Phase transitions in vector-gluon models: A solution to the U(3) problem*

Paul Langacker and Heinz Pagels
The Rockefeller University, New York, New York, 10021
(Received 10 October 1973)

We have examined the realizations of chiral symmetry in the Abelian vector-gluon model, in the absence of gluon self-energy. The chiral symmetry is broken spontaneously in the vacuum. A vacuum stability condition on the gluon coupling, g, and the gluon-fermion mass ratio μ/m of the form $f_\pi^2(g^2, \mu^2/m^2) > 0$ is obtained as a necessary condition for a Nambu-Goldstone realization with bound-state Goldstone bosons. Here f_π is the decay constant of the bound-state Goldstone boson. It is shown that for those channels that do not communicate with pairs of vector gluons this condition is satisfied in the weak-coupling limit, $g^2 \to 0$. For axial-vector currents that satisfy anomalous Ward identities and communicate with pairs of gluons we show that the renormalized integral equations for f_π^2 do not possess solutions. The failure to find a Goldstone solution for these channels is associated with the fact that these axial-vector currents have an anomalous dimension greater than three. Instead of Goldstone bosons one finds that axial-vector-current conservation breaks down and the symmetry is broken explicitly. Hence, there is no Goldstone boson associated with "axial baryonic charge." This also answers in the negative the old question of whether electrodynamics can support a Goldstone mode. We also calculate $g_A - 1$ to order g^2 and discuss phase transitions between the Wigner-Weyl mode, the Goldstone mode, and the mode in which the symmetry is explicitly broken. We also speculate on the implications of this work for gauge theories of the strong interactions.

I. INTRODUCTION

A. The U(3) problem

Suppose one takes as a model of the strong interactions a triplet of fermions interacting with a massive neutral vector gluon. There are various virtues of this model and related models, which we will shortly describe; however, there is also a problem. That problem is that the formal symmetry is U(3)× U(3) instead of SU(3) × SU(3)× U(1). If one assumes that the U(3)× U(3) symmetry is realized with a U(3)-symmetric vacuum then one has nine, not the desired eight, Nambu-Goldstone bosons. This is because beside the usual octet of conserved axial-vector currents (in the symmetry limit) the ninth axial-vector current, "axial baryonic charge" is also conserved. In the SU(3)× SU(3) Σ model this problem is avoided by the introduction of the trilinear determinantal interaction which breaks U(3)× U(3) symmetry down to SU(3)× SU(3)× U(1). However, in the vector-gluon model, which avoids the introduction of elementary spinless fields, this is not possible. This is the U(3) problem.

Actually this problem of the extra U(1) symmetry is nine years old. It was first posed by Nambu.[1] He asked why quantum electrodynamics (QED) does not have a zero-mass ground-state meson. The point is that in a formulation of QED[2,3] in which one ignores photon self-energy graphs, the bare electron mass m_0 vanishes as the cutoff $\Lambda \to \infty$. Hence one might conclude that the axial-vector current $\bar\psi\gamma_\mu\gamma_5\psi$ is conserved ($\partial_\mu A_\mu = 2im_0\bar\psi\gamma_5\psi = 0$) because the bare mass vanishes. If the physical electron mass m does not vanish, this conservation of axial-vector current entails the existence of a zero-mass Nambu-Goldstone boson. Clearly, such a state is not desired in electrodynamics.

Subsequently it was pointed out by Baker and Johnson[4] that although the bare mass m_0 vanishes as the cutoff becomes large, the matrix elements of the operator $\bar\psi\gamma_5\psi$ could diverge in just such a manner that the matrix elements of $\partial_\mu A_\mu = 2im_0\bar\psi\gamma_5\psi$ do not vanish. Then the symmetry is explicitly broken and one avoids the conclusion of massless excitations. It was also pointed out[5,6] that there exists a Goldstone alternative in the vector-gluon model. Either the axial-vector current is conserved and one has Goldstone bosons as bound states or axial-vector-current conservation actually breaks down and the symmetry does not exist in the first place. The Goldstone alternative is formally implemented in the boundary conditions to the homogeneous Dyson-Schwinger equations.[7]

The main point of the present article is to establish the criterion, stated as a condition on the coupling constant of the model, for which branch of the Goldstone alternative one must take. It turns out that for small values of the coupling constant e^2 electrodynamics can not have the Goldstone mode. Similarly in the quark-triplet neutral-vector-gluon model the ninth axial-vector current cannot have an associated Goldstone boson, while the octet of axial-vector currents can. This is precisely the desired solution. Before describing our solution to this problem we will digress with a discussion of the Σ model as most of our intuition about spontaneous vacuum symmetry breaking has been developed in terms of this model.

B. The Σ model and the vector-gluon model

The Σ model[8] has the virtue of algebraic simplicity. In the tree approximation most of the interesting features associated with spontaneous breaking can be realized. The price one pays for this simplicity is the introduction of elementary scalar and pseudoscalar fields. Such scalar fields will contribute to the electric current and produce a nonvanishing, possibly large, component to the longitudinal cross sections in electroproduction. This is in conflict with SLAC experiments at the present energy regime, so the Σ model is probably irrelevant to real physics.

The principal idea in the Σ model, or any model of spontaneous breaking, is that the stable vacuum state may not have the symmetry of the full Hamiltonian. In the Σ model (a field-theory version of Landau's theory of spontaneous magnetization) one has a potential which in the SU(2) × SU(2)-symmetric version of the model is given by

$$V(\sigma, \pi) = \mu^2(\sigma^2 + \vec{\pi}^2) + \lambda(\sigma^2 + \vec{\pi}^2)^2, \quad \lambda > 0$$

where $\sigma, \vec{\pi}$ are the usual σ and pion fields transforming like $(\frac{1}{2}, \frac{1}{2})$ under the group. In the tree approximation if one minimizes the vacuum energy one finds for the order parameter $f \equiv \langle\sigma\rangle_0$, assuming that it is not zero,

$$f^2 = -\mu^2/2\lambda > 0. \tag{1.1}$$

What we want to emphasize is that this resulting equation is for the square of f, not f itself. Since $f^2 > 0$ we require $\mu^2 < 0$ in order to have the solution which exhibits spontaneous breaking, $f \neq 0$. So for certain domains of the parameters μ^2 and λ the stable vacuum does not have this symmetry of the Hamiltonian and the pion will have the role of a Nambu-Goldstone boson. For values of the parameters for which $f^2 < 0$ we do not satisfy the stability condition (1.1), and we must pick the other branch of the Goldstone alternative, $f = 0$, which for this model implies parity doubling of the σ and π.

While these features of the Σ model are well known, it is of interest to see how they are maintained when we now turn our attention to the vector-gluon models. The possibility of a chiral-symmetry realization with bound-state Goldstone bosons has already been discussed by one of us.[7]

The vector-gluon model has many virtues over that of the Σ model as a model of strong interactions. It brings together the concepts of chirality, partial conservation of axial-vector current (PCAC), and soft-pion theorems on one hand, with quark-model spectroscopy, parton-model notions, and approximate scaling behavior on the other hand. Radiative corrections to the strong interactions are completely finite in this model[6]; the usual divergence of $\ln\Lambda^2$ is effectively replaced by $(4\pi)^2/3g^2$, with g the vector-gluon coupling constant. The study of such models of the strong interactions receives a further impetus from gauge theories of the weak interactions, for one can avoid the presence of parity violation and strangeness violation of order $\alpha = \frac{1}{137}$ in the strong interactions by redefining the fermion fields.[9]

The SU(3)×SU(3) Σ model, even in the symmetry limit, has many free parameters. Various phenomenological fits to experimental data depend sensitively on these parameters. We think it would be of interest to examine the data in terms of the gluon model, as there is evidently only one coupling constant and the mass of the gluon as free parameters. Further, it is difficult to see how the phenomena described by Carruthers and Haymaker[10] would emerge in the vector-gluon-type models. In short, the vector-gluon model as we present it here is a new vehicle for the study of chiral-symmetry phenomena and deserves further investigation.

The main burden of this article will be to examine the phase transition problem in the vector-gluon model. In particular we wish to examine the necessary conditions on the gluon coupling constant g and gluon-fermion mass ratio μ/m such that we have a Goldstone phase. This necessary conditon turns out to be similar to that of the Σ model, Eq. (1), and is of the form

$$f_\pi^2(g^2, \mu^2/m^2) > 0, \tag{1.2}$$

where f_π, the order parameter, is the usual decay constant. We conjecture that this is also a sufficient condition. The phase transition curve in parameter space is then $f_\pi^2(g^2, \mu^2/m^2) = 0$. For values of $g^2, \mu^2/m^2$ not satisfying (1.2), one does not have a Goldstone phase; rather what happens is that the formal conservation of axial-vector current breaks down and $f_\pi = 0$. (An alternative possibility is that the fermions become massless in this normal phase.)

For values of $g^2, \mu^2/m^2$ satisfying (1.2) we can calculate f_π or, by the Goldberger-Treiman relation $g_\pi = g_A m/f_\pi$, the fermion-pion coupling constant. Since $f_\pi \simeq 90$ MeV is known, this serves to establish that in quark-gluon models with chiral symmetry realized with Goldstone bosons the coupling g and masses μ and m are not independent parameters.

For quantum electrodynamics, for which $\mu=0$ and $g=e$, $e^2/4\pi \simeq \frac{1}{137}$, we will show that the integral equations for the order parameter, f, do not possess solutions. Consequently $f=0$ and there is no Goldstone mode in electrodynamics. The reason for this is that the axial-vector current communicates with the two-photon channel and has a dimension greater than the canonical value,[11] given by $3+2\chi(e^2)$, where $\chi(e^2)=\frac{3}{4}(\alpha/\pi)^2+O(\alpha^3)$. For $\chi(e^2)>0$, as is valid for small coupling, the renormalized integral equation for f^2 diverges like $(\Lambda^2)^\chi$ and no solution exists. Hence one must pick the other branch of the Goldstone alternative, $f=0$, and axial-vector current conservation breaks down. This answers in the negative Nambu's question[1] of whether there exists a Goldstone boson in $m_0=0$ electrodynamics. The failure of QED to exhibit the Goldstone mode is intimately associated with the presence of an anomalous term in the axial-vector Ward identities.[12,13]

This also solves the U(3) problem for the quark-triplet neutral-vector-gluon model. The model is realized with eight not nine Goldstone bosons. The reason there is no Goldstone boson for the current transforming like $\lambda_0\gamma_\mu\gamma_5$ is that this current communicates with the two-vector-gluon channel and no nontrivial solutions exist for $f_9{}^2$. Instead one has $f_9=0$, there is no Goldstone boson, and the axial-vector current conservation breaks down. The other eight axial-vector currents $\lambda_a\gamma_\mu\gamma_5$, $a=1,2,\ldots,8$ do not communicate with the two-vector-gluon channel, and one finds for small coupling constants that $f^2>0$, so there can be Goldstone bosons associated with these currents being conserved. This is the desired solution.

