϕ_e, which may arise from the coupling of ϕ to other meson representations (χ) also acquiring a nonvanishing VEV. The presence of such coupling terms as counterterms in the Lagrangian is sometimes forced by the requirement that the theory be renormalizable, i.e., by the need to absorb divergences which appear in amplitudes involving both ϕ and χ. If no direct coupling between ϕ and χ was introduced in the original Lagrangian, the amplitudes in question could obviously still be nonvanishing and actually formally divergent as a result of loop contributions present in some order of the gauge interaction. Unfortunately, this is precisely what happens in the case (under discussion) of Weinberg's model. If the electron in this model is to acquire its mass via radiative corrections involving the muon, it is necessary that there be a certain direct mass mixing between left-handed (W_L) and right-handed (W_R) gauge fields induced by some scalar-meson representation. But then there exist superficially divergent two-loop diagrams considered by Georgi and Glashow[1] (see Fig. 1), with four external elementary meson legs including one ϕ_e whose renormalization indeed requires the introduction of those unwanted (destabilizing) Lagrangian terms we referred to above. Thus, the original choice of zeroth-order lepton mass spectrum appears to be inconsistent with simple stability criteria and ϕ_e acquires an incalculable VEV.[5]

Georgi and Glashow, in their work, suggest, as a way to circumvent the above-mentioned obstacle, enlarging the gauge group to $SU(3)\times SU(3)\times SU(3)$; it is then possible to avoid a direct mass mixing between W_L and W_R. This prescription cures the specific difficulty addressed by Georgi and Glashow associated with the particular perturbation-theory diagrams they consider. However, even if we assume that the introduction of the unwanted (destabilizing) Lagrangian terms is not forced upon this model in any other way, so that there does not appear to be any obvious inconsistencies with the choice made of VEV, such a solution may be regarded as too *ad hoc* to constitute anything more than a technical victory. The latter criticism may also apply to the $SU(3)\times U(1)$ model proposed by the same authors. In summary, the requirement that the electron mass be calculable as a result of the perturbative technique of electron mass generation mentioned above leads to constraints which are selective enough to make it apparently very hard to implement those ideas and still obtain acceptable models.

While the role of very discriminatory criteria may be regarded as quite useful in limiting the proliferation of models, we must clearly entertain the possibility that our presumption that the lepton mass spectrum is a perturbation-theory effect off a perturbative spontaneously broken zeroth-order condition does not happen to be true in the real world. This would not necessarily imply that the electron and the muon masses are independent parameters since a relationship between the two could well exist as an entirely nonperturbative effect, i.e., completely outside of the realm of perturbation theory. This note is aimed at discussing some of the salient features of such a possibility in the general framework of gauge theories.[6]

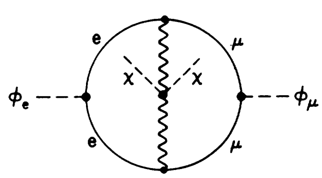

FIG. 1. Superficially divergent Feynman diagram leading to the loss of the zeroth-order masslessness of the electron in Weinberg's SU(3)×SU(3) model with elementary Higgs fields.

The idea that the solution to the electron-muon puzzle could be of nonperturbative character is relatively old. It was originally formulated in the framework of the ordinary electrodynamic interaction of muons and electrons by Baker and Glashow.[7] The starting point for these authors (as well as for many others interested in nonperturbative phenomena of spontaneous symmetry breaking) is the consideration of the Dyson-Schwinger equations. They observe that, owing to the nonlinear nature of these equations, nonperturbative solutions to the theory will not in general possess the symmetries of the equations themselves and adopt the philosophy that the electron-muon splitting is a reflection of this property.

