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$$\begin{aligned} B^{mn}_\mu R^*_{u_n} L_{u_m} w_L &= B^{mn}_\mu R^*_{u_n} L_{u_m}(w_0 u_0 + w_0^* u_0^*) \\ &= B^{mn}_\mu w_0^*(u_m u_0^*) u_n^* \\ &= -B^{mn}_\mu w_0^* \delta_{mn} u_0 \\ &= 0. \end{aligned}$$

A systematic study of the octonionic field operators acting in an octonionic Hilbert space will be given elsewhere. In the rest of this paper we shall assume that all the field operators act from the left.

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Parton models and asymptotic freedom

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We generalize the scale-invariant parton model to describe fixed-point and asymptotically free gauge and nongauge theories. Q^2-dependent scaling laws for the moments of νW_2 are derived. In fixed-point theories the moments scale as powers of Q^2, while in asymptotically free theories they scale as powers of $\ln Q^2$. The behaviors of the elastic form factors, the ratio σ_S/σ_T, and the mean-squared transverse momentum of hadron secondaries are discussed in the various theories. The experimental study of these quantities should distinguish clearly between the conventional parton-model and asymptotically free theories of strong interactions.

I. INTRODUCTION

It has become clear that the light-cone behavior of renormalizable field theories must differ from the light-cone behavior of free fields. This is due to the infinite renormalizations necessary to define physical quantities in field theories with dimensionless coupling constants. Roughly speaking, the resolving power of an external probe can never be sufficient to uncover all the structure in the interacting fields. To study the short-distance character of interacting fields one defines a coupling constant $g(\lambda)$ which determines the strength of interactions between quanta of the field theory having spacelike momenta $\lambda\hat{p}$, where $\hat{p}$ is some arbitrary spacelike reference momentum.[1] $g(\lambda)$ is thus a dimensionless measure of the deviations from free-field behavior at length scales λ^{-1}. One popular assumption is that $g(\lambda)$ tends to a nonzero constant for large λ, i.e., $g(\lambda)$ tends to a fixed point[2] g^*. Then the interacting theory is scale-invariant at short distances. However, recent perturbative studies of the renormalization group have shown that in certain theories $g(\lambda)$ can vanish as λ increases.[3] This rate is, however, at most logarithmic, like $(\ln\lambda)^{-1}$. The approach of $g(\lambda)$ to zero is therefore so slow that interactions at short distances in these "asymptotically free" renormalizable theories are not generally negligible. This fact contrasts sharply with super-renormalizable theories (upon which parton models are based) in which $g(\lambda)$ vanishes as a power of λ as λ increases. It should be clear that an intuitive but conceptually adequate approach to short-distance behavior of renormalizable field theories must account for the fact that these theories have significant interactions on length scales ranging all the way to zero.[4]

We have recently formulated a parton model which incorporates the subtleties of renormalizable field theory.[5] This approach has been used to discuss the physics which accompanies a fixed point of the renormalization group. It is the purpose of this article to extend our previous considerations to theories which are asymptotically free at short distances. Most of the results obtained in this article have been found by more formal methods.[6] Our approach is intended to expose the simple physics behind the results. To begin we review our general approach. A linear iterative integral equation for νW_2 is obtained which governs the function's Q^2 dependence as the resolving power of an external virtual photon increases. The kernel function of the integral describes the short-distance structure in the theory and controls the Q^2-dependent scaling laws for the moments of νW_2. Four different types of short-distance behavior are discussed: fixed-point short-distance behavior in gauge and nongauge theories and asymptotically free gauge and nongauge theories. In all cases the area under νW_2 is asymptotically independent of Q^2 while other moments of νW_2 either fall as powers of Q^2 (fixed-point theories) or as powers of logarithms of Q^2 (asymptotically free theories). Asymptotically free gauge theories are discussed in some detail. Elastic form factors are found to vanish slightly faster than any power of Q^2 at truly asymptotic Q^2. σ_S/σ_T should vanish in the deep-inelastic region roughly as $(\ln Q^2)^{-1}$. The total mean-squared momentum, of hadronic secondaries, transverse to the direction of the virtual photon, should grow as $Q^2/\ln Q^2$. These results are contrasted with the conventional pointlike-parton model and scale invariance at a fixed point.

