Keeping only terms of orders ϵ^{-1} and ϵ^{0}, and integrating over all directions of the vector n^α with the help of the integral relations $\int n^\alpha d\Omega = 0$ and $(4\pi)^{-1}\int n^\alpha n^\beta d\Omega = \frac{1}{3}(g^{\alpha\beta} + c^{-2}u^\alpha u^\beta)$, we find for the bound momentum [see also formula (3.20) of Ref. 1]

$$P_B^\alpha(\tau) = \lim_{\epsilon \to 0} \frac{q^2}{8\pi c^2 \epsilon} u^\alpha(\tau) - \frac{q^2}{6\pi c^3} a^\alpha(\tau). \tag{19}$$

As stressed by Teitelboim, the final result (19) only depends on the state of the charge at the present time, although the whole past history of the charge contributes to the integral (8). Principally, this is due to the mathematical property (9), which seems to be quite general. For example, a relation like this can also be proved for particles which carry multipole moments of arbitrary order.[8] This makes it possible to extend, in a unique way, the definition of the bound momentum for a charge to particles with a more complicated electromagnetic structure.

*Publication assisted by the Stichting voor Fundamenteel Onderzoek der Materie (F.O.M.), which is financially supported by the Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek (Z.W.O.), The Netherlands.

[1]C. Teitelboim, Phys. Rev. D 1, 1572 (1970).
[2]F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965).
[3]A. Schild, J. Math. Anal. Appl. 1, 127 (1960).
[4]M. H. L. Pryce, Proc. R. Soc. Lond. A168, 389 (1938).
[5]Harish-Chandra, Proc. R. Soc. Lond. A183, 142 (1944). See formula (91).
[6]J. Aharoni, *The Special Theory of Relativity* (Clarendon, Oxford, 1965), p. 109.
[7]P. A. M. Dirac, Proc. R. Soc. Lond. A167, 148 (1938).
[8]Ch. G. van Weert, Physica (to be published).

PHYSICAL REVIEW D VOLUME 9, NUMBER 2 15 JANUARY 1974

Adiabatic regularization of the energy-momentum tensor of a quantized field in homogeneous spaces*

Leonard Parker and S. A. Fulling
Physics Department, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201
(Received 9 October 1973)

In the theory of a quantized scalar field interacting with the classical Einstein gravitational field, the formal expression for the energy-momentum tensor has infinite expectation values. We propose a procedure for defining, in certain cosmological models, suitable finite expectation values of this tensor, when the mass of the scalar matter field does not vanish. Our method uses the decomposition of the scalar field into modes permitted by the symmetry of the models. The identification of the divergent terms, which are to be subtracted mode by mode from the formal tensor, follows in a natural manner from the identification of physically relevant creation and annihilation operators under conditions of arbitrarily slow (adiabatic) time dependence of the metric. The extension of the results to periods of strong time dependence is accomplished with the aid of the requirement that the four-divergence of the regularized energy-momentum tensor remain zero at all times. The energy-momentum tensor obtained by adiabatic regularization is the same as that obtained by the n-wave regularization procedure of Zel'dovich and Starobinsky, although the two methods are conceptually quite different. In this paper we apply the adiabatic-regularization method to the minimally coupled scalar field with positive mass in the Robertson-Walker universes. Later papers will concern extensions to conformal coupling, anisotropic metrics, and massless fields, as well as a possible physical interpretation of the regularization procedure in terms of renormalization of coupling constants in Einstein's equation.

I. INTRODUCTION

In theories involving quantized fields, such as quantum electrodynamics, the formal expressions for the observables of the theory often possess infinite expectation values. Methods for obtaining suitable finite observables from the formal expressions fall into two main categories. One is regularization, in which the divergent quantities are replaced by well-defined expressions in a manner consistent with the physical basis of the theory. The other is renormalization, in which the infinities are absorbed into the physical constants, such as charge and mass, or are canceled by suitable counterterms in the Lagrangian.

Divergent formal expressions arise at the most fundamental level in the theory of a quantized matter field interacting with a classical gravitational field through Einstein's equation. In such a theory, expectation values of the energy-momentum tensor of the quantized field, which act as the source of the gravitational field, are formally divergent.[1] In this paper we propose a procedure, called adiabatic regularization, for defining suitable expectation values in the context of certain homogeneous cosmological models. The essential point is the identification of the contributions of the vacuum, which are subtracted to obtain a finite result.

Utiyama and DeWitt[2] have given a renormalization prescription which applies to asymptotically Minkowskian spacetimes, but has not yet been extended to the non-Minkowskian boundary conditions which appear in most cosmological problems. In the context of the general homogeneous, nonrotating, spatially flat metric, Zel'dovich and Starobinsky[3] have offered a method ("n-wave regularization") for defining a finite tensor by subtracting the leading terms in an asymptotic expansion of the energy-momentum tensor of particles whose mass and momentum approach infinity.

In our adiabatic regularization procedure, the vacuum energy density and pressure are determined by considering the limit of an arbitrarily slow time dependence of the metric, in which limit a time-independent physical vacuum state can be defined. The resulting expression for the regularized energy-momentum tensor is extended to the case of arbitrarily strong time dependence of the metric by means of the requirement that the four-divergence of the tensor vanish. The method should be applicable to any metric of sufficient symmetry to allow a decomposition of the quantized field into modes. It should also apply to fields of any spin. As we show in Sec. V, our regularized tensor is the same as the one which would be obtained by the method of Zel'dovich and Starobinsky, although the concepts involved in adiabatic regularization are quite different, and perhaps of more direct physical significance.

In the well-known theory of a free field in Minkowski space, the infinities disappear when the vacuum expectation value of the energy-momentum tensor is subtracted mode by mode from the full formal expression for the tensor,[4] the physical justification being that such vacuum contributions are unobservable. For a time-dependent metric, gravitationally induced particle creation makes the correspondence between physical particles and creation or annihilation operators time-dependent and inherently ill defined,[5] so that it is not at all clear how to determine the vacuum terms to be subtracted. The seemingly natural procedure of using the creation and annihilation operators which instantaneously diagonalize the Hamiltonian to define an instantaneous vacuum[6] does not succeed in preventing expectation values of such quantities as particle density and energy density from becoming infinite in the course of the time development.[7]

The present work develops ideas propounded some time ago by one of us,[8] and applies them to the problem of identifying the correct vacuum subtractions involved in regularization. The basic assumption is the minimization postulate, which leads to the conclusion that at any given time during an adiabatic change of the metric the density of newly created physical particles vanishes faster than any finite power of a slowness parameter when that parameter approaches zero. The implementation of this postulate in the form of Eq. (3.4) determines the physical Hilbert space of state vectors even for a rapidly changing metric, provided the variation is smooth. It also determines the form of the vacuum subtractions, by specifying the annihilation operators which correspond to physical particles in the adiabatic limit of arbitrarily slow time dependence of the metric. Those operators are obtained in Sec. III with the aid of a higher-order WKB approximation, and are used in Sec. IV to find the correct vacuum subtractions and the regularized energy-momentum tensor. The adiabatic operators do not diagonalize the Hamiltonian during a period of finite time dependence, because the time variation of the metric induces a shift in the effective frequency or energy of each mode.

In this paper we carry out adiabatic regularization in detail for the three types of Robertson-Walker metric (closed, flat, and open) with a quantized scalar field of nonzero mass minimally coupled to the gravitational field [Eqs. (2.6) and (2.7)]. A later paper[9] will treat the modified coupling which makes the scalar field equation conformally invariant when the mass is zero. (In the conformal case, no infrared-divergence problem arises in the Robertson-Walker metrics when the mass vanishes.) The method will also be applied in that paper to the general homogeneous and nonrotating (but anisotropic) universe with flat three-space (cf. Ref. 3). We have found that our regularization procedure can be given a renormalization interpretation (cf. Ref. 2) in certain cases, such as the Robertson-Walker universe with flat three-space.[10] Also under investigation is the possibility of extending adiabatic regularization to more general homogeneous metrics, such as the mixmaster universe, for which the field decomposes not into single modes, but into finite multiplets of coupled modes.[11]

Throughout this paper we use units such that $\hbar = c = 1$. The summation convention is in force over pairs of Greek (spacetime) indices, but not over Latin (three-space) indices. We use the metric signature (+ − − −) and the conventions $R^{\alpha}_{\ \beta\gamma\delta} = \left\{ {\alpha \atop \beta\gamma} \right\}_{,\delta} - \cdots$ and $R_{\mu\nu} = R^{\alpha}_{\ \mu\alpha\nu}$.

II. FIELD EQUATIONS AND ENERGY-MOMENTUM TENSOR

For a Robertson-Walker metric[12]

$$ds^2 = dt^2 - a(t)^2 \sum h_{jk}\, dx^j dx^k, \tag{2.1}$$

the Einstein equation

$$R_\mu{}^\nu - \tfrac{1}{2} g_\mu{}^\nu R + \Lambda g_\mu{}^\nu = -8\pi G \langle T_\mu{}^\nu \rangle \tag{2.2}$$

takes the form

$$\left(\frac{\dot a}{a}\right)^2 + \frac{\epsilon}{a^2} - \frac{1}{3}\Lambda = \frac{8\pi G}{3}\rho, \tag{2.3}$$

$$2\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 + \frac{\epsilon}{a^2} - \Lambda = -8\pi G P, \tag{2.4}$$

where ϵ is equal to +1, 0, or −1, for the case of the closed, flat, or open (hyperbolic) three-space, respectively. The cosmological constant Λ is usually set equal to zero. In a semiclassical theory the energy density ρ and pressure P are the expectation values

$$\rho = \langle T_0{}^0 \rangle, \quad P\delta_i{}^j = -\langle T_i{}^j \rangle \tag{2.5}$$

in a pure or mixed state of the proper symmetry. As explained in Sec. I, the expressions suggested by canonical field theory for ρ and P are divergent and must be replaced in a consistent theory by finite regularized expressions, the specification of which is our goal.