We examine the question of phase transitions. For small coupling constants one can study the question with some confidence. However, there may be critical values of the coupling constant for which the phase changes from a Goldstone mode to the mode in which axial-vector-current conservation breaks down or the fermion becomes massless. The phase, presumably relevant for real physics, of eight Goldstone bosons is most simply accomplished if the coupling constant is reasonably small; this suggests that leading-order Bethe-Salpeter approximation or perturbation theory in the kernel is valid.

We also calculate to leading order in g^2 the axial-vector renormalization constant g_A for the fundamental fermions. It turns out that $g_A<1$ to this order. This suggests that the nucleonic $G_A/G_V\simeq 1.25$, which in a free-quark model is $\frac{5}{3}$, is modified in the presence of interactions to be $\frac{5}{3}g_A<\frac{5}{3}$ in closer agreement with the observed number. If $g_A=0.75$ the agreement is precise.

Finally in our concluding section we remark on the Goldstone realization for asymptotically stable gauge theories of the strong interaction[14] such as implemented by a Yang-Mills "colored" octet of gluons.[15] We also discuss the relation of our work to the bound-state Higgs models or "Higgs mechanism without Higgs scalars" of Jackiw and Johnson[16] and Cornwall and Norton.[17]

It is our primary purpose in this work to construct a quark model consistent with the requirements of a chiral symmetry realized with Nambu-Goldstone bosons. We believe that this should be an essential feature of any future theory of strong interactions. What is left completely unanswered in this work is how the quarks are contained in the hadrons. In the present SU(3)× SU(3) model the quarks are evidently real. Further, it is not clear how to construct other bound states besides the ground-state octet, whether such bound states exist, or if they lie on approximately linear Regge trajectories. Finally we have not treated symmetry breaking—although that is not an essential difficulty.

II. THE VECTOR-GLUON MODEL

The model we will examine is massive electrodynamics with a gluon (photon) mass μ and zero bare fermion mass. We will also introduce an internal symmetry U(2)× U(2) with the fermions transforming as a doublet under SU(2) and the gluon as a singlet. This symmetry is trivially generalizable to U(3)× U(3). The model we consider will be in the absence of gluon self-energy insertions,[3] so we do not consider vacuum-polarization effects. This may be equivalent to assuming that an eigenvalue condition is satisfied on the gluon coupling g so that gluon propagators are asymptotically free. Further, in actual calculations we will resort to the leading-order Bethe-Salpeter approximation, although our general conclusions will be independent of any such approximations.

The formal symmetry of the Lagrangian is chiral U(2)× U(2) with formally conserved currents $\bar\psi\gamma_\mu\frac{1}{2}\tau_a\psi$, $\bar\psi\gamma_\mu\psi$, $\bar\psi\gamma_\mu\gamma_5\frac{1}{2}\tau^a\psi$, $\bar\psi\gamma_\mu\gamma_5\psi$. Although the bare fermion mass vanishes, the physical fermion mass need not vanish, provided that we impose a nontrivial boundary condition, $\Sigma(m^2)=m$, on the fermion self-energy $\Sigma(p^2)$. Consequently the chiral symmetry is broken in the vacuum by imposing this nontrivial boundary condition on the homogeneous Dyson-Schwinger equation for $\Sigma(p)$.[7]

In the approximations described above, the Schwinger-Dyson equation for the fermion propagator (the gap equation) in the Landau gauge reads[2]

$$\Sigma(p^2)=\frac{-3g^2 i}{(2\pi)^4}\int\frac{d^4k\,\Sigma(k^2)}{(k^2-m^2)[(k-p)^2-\mu^2]}\,, \tag{2.1}$$

with the boundary condition

$$\Sigma(m^2)=m\ .$$

It is because (2.1) can possess nontrivial solutions that, in spite of the bare mass vanishing, the physical mass need not vanish. For real electrodynamics with $\mu=0$ the solution to (2.1) is well known.[18] The solution for $\mu\neq 0$ is not known.

The asymptotic behavior of the fermion propagator in the Landau gauge and in the absence of gluon self-energy insertions is given by

$$\Sigma(p^2)\underset{-p^2\to\infty}{\sim} mA(g^2,\mu^2/m^2)(-p^2/m^2)^{-\epsilon(g^2)}\,, \tag{2.2}$$

with[4] $\epsilon(g^2)=3g^2/(4\pi)^2+\frac{3}{2}g^4/(4\pi)^4+\cdots$ and $A(0,\mu^2/m^2)=1$. We will assume $\epsilon(g^2)>0$. It is interesting to consider the weak-coupling limit $g^2\to 0$. Then as $g^2\to 0$ we see from (2.2) that the integral (2.1) diverges like $1/g^2$, so

$$\Sigma(p^2)\underset{g^2\to 0}{\sim}\left(\frac{-3g^2 i}{(2\pi)^4}\right)\left(\frac{i(2\pi)^4 m}{3g^2}\right)=m\ . \tag{2.3}$$

It is typically the appearance of coupling constants in the denominator to cancel ones in the numerator that characterizes spontaneously broken vacuum symmetries in the weak-coupling limit. In this way the fermion acquires a mass by spontaneous breaking. To see if this entails Goldstone bosons one must examine the Goldstone alternative.

If one considers the cut-off version of the model the divergence of the axial-vector current is given by $\partial_\mu A^a_\mu(x)=2im_0(\Lambda)\bar\psi_\Lambda\gamma_5\frac{1}{2}\tau^a\psi_\Lambda$. Although one can show[2,3] that as $\Lambda\to\infty$, $m_0(\Lambda)\to m(\Lambda^2/m^2)^{-\epsilon}\to 0$, one cannot conclude that matrix elements of $\partial_\mu A^a_\mu$ vanish, necessarily entailing a Goldstone boson. This is because the matrix elements of the operator $\bar\psi_\Lambda\gamma_5\frac{1}{2}\tau_a\psi_\Lambda$ can behave like $(\Lambda^2/m^2)^\epsilon$ as $\Lambda^2\to\infty$ so that matrix elements of $\partial_\mu A^a_\mu$ need not vanish.

To elucidate this further we introduce the unrenormalized axial-vector vertex

$$S(p')\,{}^5\Gamma^a_\mu(p',p)S(p)=-\int d^4x\,d^4y\,e^{ip'\cdot x}e^{-ip\cdot y}\times\langle 0|(\psi(x)A^a_\mu(0)\bar\psi(y))_+|0\rangle \tag{2.4}$$

and the divergence

$$S(p')\,2m_0\,{}^5\Gamma^a_D(p',p)S(p)=+i\int d^4x\,d^4y\,e^{ip'\cdot x}e^{-ip\cdot y}\times\langle 0|(\psi(x)\partial_\mu A^a_\mu(0)\bar\psi(y))_+|0\rangle\ . \tag{2.5}$$

These cutoff-dependent quantities (the cutoff is introduced as a regulator of the photon propagator) are rendered finite by multiplication by Z_A and Z_D:

$$\begin{aligned}Z_A\,{}^5\Gamma^a_\mu(p',p)&={}^5\tilde\Gamma^a_\mu(p',p)\,,\\ Z_D\,2m_0\,{}^5\Gamma^a_D(p',p)&=2m\,{}^5\tilde\Gamma^a_D(p',p)\,.\end{aligned} \tag{2.6}$$

We will denote renormalized, cutoff-independent quantities with a tilde. The unrenormalized fermion propagator is specified by

$$S^{-1}(p)=Z_2^{-1}\tilde S^{-1}(p)\underset{p\to m}{\sim}Z_2^{-1}(\not p-m)\,,$$

where $\tilde S(p)$ is the renormalized propagator. Z_A is defined by

$$\bar u(p')\,{}^5\Gamma^a_\mu(p',p)u(p)\underset{p'\to p}{\sim}Z_A^{-1}\bar u(p')\gamma_\mu\gamma_5\tfrac{1}{2}\tau^a u(p)\,. \tag{2.7}$$

The axial-vector renormalization constant g_A is defined by

$$\langle p'|A^a_\mu(0)|p\rangle\underset{p'\to p}{\sim}\bar u(p')[g_A\gamma_\mu\gamma_5\tfrac{1}{2}\tau^a+(\text{pole terms})]u(p)\,. \tag{2.8}$$

From this definition by contracting out the fermions and using (2.4) and (2.7) one has

$$g_A=Z_2/Z_A\,. \tag{2.9}$$

The axial-vector currents transforming like $\gamma_\mu\gamma_5\frac{1}{2}\tau^a$ do not have anomalies in the neutral-vector-gluon model. The Ward identity is

$$q^\mu\,{}^5\Gamma^a_\mu(p',p)=2m_0\,{}^5\Gamma^a_D(p',p)+S_F^{-1}(p')\gamma_5(\tfrac{1}{2}\tau^a)+\gamma_5(\tfrac{1}{2}\tau^a)S_F^{-1}(p)\,. \tag{2.10}$$

Upon renormalizing

$$q^\mu\,{}^5\tilde\Gamma^a_\mu(p',p)=2m\,{}^5\tilde\Gamma^a_D(p',p)\left(\frac{Z_A}{Z_D}\right)+\left(\frac{Z_A}{Z_2}\right)[\tilde S_F^{-1}(p')\gamma_5(\tfrac{1}{2}\tau^a)+\gamma_5(\tfrac{1}{2}\tau^a)\tilde S_F^{-1}(p)]\,, \tag{2.11}$$

where the ratios (Z_A/Z_D) and (Z_A/Z_2) can be shown to be cutoff-independent.[19] Further, it can be shown that ${}^5\tilde\Gamma_D(p',p)$ satisfies the homogeneous integral equation[6]

$${}^5\tilde\Gamma^a_D=\int\tilde S\,{}^5\tilde\Gamma^a_D\tilde S\tilde K\,, \tag{2.12}$$

where $\tilde{K}$ is the renormalized fermion-fermion scattering kernel. The Goldstone alternative now depends on the choice of boundary conditions required to solve the homogeneous equation (2.12).