II. REVIEW OF SCALE-INVARIANT PARTON MODEL

To begin, consider a deep-inelastic experiment[7] performed with kinematic variables Q and $\sqrt{\nu}$ each between the values 1 and λ. Here λ denotes a momentum where renormalization effects become important. We can describe the dynamics of the hadron by a cutoff Hamiltonian H_1 whose degrees of freedom are the transverse momenta and longitudinal fractions of constituent partons. In the cutoff theory partons cannot have relative transverse momenta in excess of λ. In other words, partons of the cutoff theory are not found closer than a distance λ^{-1} in the transverse plane.[5] Suppose now that the momentum of the probe is increased above λ, so that distances shorter than λ^{-1} can be resolved. The bare partons of the Hamiltonian H_1 are replaced by a distribution of the constituents described by the Hamiltonian H_2. A convenient choice for the cutoff of H_2 is λ^2. This formulation of the field theory may be carried further[5] so that scattering experiments involving $(Q^2)^{1/2} \sim \lambda^N$ are described by the constituents of a Hamiltonian H_N having a cutoff λ^N. The partons of length scale N-1 may be understood as dynamically bound clusters of partons of the Nth scale. Knowing the Hamiltonian H_N we can always solve for the Hamiltonian H_{N-1} in the same way that intermolecular forces can be derived from atomic forces. In the usual renormalizable field theories the Hamiltonians are characterized by a set of dimensionless coupling constants $\{g_i\}$. The properties of the theory at small distances are then determined by the variation of the $\{g_i\}$ as we vary from one scale to the next.

At every length scale we describe a deep-inelastic scattering process using a parton model appropriate to that length scale, i.e., $(Q^2)^{1/2} \sim \lambda^N$. The matching between the scale N and the momentum q reflects the fact that the parton model will work only if the probing wavelengths are smaller than the mean transverse separations between partons but larger than the size of the parton. Thus, in this picture,[8]

$$\nu W_2(\eta, Q^2) = \eta \frac{dn}{d\eta}\bigg|_{N=\ln Q/\ln\lambda}, \quad \eta = Q^2/2p\cdot q \tag{1}$$

where $dn/d\eta|_N$ is the longitudinal-momentum distribution of partons of transverse size λ^{-N}.

The modern formulation of the renormalization group[1,2,4] suggests that we do not attempt to compute the distribution of partons of type N in the hadron directly, but instead formulate a recursion relation connecting the distributions of partons at adjacent levels. To do this we must know the makeup of constituents of type N in terms of those of type $N+1$. In particular, we must know how the longitudinal fraction of a constituent of type N is shared by constituents of type $N+1$. We introduce a function $f_{N+1,N}(\beta/\eta)/(\beta/\eta)$ which gives the probability per unit β/η to find a parton of type $N+1$ and longitudinal fraction β in a constituent of type N and longitudinal fraction η. Then the distribution, $F_2(\beta, N+1)$, of constituents of type $N+1$ having longitudinal fraction β, satisfies the equation[5]

$$\frac{F_2(\beta, N+1)}{\beta}=\int_\beta^1 \frac{f_{N+1,N}(\beta/\eta)}{(\beta/\eta)}\ \frac{F_2(\eta, N)}{\eta}\ \frac{d\eta}{\eta}\ . \qquad (2)$$

It is often more convenient to analyze Eq. (2) when it is written in terms of the rapidity variable

$$y=\ln\eta \qquad (3)$$

so that

$$F_2(y, N+1)=\int f_{N+1,N}(y-y')F_2(y', N)dy'\ . \qquad (4)$$

The kernel function $f_{N+1,N}$ is constrained by longitudinal-momentum conservation—the sum of the longitudinal fractions of the $(N+1)$ constituents in a constituent of type N should be the longitudinal fraction of the N constituent,

$$\int f_{N+1,N}(\eta)d\eta=1\ . \qquad (5)$$

Equations (4) and (5) will play a central role in the following.