We consider the model in which the matter in the universe is represented by a neutral scalar field $\phi(x, t)$ $[x = (x^1, x^2, x^3),\ t = x^0]$ characterized by the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}(-g)^{1/2}(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - m^2\phi^2). \tag{2.6}$$

The corresponding field equation is

$$(\nabla_\mu \nabla^\mu + m^2)\phi = 0, \tag{2.7}$$

where ∇_μ denotes the covariant derivative. We assume that m^2 is nonzero. The conformal coupling of the scalar field, in which m^2 is replaced by $m^2 + \frac{1}{6}R$, R being the four-dimensional scalar curvature, will be discussed in a sequel to this paper (Ref. 9).

The general Hermitian-operator solution of the field equation can be written as a sum or integral over modes in the form

$$\phi(x, t) = \int d\tilde\mu(\underline{k})[A_{\underline{k}} \mathcal{Y}_{\underline{k}}(x)\psi_k(t) + \text{H.c.}], \tag{2.8}$$

where H.c. denotes the Hermitian conjugate and $\mathcal{Y}_{\underline{k}}$ is an eigenfunction of the covariant Laplacian associated with h_{jk}:

$${}^{(3)}\Delta \mathcal{Y}_{\underline{k}}(x) = -k^2 \mathcal{Y}_{\underline{k}}(x). \tag{2.9}$$

These functions and the associated indices $\underline{k}$, for the three values of ϵ, are described in Appendix A; note particularly Eqs. (A8), which give the numbers k in Eq. (2.9), and Eq. (A9), which defines the measure $\tilde\mu(\underline{k})$.

The canonical commutators of the field and its conjugate momentum π lead in the usual way to the commutation relations

$$[A_{\underline{k}}, A_{\underline{k}'}] = 0, \quad [A_{\underline{k}}, A^\dagger_{\underline{k}'}] = \delta(\underline{k}, \underline{k}'), \tag{2.10}$$

provided that Eq. (2.14) below is satisfied. We denote by $|0_A\rangle$ a normalized vector which is annihilated by all the $A_{\underline{k}}$. A basis for a Fock space may be built up by operating on $|0_A\rangle$ with the $A^\dagger_{\underline{k}}$.

The time-dependent function $\psi_k(t)$ appearing in Eq. (2.8) satisfies the equation

$$\partial_\tau{}^2\psi_k + a^6\omega_k{}^2\psi_k = 0, \tag{2.11}$$

where

$$\omega_k = \left(\frac{k^2}{a^2} + m^2\right)^{1/2} \tag{2.12}$$

and

$$\tau = \int^t a^{-3}\,dt', \quad \partial_\tau = \frac{d}{d\tau} = a^3\partial_0. \tag{2.13}$$

The equivalence of the relations (2.10) with the canonical commutation relations of ϕ and π requires a particular value for the conserved Wronskian:

$$\psi_k \partial_\tau \psi_k^* - \psi_k^* \partial_\tau \psi_k = i. \tag{2.14}$$

Additional conditions are needed to determine ψ_k completely. Different choices of ψ_k satisfying Eqs. (2.11) and (2.14) determine different operators $A_{\underline{k}}$ and vectors $|0_A\rangle$. The $A_{\underline{k}}$ do not depend on time. In general, no choice of the $A_{\underline{k}}$ corresponds to the annihilation operators of physical particles, since the particle number is not constant in time-dependent gravitational fields. In Sec. III we will restrict the solution ψ_k (and hence $A_{\underline{k}}$ and $|0_A\rangle$) by requiring that ψ_k be a generalized positive-frequency solution [see Eq. (3.4)] to any finite order in a slowness parameter T^{-1}. Although the corresponding $A_{\underline{k}}$ cannot be associated with definite observable particles when the metric is rapidly changing, they do correspond to physical particles in the adiabatic limit ($T\to\infty$), as well as in the limit of large k (with T fixed). We will often refer to the $A_{\underline{k}}$ specified in that way as adiabatic particle operators.

The energy-momentum tensor $T_{\mu\nu}$ [$=2(-g)^{-1/2}\times\delta\mathcal{L}/\delta g^{\mu\nu}$] has the diagonal mixed components

$$T_0{}^0=\tfrac{1}{2}\left[(\partial_0\phi)^2+a^{-2}\sum_{i=1}^{3}(\partial_i\phi)^2+m^2\phi^2\right], \qquad (2.15)$$

$$T_j{}^j=-\tfrac{1}{2}\left[(\partial_0\phi)^2+2a^{-2}(\partial_j\phi)^2 -a^{-2}\sum_{i=1}^{3}(\partial_i\phi)^2-m^2\phi^2\right] \qquad (2.16)$$

(no sum on j). We take the expectation values of these quantities in a homogeneous and isotropic state, pure or mixed.[13] For the state $|0_A\rangle$, one finds for the energy density

$$\rho_0=\langle 0_A|T_0{}^0|0_A\rangle =(4\pi^2)^{-1}\int d\mu(k)(|\partial_0\psi_k|^2+\omega_k{}^2|\psi_k|^2), \qquad (2.17)$$

where

$$\int d\mu(k)=\begin{cases}\displaystyle\int_0^\infty dk\,k^2 & \text{for } \epsilon=0,\\ \displaystyle\sum_l (l+1)^2 & \text{for } \epsilon=1,\\ \displaystyle\int_0^\infty dq\,q^2 & \text{for } \epsilon=-1\end{cases} \qquad (2.18)$$

(see Appendix A). For a more general homogeneous and isotropic state one finds

$$\rho=\rho_0+(4\pi^2)^{-1}\int d\mu(k) \times\left(2\langle A_{\underline{k}}^\dagger A_{\underline{k}}\rangle(|\partial_0\psi_k|^2+\omega_k{}^2|\psi_k|^2) +2\,\mathrm{Re}\{\langle A_{\underline{k}}A_{-\underline{k}}\rangle[(\partial_0\psi_k)^2+\omega_k{}^2\psi_k{}^2]\}\right), \qquad (2.19)$$

where Re denotes the real part. Equation (A12) has been used here. The matrix elements $\langle A_{\underline{k}}^\dagger A_{\underline{k}}\rangle$ and $\langle A_{\underline{k}}A_{-\underline{k}}\rangle$ must depend only on the scalar k, because of the assumed symmetry of the state. The analogous expressions for the pressure are

$$P_0=-\langle 0_A|T_1{}^1|0_A\rangle =(4\pi^2)^{-1}\int d\mu(k)\left[|\partial_0\psi_k|^2-\left(\frac{k^2}{3a^2}+m^2\right)|\psi_k|^2\right] \qquad (2.20)$$

and

$$P=P_0+(4\pi^2)^{-1}\int d\mu(k)\left(2\langle A_{\underline{k}}^\dagger A_{\underline{k}}\rangle\left[|\partial_0\psi_k|^2-\left(\frac{k^2}{3a^2}+m^2\right)|\psi_k|^2\right]+2\,\mathrm{Re}\left\{\langle A_{\underline{k}}A_{-\underline{k}}\rangle\left[(\partial_0\psi_k)^2-\left(\frac{k^2}{3a^2}+m^2\right)\psi_k{}^2\right]\right\}\right). \qquad (2.21)$$

When $\epsilon=0$ or $\epsilon=-1$, the calculations leading to Eqs. (2.19) and (2.21) are more subtle than for the closed universe. Because of the infinite volume of the universe, a homogeneous state $|\Psi\rangle$ of nonvanishing energy and pressure necessarily represents infinitely many particles. Such a state[14] cannot belong to the Fock space of the operators $A_{\underline{k}}$, and the expectation values $\langle\Psi|A_{\underline{k}}^\dagger A_{\underline{k}}|\Psi\rangle$, etc., are generally infinite. Equations (2.19) and (2.21) remain valid, however, if $\langle A_{\underline{k}}^\dagger A_{\underline{k}}\rangle$ is interpreted in the static limit as the expectation value of the number of particles per unit volume in $\underline{k}$ space and per $(2\pi)^3$ units of coordinate volume [physical volume $(2\pi a)^3$] in x space, and if $\langle A_{\underline{k}}A_{-\underline{k}}\rangle$ is similarly "renormalized."[15] The correctness of this normalization can be seen by examining the $\langle A_{\underline{k}}^\dagger A_{\underline{k}}\rangle$ term in ρ [Eq. (2.19)] for the case of a static universe, with

$$\psi_k(t)=(2a^3\omega_k)^{-1/2}e^{-i\omega_k t}. \qquad (2.22)$$

We remark, incidentally, that for the purposes of a semiclassical calculation where the main interest is in the solution $a(t)$ for the metric, a state may be taken to be defined by its quadratic expectation values $\langle A_{\underline{k}}^\dagger A_{\underline{k}}\rangle$ and $\langle A_{\underline{k}}A_{-\underline{k}}\rangle$.