Suppose we pick a trivial boundary conditiom ${}^5\tilde{\Gamma}_D(p,p)=0$ corresponding to matrix elements of the divergence of the axial-vector current actually being conserved. Then taking $q_\mu=(p'-p)_\mu\to 0$ in (2.11), one finds there exists a zero-mass state which appears as a pole in the axial-vector vertex

$$ {}^5\tilde{\Gamma}_\mu(p',p) \underset{p'\to p}{\sim} -\frac{q_\mu\gamma_5\tau^a}{q^2}\, g_A{}^{-1}\Sigma(p)\,. \qquad (2.13)$$

This is the Nambu-Goldstone pion. If we further define the quantities

$$ S(p')\gamma_5\tau^a G(p',p)S(p)=\int d^4x\, e^{ip'\cdot x}\langle\pi^a(p-p')|(\psi(x)\bar{\psi}(0))_+|0\rangle|_{(p'-p)^2=0} \qquad (2.14)$$

and

$$\langle 0|A_\mu^a(0)|\pi^b(k)\rangle = if_\pi k_\mu\delta^{ab} \qquad (2.15)$$

in terms of the zero-mass bound state $|\pi^a(k)\rangle$, then we obtain from (2.4) and (2.5) the result

$$ q^2\,{}^5\Gamma_\mu^a(p',p) \underset{q^2\to 0}{\sim} -q_\mu f_\pi\gamma_5\tau^a G(p',p)\,. \qquad (2.16)$$

From this result defining the decay constant, f_π, and pion-fermion coupling, $G(p',p)$, we obtain from (2.13) and (2.6) the relation

$$ Z_A f_\pi G(p,p)=g_A{}^{-1}\Sigma(p)\,. \qquad (2.17)$$

This is the Goldberger-Treiman formula. We also find that since $g_A{}^{-1}\Sigma(p)$ is cutoff-independent so is

$$\tilde{G}(p',p)=Z_Af_\pi G(p',p)\,. \qquad (2.18)$$

$\tilde{G}(p',p)$ is a useful quantity in our analysis. It can be decomposed into invariant functions according to

$$\tilde{G}(k,k+q)=\tilde{g}_1(k,k+q)+\not{q}\,\tilde{g}_2(k,k+q) +\not{k}\,k\cdot q\,\tilde{g}_3(k,k+q)+\not{k}\not{q}\,\tilde{g}_4(k,k+q)\,, \qquad (2.19)$$

where $q^2=0$. The physical pion-quark coupling constant g_π is related by $f_\pi g_\pi=mg_A=g_A(\tilde{g}_1+2m\tilde{g}_2-2m^2\tilde{g}_4)$, evaluated on-shell.

From the definition (2.16) and (2.18) of $\tilde{G}$ and from the integral equation for ${}^5\Gamma_\mu^a$,

$$ {}^5\Gamma_\mu^a=\gamma_\mu\gamma_5\tfrac{1}{2}\tau^a+\int\tilde{S}\,{}^5\Gamma_\mu^a\tilde{S}\tilde{K}\,, \qquad (2.20)$$

one finds that $\tilde{G}$ satisfies a homogeneous integral equation

$$\gamma_5\tau^a\tilde{G}=+\int\tilde{S}\gamma_5\tau^a\tilde{G}\tilde{S}\tilde{K} \qquad (2.21)$$

subject to the boundary condition (2.17)

$$\tilde{G}(p,p)=g_A{}^{-1}\Sigma(p)\,. \qquad (2.22)$$

We note that the quantities in the integral equation (2.21) and the boundary condition (2.22) are all cutoff-independent. We assume that there exist solutions to this integral equation. This completes the analysis of the Ward identity for the Goldstone mode.

Alternatively we could have assumed there were no Goldstone bosons and $f_\pi=0$. Then the Ward identity (2.11) would inform us that as $q_\mu\to 0$

$$ 2m\,{}^5\tilde{\Gamma}_D^a(p,p)=\left(\frac{Z_D}{Z_2}\right)\Sigma(p)\gamma_5\tau^a\,. \qquad (2.23)$$

This is the boundary condition for the integral equation (2.12). Hence in the absence of Nambu-Goldstone bosons the formal conservation of the axial-vector current breaks down and matrix elements of $\partial_\mu A_\mu$ do not vanish. In this instance the symmetry is explicitly broken.

There is of course a rather uninteresting third possibility: The current is conserved (${}^5\Gamma_D=0$), there is no Goldstone boson ($f_\pi=0$), and the physical fermion mass vanishes $[\Sigma(p)=0]$.

We now turn to the question of which of these alternatives is actually realized.

III. CALCULATION OF f_π AND THE STABILITY CONDITION

It is not possible to have both Goldstone alternatives realizable for all values of the parameters g, μ, and m since the integral equations of the model may not possess solutions for all values of the parameters. For example, in the Σ model one has the constraint $-\mu^2/2\lambda>0$, described in the Introduction, which is necessary for the Goldstone mode. The question before us is to determine which branch of the alternative is actually realized for the vector-gluon model.

Let us assume that the axial-vector current is conserved so ${}^5\tilde{\Gamma}_D^a(p',p)=0$ and we have a bound-state Goldstone pion. The meson decay constant f_π is nonvanishing and specified by

$$\langle 0|A_\mu^a(0)|\pi^b(k)\rangle = i\delta^{ab}k_\mu f_\pi\,. \qquad (3.1)$$

Next consider the current correlation function for the unrenormalized axial-vector current

$$ {}^5\Delta^{ab}_{\mu\nu}(q) = -i\int d^4x\, e^{-iq\cdot x} \times\langle 0|(A^a_\mu(x)A^b_\nu(0))_+|0\rangle \,. \tag{3.2}$$

Matrix elements of $A^a_\mu(x)$ are finite, provided we introduce the multiplicative factor Z_A/Z_2 according to $A^a_\mu \to (Z_A/Z_2)A^a_\mu$ for each axial-vector current in the matrix element. However, subtractive renormalizations are also required in dealing with ${}^5\Delta^{ab}_{\mu\nu}(q)$ because of quadratic divergences. Crewther, Shei, and Yan[11] define the cutoff-independent amplitude ${}^5\tilde\Delta^{ab}_{\mu\nu}(q)$ according to

$$ {}^5\tilde\Delta^{ab}_{\mu\nu}(q) = (Z_A/Z_2)^2\ {}^5\Delta^{ab}_{\mu\nu}(q) - (q_\mu q_\nu - g_{\mu\nu}q^2)S_1(\Lambda) - g_{\mu\nu}[q^2 S_2(\Lambda) + S_3(\Lambda) + S_4(\Lambda)]\,, \tag{3.3}$$

where the $S_i(\Lambda)$ are given by them in their paper. In our application the cutoff dependence of $S_i(\Lambda)$ is of no concern if we extract the residue of the pion-pole term in (3.2). Using (3.1) we have

$$\delta^{ab} f_\pi^{\ 2} q_\mu q_\nu \underset{q^2\to 0}{\sim} q^2\, {}^5\Delta^{ab}_{\mu\nu}(q) \underset{q^2\to 0}{\sim} (Z_2/Z_A)^2 q^2\, \tilde\Delta^{ab}_{\mu\nu}(q)\,. \tag{3.4}$$

For later convenience we will introduce

$$\tilde f_\pi = (Z_A/Z_2) f_\pi\,, \tag{3.5}$$

which is cutoff-independent even in the presence of anomalies (to be considered in Sec. IV). In the present nonanomalous case, $Z_A/Z_2 = g_A^{-1}$ is a finite number. We emphasize that it is the unrenormalized current which is associated with the weak interactions and which obeys canonical commutation relations.

The Schwinger-Dyson equation for ${}^5\Delta^{ab}_{\mu\nu}$ is

$$\begin{aligned} {}^5\Delta^{ab}_{\mu\nu}(q) &= -i\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}[\gamma_\mu\gamma_5\tfrac{1}{2}\tau^a S(k)\,{}^5\Gamma^b_\nu(k,k+q)S(k+q)] \\ &= -i\,\frac{Z_2^{\ 2}}{Z_A}\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}[\gamma_\mu\gamma_5\tfrac{1}{2}\tau^a\tilde S(k)\,{}^5\tilde\Gamma^b_\nu(k,k+q)\tilde S(k+q)]\,. \end{aligned}\tag{3.6}$$

Using (3.4), (3.5), (2.16), (2.18), and (3.6) one obtains the result for $\tilde f_\pi$:

$$\tilde f_\pi^{\ 2} q_\mu = -Z_A\, i\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}[\gamma_\mu\gamma_5\tilde S(k)\gamma_5\tilde G(k,k+q)\tilde S(k+q)]\Big|_{q^2=0}\,. \tag{3.7}$$

We introduced the quantity $\tilde G(k,k+q)$ before and it is given by the solution to the integral equation (2.21) subject to the boundary condition $\tilde G(p,p) = g_A^{-1}\Sigma(p)$. As in the Σ model the assumption of the existence of a broken vacuum symmetry leads to an equation for $f_\pi^{\ 2}$. In general (3.7) specifies $\tilde f_\pi^{\ 2}$ as a function of g^2 and μ^2/m^2. Vacuum stability requires

$$f_\pi^{\ 2}(g^2,\mu^2/m^2) > 0. \tag{3.8}$$

Otherwise $\tilde f_\pi = 0$ and the Goldstone mode is absent.

In general it is difficult to calculate $\tilde f_\pi^{\ 2}(g^2,\mu^2/m^2)$, but for small values of the coupling constant we can compute it exactly.