More precisely, Baker and Glashow consider electrodynamics without bare-mass terms

$$\mathcal{L} = i\bar{\psi}_e\gamma^\lambda\partial_\lambda\psi_e + i\bar{\psi}_\mu\gamma^\lambda\partial_\lambda\psi_\mu + e_0 A_\lambda(\bar{\psi}_e\gamma^\lambda\psi_e + \bar{\psi}_\mu\gamma^\lambda\psi_\mu)\,, \tag{1}$$

where ψ_e, ψ_μ, and A_λ are the electron, muon, and photon fields, respectively, and e_0 is the bare electric charge. They seek solutions to the renormalized Dyson-Schwinger equations such that the renormalized electron Green's function G_e has a pole at the physical electron mass m_e, and the renormalized muon Green's function G_μ has a pole at the physical muon mass m_μ (to be determined):

$$G_e \approx (\gamma\cdot p + m_e)^{-1}$$

and

$$G_\mu \approx (\gamma\cdot p + m_\mu)^{-1}\,.$$

Of course, since available techniques do not allow them to find solutions to the full system of coupled nonlinear Dyson-Schwinger equations, they are forced to consider approximations to these equations. An approximation which is undoubtedly too crude consists of neglecting in the integral equations for the self-energy parts, $\Sigma_e(\gamma\cdot p)$ and $\Sigma_\mu(\gamma\cdot p)$, all spectral-function parts. Thus they replace the exact vertex operator Γ_μ by the bare vertex γ_μ, the exact photon propagator $D(q^2)$ by $1/q^2$, and ignore the $\gamma\cdot p$ dependences of $\Sigma(\gamma\cdot p)$ [Fig. 2(a)]. This pole approximation reduces what is an integral-equation problem to an algebraic one. Since divergent integrals appear, a cutoff parameter Λ must also be introduced. In the limit $\Lambda \gg m_e$ and $\Lambda \gg m_\mu$ one then finds

$$\begin{aligned} m_e &= (3\alpha/4\pi)m_e \ln(\Lambda^2/m_e^2)\,, \\ m_\mu &= (3\alpha/4\pi)m_\mu \ln(\Lambda^2/m_\mu^2)\,. \end{aligned} \tag{3}$$

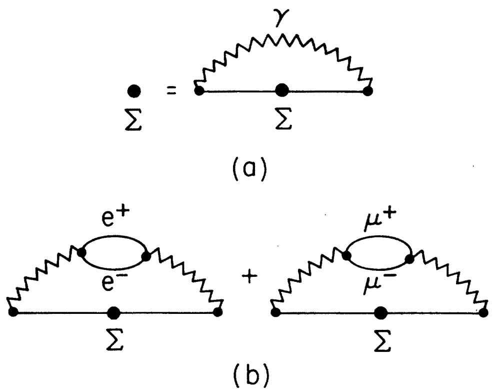

FIG. 2. (a) Pole approximation to the Dyson-Schwinger equation for the self-energy of a fermion in conventional QED. (b) Inclusion of spectral-function corrections to the photon propagator.

These equations are clearly symmetric in m_e and m_μ and do yield the perturbative solution $m_e = m_\mu = 0$. But they possess nonsymmetric solutions as well, for example,

$$m_e = 0, \quad m_\mu = \Lambda e^{-2\pi/3\alpha} \quad . \tag{4}$$

There is however a very obvious and substantial objection to Eqs. (3): the lack of mutual coupling between m_e and m_μ. In the present context of quantum electrodynamics (QED) one could hope to find a remedy to this by including spectral-function terms in the photon propagator such as the ones coming from $\mu^+\mu^-$ and e^+e^- intermediate states [Fig. 2(b)]. This would lead to

$$\begin{aligned} m_e &= (3\alpha/4\pi)m_e \ln(\Lambda^2/m_e^2) \\ &\quad + \alpha^2 m_e F(\Lambda/m_e, \Lambda/m_\mu), \\ m_\mu &= (3\alpha/4\pi)m_\mu \ln(\Lambda^2/m_\mu^2) \\ &\quad + \alpha^2 m_\mu F(\Lambda/m_e, \Lambda/m_\mu), \end{aligned}$$

and to solutions perhaps more interesting than Eq. (4). There will however always remain a major objection to the implementation of the Baker-Glashow program in the context of pure QED: the lack of any obvious compelling reason for the introduction of a muon in the first place. It is very tempting at this time to associate this disturbing conceptual difficulty with the Abelian nature of pure QED.