Since the density of physical particles is presumably finite at all times, it is reasonable to assume, after the $A_{\underline{k}}$ have been properly chosen, that for any physically realizable state the expectation values in the second terms of Eqs. (2.19) and (2.21) approach zero sufficiently rapidly at large k that the integrals are convergent. The terms ρ_0 and P_0 arise as a consequence of the commutators (2.10). They are generally not the energy density and pressure associated with a physical vacuum, since the $A_{\underline{k}}$ do not correspond to physical particles except in the adiabatic limit. Therefore, although ρ_0 and P_0 contain all the divergences, they cannot be simply subtracted as in a free-field theory in Minkowski space. In what follows we shall argue that the form of the correct vacuum subtraction[16] is determined by the first three terms in the adiabatic asymptotic expansions of the integrands of Eqs. (2.17) and (2.20), the ψ_k being those associated with the adiabatic particle operators.

III. PHYSICAL PARTICLES IN AN ADIABATICALLY CHANGING METRIC

An analysis of the concept of particle number in an expanding universe has been given in Ref. 5. Although the discussion was in the context of the Robertson-Walker metric with Euclidean three-space, it also is applicable to the closed and open cases. The physical particle number is, in principle, not precisely defined when expansion-induced particle creation is significant. The minimization postulate [Ref. 5(b)] was proposed as a means of reaching the best definition of the physically relevant creation and annihilation operators, to within the uncertainties intrinsic to the measurement process. According to the minimization postulate, particle operators for each mode should be chosen, subject to constraints which must be determined by physical reasoning,[17] to minimize the time rate of change of the average particle number and its derivatives.[18] The identification of physical particles becomes exact only in the limit of an adiabatic (slow) expansion, or for sufficiently high mode number k.

The minimization postulate was implemented (although not given that name) in Ref. 5(a) in order to put an upper bound on the creation rate for pions, electrons, and protons in the present expanding universe.[19] To carry out that program a higher-order WKB approximation to the time dependence of the scalar field was developed through an iterative procedure called successive adiabatic approximation.[20] Essentially the same extended WKB approximation for solutions of equations like (2.11) has recently been given in explicit form to all orders by Chakraborty.[21] This approach forms the basis of our present discussion.

To talk about the adiabatic limit with mathematical precision, we generalize Eq. (2.11) by replacing $a(\tau)$ in the $a^6(k^2/a^2+m^2)$ term by $a(\tau/T)$. In the limit of large T the quantity $a(\tau/T)$ and its derivatives will necessarily be slowly varying functions of τ. [We are assuming that $a(\tau)$ is a smooth (C^∞) function.] By changing to the variable $\tau_1=\tau/T$ in the parametrized equation [cf. Eq. (3.1)], one easily sees that an equivalent way of introducing the large parameter T into Eq. (2.11) is by associating a factor of T^{-1} with each derivative ∂_τ. Therefore, in an asymptotic expansion in inverse powers of T (τ_1 being regarded as independent of T), each factor of T^{-1} will be associated with a time derivative. The adiabatic expansion of the solution of the original equation (2.11) is obtained by setting $T=1$ in the more general expansion. The order of each term may be identified after the parameter T is suppressed, by recalling that each derivative ∂_τ is associated with a factor of T^{-1}. We will often speak of expansion in powers of T^{-1}, and of the adiabatic limit $T\to\infty$, without introducing T explicitly.

The particle number is an adiabatic invariant, remaining constant in the limit of an infinitely slow expansion. [Note that passing to the adiabatic limit is not the same thing as specializing to a static metric; $a(\tau/T)$ never becomes a constant function globally.] In this limit, therefore, particle number ought to have a definite physical meaning. More precisely, if two static epochs are separated by a period of adiabatic expansion, the density of particles created by the expansion (which is well defined) approaches zero faster than any power of T^{-1} in the limit that T approaches infinity.[22] This behavior permits us to define physically acceptable particle operators during the expansion, up to any finite order of T^{-1}.

From the parametrized equation in the form

$$\frac{d^2}{d\tau_1^2}\psi_k+T^2a(\tau_1)^6\left[\frac{k^2}{a(\tau_1)^2}+m^2\right]\psi_k=0\,, \tag{3.1}$$

it is clear that the problem of expanding ψ_k in the limit of large T is mathematically equivalent to that of finding an asymptotic expansion in the limit of large k and m. Such an expansion will remain valid for $k\to\infty$ with m fixed. For a given mode $\underline{k}$, an asymptotic expansion in powers of T^{-1} will be valid if $T\omega_k$ is large with respect to unity.[23] Each factor of T^{-1} in the terms of such an asymptotic expansion will be associated with a factor of k^{-1} in the limit of large k. This feature ensures the convergence of the integrals defining the regularized energy density and pressure, which will be obtained by subtracting the three leading terms in the adiabatic expansions of the integrands of ρ_0 and P_0 [see Eqs. (4.16) and (4.20)].

On the other hand, since the mass m is assumed nonzero, there is a lower bound on the mode energies $(k^2/a^2+m^2)^{1/2}$. Hence as $T\to\infty$ the adiabatic expansions are uniformly valid for all the modes at once. This fact will also play an important role in the argument of Sec. IV.

We turn now to the actual construction of adiabatic approximations to the solutions of Eq. (2.11).[24] The general solution can be written to any given finite order in T^{-1} as a superposition of positive- and negative-frequency generalized WKB functions:

$$\psi_k = (2a^3 W_k)^{-1/2}\left[\alpha_k \exp\left(-i\int^t W_k\, dt'\right) + \beta_k \exp\left(i\int^t W_k\, dt'\right)\right], \quad (3.2)$$

where the real function W_k approaches ω_k in the adiabatic limit and is chosen to make α_k and β_k constant to the desired order in T^{-1}. (The detailed construction of W_k is described in Sec. IV.) The Wronskian condition (2.14) requires that

$$|\alpha_k|^2 - |\beta_k|^2 = 1 \quad (3.3)$$

to that order of approximation. [We have reverted to the variable t through Eq. (2.13).]

We argue now that, for a sufficiently slow and smooth expansion, the operators $A_{\underline{k}}$ in Eq. (2.8) correspond to physical particles when ψ_k is chosen to be a solution which is approximated to sufficient order, say T^{-2n}, by the expression (3.2) with $\alpha_k = 1$ and $\beta_k = 0$:

$$\psi_k = (2a^3 W_k)^{-1/2} \exp\left(-i\int^t W_k\, dt'\right) + O(T^{-2n-1}). \quad (3.4)$$

More precisely, the identification of $A_{\underline{k}}$ with physical particles is good to the order $2n$ of the approximation used in Eq. (3.4), and it becomes exact in the adiabatic limit $T\rightarrow\infty$ as well as in the limit $k\rightarrow\infty$ with T fixed. It can be shown that for the closed universe the Hilbert spaces constructed using the above $A_{\underline{k}}$ for any values of $n \geq 1$ are the same (i.e., carry unitarily equivalent representations). If Eq. (3.4) holds at one time to a given order, it will be satisfied to that order at all times provided $a(\tau)$ is a smooth function.

The reason for imposing Eq. (3.4) can be explained by considering a smooth expansion possessing static periods. During any time interval in which $a(\tau)$ is constant, the particle interpretation of the field theory is clear[25]; W_k reduces to ω_k, and the $A_{\underline{k}}$ in Eq. (3.2) correspond to physical particles if β_k is chosen to vanish. During a later static interval, as a result of the change in $a(\tau)$, the quantity $|\beta_k|^2$, which is a measure of the number of created particles in mode $\underline{k}$ [see, e.g., Ref. 5(b)], will no longer vanish. However, it will decrease faster than any power of the adiabatic parameter T^{-1}, so that $|\beta_k|^2 \equiv 0$ is a valid asymptotic representation of it. Consequently, the $A_{\underline{k}}$ corresponding to the choice of ψ_k in Eq. (3.4) accurately represent physical particles to order $2n$ in T^{-1} during any static interval (i.e., there is no particle creation to that order). The minimization postulate then suggests strongly that the $A_{\underline{k}}$ represent the physical particles also at times when the metric is changing, up to the order $2n$ specified in Eq. (3.4). Finally, the same must then be true for a metric without static periods. Since m is nonzero, these considerations are valid for all modes in the limit $T\rightarrow\infty$, and they apply in any case for large k. Support for the validity of these ideas is given by their successful application to the regularization of the energy-momentum tensor (Sec. IV).

The time independence of the $A_{\underline{k}}$ does not contradict the phenomenon of particle creation, since the creation rate, to the extent that it is well defined, falls off faster than any order of T^{-1}, whereas the identification of $A_{\underline{k}}$ with the physical particles is valid only to finite orders.