As we have assumed that the axial-vector currents correspond to nonanomalous channels we can compute in the Landau gauge which makes Z_2 finite and therefore Z_A is finite also. Further, in this gauge, we may write $\tilde S(p) = [\not p + \Sigma(p)]/(p^2 - m^2)$ which has the correct asymptotic behavior. Using this and the decomposition (2.19) for $\tilde G$ in terms of the invariant amplitudes one obtains

$$\begin{aligned}\tilde f_\pi^{\ 2} = -\frac{4iZ_A}{(2\pi)^2}\int\frac{d^4k}{(k^2-m^2)^2}\Big[&\Big(\Sigma(k^2) - \frac{k^2}{2}\frac{d}{dk^2}\Sigma(k^2)\Big)\tilde g_1(k,k) \\ &+ [\Sigma^2(k^2) + \tfrac{1}{2}k^2]\,\tilde g_2(k,k) + k^2[\Sigma^2(k^2) - k^2]\,\tilde g_3(k,k) - \tfrac{3}{2}k^2\,\Sigma(k^2)\tilde g_4(k,k)\Big]\,.\end{aligned}\tag{3.9}$$

It can be shown from the integral equations for the $\tilde g_i(k,k)$ that $\tilde g_{2,3,4}$ contribute terms of $O(g^2)$ larger to $\tilde f_\pi^{\ 2}$ than $\tilde g_1$ so we may ignore them in the weak-coupling limit. Similarly $k^2 d\ln\Sigma(k^2)/dk^2$ is of $O(g^2)$. Hence, using $\tilde g_1(k,k) = g_A^{-1}\Sigma(k)$ one has

$$\tilde f_\pi^{\ 2} = -\frac{4iZ_A}{(2\pi)^4 g_A}\int\frac{d^4k\,\Sigma^2(k^2)}{(k^2-m^2)^2}\,. \tag{3.10}$$

This equation implies $\tilde f_\pi^{\ 2} > 0$ since the integrand (after transforming to Euclidean space) is positive-definite. However, this result was based on perturbative-type approximations; for the general expression (3.7) we have not obtained any such general positivity requirement. Hence there may be values of the coupling constant g^2 and μ^2/m^2 such that the stability condition (3.8) is not satisfied.

For small coupling constants for which (3.10) is valid, we find that since $\Sigma(k^2) \sim m(-k^2/m^2)^{-3g^2/(4\pi)^2}$ as $-k^2 \to \infty$, $g^2 \to 0$

$$f_\pi{}^2 = g_A{}^2 \tilde{f}_\pi{}^2 \underset{g^2 \to 0}{\sim} \frac{2m^2}{3g^2} > 0 . \tag{3.11}$$

We conclude that in channels which are free of anomalies, the stability condition for the vacuum (3.8) is satisfied for sufficiently small values of the coupling constant. Hence the necessary condition for the existence of the Goldstone mode is fulfilled. We conjecture that this condition is also sufficient.

For larger values of the coupling there may be a phase transition. The boundary of the phase regions is given by

$$\tilde{f}_\pi{}^2(g^2, \mu^2/m^2) = 0 .$$

We will discuss possible phase transitions in Sec. V.

From the defining equation (3.7) and the axial-vector current Ward identity we can use manipulations introduced by Jackiw and Johnson[16] to obtain the alternate exact relation

$$g_{\mu\nu} f_\pi{}^2 \delta_{ab} = -iZ_2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}[\tilde{S}(k)\tfrac{1}{2}\tau^a \gamma_\mu \tilde{S}(k)\tilde{\Gamma}_\nu^b(k,k) - g_A \tilde{S}(k)\tfrac{1}{2}\tau^a\gamma_\mu\gamma_5\tilde{S}(k)\,{}^5\tilde{\Gamma}_\nu^{b,\mathrm{reg}}(k,k)] , \tag{3.12}$$

where $\tilde{\Gamma}_\nu^b$ is the renormalized vector-current vertex and ${}^5\tilde{\Gamma}_\nu^{b,\mathrm{reg}}$ is the nonpole part of the renormalized axial-vector vertex. If we approximate $Z_2 \simeq g_A \simeq 1$, $\tilde{\Gamma}_\nu^b \simeq \gamma_\nu \frac{1}{2}\tau^b$, ${}^5\tilde{\Gamma}_\nu^{b,\mathrm{reg}} \sim \gamma_\nu\gamma_5\frac{1}{2}\tau^b$, $\tilde{S}(k) \simeq [\not{k} + \Sigma(k)]/(k^2 - m^2)$ in (3.12), we recover the results (3.10) and (3.11) for the leading contribution to $f_\pi{}^2$ in the weak-coupling limit.

IV. AXIAL-VECTOR ANOMALY

In this section we will consider the case of those axial-vector currents which satisfy Ward identities with an anomalous term present.[12,13] This occurs if the axial-vector current communicates with the two-vector-gluon channel. An example is the axial-vector current in electrodynamics; another is the ninth axial-vector current in the quark-triplet vector-gluon model.

A. Analysis of the Ward identity

Let us consider the axial-vector current in electrodynamics. For the unrenormalized irreducible vertices one has the Ward identity[13]

$$q^\mu\,{}^5\Gamma_\mu(p',p) = 2m_0\,{}^5\Gamma_D(p',p) + S_F{}^{-1}(p')\gamma_5 + \gamma_5 S_F{}^{-1}(p) + F(p',p) . \tag{4.1}$$

Here ${}^5\Gamma_\mu(p',p)$ is the vertex corresponding to the axial-vector current $\bar\psi\gamma_\mu\gamma_5\psi$, $2m_0\,{}^5\Gamma_D(p',p)$ to $2m_0\bar\psi\gamma_5\psi$, and $S_F{}^{-1}(p)$ is the unrenormalized fermion propagator. These quantities, since they are bilinear in the fermion fields, are rendered cutoff-independent by multiplications:

$${}^5\tilde{\Gamma}_\mu(p',p) = Z_A\,{}^5\Gamma_\mu(p',p) ,$$

$$2m\,{}^5\tilde{\Gamma}_D(p',p) = Z_D\,2m_0\,{}^5\Gamma_D(p',p) ,$$

$$\tilde{S}_F(p) = Z_2{}^{-1} S_F(p) .$$

$F(p',p)$ is the anomalous term corresponding to the interaction $-i\alpha/4\pi : F_{\mu\nu}{}^*F_{\mu\nu}:$.

The analysis for the anomalous Ward identity in electrodynamics without a Goldstone boson has been given by Adler.[13] Here we want to see if this is the only possibility. Let us assume that the "normal" term in the axial-vector divergence $2m_0\bar\psi\gamma_5\psi$ vanishes, corresponding to an axial-vector current conserved up to the anomaly.

If $2m_0\,{}^5\Gamma_D(p',p) = 0$ and the fermion has a nonzero physical mass (which we will always assume) then we must have a Goldstone boson. To see this fact suppose to the contrary that there is no Goldstone boson. Then (4.1) implies as $q_\mu \to 0$, $S_F{}^{-1}(p)\gamma_5 + \gamma_5 S_F{}^{-1}(p) + F(p,p) = 0$. But if there is no zero-mass particle then the anomalous term vanishes at $p' = p$ so $F(p,p) = 0$. Hence we have $Z_2{}^{-1}\Sigma(p) = 0$ or $\Sigma(m^2) = m = 0$. Consequently, we must have a Goldstone pole if $2m_0\,{}^5\Gamma_D(p',p) = 0$ and $m \neq 0$.

The vertex ${}^5\Gamma_\mu(p',p)$ thus has a Goldstone pole:

$$q^2\,{}^5\Gamma_\mu(p',p) \underset{q^2 \to 0}{\longrightarrow} -q_\mu f \gamma_5 2G(p',p) . \tag{4.2}$$

As before we will introduce the cutoff-independent amplitude $\tilde{G}(p',p) = Z_A f G(p',p)$ which satisfies a homogeneous integral equation similar to (2.21).

Next we note that since there is a Goldstone boson, $F(p',p)$ defined by

$$S(p')F(p',p)S(p) = +i\int d^4x\, d^4y\, e^{ip'\cdot x} e^{-ip\cdot y} \left\langle 0 \left| \left(\psi(x)\frac{\alpha}{4\pi} : F_{\mu\nu}(0)\,{}^*F_{\mu\nu}(0) : \bar\psi(y) \right)_+ \right| 0 \right\rangle \tag{4.3}$$

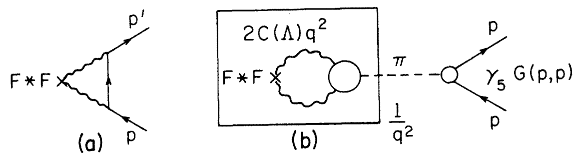

FIG. 1. (a) Typical anomaly diagram which vanishes for $p'-p=0$. (b) Pole contribution to anomaly which survives when $p'-p=0$.

does not vanish as $p'\to p$. Diagrams of Fig. 1(a) do vanish in this limit but diagrams of Fig. 1(b) do not. This is because the pole term of $O(1/q^2)$ just cancels the $O(q^2)$ contribution of the $\langle 0|F_{\mu\nu}(0)\,{}^*F_{\mu\nu}(0)|\pi\rangle$ amplitude [shown in the box of Fig. 1(b)] as $q\to 0$. The residue of the pole term is proportional to $\gamma_5 G(p,p)$ times a divergent quantity $C(\Lambda)$ from the integral over the photon loops. Here Λ^2 is the cutoff introduced as a regulator mass of the photons. We conclude that since these are the only nonvanishing terms in the $q_\mu\to 0$ limit the anomalous term is of the form

$$F(p',p)=2C(\Lambda)q^2\left(\frac{1}{q^2}\right)\gamma_5 G(p,p)$$
$$=2\gamma_5 C(\Lambda)G(p,p)\,. \tag{4.4}$$

In the $q\to 0$ limit the Ward identity (4.1) reads

$$fG(p,p)=Z_2^{-1}\Sigma(p)-C(\Lambda)G(p,p)\,. \tag{4.5}$$

Since this result is valid for all p, $\Sigma(p)/G(p,p)$ is independent of p. We will define

$$\frac{\Sigma(p)}{Z_A fG(p,p)}=\frac{\Sigma(p)}{\tilde G(p,p)}=\bar g_A\,, \tag{4.6}$$

and since $\tilde G(p,p)$ and $\Sigma(p)$ are cutoff-independent so is $\bar g_A$. We also have as the boundary condition for the homogeneous integral equations for $\tilde G(p',p)$ that $\bar g_A\tilde G(p,p)=\Sigma(p)$. From (4.5) and (4.6) we obtain for $C(\Lambda)$

$$C(\Lambda)=f\left(\bar g_A\frac{Z_A}{Z_2}-1\right). \tag{4.7}$$

An important characteristic of axial-vector currents with anomalous Ward identities is that

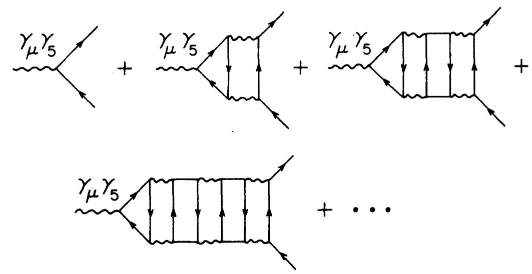

FIG. 2. Diagrams responsible for the leading divergence in Z_A/Z_2.