This is then the first objection that a reformulation of the Baker-Glashow program in the context of non-Abelian gauge theories could clearly solve. The solution is in fact the same as the one already mentioned in discussing the perturbative approach, i.e., that the *a priori* existence of a dynamical group implies the *a priori* existence of a minimal number of independent "elementary" fields to make up the fundamental representation of the group.

Besides this conceptual reason, there also appear to be other reasons of more practical nature that suggests pursuing the idea of Baker and Glashow within gauge theories rather than pure QED. One of these reasons is the dependence of Eqs. (3) or (5) on the cutoff Λ. If we are to compute the electron-muon mass ratio unambiguously, in terms of the fundamental parameters of the theory (α) alone, the cutoff dependence of the approximate equations under study must clearly not leak into a cutoff dependence of the approximate solution for that ratio. It is by no means clear whether this can be achieved for nontrivial solutions in QED. On the other hand, within the context of gauge theories one can obtain an approximate self-consistent set of equations for the electron and muon masses which is unambiguous, i.e., independent of external cutoffs (the role of cutoff being assumed by physical gauge-boson masses) provided a careful choice is made of the gauge group and of the lepton representation.

Finally, let us notice a further advantage of very practical nature which, in the nonperturbative approach to the electron-muon problem, definitely favors the non-Abelian gauge theory framework over pure QED: In the former case, when electron and muon are representation partners, there exists a direct coupling between the equations for the masses of these particles in the simplest approximation to the Dyson-Schwinger equations, which does not retain any spectral function correction to the gauge-boson propagators. This allows one the freedom of tentatively assuming that the effect of these corrections may be consistently ignored if one is only interested in approximate relations, thus leading to considerable technical simplifications.

Having made these remarks, let us now consider the class of gauge theories based on chiral $SU(n)\times SU(n)$ groups with the left-handed and right-handed components of the known and (possibly) unknown leptons assigned to the $(1, \bar{n})$ and $(n, 1)$ fundamental vector representations, respectively. We shall set the coupling constants of the left-handed (W_L) and right-handed (W_R) gauge fields to the fermions equal ($g_L = g_R = g$) so that the Lagrangian is parity-invariant and we have one-coupling-constant theories. We chose to restrict ourselves to this class of theories in the following discussion to have definite models in mind, in spite of the fact that our considerations have a wider range of applicability. In particular, besides Weinberg's $SU(3)\times SU(3)$ model and models based on $SU(4) \times SU(4)$ gauge symmetry, we[8] have examined in detail the possible implementation of the idea of nonperturbative lepton mass generation in an interesting class of models[9] in which the left-handed components of leptons and antileptons ($e^-, e^+; \mu^-, \mu^+; \nu, \bar{\nu}; \ldots$) share the same representation of an SU(n) group.

The exact proper self-energy parts of our leptons satisfy a nonlinear coupled system of integral equations whose kernels are expressible in terms of the exact vertex functions and gauge-boson propagators [Fig. 3(a)]. Suppose, now, that these quantities (and, therefore, the kernels) were known in some approximation corresponding to a spontaneously broken solution of the theory. Then, assuming that the approximation of the kernels does not alter completely the nature of the solutions to the integral equations, we could use these to infer the implications of the spontaneous symmetry breaking, as manifested in the gauge-boson sector, on the lepton mass spectrum. Since the most striking manifestation of the occurrence of spontaneous symmetry breaking in the gauge-boson sector is the acquisition of masses and the appearance of mass splittings between these particles, taking an optimistic attitude, one should hope that knowledge of the pole approximation to the gauge-boson propagators would be enough to obtain the rough pattern in the generation of the lepton masses.[10] This is the approach we tentatively wish to pursue. The approximation we are proposing amounts to substituting the system of integral equations corresponding to Fig. 3(a) with an algebraic system corresponding to the set of diagrams of Fig. 3(b). The latter is similar to the set of diagrams one encounters in lowest-order perturbation theory for the fermion self-energies except that the gauge-boson and fermion propagators now have poles corresponding to the physical masses rather than at zero mass. Notice that such an approximation is not generally gauge-invariant and must therefore be accompanied by the