As noted earlier, if the arbitrariness in the $A_{\underline{k}}$ is restricted by imposing Eq. (3.4) to order T^{-2} or higher, the Fock space generated from $|0_A\rangle$ by the $A_{\underline{k}}^\dagger$ (i.e., spanned by eigenstates of the number operators $A_{\underline{k}}^\dagger A_{\underline{k}}$ with a finite total number of particles) is uniquely determined[26] in the case $\epsilon = 1$. This Fock space is (the closure of) the space of physically realizable states. The assumption that $\rho-\rho_0$ and $P-P_0$ in Eqs. (2.19) and (2.21) are finite now emerges as a restriction which may be imposed consistently at all times on the number of physical particles present in the high-energy modes. (The last statement is true also for the flat and open universes.[27])

The minimization postulate evidently implies that Eq. (3.4) should be imposed on the solution ψ_k to all orders n, because then the $A_{\underline{k}}$ yield no particle creation to any finite order in T^{-1}. (However, to carry out the regularization procedure it is sufficient to require that condition only to order T^{-4}.) Note that solutions ψ_k exist satisfying Eq. (3.4) to all orders, even though for any finite T the sequence of functions W_k corresponding to increasing values of n does not generally converge. One considers that solution $\psi_k^{(T)}(\tau_1)$ of the parametrized equation (3.1) which is to reduce to $\psi_k(\tau)$ when $T=1$, and demands that its asymptotic expansion in powers of T^{-1} agree with that of Eq. (3.4) to each order. (The explicit expression for W_k is given to every order in Ref. 21.) These considerations refer to all times τ, not just periods when $a(\tau)$ is slowly varying [provided only that $a(\tau)$ is a smooth function]. The remaining arbitrariness in the particular solution ψ_k (namely, the part of ψ_k which vanishes faster than any power of T^{-1}) is connected with the creation of real particles. Complete specification of ψ_k would require a physically motivated initial condition, consistent with Eq. (3.4) and with the particular circumstances of the problem under study.

IV. IDENTIFICATION OF VACUUM TERMS AND DEFINITION OF REGULARIZED ENERGY DENSITY AND PRESSURE

In accordance with the discussion in Sec. III, we can assume that the particular solution ψ_k which appears in Eq. (2.8) satisfies Eq. (3.4) with n at least as large as 2. Making use of Eq. (3.4) to evaluate the contribution of each mode to ρ_0 and P_0 in Eqs. (2.17) and (2.20), we find

$$\rho_0 = (4\pi^2)^{-1} \int d\mu(k)\rho_0(k), \qquad (4.1)$$

with

$$\begin{aligned}\rho_0(k) &= |\partial_0\psi_k|^2 + \omega_k^2|\psi_k|^2 \\ &= (2a^3W_k)^{-1}\{\omega_k^2 + W_k^2 + \tfrac{1}{4}[\partial_0 \ln(a^3W_k)]^2\} \\ &\quad + O(T^{-2n-2}), \end{aligned} \qquad (4.2)$$

and

$$P_0 = (4\pi^2)^{-1}\int d\mu(k)P_0(k), \qquad (4.3)$$

with

$$\begin{aligned}P_0(k) &= |\partial_0\psi_k|^2 - \left(\frac{k^2}{3a^2} + m^2\right)|\psi_k|^2 \\ &= (2a^3W_k)^{-1}\{-\tfrac{1}{3}\omega_k^2 - \tfrac{2}{3}m^2 + W_k^2 \\ &\quad + \tfrac{1}{4}[\partial_0\ln(a^3W_k)]^2\} + O(T^{-2n-2}). \end{aligned} \qquad (4.4)$$

We wish to obtain the leading terms in the asymptotic series for ρ_0 and P_0 in powers of T^{-2}, since we want to study the adiabatic limit $T\to\infty$, and since the divergences in ρ and P are concentrated in those terms up to order T^{-4}. To calculate these terms we may use the generalized WKB approximation good to order T^{-4}, which is obtained (Ref. 21) by letting

$$W = \omega(1+\epsilon_2)^{1/2}(1+\epsilon_4)^{1/2}, \qquad (4.5)$$

where[28]

$$\epsilon_2 = -(a^3\omega)^{-3/2}\partial_\tau\{(a^3\omega)^{-1}\partial_\tau[(a^3\omega)^{1/2}]\}, \qquad (4.6)$$

$$\begin{aligned}\epsilon_4 &= -(a^3\omega)^{-1}(1+\epsilon_2)^{-3/4} \\ &\quad \times \partial_\tau\{(a^3\omega)^{-1}(1+\epsilon_2)^{-1/2}\partial_\tau[(1+\epsilon_2)^{1/4}]\}. \end{aligned} \qquad (4.7)$$

We have dropped the subscript k for convenience. We refer to ϵ_2 and ϵ_4 as the adiabatic frequency corrections. Note that ϵ_2 is of order T^{-2} and that ϵ_4 contains terms of order T^{-4} and higher. The terms of higher order than T^{-4} make no contribution to the divergent integrals. We will write $\epsilon_{4(4)}$ for the terms in ϵ_4 of order T^{-4} precisely. Explicit expressions for ϵ_2, $\dot\epsilon_2$, $\ddot\epsilon_2$, and $\epsilon_{4(4)}$ are given in Appendix B.

Let

$$\begin{aligned} Z &= 3\frac{\dot a}{a} + \frac{\dot\omega}{\omega} \\ &= \frac{\dot a}{a}\,\omega^{-2}(2\omega^2 + m^2), \end{aligned} \qquad (4.8)$$

where the dot denotes differentiation with respect to t. Expanding in powers of T^{-2} and retaining only terms relevant to the divergent integrals, we have

$$W^{-1} \approx \omega^{-1}(1 - \tfrac{1}{2}\epsilon_2 - \tfrac{1}{2}\epsilon_{4(4)} + \tfrac{3}{8}\epsilon_2^2), \qquad (4.9)$$

$$\left[\frac{d}{dt}\ln(a^3W)\right]^2 \approx Z^2 + Z\dot\epsilon_2. \qquad (4.10)$$

Substituting into Eqs. (4.2) and (4.4), one finds to fourth order

$$\begin{aligned}\rho_0(k) &= (a^3\omega)^{-1}[\omega^2 + \tfrac{1}{8}Z^2 + \tfrac{1}{8}Z\dot\epsilon_2 - \tfrac{1}{16}Z^2\epsilon_2 \\ &\quad + \tfrac{1}{8}\omega^2\epsilon_2^2 + O(T^{-6})] \end{aligned} \qquad (4.11)$$

and

$$P_0(k) = (a^3\omega)^{-1}[\tfrac{1}{3}(k^2/a^2) + \tfrac{1}{8}Z^2 + \tfrac{1}{3}\epsilon_2(\omega^2 + \tfrac{1}{2}m^2) + \tfrac{1}{8}Z\dot\epsilon_2 - \tfrac{1}{16}Z^2\epsilon_2 + \tfrac{1}{3}\epsilon_{4(4)}(\omega^2 + \tfrac{1}{2}m^2) - \tfrac{1}{8}\epsilon_2^2(\omega^2 + m^2) + O(T^{-6})]. \qquad (4.12)$$

It is important to note that although a term like $\epsilon_2^2 m^2$ by itself leads to a convergent integral in P_0, it cannot be dropped from consideration here, because it is of order T^{-4} and there are terms of that order which do diverge. The condition of the vanishing of the covariant four-divergence $\nabla_\nu T_\mu{}^\nu$, which in a Robertson-Walker metric reduces to the equation

$$\partial_0(\rho a^3) + P\partial_0(a^3) = 0, \qquad (4.13)$$

is satisfied in the integrands of ρ_0 and P_0 for each order in T^{-1} separately, as well as for each mode. Terms of a given order should, therefore, be treated as a whole, so as to obtain after vacuum subtraction a unique regularized energy-momentum tensor which possesses this fundamental property, required for consistency of the Einstein equations.

Hence the contributions to ρ_0 and P_0 from the first three orders of T^{-2} in the adiabatic expansions of the integrands are

$$\rho_{\text{div}} = (4\pi^2)^{-1}\int d\mu(k)(a^3\omega)^{-1}(\omega^2 + \tfrac{1}{8} Z^2 + \tfrac{1}{8} Z\dot{\epsilon}_2 - \tfrac{1}{16} Z^2\epsilon_2 + \tfrac{1}{8}\omega^2\epsilon_2{}^2) \tag{4.14}$$

and

$$P_{\text{div}} = (4\pi^2)^{-1}\int d\mu(k)(a^3\omega)^{-1}[\tfrac{1}{3}(k^2/a^2) + \tfrac{1}{8} Z^2 + \tfrac{1}{3}\epsilon_2(\omega^2 + \tfrac{1}{2}m^2) + \tfrac{1}{8} Z\dot{\epsilon}_2 - \tfrac{1}{16} Z^2\epsilon_2 + \tfrac{1}{3}\epsilon_{4(4)}(\omega^2 + \tfrac{1}{2}m^2) - \tfrac{1}{8}\epsilon_2{}^2(\omega^2 + m^2)]\,. \tag{4.15}$$

Note that $\epsilon_{4(4)}$ appears only in the pressure term.