Z_A/Z_2 is not cutoff-independent. In our considerations in electrodynamics to leading order in α, one has[13]

$$\frac{Z_A}{Z_2}=1+\frac{3\alpha^2}{4\pi^2}\ln(\Lambda^2/m^2)+O(\alpha)\times\text{finite}$$
$$+O(\alpha^2)\times\text{finite}\,. \tag{4.8}$$

It is now known that the axial-vector current develops an anomalous dimension[11]

$$2\chi=\frac{3\alpha^2}{2\pi^2}+O(\alpha^3) \tag{4.9}$$

and that, according to the analysis of Crewther, Shei, and Yan[11]

$$Z_A/Z_2=(\Lambda^2/m^2)^\chi+(\text{less singular terms}) \tag{4.10}$$

so that the divergences in (4.8) sum up to (4.10). The relevant diagrams are shown in Fig. 2. In the case there is no anomaly $C(\Lambda)=0$ and $\bar g_A=g_A=Z_2/Z_A$, consistent with the conclusion from the Ward identity (4.7).

In the anomalous case we can check the Ward identity (4.7) in perturbation theory. We conclude from (4.7) and (4.8) that to leading order

$$C(\Lambda)=f\,\frac{3\alpha^2}{4\pi^2}\ln(\Lambda^2/m^2)+O(\alpha)\times\text{finite}\,. \tag{4.11}$$

On the other hand, we can calculate $C(\Lambda)$ directly to this order by calculation of the fermion triangle contribution to Fig. 1(b). One has to leading order

$$C(\Lambda)q^2=\frac{3\alpha^2}{4\pi^2}\ln\frac{\Lambda^2}{m^2}\,\frac{Z_2^2}{Z_A^2 f}\,q^\mu\left(-iZ_A\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}[\gamma_\mu\gamma_5\tilde S(k)\gamma_5 2\tilde G(k,k+q)\tilde S(k+q)]\right).$$

Using the analog of (3.7) (which is still valid) the quantity in large parentheses is just $q_\mu\bar f^2$, so that $C(\Lambda)=f(3\alpha^2/4\pi^2)\ln(\Lambda^2/m^2)$, in agreement with (4.11).

B. Stability condition for the anomalous channel

Before we can conclude that Goldstone bosons are present we must examine the question of vacuum stability and whether solutions to the integral equations exist for the Goldstone mode. The analysis we gave in Sec. III is also valid in the presence of the anomaly and we have for $\bar f=(Z_A/Z_2)f$ the result from Eq. (3.7):

$$\tilde{f}^2 q_\mu = -Z_A i \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr}[\gamma_\mu \gamma_5 \tilde{S}(k) \gamma_5 2\tilde{G}(k, k+q) \tilde{S}(k+q)]\Big|_{q^2=0} . \tag{4.12}$$

Here $\tilde{G}$ is defined as the solution to the homogeneous integral equation

$$\gamma_5 \tilde{G} = \int \tilde{S}\gamma_5 \tilde{G}\tilde{S}\tilde{K} + O(q^2) \tag{4.13}$$

subject to the boundary condition $\tilde{G}(p,p) = \bar{g}_A{}^{-1}\Sigma(p)$.

Contrary to the nonanomalous case the factor Z_A in Eq. (4.12) diverges even in the Landau gauge. Further, there are potential overlapping divergences in the integral representation (4.12). To calculate $\tilde{f}^2$ we will utilize the method given by Jackiw and Johnson in the appendix of their paper[16] to obtain a representation for $\tilde{f}^2$ in terms of renormalized amplitudes and free of overlapping divergences. The method of Jackiw and Johnson is applicable here, in spite of the differences between their model and ours, since it gives the residue of the pole term in the axial-vector current correlation function.

Some care must be exercised in obtaining the final representation for $\tilde{f}^2$ since in our application, we allow for the possibility of contributions from the two-gluon (anomalous) channel. (Jackiw and Johnson have removed, by doubling the number of fermions, all anomalous contributions in their treatment.) In obtaining the representation for $\tilde{f}^2$ one will encounter the axial-vector–two-photon triangle graphs which, as is well known, have the ambiguity that their value depends on how the momentum is routed through the graph. Such ambiguity, however, is completely removed once we specify that the triangle graph is always to be computed by imposing gauge invariance for the photon legs. This is equivalent to specifying the regulator value of the triangle amplitude.[13] With this proviso the representation we obtain for $\tilde{f}^2$ is identical to that given by Jackiw and Johnson.[16] For the reader interested in details this representation for $\tilde{f}^2$, shown in Figs. 3(a) and 3(b), 3(b), is rederived in our appendix. Our notation in Figs. 3(a) and 3(b) is that the slash denotes differentiation with respect to the momentum transfer q_ν flowing through the amplitude and then setting q_ν to zero. ${}^5\tilde{\Gamma}^R_\mu(k,k)$ denotes the regular or nonpole part of ${}^5\tilde{\Gamma}_\mu(k,k)$. For the anomalous channel this vertex has the asymptotic behavior (in the Landau gauge)

$${}^5\tilde{\Gamma}^R_\mu(k,k) \underset{-k^2\to\infty}{\sim} \gamma_\mu\gamma_5(-k^2/m^2)^\chi . \tag{4.14}$$

First we examine the diagram of Fig. 3(a) in the weak-coupling limit. Here $\tilde{G}(p,p) = \Sigma(p)\bar{g}_A{}^{-1}$ so that this diagram is the same as for the nonanomalous case except that the axial-vector vertex behaves like (4.14) instead of just $\gamma_\mu\gamma_5$. One finds, as in (3.10),

$$\tilde{f}^2_{(3a)} = \frac{-8i}{(2\pi)^4 \bar{g}_A} \int \frac{d^4k(-k^2/m^2)^\chi \Sigma^2(k^2)}{(k^2-m^2)^2} \underset{e^2\to 0}{\sim} 4m^2/3e^2 . \tag{4.15}$$

The integral (4.15) converges provided $2\epsilon(e^2) > \chi(e^2)$ which is certainly valid in the weak-coupling limit, relevant for QED. For stronger couplings for which $2\epsilon < \chi$, $\tilde{f}^2$ does not exist and there may be a phase transition to the mode for which $\tilde{f}=0$. We discuss possible phase transitions in the next section.

On the basis of Fig. 3(a) alone there is no reason to rule out a Goldstone mode for QED. Next we examine Fig. 3(b). For kernels $\tilde{K}$ not containing the two-photon state we find that the integrals are sufficiently convergent for $\tilde{f}^2$ to exist. These contributions to $\tilde{f}^2$ behave at worst as constants as $e^2 \to 0$.

However, for the kernel with the anomalous channel [shown in Fig. 3(c) in skeleton expansion] we find that its contribution to $\tilde{f}^2$, $\tilde{f}_A{}^2$ diverges for $\chi > 0$. Consequently, there does not exist a solution to the renormalized integral equations for $\tilde{f}^2$ if the anomalous channel is present.

To see this we differentiate $\tilde{K}_A$ as is required

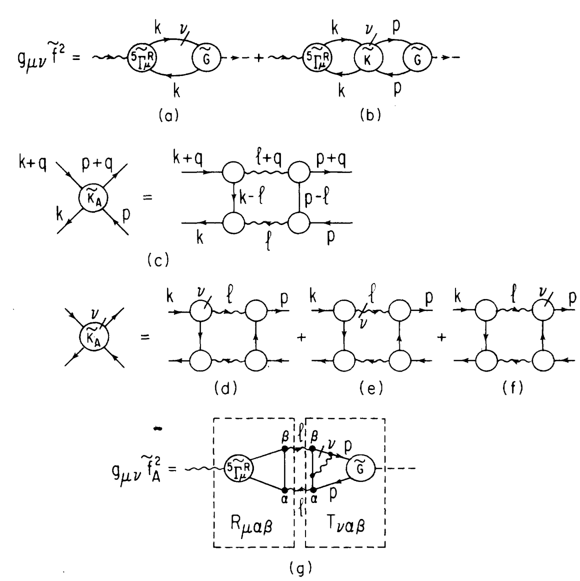

FIG. 3. (a) and (b) The two terms in the formal expression for $\tilde{f}^2$. (c) The anomalous kernel. (d)–(f) The derivative of the anomalous kernel. (g) The leading contribution to $\tilde{f}_A{}^2$.

in Fig. 3(b) and this is shown in Fig. 3(c). The graph with crossed photons contributes equally. There arise the terms shown in Figs. 3(d), 3(e), and 3(f). The terms corresponding to Fig. 3(d) and 3(e) if substituted into the integrals for $\tilde{f}^2$ shown in Fig. 3(b) actually give a vanishing contribution. The reason for this is that the triangle amplitude involving $\tilde{G}\gamma_5$ and the two photons vanishes if the photons have momenta that are proportional, as is the case here. There remains the contribution of 3(f) which if substituted into 3(b) gives, to lowest order in the couplings in the kernel, the diagram shown in Fig. 3(g). The axial-vector vertex triangle denoted by $R_{\mu\alpha\beta}(l)$ and the pseudoscalar vertex $T_{\nu\alpha\beta}(l)$ are shown in Fig. 3(g).

For the anomalous channel contribution to $\tilde{f}^2$ we finally obtain from Fig. 3(g) the expression

$$g_{\mu\nu}\tilde{f}_A{}^2 = i\int\frac{d^4l}{(2\pi)^4}\,\frac{1}{l^4}\,T_{\nu\alpha\beta}(l)R_{\mu\alpha\beta}(l)\,, \qquad (4.16)$$

where l is the photon-loop momentum. The amplitudes in (4.16) are completely specified according to

$$R_{\mu\alpha\beta} = \epsilon_{\mu\alpha\beta\lambda}\,l_\lambda R(l^2)\,,$$
$$T_{\nu\alpha\beta} = \epsilon_{\nu\alpha\beta\lambda}\,l_\lambda T(l^2)\,, \qquad (4.17)$$

so that

$$\tilde{f}_A{}^2 = -\frac{3}{32\pi^2}\int_0^\infty dl^2 R(-l^2)\,T(-l^2) \qquad (4.18)$$

in terms of the invariant functions $R(l^2)$, $T(l^2)$.