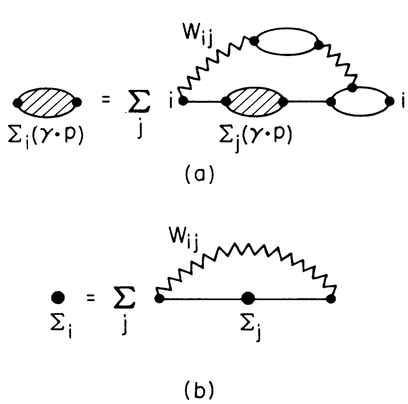

FIG. 3. (a) Dyson-Schwinger equations for the fermion self-energies in gauge theories. (b) Pole approximation to the equations of (a).

specification of a restricted set of (renormalizable) gauges. This will include, in particular, the Feynman gauge in which the gauge-boson propagators take the purely diagonal form $g_{\mu\nu}(q^2-M^2)^{-1}$ and in which our considerations are most easily carried out.

In the Feynman gauge, each of the nonvanishing diagrams in Fig. 3(b) is logarithmically divergent. However, owing to the chiral nature of the gauge theories under consideration, there is a minimum of two gauge bosons, i.e., a mixture of left-handed (W_L) and right-handed (W_R) fields, contributing to the self-mass of lepton i for any fixed virtual lepton j. Hence, mutual cancellation of divergences between diagrams with the same intermediate lepton j may then arise when these contributions are summed. The resulting finite self-masses depend on the angle of mixing between W_L and W_R, in absence of which they clearly vanish. Let us assume, for the simplicity of the discussion, maximal mixing, so that the eigenstates of the gauge-boson mass matrix (gauge bosons of definite mass) are purely vector and axial-vector fields, W_1 and W_2, respectively. In the limit in which their masses are large compared to the physical mass of the virtual lepton j, their contribution to any lepton self-mass is proportional to the logarithm of their mass ratio, $\ln(M_1/M_2)$. We also recall that if $W_1=\gamma$ is the photon, then the contribution of $W_1=\gamma$ and $W_2=Z$ is proportional to the logarithm of the ratio of the Z mass to the mass of the lepton $j=i$, $\ln(M_Z/m_i)$.

We, therefore, finally write down the system of equations corresponding to Fig. 3(b) in the form

$$m_i=\sum c_{ij}m_j\,,$$
$$c_{ij}\propto\alpha\ln(M_{ij}^{(1)}/M_{ij}^{(2)})+\alpha\delta_{ij}\ln(M_Z/m_i)\,, \tag{6}$$

where we have ignored numerical factors generally of order 1 depending on quantities of the type of Weinberg's angle which are related to the geometry of the gauge group, and we have assumed that, except for the photon, all of the gauge bosons have masses far larger than any of the leptons. The photon-Z contribution is then the only contribution which communicates the notion of the mass scale existing in the gauge-boson sector to the world of leptons. In fact, if we ignore this contribution, the system of Eqs. (6) reduces to a linear homogeneous system whose solutions are determined only up to over-all constant factors.

In general, the system of Eqs. (6) will possess nontrivial solutions only if the obvious eigenvalue equation

$$\det\|c_{ij}-\delta_{ij}\|=0 \tag{7}$$

is satisfied. We are clearly interested in solutions that fulfill the following requirements: (a) The electron-muon mass ratio has the correct value, of order α; (b) presently unobserved leptons, if they are required by the choice of gauge group and of lepton representation, have masses large enough to justify their having escaped observation; (c) neutrinos are massless as a manifestation of a residual unbroken (discrete) symmetry.