As remarked in Sec. III, the dependence of a term in the expansions (4.11) and (4.12) on k, at large k, is closely related to its order in T^{-1}. Consequently (as one may see by inspection), the integrals in Eqs. (4.14) and (4.15) of the terms of order T^0, T^{-2}, and T^{-4} diverge at the upper limit as ∞^4, ∞^2, and $\ln\infty$, respectively, while higher-order terms lead to convergent integrals. Therefore, if one considers the adiabatic limit ($T\to\infty$) of ρ and P, the terms of higher order than T^{-4} unambiguously approach zero. However, the lower-order terms diverge for each finite value of T. [This behavior occurs even though the terms of order T^{-2} and T^{-4} appearing in the integrands of Eqs. (4.14) and (4.15) vanish in the adiabatic limit for each value of k.]

In the adiabatic limit, the regularized expressions for ρ and P clearly should approach their static values. The static Robertson-Walker universe with flat three-space is simply Minkowski space, for which ρ_{reg} and P_{reg} are given by the normal-ordered expressions for ρ and P [i.e., the second terms in Eqs. (2.19) and (2.21)]. In the closed and open static universes as well, the particle interpretation appears to be clear. Thus, as a working hypothesis (see Sec. VI) we assume that the normal-ordered expressions for ρ and P give the regularized energy density and pressure whenever $a(t)$ is constant (and ψ_k is the positive-frequency solution). For the static metric, therefore, the entire quantities ρ_0 and P_0 are to be attributed to the vacuum, and subtracted. To approach those same normal-ordered expressions in the adiabatic limit, we must subtract ρ_{div} from ρ in the expression for ρ_{reg} (and similarly for P_{reg}) before letting T approach infinity. (The subtraction is to be carried out for each mode before integration.)

This identification of vacuum subtractions is consistent with the viewpoint that the operators $A_{\mathbf{k}}$ specified by the minimization postulate, in the form of Eq. (3.4), correspond to the physical particles in the adiabatic limit. The adiabatic correspondence implies more than just that $A_{\mathbf{k}}$ annihilates physical particles when $a(t)$ is static. Through Eq. (3.4) it determines what limiting expressions will be approached when T (or k) is allowed to approach infinity, and hence the form of the terms which must be attributed to the physical vacuum.

Thus, in the adiabatic limit ρ_{div} and P_{div} give the correct vacuum subtractions. These subtractions apply, in particular, during an interval when $a(t)$ is slowly varying (or, more precisely, smoothly approaches a constant). These quantities must therefore be subtracted at all times, since otherwise the condition that the regularized tensor have vanishing four-divergence [Eq. (4.13)] would be violated at the time when the form of the subtraction changed. For consistency this conclusion clearly must hold in general, even when $a(t)$ does not possess an interval of slow variation. Hence we assume that ρ_{div} and P_{div} are the correct vacuum subtractions at all times, even when $a(t)$ varies rapidly (but smoothly).[29]

We thus find for the regularized energy density

$$\rho_{\text{reg}} = (4\pi^2)^{-1}\int d\mu(k)[\rho_0(k) + \rho_1(k) - \rho_{\text{vac}}(k)], \tag{4.16}$$

where

$$\rho_0(k) = |\partial_0\psi_{\mathbf{k}}|^2 + \omega_{\mathbf{k}}{}^2|\psi_{\mathbf{k}}|^2, \tag{4.17}$$

$$\rho_1(k) = 2\langle A_{\mathbf{k}}^\dagger A_{\mathbf{k}}\rangle\rho_0(k) + 2\operatorname{Re}\{\langle A_{\mathbf{k}} A_{-\mathbf{k}}\rangle[(\partial_0\psi_{\mathbf{k}})^2 + \omega_{\mathbf{k}}{}^2\psi_{\mathbf{k}}{}^2]\}\,, \tag{4.18}$$

and

$$\rho_{\text{vac}}(k) = a^{-3}\omega^{-1}(\omega^2 + \tfrac{1}{8} Z^2 + \tfrac{1}{8} Z\dot{\epsilon}_2 - \tfrac{1}{16} Z^2\epsilon_2 + \tfrac{1}{8}\omega^2\epsilon_2{}^2)\,. \tag{4.19}$$

For the regularized pressure we have

$$P_{\text{reg}} = (4\pi^2)^{-1}\int d\mu(k)[P_0(k) + P_1(k) - P_{\text{vac}}(k)], \tag{4.20}$$

where

$$P_0(k) = |\partial_0\psi_{\mathbf{k}}|^2 - \left(\frac{k^2}{3a^2} + m^2\right)|\psi_{\mathbf{k}}|^2, \tag{4.21}$$

$$P_1(k) = 2\langle A_{\mathbf{k}}^\dagger A_{\mathbf{k}}\rangle P_0(k) + 2\operatorname{Re}\left\{\langle A_{\mathbf{k}} A_{-\mathbf{k}}\rangle\left[(\partial_0\psi_{\mathbf{k}})^2 - \left(\frac{k^2}{3a^2} + m^2\right)\psi_{\mathbf{k}}{}^2\right]\right\}, \tag{4.22}$$

and

$$P_{\text{vac}}(k)=a^{-3}\omega^{-1}\left[\tfrac{1}{3}(k^2/a^2)+\tfrac{1}{8}Z^2+\tfrac{1}{3}\epsilon_2(\omega^2+\tfrac{1}{2}m^2)+\tfrac{1}{8}Z\dot{\epsilon}_2-\tfrac{1}{16}Z^2\epsilon_2+\tfrac{1}{3}\epsilon_{4(4)}(\omega^2+\tfrac{1}{2}m^2)-\tfrac{1}{8}\epsilon_2{}^2(\omega^2+m^2)\right]. \tag{4.23}$$

These expressions apply even when the metric is strongly time-dependent, at which times $\rho_0(k)$ and $\rho_{\text{vac}}(k)$ [and similarly $P_0(k)$ and $P_{\text{vac}}(k)$] may be quite different from one another. The quantities ρ_{reg} and P_{reg} are the regularized energy density and pressure which should appear in the Einstein equations (2.3) and (2.4).

V. COMPARISON WITH THE METHOD OF ZEL'DOVICH AND STAROBINSKY

In Ref. 3 Zel'dovich and Starobinsky discuss a conformally coupled scalar field in the general homogeneous, nonrotating, spatially flat cosmological model. They employ a procedure of "n-wave regularization," which they introduce as a modification of the Pauli-Villars regularization in quantum electrodynamics.[30] The method we have developed in this paper gives the same results as theirs. (It will be applied specifically to their problem in Ref. 9.) The conceptual basis we have presented, however, is quite different. Hence it is worthwhile to point out why the two methods are mathematically equivalent.[31]

Consider the ψ_k of Eq. (3.4) (without setting $T=1$) as a function of τ_1—i.e., as a solution of Eq. (3.1). Then a, ω_k, $T^2\epsilon_2$, and $T^4\epsilon_{4(4)}$ are independent of T, as can be seen by substituting $a(\tau/T)$ for $a(t)$ in Eqs. (4.6) and (4.7), performing the differentiations, and then setting $\tau/T=\tau_1$. Then for the three leading terms in $|\psi_k|^2$, for instance, we have from Eqs. (3.4) and (4.9)

$$|\psi_k|^2\approx(2a^3\omega_k)^{-1}(1-\tfrac{1}{2}\epsilon_2-\tfrac{1}{2}\epsilon_{4(4)}+\tfrac{3}{8}\epsilon_2{}^2), \tag{5.1}$$

where $-\frac{1}{2}\epsilon_2$ is proportional to T^{-2} and the next two terms are proportional to T^{-4}. That is, $|\psi_k|^2$ has the form $\Psi_0+T^{-2}\Psi_2+T^{-4}\Psi_4+O(T^{-6})$, where the Ψ's are independent of T. Clearly,

$$\Psi_0=\lim_{T\to\infty}|\psi_k|^2,$$

$$\Psi_2=\lim_{T\to\infty}\frac{d}{d(T^{-2})}|\psi_k|^2,$$

$$\Psi_4=\frac{1}{2}\lim_{T\to\infty}\frac{d^2}{d(T^{-2})^2}|\psi_k|^2.$$

The analogous statements hold for $|\partial_0\psi_k|^2$ and hence for $\rho_0(k)$ and $P_0(k)$ as defined by Eqs. (4.2) and (4.4).

Secondly, we use the equivalence between large T and large k and m pointed out in Sec. III. Writing $\psi^{(T)}(k,m)$ to indicate explicitly the dependence of ψ_k on T and m, one easily sees that

$$|\psi^{(T)}(k,m)|^2=T|\psi(Tk,Tm)|^2\quad(\psi\equiv\psi^{(1)}),$$

where the factor T comes from the factor ω^{-1} in Eq. (5.1). Similarly, we have

$$|\partial_0\psi^{(T)}(k,m)|^2=T^{-1}|\partial_0\psi(Tk,Tm)|^2,$$

and hence, from Eqs. (4.2) and (4.4),

$$\rho_0^{(T)}(k,m)\equiv|\partial_0\psi^{(T)}(k,m)|^2+\omega_k{}^2|\psi^{(T)}(k,m)|^2=T^{-1}\rho_0(Tk,Tm),$$

$$P_0^{(T)}(k,m)=T^{-1}P_0(Tk,Tm).$$

Now define

$$T_\mu{}^{\nu(T)}(k,m)=T^{-1}T_\mu{}^\nu(Tk,Tm), \tag{5.2}$$

where

$$T_\mu{}^\nu=(4\pi^2)^{-1}\int d\mu(k)\,T_\mu{}^\nu(k,m). \tag{5.3}$$

Then, combining the considerations of the preceding two paragraphs, one obtains

$$\langle 0_A|T_\mu{}^\nu(k,m)|0_A\rangle_{\text{div}}=\lim_{T\to\infty}\left\langle 0_A\left|\left[T_\mu{}^{\nu(T)}(k,m)+\frac{d}{d(T^{-2})}T_\mu{}^{\nu(T)}(k,m)+\frac{1}{2}\frac{d^2}{d(T^{-2})^2}T_\mu{}^{\nu(T)}(k,m)\right]\right|0_A\right\rangle \tag{5.4}$$

as a formula for the divergent vacuum terms defined in Eqs. (4.14) and (4.15). This equation defines the n-wave procedure of Zel'dovich and Starobinsky. [Cf. their Eq. (21). They write n instead of T.] In Ref. 3, Eq. (5.4) was supplemented by the condition that ψ_k reduce to a positive-frequency solution during some time interval when the metric is static. That boundary condition attains the effect of our Eq. (3.4), in the special case that such a static time interval exists. The increased generality of our method, which does not assume a static interval, may be important in applications.