The axial-vector vertex triangle with the axial-vector vertex given by (4.15) can be calculated by requiring gauge invariance for $R_{\mu\alpha\beta}$ for photons with arbitrary momenta. One finds for the asymptotic behavior of the invariant amplitude

$$R(-l^2) \underset{l^2\to\infty}{\sim} -i\,\frac{e^2}{2\pi^2}\,(l^2/m^2)^\chi\,. \qquad (4.19)$$

The asymptotic behavior of $T_{\nu\alpha\beta}(l)$ may be established by a much more laborious calculation. We have carried out this computation in the Landau gauge using the correct asymptotic forms $\tilde{S}(p) = [\not{p} - \Sigma(p)]^{-1}$, $\Sigma(p) = m(-p^2/m^2)^{-\epsilon}$, and $\gamma_5\tilde{G}(p,p) = \gamma_5\tilde{g}_A{}^{-1}\Sigma(p) \approx \gamma_5\Sigma(p)$. The result is that $l^2T(-l^2) \to_{l^2\to\infty}$ (constant). That is, the damping factor $(-p^2/m^2)^{-\epsilon}$ from $\tilde{G}(p,p)$ does not propagate through the diagram to damp $l^2T(-l^2)$. The exact result is

$$l^2T(-l^2) \underset{\substack{l^2\to\infty \\ e^2\to 0}}{\sim} \frac{-e^4m^2i}{48\pi^2\epsilon} + O(e^4) + O\left(\frac{e^4}{\epsilon}\left(\frac{l^2}{m^2}\right)^{-2\epsilon}\right)$$
$$= \frac{-ie^2m^2}{9\pi^2}\,. \qquad (4.20)$$

The factor $1/\epsilon$ in (4.20) is a manifestation of the property that the potential logarithmic factors, $\ln l^2$, are replaced by $1/\epsilon$ in finite QED.

From (4.18), (4.19), and (4.20) we find that $\tilde{f}_A{}^2$ diverges:

$$\tilde{f}_A{}^2 = \frac{m^2e^4}{3\times 2^6\pi^6}\int_0^\infty \frac{dl^2}{l^2}\left(\frac{l^2}{m^2}\right)^\chi\,. \qquad (4.21)$$

If we regulate the photon propagator in (4.21) with a mass Λ, we find

$$\tilde{f}_A{}^2 \to \frac{m^2e^4}{3\times 2^6\pi^6}\times\begin{cases}\dfrac{1}{\chi}\left(\dfrac{\Lambda^2}{m^2}\right)^\chi, & \chi>0\\ \ln\dfrac{\Lambda^2}{m^2}, & \chi=0\\ \text{finite}, & \chi<0\,.\end{cases} \qquad (4.22)$$

We have found no other diagrams that could cancel this divergence.

This divergence in the renormalized integral equations for $\chi \geq 0$ is associated with the fact that the axial-vector current has dimension greater than 3. Does this divergence in $\tilde{f}$ really destroy the possibility of a Goldstone state? It might be argued that since $\tilde{f}$ is not a measurable quantity (unless the weak interactions couple to neutral axial-vector currents), who cares if it does not exist? The homogeneous integral equations for the Goldstone boson coupling $\tilde{G}$ are completely convergent and well behaved in the presence of the anomalous channel. The difficulty with this line of argument is not that the homogeneous integral equation do not have solutions but in the boundary condition we impose, which is where the divergent quantity $\tilde{f}_A$ enters. For example, the renormalized coupling constant for the Goldstone boson to the fermion is given by $\tilde{g}_\pi = \tilde{g}_A m/\tilde{f}$ and this quantity actually vanishes as $\Lambda\to\infty$, $\tilde{f}\to\infty$.

The failure of $\tilde{f}^2$ to exist for $\chi>0$ has the physical consequence that the Goldstone mode cannot exist in electrodynamics. Instead we must choose the other branch of the Goldstone alternative $\tilde{f}=0$ and axial-vector current conservation breaks down. Similarly, there are no Goldstone bosons associated with axial-vector currents in vector-gluon models which communicate with the two-vector-gluon channel. As the ninth axial-vector current in the quark-triplet model has this property, there is no ninth Goldstone boson and this current is not conserved. *Our general conclusion is that for axial-vector currents which communicate with two vector mesons and satisfy anomalous Ward identities, there are no associated Goldstone bosons for sufficiently small values of the coupling constant.* This is our solution to the U(3) problem.

V. GLUON-MODEL PHASE TRANSITIONS

It is interesting to examine what possible phase transitions might occur in vector-gluon models. We will hold μ^2/m^2 fixed and consider the behavior of the model as a function of g^2, the coupling constant. The three phases we will consider are the Goldstone mode, the mode for which the physical fermion mass $m=0$, $f=0$ and the axial-vector current is conserved, and the mode in which axial-vector current conservation is violated and $f=0$.

First we consider the behavior of the critical exponent corresponding to the asymptotic behavior of $\Sigma(p^2)$, which we denote by $\epsilon(g^2)$ $[\Sigma(p^2) \to (-p^2/m^2)^{-\epsilon(g^2)}$ for large $-p^2]$. For small g^2, $\epsilon(g^2)>0$. Suppose that for $g^2>g_\epsilon{}^2$, $\epsilon(g^2)<0$ defining the critical coupling $g_\epsilon{}^2$. Then the integral equation for the fermion propagator, the gap equations, will not have a nontrivial solution. So for $g^2>g_\epsilon{}^2$ we have $m=0$ and the chiral symmetry is realized by massless fermions.

Next consider the decay constant f_π associated with Goldstone bosons whose quantum numbers do not communicate with the two-gluon channel. According to (3.11) $f_\pi{}^2(g^2)$ has a pole at $g^2=0$. As g^2 approaches $g_\epsilon{}^2$, $f_\pi{}^2$ will again diverge as $1/\epsilon(g^2)$ since both $\Sigma(k)$ and $\tilde{g}_1(k,k)$ behave as $(-k^2/m^2)^{-\epsilon(g^2)}$ for large k^2. The residue of this pole could be either positive or negative: Neither the normalization of Σ nor the behavior of $\tilde{g}_{2,3,4}$ in (3.9) (which can also contribute to the pole) is known for $g^2\to g_\epsilon{}^2$. These two possibilities are shown in Figs. 4(a) and 4(b), respectively. In Fig 4(a) the axial-vector current is everywhere conserved. For $g^2<g_\epsilon{}^2$ the Goldstone mode (indicated by G) is present and $f_\pi{}^2>0$. For $g^2>g_\epsilon{}^2$ we are in the Wigner-Weyl mode (indicated by W): $f_\pi{}^2=0$, $m=0$. In Fig. 4(b) we have $f_\pi{}^2>0$ for $g^2<g_f{}^2$. Between $g_f{}^2$ and $g_\epsilon{}^2$ the divergence of the axial-vector current no longer vanishes (indicated by D). In this case $f_\pi{}^2=0$ and the matrix element of $\partial\cdot A^a$ between fermion states at zero momentum transfer has a form factor $2mg_A(g^2)$ [we assume $g_A(g^2)\neq 0$ and $m\neq 0$]. For $g^2>g_\epsilon{}^2$ we again have $\partial\cdot A^a=0$ and are in the Wigner mode. There are of course more complicated possibilities, such as the one shown in Fig. 4(c).

Finally we consider the case of those axial-vector currents which communicate with the two-gluon channel. For small g^2, $\chi(g^2)>0$ and we define $g_\chi{}^2$ so that $\chi(g^2)<0$ for $g^2>g_\chi{}^2$. This is shown in Fig. 5(a). A possible behavior of the associated (renormalized) decay constant $\tilde{f}^2(g^2)$ is shown in Fig. 5(b). For $g^2<g_\chi{}^2$, axial-vector current conservation is violated and $f^2=0$. For $g_\chi{}^2<g^2<g_\epsilon{}^2$ there is a Goldstone mode, with $\tilde{f}^2\sim -1/\chi(g^2)$ for g^2 near $g_\chi{}^2$. For $g^2>g_\epsilon{}^2$ the Wigner mode is present. If $g_\chi{}^2>g_\epsilon{}^2$ there is no Goldstone mode. This is shown in Fig. 5(c).

The crucial point about all these critical values of the coupling $g_\epsilon{}^2$, $g_\chi{}^2$, and $g_f{}^2$ is that for g^2 smaller than any of them we have the desired symmetry of the world of real hadrons. This at least suggests that g^2 is small enough to permit perturbation expansions for the Bethe-Salpeter kernel.

It would be quite interesting to examine the behavior of hadron amplitudes when the coupling constant approaches a critical value. Amplitudes are apparently nonanalytic and perhaps diverge at phase transition points. If there is a phase transition of the type described here in QED for some value $\alpha=\alpha_c$ of the coupling then we do not expect analytic perturbation theory to converge in the neighborhood of α_c. Whether this influences expansions about the origin, $\alpha=0$, of course depends on the physical value of α_c.

It would also be useful to study the Bethe-Salpeter equations for the invariant functions $\tilde{g}_i$ defined in (2.19) for both the anomaly and non-anomaly cases. One may be able to distinguish, for example, between the options in Fig. 4. If Fig. 4(b) were correct, then the small physical value of $f_\pi \simeq 90$ MeV on the scale of hadron masses would suggest that real physics is close to a phase transition point. If one had some idea as to the behavior of hadron amplitudes in the neigh-

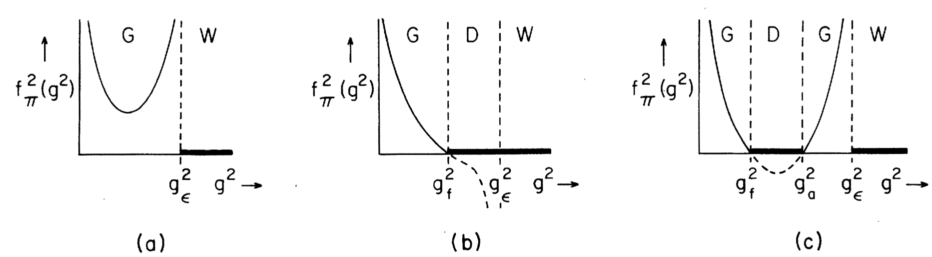

FIG. 4. Three possible behaviors of $f_\pi{}^2(g^2)$ in the absence of anomalies. G indicates the Goldstone phase ($f_\pi{}^2>0$, $m\neq 0$), W indicates the Wigner-Weyl phase ($f_\pi{}^2=0$, $m=0$), and D indicates the phase in which $f_\pi{}^2=0$, $m\neq 0$ and axial-vector-current conservation is violated.