We seem to be able to satisfy all of these requirements [including Eq. (7)] in a variety of models[8] if we postulate the existence of at least one gauge boson of ultralarge mass (this is, in fact, so large that it is difficult to conceive how such particles could ever play a role if not as highly virtual states), since this is the only obvious way we can make one of the c_{ij} of order 1 and solve Eq. (7). In the natural basis defined by the physical leptons, an ultraheavy gauge boson should correspond to a diagonal generator which couples only to the most massive lepton and to the neutrinos, but not to the electron. The electron, if the muon is the most massive lepton, would then acquire its mass mostly through an ordinary matrix element c_{ij} (of order α) which couples it to the muon. If, on the other hand, some other lepton L rather than the muon is the most massive lepton and the muon itself acquires its mass of order α mostly through its coupling to L, to obtain the right pattern of masses we must make sure that in the system of Eqs. (6) there be no direct coupling (of order larger than α^2) between the mass of the electron and the mass of L. This coupling could actually be identically vanishing if, e.g., the corresponding left-handed and right-handed gauge bosons were prevented from mixing.

To summarize, we have argued in favor of a nonperturbative approach to the mass spectrum of leptons. Its use enables us to bypass obstacles, mainly connected with the stability of certain choices of zeroth order VEV, encountered in attempts to solve the problem of the electron-muon mass ratio perturbatively in theories with elementary Higgs fields. In its crudest form, the nonperturbative approach proposed here was based on a simple approximation to the Dyson-Schwinger equations for the fermion self-energy parts of a chiral gauge theory. We thus have reformulated the Baker-Glashow program for computing the electron-muon mass ratio in conventional QED. This reformulation solves various difficulties, both conceptual and practical, of the original program; e.g., *a priori* understanding of the existence of the muon, dependence on external cutoffs, and lack of direct functional coupling between the muon and electron masses. To be sure, the approach proposed here is not without difficulties of its own, as many of us will believe that the condition for the existence of ultraheavy gauge bosons is an unacceptably high price to pay for the computability of the electron-muon mass ratio. However, such a condition could well be a feature of the approximation, which can be remedied by a more sophisticated approximation without spoiling other desirable features of the general scheme.

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[1]H. Georgi and S. L. Glashow, Phys. Rev. D 7, 2457 (1973); 6, 2977 (1972).

[2]S. Weinberg, Phys. Rev. D 5, 1962 (1972).

[3]For a review see, for example, E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973).

[4]The requirement of a zeroth-order asymmetry in the present case should be contrasted with the zeroth-order symmetry requirement one enforces to ensure finiteness and computability of the proton-neutron mass difference. See, for instance, R. N. Mohapatra and P. Vinciarelli, Phys. Rev. Lett. 30, 804 (1973).

[5]$\langle\phi_e\rangle$ may actually remain calculable in principle and very small if the scalar representation ϕ corresponds to dynamical bound states rather than being elementary since then the diagram of Fig. 1, because of the presence of form factors at some of its vertices, will be convergent. In the rest of our present discussion we shall ignore this interesting possibility [T. Goldman and P. Vinciarelli, SLAC report (unpublished)] which is not referred to by Georgi and Glashow, and which could take care of their objection to the calculability of the electron mass in Weinberg's model. One would, however, have to assume that dynamical symmetry breaking occurs giving rise to physical Higgs bosons.

[6]Nonperturbative mechanisms of symmetry breaking and mass generation in gauge theories have recently been considered by S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973); H. Pagels, *ibid.* 7, 3689 (1973); R. Jackiw and K. Johnson, *ibid.* 8, 2386 (1973); J. Cornwall and R. Norton, *ibid.* 8, 3338 (1973).

[7]M. Baker and S. L. Glashow, Phys. Rev. 128, 2462 (1962).

[8]T. Goldman and P. Vinciarelli (unpublished).

[9]Models of this type have been investigated by J. Bjorken (private communication).

[10]The mechanisms by which the gauge fields acquire masses will not be discussed in this paper. See, e.g., Ref. 6.