III. FIXED-POINT AND ASYMPTOTICALLY FREE THEORIES

A. Fixed-point theories

In this case the dimensionless coupling constants $\{g_i\}$ tend to finite nonzero values as N grows, and the dynamics becomes identical at every length scale (apart from a rescaling of lengths and times). In particular the kernel function $f_{N+1,N}$ describing N-type constituents in terms of $(N+1)$-type constituents becomes independent of N. Then Eq. (4) simplifies to

$$F_2(y, N+1)=\int f(y-y')F_2(y', N)dy'\ . \qquad (6)$$

This equation can be solved by Laplace transforms. Define the αth moment of $F_2(\eta, N)$,

$$M_\alpha(N)=\int_0^1 \eta^\alpha F_2(\eta, N)d\eta/\eta$$
$$=\int e^{\alpha y}F_2(y, N)dy \qquad (7)$$

and the αth moment of f,

$$m_\alpha=\int \eta^\alpha f(\eta)\frac{d\eta}{\eta}$$
$$=\int e^{\alpha y}f(y)dy. \qquad (8)$$

Substituting into Eq. (6) we have

$$M_\alpha(N+1)=m_\alpha M_\alpha(N)\ . \qquad (9)$$

To solve this iterative equation we must supplement it with boundary conditions at small N which describe the large-scale character of hadronic structure. Choose boundary conditions[5]

$$M_\alpha(N=0)=M_\alpha\ , \qquad (10)$$

where M_α are the moments of νW_2 in the first scaling region $(1<Q^2<\lambda^2)$. The solution of Eq. (9) then becomes

$$M_\alpha(N)=(m_\alpha)^N M_\alpha. \qquad (11)$$

Using the relation between the momentum transfer Q and the scale N, $N=\ln Q/\ln\lambda$, allows the Q^2 dependence in Eq. (11) to be made explicit,

$$M_\alpha(Q^2)=(m_\alpha)^{\ln Q/\ln\lambda}M_\alpha$$
$$=Q^{\ln m_\alpha/\ln\lambda}M_\alpha\ . \qquad (12)$$

From the definition of $M_\alpha(N)$ in Eq. (7) we have finally

$$\int_0^1 \eta^{\alpha-1}\nu W_2(\eta, Q^2)d\eta\approx(Q^2)^{-d_\alpha}M_\alpha\ , \qquad (13)$$

where

$$d_\alpha=-\ln m_\alpha/\ln\lambda\ .$$

In other words, the moments of νW_2 are power-behaved in Q^2 with α-dependent powers. The $\alpha=1$ moment—the area under νW_2—is special, however. It follows from Eq. (5) that $m_1=1$, so that $d_{(\alpha=1)}=0$. Therefore, the area under νW_2 does not change as Q^2 varies,

$$\int_0^1 \nu W_2(\eta, Q^2)d\eta=\text{const}\ . \qquad (14)$$

Furthermore, the positivity of f insures that the sequence $\{d_\alpha\}$ is nondecreasing. So, aside from the possibility that $d_\alpha=0$ for all α, all the moments of νW_2 except the first vanish as powers of Q^2. Therefore, at truly asymptotic values of Q^2, νW_2 becomes a δ function at $\eta=0$ with a weight given by Eq. (14).

Two examples of fixed-point theories will now be discussed.

1. Nongauge theories. Typically when H_{N+1} is solved there is a nonzero probability that a constituent of type N consists of just one constituent of type $N+1$. This probability we call Z. The constant Z is similar to the wave-function renormalization constant in a finite field theory. Since the field theory has been cut off in transverse momentum, the wave-function renormalization constant Z is generally expected to be a finite number. However, if the transverse cutoff theory is still divergent due to integrations over longitudinal momenta, the probability Z may in fact be zero. In field theories involving spin-0 and spin-$\frac{1}{2}$ quanta there are no divergences associated with longitudinal momenta in the infinite-momentum frame. Then f has the form

$$f(y)=Z\delta(y)+\text{smooth function of } y, \qquad (15)$$

where the δ function represents the possibility that only one parton of type $N+1$ carries the total longitudinal fraction of the constituent of type N. From Eqs. (8) and (15) it follows that the high moments of $f(y)$ behave as

$$m_\alpha \sim Z \qquad (16)$$

as $\alpha\to\infty$. From Eq. (12) this means that the high moments of νW_2 behave as

$$\int \eta^{\alpha-1}\nu W_2 d\eta \to (Q^2)^{\ln Z/\ln\lambda}. \qquad (17)$$

From this result and Eq. (14) we see that the quantities d_α begin at $d_{(\alpha=1)}=0$ and increase to $d_{(\alpha\to\infty)}=\ln Z/\ln\lambda$. Other consequences and a more detailed analysis of these theories can be found in Ref. 5.