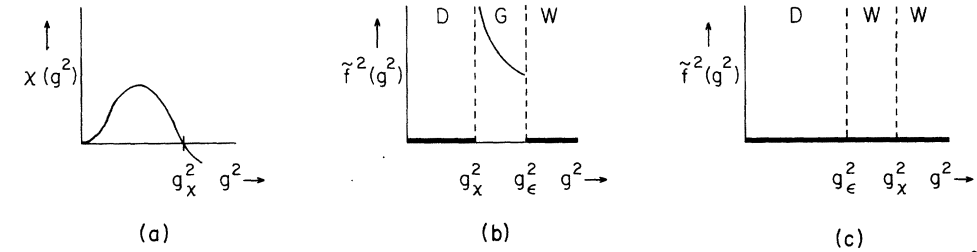

FIG. 5. (a) A possible behavior of the anomalous dimension $\chi(g^2)$. (b) and (c) Two possible behaviors for $\tilde{f}^2(g^2)$ in the presence of anomalies.

borhood of a phase transition point then it might be possible to exploit the small value of f_π. In our present state of ignorance, however, this is not possible.

VI. AXIAL-VECTOR RENORMALIZATION

Because the vector current is conserved it is well known that the interactions do not renormalize the vector form factor. For the axial-vector current there is a nontrivial renormalization and for axial-vector currents without anomalies this renormalization is finite. Here we will calculate $g_A = Z_2/Z_A$ in the vector-gluon model to order g^2.

Consider the diagrams of Fig. 6, where we have already subtracted out the mass counterterms. They contribute to the renormalized vector vertex

$$\bar{u}(p)\, Z_2^{-1}\gamma_\mu[1+(Z_V^{-1}-1)-2(Z_2^{-1}-1)]\,\tfrac{1}{2}\tau^a u(p) = (Z_2/Z_V)\bar{u}(p)\gamma_\mu\tfrac{1}{2}\tau^a u(p)+O(g^4)\,. \quad (6.1)$$

The factor Z_2^{-1} comes from the wave-function renormalization. Since the Ward identity implies $Z_2 = Z_V$ the vector current is unrenormalized. The renormalization of the axial-vector current to $O(g^2)$ is given by a similar set of diagrams and contributes

$$\bar{u}(p)\, Z_2^{-1}\gamma_\mu\gamma_5[1+(Z_A^{-1}-1)-2(Z_2^{-1}-1)]\,\tfrac{1}{2}\tau^a u(p) = (Z_2/Z_A)\bar{u}(p)\gamma_\mu\gamma_5\tfrac{1}{2}\tau^a u(p)+O(g^4)\,. \quad (6.2)$$

Consequently for g_A we have

$$g_A = Z_2/Z_A = 1+(Z_A^{-1}-1)-(Z_2^{-1}-1)+O(g^4)\,. \quad (6.3)$$

For the vector renormalization one obtains to $O(g^2)$

$$\bar{u}(p)(Z_V^{-1}-1)\gamma_\mu u(p) = \bar{u}(p)\,\frac{ig^2}{(2\pi)^2}\int\frac{d^4l}{l^2-\mu^2}\left(-g_{\alpha\beta}+\frac{\lambda l_\alpha l_\beta}{l^2}\right)\gamma_\alpha\,\frac{(\not{p}+\not{l}+m)}{(p+l)^2-m^2}\,\gamma_\mu\,\frac{(\not{p}+\not{l}+m)}{(p+l)^2-m^2}\,\gamma_\beta u(p)\,. \quad (6.4)$$

A similar expression with $\gamma_\mu\to\gamma_\mu\gamma_5$ for $Z_A^{-1}-1$ has the property that $(Z_A^{-1}-1)-(Z_V^{-1}-1)$ is independent of the gauge parameter λ. Hence g_A is gauge-independent. A straightforward calculation then gives us

$$g_A - 1 = -\frac{1}{\pi}\left(\frac{g^2}{4\pi}\right)\int_0^1\frac{dz\, z^3}{z^2+(\mu^2/m^2)(1-z)}\,. \quad (6.5)$$

The result (6.5) is infrared finite. For $\mu^2=0$ one has

$$g_A - 1 = -\frac{1}{2\pi}(g^2/4\pi)\,. \quad (6.6)$$

For arbitrary μ^2/m^2 (6.5) implies that $g_A - 1 < 0 + O(g^4)$. This suggests that the nucleonic $G_A/G_V \simeq 1.25$ which in a free quark-triplet model is $G_A/G_V = \frac{5}{3}$ will get modified in the right direction by the interactions due to neutral vector gluons. We would have $G_A/G_V \simeq \frac{5}{3}g_A < \frac{5}{3}$. For $g_A = 0.75$ the agreement would be precise.

Suppose the gluon coupling is small and gluon mass μ is small so (6.6) is valid. Then with $g_A = 0.75$, using our result for $\tilde{f}_\pi^{\,2}$ given by (3.11) we obtain for the fermion mass

$$m = [12\pi^2(1-g_A)f_\pi^{\,2}]^{1/2} \simeq 0.5\ \text{GeV} \quad (6.7)$$

for $f_\pi = g_A\tilde{f}_\pi \simeq 90$ MeV. We do not take this lowest-order calculation very seriously, since it requires $g^2/4\pi^2 \simeq \frac{1}{2}$ and since $(\mu/m)^2$ is probably large. However, (6.7) is of a reasonable magnitude. In general the calculation of g_A must be compatible with the calculation of f_π and this will impose constraints on the parameters of the model.

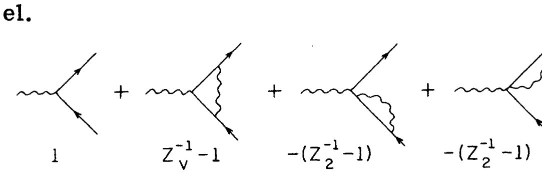

FIG. 6. Lowest-order diagrams contributing to the renormalization of the vector vertex.

VII. CONCLUDING REMARKS

We have seen that in an Abelian vector-gluon model one can have the desired realization of chiral symmetry. Most of our results are predicated on anomalous dimensional behavior of amplitudes. This follows if, as we have done, one considers the model in the absence of vacuum polarization or if some eigenvalue condition is satisfied by the coupling constant.

If we now turn to gauge models of strong interactions such as that based on $SU(3)\times SU(3)\times SU'(3)$ with a "colored" octet of Yang-Mills gauge vector bosons[15] then we lose the anomalous dimensional power behavior in the momenta. In such asymptotically free theories[14] one typically finds from an analysis of the Callan-Symanzik[20] scaling equations that amplitudes have power behavior in the logarithm of the momenta.

The ninth axial-vector current, a "color" singlet, will have a divergence with an anomalous term proportional to $F^a_{\mu\nu}\,{}^*F^a_{\mu\nu}$, where $F^a_{\mu\nu}=\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + iC^{abc}A^b_\mu A^c_\nu$ and A^a_μ is the "color" gauge field. One might expect that because of the additional term $C^{abc}A^b_\mu A^c_\nu$ matrix elements of $F^a_{\mu\nu}\,{}^*F^a_{\mu\nu}$ do not vanish at zero momentum transfer and hence the naive Goldstone theorem is violated and there is no Goldstone boson. However, one may easily verify that $F^a_{\mu\nu}\,{}^*F^a_{\mu\nu}$ is still a total divergence and hence its matrix elements vanish at zero momentum transfer if there is no Goldstone boson. The zero-mass ghost states that are required in gauge theories will not change this circumstance since they have the wrong parity and do not couple to fermions as is required in order to eliminate the factor proportional to the momentum transfer in the matrix elements of $F^a_{\mu\nu}\,{}^*F^a_{\mu\nu}$.

In asymptotically free gauge theories of the strong interactions one finds that Z_A/Z_2 is finite:

$$Z_A/Z_2 \underset{\Lambda^2\to\infty}{\sim} \exp\left[-\frac{C_A}{\ln(\Lambda^2/m^2)}\right] \to \text{const}\,, \tag{7.1}$$

where C_A is a positive number depending on the group. Here Z_A is the renormalization constant associated with the ninth axial-vector current. From the discussion given in Sec. IV for the Abelian case, if we examine the integral equations for $\bar{f}^2$, assuming there is a Goldstone boson in this channel, then as in the Abelian case for $\chi=0$ we will again encounter a divergence in $\bar{f}^2 \sim \ln(\Lambda^2/m^2)$. So on this basis we speculate that no solution exists for Goldstone bosons in this channel. Instead axial-vector current conservation breaks down just as in electrodynamics.

For those "color" singlet axial-vector currents transforming like SU(3) octets, there are no anomalies in the color gluon model. However, it is not clear if one can have Goldstone bosons in these channels since it is difficult to calculate $\bar{f}^2$ for these channels. In the Abelian case this calculation was facilitated by the fact that the integral equations for $\bar{f}^2$ diverged as $1/g^2$ as we took the weak-coupling limit. However, in the color gauge models one has for the fermion propagator

$$|\Sigma(p^2)| \underset{-p^2\to\infty}{\sim} [\ln(-p^2/m^2)]^{-c_\Sigma}\,, \tag{7.2}$$

where the positive number c_Σ is a number depending on the gauge group and is independent of the coupling constant. In order for the integral equations for $\Sigma(p^2)$ to have nontrivial solutions we require $c_\Sigma > 1$, so that the integrals converge which places a constraint on the possible gauge group. The contribution to $\bar{f}^2$ as given by Eq. (3.10) will satisfy $\bar{f}^2>0$. The result $\bar{f}^2>0$ can also be obtained from the general equation (3.12) provided we make the standard approximations $g_A\simeq 1$; $\bar{\Gamma}^b_\nu=\gamma_\nu\,\frac{1}{2}\tau^b$; ${}^5\bar{\Gamma}^{b,\,\text{reg}}_\nu\sim\gamma_\nu\gamma_5\,\frac{1}{2}\tau^b$, $\bar{S}(k)=[\not k+\Sigma(k)]/(k^2-m^2)$. Hence we conjecture that an octet of Goldstone bosons can exist in those channels free of anomalies.

On the basis of these speculations we conjecture that everything works out as before for gauge models of the strong interactions; however, this question deserves more detailed investigation.