VI. CONCLUDING REMARK

The main purpose of this paper is to present a general approach to defining a physically acceptable energy-momentum tensor, based on the identification of physical particles in the adiabatic limit. The details of the particular regularization ansatz proposed in Sec. IV should be regarded as subject to possible refinement on the basis of further investigation. Such work is in progress along two related fronts: (1) reinterpretation of the vacuum subtractions as renormalizations of coupling constants; (2) application of the method in special cases where the spacetime curvature vanishes. Our results to date confirm the regularization prescription of Sec. IV for the Robertson-Walker universe with flat three-space. On the other hand, the preliminary evidence indicates that in the closed and open cases the method may need to be slightly modified by including in $\rho_{\rm reg}$ and $P_{\rm reg}$, even when the metric is static, a nonvanishing vacuum energy density and pressure associated with the curvature of three-space.

ACKNOWLEDGMENTS

We are grateful to the Institute for Advanced Study, Princeton, New Jersey, for hospitality while part of this research was being done.

APPENDIX A: EIGENFUNCTIONS OF THE COVARIANT LAPLACIANS

The solutions of Eq. (2.9) for Euclidean three-space ($\epsilon = 0$) are the plane waves,

$$[\epsilon=0]:\ \mathcal{Y}_{\underline{k}}(x) = (2\pi)^{-3/2} e^{i\vec{k}\cdot\vec{x}},\quad \underline{k} = \vec{k} = (k_1, k_2, k_3) \quad (-\infty < k_j < \infty). \tag{A1}$$

For $\epsilon = 1$ (closed, spherical space) two forms are in common use. If the metric is written (so as to emphasize the homogeneity of the space) as

$$[\epsilon=1]:\ \sum h_{jk}\,dx^j dx^k = d\omega^2 + \sin^2\omega\, d\alpha^2 + \cos^2\omega\, d\beta^2, \tag{A2}$$

then a convenient set of eigenfunctions consists of functions of the form[32]

$$[\epsilon=1]:\ \mathcal{Y}_{\underline{k}}(x) = d^{l/2}_{nm}(\omega)\, e^{in\alpha} e^{im\beta},\quad \underline{k} = (l, m, n)$$
$$(l = 0, 1, \ldots;\ n, m = -\tfrac{1}{2}l, -\tfrac{1}{2}l + 1, \ldots, \tfrac{1}{2}l), \tag{A3}$$

where $d^{l/2}_{nm}$ is, up to phase, a representation function of the group SU(2). On the other hand, when the isotropy of the space about one point is exhibited by writing

$$[\epsilon=1]:\ \sum h_{jk}\,dx^j dx^k = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{A4}$$

the natural eigenfunctions have the form[33]

$$[\epsilon=1]:\ \mathcal{Y}_{\underline{k}}(x) = \Pi^{(+)}_{lJ}(\chi)\, Y^M_J(\theta,\phi),\quad \underline{k} = (l, J, M)$$
$$(l = 0, 1, \ldots;\ J = 0, 1, \ldots, l;\ M = -J, -J+1, \ldots, J), \tag{A5}$$

where the Y^M_J are the usual spherical harmonics with an appropriate phase.

The metric of the open three-space (by which we always mean that of negative curvature, $\epsilon = -1$) is

$$[\epsilon=-1]:\ \sum h_{jk}\,dx^j dx^k = d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{A6}$$

The eigenfunctions[34] are obtained by replacing χ by $i\chi$ and $l+1$ by iq in the formulas for $\epsilon = +1$. However, q is now a continuous variable and J can be arbitrarily large:

$$[\epsilon=-1]:\ \mathcal{Y}_{\underline{k}}(x) = \Pi^{(-)}_{qJ}(\chi)\, Y^M_J(\theta,\phi),\quad \underline{k} = (q, J, M)$$
$$(0 < q < \infty;\ J = 0, 1, \ldots;\ M = -J, -J+1, \ldots, J). \tag{A7}$$

The functions $\Pi^{(\pm)}$ are defined up to normalization by certain differential equations with boundary conditions. Solutions are given in Refs. 33 and 34.

The eigenvalue $-k^2$ in Eq. (2.9) is determined by

$$k = \begin{cases} |\vec{k}| & \text{if } \epsilon = 0, \\ [l(l+2)]^{1/2} & \text{if } \epsilon = 1, \\ (q^2+1)^{1/2} & \text{if } \epsilon = -1. \end{cases} \tag{A8a}$$

If we let $q = l+1$ for $\epsilon = 1$ and $q = |\vec{k}|$ for $\epsilon = 0$, Eq. (A8a) takes the unified form

$$k = (q^2 - \epsilon)^{1/2} \quad (q = 1, 2, \ldots \text{ if } \epsilon = 1;\ 0 < q < \infty \text{ if } \epsilon = 0 \text{ or } -1). \tag{A8b}$$

An arbitrary function on the three-space can be expanded in a series of eigenfunctions [cf. Eq. (2.8)]. We symbolize the summation or integration thereby involved by a measure $\tilde{\mu}(\underline{k})$:

$$\int d\tilde{\mu}(\underline{k}) = \begin{cases} \int d^3k & \text{if } \epsilon = 0, \\ \sum_{l,m,n} \text{ or } \sum_{l,J,M} & \text{if } \epsilon = 1, \\ \int_0^\infty dq \sum_{J,M} & \text{if } \epsilon = -1. \end{cases} \tag{A9}$$

We shall take the functions to be normalized so that

$$\int d^3x\, h^{1/2}\mathcal{Y}_{\underline{k}}(x)\mathcal{Y}^*_{\underline{k}'}(x)=\delta(\underline{k},\underline{k}'), \tag{A10}$$

where $h(x)$ is the determinant of $\{h_{jk}\}$ and $\delta(\underline{k},\underline{k}')$ is the δ function with respect to $\tilde{\mu}$:

$$\int d\tilde{\mu}(\underline{k}')f(\underline{k}')\delta(\underline{k},\underline{k}')=f(\underline{k}). \tag{A11}$$

In quantum field theory it is convenient to choose the phases of the $\mathcal{Y}_{\underline{k}}$ so that

$$\mathcal{Y}^*_{\underline{k}}(x)=\mathcal{Y}_{-\underline{k}}(x), \tag{A12}$$

where $-\vec{k}$ has its usual meaning when $\epsilon=0$, and in the other cases $-\underline{k}$ is defined by

$$\begin{aligned}[\epsilon=1]:&\ -(l,m,n)=(l,-m,-n)\\ &\text{ or } -(l,J,M)=(l,J,-M),\\ [\epsilon=-1]:&\ -(q,J,M)=(q,J,-M).\end{aligned} \tag{A13}$$

In deriving Eqs. (2.17) and (2.19) and the analogous formulas for the pressure, one encounters an integrand which is known to depend only on k, so that $\int d\tilde{\mu}(\underline{k})$ can be replaced by an integration over k alone [Eq. (2.18)]. Although we omit the details of these tedious calculations, we must outline how some tricky questions of normalization are handled. (See also Refs. 13 and 15.) When the volume is finite ($\epsilon=1$), one can exploit the symmetry of the quantities by integrating them over the three-space and dividing by the volume; Eqs. (2.17) and (2.19) result easily, with the aid of Eq. (A10). Equation (2.17) for $\epsilon=0$ is easy, because of the explicit and x-independent value of $|\mathcal{Y}_{\underline{k}}(x)|^2$ from Eq. (A1). For $\epsilon=-1$ no such elementary method is visible, so we use an addition theorem,

$$\sum_{JM}|\Pi^{(-)}_{qJ}(\chi)\,Y^M_J(\theta,\phi)|^2=\frac{q^2}{2\pi^2}, \tag{A14}$$

which follows from Eq. (21) of the second paper of Bander and Itzykson (Ref. 34) in the limit $\theta\to 0$. The form of $d\mu(k)$ now being established [Eq. (2.18)], one obtains Eq. (2.19) for $\epsilon=0$ and $\epsilon=-1$ in the way sketched at the end of Sec. II.

APPENDIX B: CALCULATION OF THE ADIABATIC FREQUENCY CORRECTIONS

In this appendix we evaluate the quantities ϵ_2, $\dot{\epsilon}_2$, and $\epsilon_{4(4)}$ which appear in the formulas (4.19) and (4.23) for the vacuum energy and pressure.

The definition (4.6) of ϵ_2 reduces to

$$\epsilon_2=-\tfrac{1}{4}a^6M^{-2}[\ddot{M}-\tfrac{5}{4}M^{-1}\dot{M}^2+3(\dot{a}/a)\dot{M}], \tag{B1}$$

where

$$\begin{aligned}M&\equiv a^6\omega^2=k^2a^4+m^2a^6,\\ \dot{M}&=(4k^2a^3+6m^2a^5)\dot{a},\\ \ddot{M}&=(4k^2a^3+6m^2a^5)\ddot{a}+(12k^2a^2+30m^2a^4)\dot{a}^2,\end{aligned} \tag{B2}$$

and the differentiation is with respect to t. Thus we have

$$\epsilon_2=-\omega^{-4}\left(\frac{k^2}{a^2}+\frac{3}{2}m^2\right)\frac{\ddot{a}}{a} -\omega^{-6}\left(\frac{k^4}{a^4}+3\frac{k^2}{a^2}m^2+\frac{3}{4}m^4\right)\left(\frac{\dot{a}}{a}\right)^2. \tag{B3}$$

The equations

$$\begin{aligned}\epsilon_2&=-\frac{\ddot{a}}{a}\,\omega^{-2}C_1(k,t)-\left(\frac{\dot{a}}{a}\right)^2\omega^{-2}C_2(k,t)\\ &=-2\omega^{-1}S(k,t)\end{aligned} \tag{B4}$$

relate ϵ_2 to notation used in Ref. 5.

It follows that

$$\dot{\epsilon}_2=-\omega^{-4}\left(\frac{k^2}{a^2}+\frac{3}{2}m^2\right)\frac{a^{(3)}}{a}-3\omega^{-6}\left(\frac{k^4}{a^4}+\frac{5}{2}\frac{k^2}{a^2}m^2\right)\frac{\ddot{a}\dot{a}}{a^2}+3\omega^{-8}\left(3\frac{k^2}{a^2}m^4+\frac{1}{2}m^6\right)\left(\frac{\dot{a}}{a}\right)^3 \tag{B5}$$

and

$$\begin{aligned}\ddot{\epsilon}_2=&-\omega^{-4}\left(\frac{k^2}{a^2}+\frac{3}{2}m^2\right)\frac{a^{(4)}}{a}-4\omega^{-6}\left(\frac{k^4}{a^4}+\frac{9}{4}\frac{k^2}{a^2}m^2-\frac{3}{8}m^4\right)\frac{a^{(3)}\dot{a}}{a^2}-3\omega^{-6}\left(\frac{k^4}{a^4}+\frac{5}{2}\frac{k^2}{a^2}m^2\right)\left(\frac{\ddot{a}}{a}\right)^2\\ &+3\omega^{-8}\left(\frac{k^4}{a^4}m^2+19\frac{k^2}{a^2}m^4+\frac{3}{2}m^6\right)\frac{\ddot{a}\dot{a}^2}{a^3}-9\omega^{-10}\left(-3\frac{k^4}{a^4}m^4+\frac{25}{6}\frac{k^2}{a^2}m^6+\frac{1}{2}m^8\right)\left(\frac{\dot{a}}{a}\right)^4,\end{aligned} \tag{B6}$$

where $a^{(n)}$ is the nth derivative of a.

According to Eq. (4.7), ϵ_4 is given by

$$\epsilon_4=-\tfrac{1}{4}\omega^{-2}(1+\epsilon_2)^{-2}\ddot{\epsilon}_2+\tfrac{5}{16}\omega^{-2}(1+\epsilon_2)^{-3}\dot{\epsilon}_2{}^2-\tfrac{1}{4}\omega^{-4}(1+\epsilon_2)^{-2}\dot{\epsilon}_2\left(\frac{k}{a}\right)^2\frac{\dot{a}}{a}. \tag{B7}$$

The terms in ϵ_4 of lowest order (T^{-4}) are thus

$$\epsilon_{4(4)} = -\frac{1}{4}\left(\omega^{-2}\ddot{\epsilon}_2 + \omega^{-4}\frac{k^2}{a^2}\frac{\dot{a}}{a}\dot{\epsilon}_2'\right)$$
$$= \frac{1}{4}\left[\omega^{-6}\left(\frac{k^2}{a^2}+\frac{3}{2}m^2\right)\frac{a^{(4)}}{a} + 3\omega^{-8}\left(\frac{5}{3}\frac{k^4}{a^4}+\frac{7}{2}\frac{k^2}{a^2}m^2-\frac{1}{2}m^4\right)\frac{a^{(3)}\dot{a}}{a^2} + 3\omega^{-8}\left(\frac{k^4}{a^4}+\frac{5}{2}\frac{k^2}{a^2}m^2\right)\left(\frac{\ddot{a}}{a}\right)^2\right.$$
$$\left. + 3\omega^{-10}\left(\frac{k^6}{a^6}+\frac{3}{2}\frac{k^4}{a^4}m^2-19\frac{k^2}{a^2}m^4-\frac{3}{2}m^6\right)\frac{\ddot{a}\dot{a}^2}{a^3} + 36\omega^{-12}\left(-\frac{k^4}{a^4}m^4+\frac{k^2}{a^2}m^6+18m^8\right)\left(\frac{\dot{a}}{a}\right)^4\right]. \quad \text{(B8)}$$

---

*Research supported in part under National Science Foundation Grant No. GP-38994.

[1]For discussion of this semiclassical theory see L. Parker and S. A. Fulling, Phys. Rev. D 7, 2357 (1973), and the references cited there. We remark that in that paper we avoided the problem of divergences by considering only the quantum states for which one can plausibly argue that the precise form of the vacuum terms (to be subtracted in regularization) is unimportant. We believe that application of our present results to the situation studied in that paper would not affect the conclusions.

[2]R. Utiyama and B. S. DeWitt, J. Math. Phys. 3, 608 (1962); B. S. DeWitt, Phys. Rev. 162, 1239 (1967). See also A. D. Sakharov, Dokl. Akad. Nauk SSSR 177, 70 (1967) [Sov. Phys.—Dokl. 12, 1040 (1968)]; H. Nariai, Prog. Theor. Phys. 46, 433 (1971).

[3]Ya. B. Zel'dovich and A. A. Starobinsky, Zh. Eksp. Teor. Fiz. 61, 2161 (1971) [Sov. Phys—JETP 34, 1159 (1972)].

[4]This prescription is equivalent to normal ordering the expressions with respect to the creation and annihilation operators of the particles.

[5]L. Parker, (a) Ph.D. thesis, Harvard University, 1966 (unpublished); (b) Phys. Rev. 183, 1057 (1969).

[6](a) A. A. Grib and S. G. Mamaev, Yad. Fiz. 10, 1276 (1969) [Sov. J. Nucl. Phys. 10, 722 (1970)]; (b) S. A. Fulling, Ph.D. dissertation, Princeton University, 1972 (unpublished), Chap. X; (c) B. A. Levitsky, Teor. Mat. Fiz. 8, 226 (1971) [Theor. Math. Phys. 8, 791 (1972)].

[7]This happens because the vacuum states so defined at different times belong to different Hilbert spaces (inequivalent representations of the canonical field algebra).

[8]See Ref. 5. For the case of spin one-half, see L. Parker, Phys. Rev. D 3, 346 (1971).

[9]S. A. Fulling, L. Parker, and B. L. Hu, paper in preparation.

[10]S. A. Fulling and L. Parker, paper in preparation.

[11]B. L. Hu, Phys. Rev. D 8, 1048 (1973); B. L. Hu, S. A. Fulling, and L. Parker, *ibid.* 8, 2377 (1973).

[12]The time-independent line element $\sum h_{jk}(x^1,x^2,x^3)dx^j dx^k$ refers to a three-dimensional manifold of constant curvature, appropriately normalized. See Appendix A.

[13]Spatial symmetry of the quantum state ensures that $\langle T_0{}^0\rangle$ and $\langle T_j{}^j\rangle$ will be independent of x, $\langle T_j{}^j\rangle \equiv -P$ will be independent of j, and $\langle T_\mu{}^\nu\rangle$ will equal 0 for $\mu\neq\nu$, as required for consistency in the Einstein equations for a Robertson-Walker metric. (In Ref. 1 we have explained, in the context of the closed universe, that a state which is only roughly symmetrical on a global scale may be replaced by the symmetrical mixed state obtained by averaging it over all "orientations" in three-space.) Since $\langle T_0{}^0\rangle$ and $\langle T_j{}^j\rangle$ are independent of x, they are equal to their average values over the three-space. In evaluating ρ_0 and P_0 [Eqs. (2.17) and (2.20)] for the cases $\epsilon=\pm 1$, such an averaging is used to justify an integration by parts which allows one to evaluate the terms involving space derivatives without explicit knowledge of the first-order partial derivatives of the $\mathcal{Y}_k$'s. Averaging is also essential in arriving at Eqs. (2.19) and (2.21) in all three cases. When the volume of the universe is infinite ($\epsilon=0$ or $\epsilon=-1$), such averages should be defined by an appropriate limiting process; we have justified our statements for $\epsilon=0$ in that way, and are confident of the results for $\epsilon=-1$, obtained less rigorously, because of their very close analogy to the results for the other two cases. The factor $(4\pi^2)^{-1}$ arises in ρ_0 and P_0 for that case directly from the addition theorem (A14), without averaging.