Finally we remark on the relation of our work to that of Jackiw and Johnson[16] and Cornwall and Norton.[17] These authors have constructed models with real axial-vector-meson gluon coupling to an axial-vector current (which is conserved). Although the axial-vector meson starts out massless in the bare theory, it acquires a mass through the interaction without benefit of introducing explicit scalar fields. The point is that in their model, as in ours, the axial-vector current correlation function develops a zero-mass pole corresponding to a Goldstone boson which decouples from physical hadron amplitudes. The residue of this pole is the mass squared of the decay constant, so the formal connection between these models is the correspondence

$$g^2 f_\pi^{\,2} \to \mu_B^{\,2}\,, \tag{7.3}$$

where g^2 is a coupling constant of the gluon and μ_B is the gluon mass. The calculations of $\mu_B^{\,2}$ and $g^2 f_\pi^{\,2}$ are formally identical and for weak couplings are positive, as is required if these mechanisms are to work.

In spite of the formal similarities the physics is quite different in these models. The massless Goldstone state for our formulation of the model is a real physical state and couples to hadron amplitudes. Further the vector-gluon mass gets put in by hand. In the work of Jackiw and Johnson[16] and Cornwall and Norton[17] the massless excitations must decouple as they are unphysical and should not contribute to intermediate states in unitarity sums. Also the axial-vector-meson mass is an output of the model. In general, however, models of the strong interactions based on axial-vector gluons (rather than vector gluons) have the problem that bound states will be charge-conjugation-degenerate in conflict with the observed hadron spectrum.

ACKNOWLEDGMENT

One of us (H.P) would like to thank Academician A. Logunov for the warm hospitality of the Institute of Theoretical Physics, Serpukov, U. S. S. R., where part of this work was done. He would like to thank Professor B. Zumino for the opportunity to visit the theoretical division at CERN, and he would also like to acknowledge useful discussions with visitors to the Aspen Center for Physics (1973), especially K. Johnson. The other author (P. L.) would like to thank Professor J. C. Polkinghorne for the hospitality of D. A. M. T. P., University of Cambridge, Cambridge, England, and to acknowledge useful discussions with R. Jackiw.

APPENDIX

Here we will derive the representation for $\tilde{f}^2$ shown in Figs. 3(a) and 3(b). Our procedure is identical to that given in the Appendix of Jackiw and Johnson[16] and is repeated here for the convenience of the reader.

Our starting expression for $\tilde{f}^2 q_\mu$ (4.12) and integral equation for $\tilde{G}$ are shown diagrammatically in Figs. 7(A1) and 7(A2). The result of differentiating these expressions is shown in Figs. 7(A3) and 7(A4). Here differentiation with respect to q_ν is denoted by a slash. The next step is to use the integral representation for

$$Z_A \gamma_\mu \gamma_5 = {}^5\tilde{\Gamma}^R_\mu - \int \tilde{S}\, {}^5\tilde{\Gamma}^R_\mu \tilde{S} \tilde{K},$$

as is shown in Fig. 7(A5). Here ${}^5\tilde{\Gamma}^R_\mu(k,k)$ is the

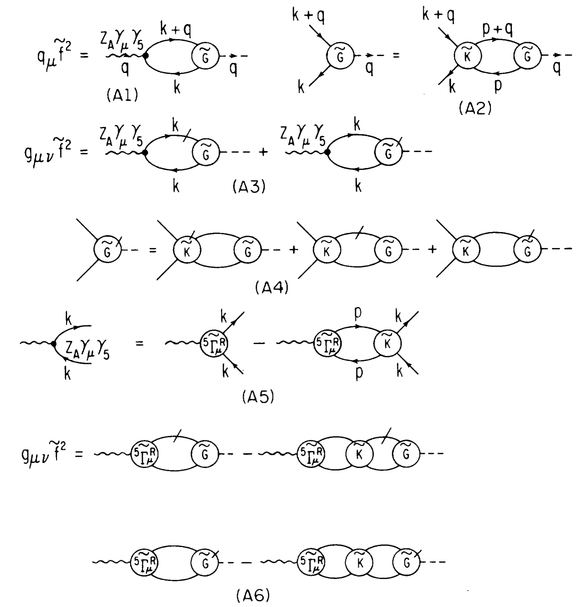

FIG. 7. (A1) The starting expression for $q_\mu \tilde{f}^2$. (A2) The integral equation for $\tilde{G}(k,k+q)$. (A3) The starting expression for $\tilde{f}^2 g_{\mu\nu}$. (A4) The integral equation for $\tilde{G}' = \partial \tilde{G}(k,k+q)/\partial q_\nu|_{q=0}$. (A5) The integral equation for for the regular part of ${}^5\tilde{\Gamma}_\mu(k,k)$. (A6) A representation for $\tilde{f}^2$ free from overlapping divergences.

regular part of ${}^5\tilde{\Gamma}_\mu(k,k)$ with the Goldstone pole subtracted out. In writing this representation for Z_A we understand that if one encounters the ambiguous triangle graph then the regulator, or gauge-invariant, value is to be used. Substituting (A5) into (A3) we obtain (A6). The final step consists of substituting the solution for the integral equation for $\tilde{G}$ given by (A4) into the third diagram of (A6). One finds cancellations among the various diagrams and one is left with the integral representation shown in Figs. 3(a) and 3(b) of the text. The cancellations among these diagrams occur in spite of the routing problems associated with the ambiguous graphs if one specifies the condition of gauge invariance for all triangles. Then routing differences can be ignored and the cancellations can be guaranteed.

---

*Work supported in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-2232.

[1]Y. Nambu, Phys. Lett. 9, 214 (1964).

[2]K. Johnson, M. Baker, and R. Willey, Phys. Rev. 136, B1111 (1964).

[3]S. L. Adler and W. Bardeen, Phys. Rev. D 4, 3045 (1971); 6, 734(E) (1972).

[4]M. Baker and K. Johnson, Phys. Rev. D 3, 2516 (1971).

[5]K. Johnson, in *Proceedings of the Ninth Latin American School of Physics*, Santiago, Chile, 1967, edited by I. Saavedra (Benjamin, New York, 1968).

[6]H. Pagels, Phys. Rev. Lett. 28, 1482 (1972).

[7]H. Pagels, Phys. Rev. D 7, 3689 (1973).

[8]M. Gell-Mann and M. Lévy, Nuovo Cimento 26, 53 (1960).

[9]S. Weinberg, Phys. Rev. D 8, 605 (1973).

[10]P. Carruthers and R. W. Haymaker, Phys. Rev. Lett. 27, 455 (1971); Phys. Rev. D 4, 406 (1971); 4, 1808 (1971).

[11]R. J. Crewther, S.-S. Shei, and T.-M. Yan, Phys.

Rev. Lett. 31, 854 (1973); Phys. Rev. D 8, 3396 (1973); see also R. Köberle and N. K. Nielson, Phys. Rev. D 8, 660 (1973).
[12]J. S. Bell and R. Jackiw, Nuovo Cimento 60A, 47 (1969).
[13]S. L. Adler, Phys. Rev. 177, 2426 (1969).
[14]D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1346 (1973); Phys. Rev. D 8, 3633 (1973); 9, 980 (1974); H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973); G. t' Hooft (unpublished).
[15]H. Fritzsch and M. Gell-Mann, in *Proceedings of the Conference on High Energy Physics, Chicago-Batavia, Ill., 1972,* edited by J. D. Jackson and A. Roberts (NAL, Batavia, Ill., 1973), Vol. 2, p. 135.
[16]R. Jackiw and K. Johnson, Phys. Rev. D 8, 2386 (1973); see also the lectures by R. Jackiw delivered at the International School of Subnuclear Physics, Erice, 1973 (unpublished).
[17]J. M. Cornwall and R. E. Norton, Phys. Rev. D 8, 3338 (1973).
[18]A. D. Maris, V. E. Herscovitz, and G. Jacob, Phys. Rev. Lett. 12, 313 (1964).
[19]G. Preparata and W. I. Weisberger, Phys. Rev. 175, 1965 (1968), Appendix C; see also Ref. 13.
[20]C. G. Callan, Phys. Rev. D 2, 1541 (1970); K. Symanzik, Commun. Math. Phys. 18, 227 (1970).

PHYSICAL REVIEW D VOLUME 9, NUMBER 12 15 JUNE 1974

Simple construction of the physical-state projection operators in dual models*

P. Ramond
Physics Department, Yale University, New Haven, Connecticut 06520
(Received 21 December 1973)

We present a simple way of understanding the Brink-Olive projection operators by relating them to the light-cone Hamiltonian.

The purpose of this short technical note is to present an alternate method for building the projection operators recently introduced by Brink and Olive.[1] This method has the advantage of being purely algebraic in nature and it will hopefully shed some light on the simple structures that underlie the projection operators. We first remark that the Hamiltonian that corresponds to time translations in light-cone coordinates is given by

$$H \equiv P^+ \equiv \frac{1}{\sqrt{2}}(p^0 + p^3)\,, \tag{1}$$

where p^0 and p^3 are the momentum components in the time and z directions, respectively.

We start by considering the Veneziano model. In this model, all quantities of interest can be derived from a generalized four-velocity vector $U_\gamma(\tau)$ which obeys a proper-time constraint[2]

$$U_\gamma(\tau)U^\gamma(\tau) = -1\,. \tag{2}$$

Note that the sign on the right-hand side indicates the tachyonic character of the constraint. One can show that the use of (2) leads to the consistent construction of a Poincaré group only where there are 26 dimensions.

On the other hand, by use of a useful principle introduced earlier,[3] the physical momentum p_γ is given by

$$p_\gamma = \frac{1}{\sqrt{\alpha'}}\langle U_\gamma\rangle\,, \tag{3}$$

where the angular brackets denote the average over the proper time τ. Here α' is a constant with dimensions of (length)^2. It corresponds to the slope of the Regge trajectory.

What we are going to do is to compute the light-cone Hamiltonian (1) using the constraint (2), which we call H_c. We have

$$H_c = \frac{1}{\sqrt{\alpha'}}\langle U_c^+\rangle\,. \tag{4}$$

The proper-time constraint in light-cone coordinates reads

$$2U^+U^- - \vec{U}_T\cdot\vec{U}_T = -1 \tag{5}$$

or

$$U^+(\tau) = \frac{1}{2}\frac{1}{U^-(\tau)}(-1 + \vec{U}_T\cdot\vec{U}_T)\,, \tag{6}$$

so that

$$H_c = \frac{1}{2\sqrt{\alpha'}}\left\langle\frac{1}{U^-(\tau)}(-1 + \vec{U}_T\cdot\vec{U}_T)\right\rangle\,. \tag{7}$$

It is convenient to introduce the longitudinal and transverse Virasoro operators through