[14]"State" must be understood here in the algebraic sense of a normalized positive linear functional (the expectation value) on an algebra of bounded local observables associated with the fields $\phi(x,t)$ or with the A_k and $A_k^\dagger$. See references cited in footnote 30 of Ref. 1. The expectation value of $A_k^\dagger A_k$ need not be defined, since $A_k^\dagger A_k$ is unbounded, global, and of a distribution nature.

[15]In the case of the flat universe ($\epsilon=0$) these quantities may be defined by the following limiting procedure: Replace the infinite three-space by a cube of coordinate length L, and write the conventional field expansion into discrete modes:

$$\phi = \sum_{\vec{k}} [a_{\vec{k}}\psi_k(t)L^{-3/2}e^{i\vec{k}\cdot\vec{x}} + \text{H.c.}],$$

where $\vec{k}=2\pi\vec{n}/L$, the components of $\vec{n}$ being integers. In the limit $L\to\infty$, $L^{-3}\sum_{\vec{k}}$ goes over into $(2\pi)^{-3}\int d^3k$. Formally, $A_{\vec{k}} = \lim_{L\to\infty}[(2\pi)^{-3/2}L^{+3/2}a_{\vec{k}}]$. One easily sees that for a homogeneous state $|\Psi\rangle$ the density of particles in $\vec{k}$ and $\vec{x}$ space is

$$(2\pi)^{-3}\langle A_{\vec{k}}^\dagger A_{\vec{k}}\rangle \equiv (2\pi)^{-3}\lim_{L\to\infty}\langle\Psi|a_{\vec{k}}^\dagger a_{\vec{k}}|\Psi\rangle.$$

The quantity $\langle A_{\vec{k}}A_{-\vec{k}}\rangle$ is then defined by analogous relations.

[16]We use this term because the subtractions are associated with the unobservable energy density and pressure of empty space, although subtraction of the expectation value with respect to a particular state vector is not necessarily implied.

[17]Any function satisfying Eqs. (2.11) and (2.14) can be written in the form

$$\psi(t)=(2a^3W)^{-1/2}\exp\left(-i\int^t W\,dt'\right),$$

with $W(t)=|\psi(t)|^{-2}/(2a^3)$ [P.C. Waterman, Am. J. Phys. 41, 373 (1973)]. However, such a choice of $W(t)$ in general violates the physical requirement that $W(t)$ reduce to $\omega=(k^2/a^2+m^2)^{1/2}$ during any period when the expansion parameter $a(t)$ becomes constant. Furthermore, rapid oscillations in the above-mentioned $W(t)$ on the scale of the de Broglie frequency will generally occur, indicating that ψ should be considered a superposition of positive- and negative-frequency solutions. The $W(t)$ provided by the extended WKB approximation [see Eqs. (3.4) and (4.5)], in contrast, reduces to ω during any static period, and varies only on the same time scale as $a(t)$.

[18]In the present paper, unlike Ref. 5, we do not deal explicitly with time-dependent particle number operators. Instead, we concentrate all the time dependence in the function ψ_k and use Eq. (3.4) as the effective embodiment of the minimization postulate to order n in T^{-2}, since it implies that there will be no particle creation to that order. The two approaches are equivalent.

[19]The results were reported in L. Parker, Phys. Rev. Lett. 21, 562 (1968).

[20]Ref. 5(a), Chap. 5 and Appendix CI. More precisely, that procedure was used to define time-dependent creation and annihilation operators which yielded no particle creation to a given order in the slowness parameter of the expansion, thus satisfying the minimization postulate to that order.

[21]B. Chakraborty, J. Math. Phys. 14, 188 (1973).

[22]Ref. 5 (a), pp. 41–50. The conclusion is based on a theorem of J. Littlewood [Ann. Phys. (N.Y.) 21, 233 (1963)]. In accordance with the discussion below, the number of particles created in mode k also falls off faster than any power of k.

[23]From a physical point of view, one may say that $a(t)$ is slowly varying with respect to a mode $\underline{k}$ if $\omega_k{}^{-1}$, the reciprocal of the de Broglie frequency, is small compared to a time, such as $(\dot{a}/a)^{-1}\propto T$, which is characteristic of the variation of $a(t)$.

[24]Cf. Refs. 5 and 21. A thorough and rigorous treatment of asymptotic expansions for equations of the form (3.1) has been given by F. W. J. Olver [Proc. Camb. Philos. Soc. 57, 790 (1961)]. Olver's expansions can be obtained from ours by expanding the two W_k-dependent factors in Eq. (3.4) in powers of T^{-1}. The approach of Refs. 5(a) and 21 is much superior for our present purposes because it leads to the identification of the effective frequency W_k, with respect to which the concept of a "positive-frequency solution" ψ_k [see Eq. (3.4)] corresponds to physical particle operators $A_{\underline{k}}$.

[25]It was shown by S. A. Fulling [Phys. Rev. D 7, 2850 (1973)] that the particle interpretation can be ambiguous in a space-time which is static with respect to two different coordinate systems, but it was also pointed out that the discrepancy in such cases is probably due to a tacit boundary condition imposed on the edge of a coordinate patch. There can be little doubt that the interpretation assumed here is correct under the present circumstances.

[26]Any two sets of operators $A_{\underline{k}}$ consistent with Eq. (3.4) to a given order will be related by a Bogolubov transformation. In the case of the closed universe the technical condition required for coincidence of the corresponding Fock spaces (i.e., unitary equivalence of the field representations) is that $\int d\mu(k)|v_k|^2<\infty$, where v_k is a coefficient in the Bogolubov transformation [see, e.g., Ref. 6(b)]. This condition, which ensures that the total number of particles in a given state is finite with respect to both definitions of the number operators, will be satisfied if the order of the approximation (3.4) is T^{-2} or higher. In the cases $\epsilon=0$ and $\epsilon=-1$ the representations will be only locally equivalent; an inequality analogous to that above [and satisfied when Eq. (3.4) holds to second order] ensures that the density of particles in x space remains finite under the Bogolubov transformation.

[27]The complications arising when $\epsilon\neq 1$ are due entirely to the infinite volume of the three-space. As remarked in Sec. II, in order to obtain states of the proper symmetry for Eqs. (2.3) and (2.4) in those cases, one must consider states whose total energy and particle number are infinite, but whose energy density and pressure are finite (and homogeneous). Thus, some states outside any given Fock space must be considered physically realizable in the context of our application. However, as outlined in Ref. 26, there is still a restriction on the density of particles in $\underline{k}$ space at large k; it leads to finite total number density, and also, when Eq. (3.4) is imposed to order T^{-4}, to finite $\rho-\rho_0$ and $P-P_0$.

[28](a) The explicit form of the extended WKB approximation correct to order T^{-3} was given in Ref. 5(a). In the notation of that reference, $\epsilon_2=-2\omega^{-1}S$. (b) The arguments that follow will be concerned with the adiabatic limit $T\to\infty$, or the high-energy limit $k\to\infty$. In either case $|\epsilon_2|$ and $|\epsilon_4|$ will be small with respect to unity, so that no problems arise with respect to the definition of the square roots in Eq. (4.5). Since W_k does not appear in the final results [Eqs. (4.16) – (4.23)], it is not necessary to define it for small T.

[29]Our argument involving the adiabatic limit does not exclude the possibility that terms in ρ_0 and P_0 of higher order in T^{-2} should be included in the vacuum subtractions. However, the condition of vanishing four-divergence does require that the number of such terms subtracted be the same at all times (i.e., they cannot be switched on or off.) We have assumed that only the minimum number of terms needed to yield the correct adiabatic limit should be subtracted. This procedure is the one which leads to the same result as Ref. 3.

[30]W. Pauli and F. Villars, Rev. Mod. Phys. 21, 434 (1949).

[31]We are not concerned here with the technical methods used to calculate the terms in the adiabatic expansions, which differ and will be discussed in Ref. 9.

[32]E. Schrödinger, *Expanding Universes* (Cambridge Univ. Press, Cambridge, England, 1956), pp. 79–86; M. Bander and C. Itzykson, Rev. Mod. Phys. 38, 330 (1966); B. L. Hu, Phys. Rev. D 8, 1048 (1973).

[33]E. Lifshitz, J. Phys. (USSR) 10, 116 (1946); E. M. Lifshitz and I. M. Khalatnikov, Adv. Phys. 12, 185 (1963), Appendix J.

[34]A. Z. Dolginov and I. N. Toptygin, Zh. Eksp. Teor. Fiz. 37, 1441 (1959) [Sov. Phys.—JETP 10, 1022 (1960)]; M. Bander and C. Itzykson, Rev. Mod. Phys. 38, 346 (1966); I. S. Shapiro, Dokl. Akad. Nauk SSSR 106, 647 (1956) [Sov. Phys.—Dokl. 1, 91 (1956)]; N. Ya. Vilenkin and Ya. A. Smorodinsky, Zh. Eksp. Teor. Fiz. 46, 1793 (1964) [Sov. Phys.—JETP 19, 1209 (1964)].