2. Non-Abelian gauge theories. To see one of the new features in gauge theories consider calculating the wave function of a charged particle to first order in perturbation theory. In zeroth order the charged particle consists of a bare quantum and the first-order correction consists of a bare charged quantum and a bare vector meson,

$$|e\rangle=\sqrt{Z}\left(|e_0\rangle+g\sum\psi(K,\eta)|e_0V_0\rangle\right), \qquad (18)$$

where g is the coupling constant, $|e_0\rangle$ and $|V_0\rangle$ represent the bare quanta, and ψ is the wave function containing spin and momentum dependence. K and η are the relative transverse momentum and longitudinal fraction of the bare vector meson. The constant Z is necessary to normalize the state $|e\rangle$. Clearly Z is given by

$$Z=\left(1+g^2\int|\psi|^2 d\vec{K}d\eta/\eta\right)^{-1}. \qquad (19)$$

For gauge theories (unlike theories without vector mesons) the integral in Eq. (19) diverges as $\eta\to 0$. Thus, even when the transverse momentum of the wave function is cut off from above and below, the probability to find a single bare charged quantum vanishes. In fact the only states which occur with finite probability contain an infinite number of low-η vector mesons. Therefore the δ function of Eq. (15) does not occur in this case. This phenomenon is formally analogous to the straggling of relativistic charged particles passing through a medium. Here the energy spectrum of the initial charged particle is smeared due to the emission of infrared photons.[9] The straggling formula giving the probability that the charged particle retains a fraction η of its longitudinal momentum while emitting soft vector mesons becomes[9]

$$f(\eta)=A(g^2)(1-\eta)^{-1+B(g^2)} \qquad (20)$$

for $\eta\approx 1$, where A and B have the perturbation-theory expansions,

$$A\propto g^2+\cdots, \quad B\propto g^2+\cdots, \qquad (21)$$

We will incorporate this straggling idea by using it as a model for the solution of the Hamiltonians H_N. That is to say, we shall assume that the structure of the charged partons of type $N-1$ are described by a sea of low-η vector partons of type N. Equation (20) is expected to be accurate for η near unity. The graphs producing the kernel $f(\eta)$ are shown in Fig. 1. The moments of $f(\eta)$ read

$$m_\alpha=A(g^2)\int\eta^{\alpha-1}(1-\eta)^{-1+B(g^2)}d\eta = A(g^2)\frac{\Gamma(\alpha)\Gamma(B)}{\Gamma(\alpha+B)} \qquad (22)$$

which for large α become

$$m_\alpha \sim A(g^2)\Gamma(B(g^2))\,\alpha^{-B(g^2)} \qquad (23)$$

so the d_α behave for large α as

$$d_\alpha \sim B(g^2)\ln\alpha/\ln\lambda^2. \qquad (24)$$

As opposed to the nongauge theories at a fixed

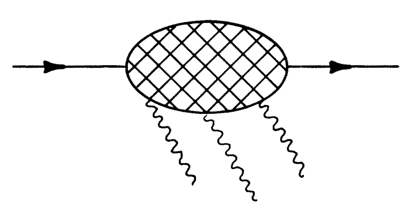

FIG. 1. Graphs controlling the $\eta\approx 1$ region of $f_{N+1,N}$ in gauge theories.

point, the higher moments of these theories suffer a more and more rapid Q^2 variation.

B. Asymptotically free theories

Of special interest are renormalizable field theories which are asymptotically free, i.e., H_N tends to a free-field Hamiltonian as $N\to\infty$. Such field theories include ϕ^4 in four dimensions with a small negative coupling constant,[10] ϕ^3 in six dimensions, and, more interesting, a large class of Yang-Mills theories with fermions in four dimensions.[3] In all of these theories the dimensionless coupling constant g_N^2 goes to zero as N^{-1} as N grows.

1. *Nongauge theories*

In this case the kernel function f is the sum of a δ function and a smooth function. For small values of the coupling constant f can be computed in perturbation theory,[11]

$$f_{N+1,N}(y)=(1-g_N^2)\delta(y)+g_N^2h(y)+O(g_N^4)\ , \quad (25)$$

where $h(y)$ is a smooth function which is given by the two-particle contributions to f. Longitudinal momentum conservation constrains h,

$$\int_0^1 h(\eta)d\eta=1\ . \quad (26)$$

In this case the iteration formula for the moments of νW_2 has explicit scale dependence,

$$M_\alpha(N+1)=m_\alpha(N+1,N)M_\alpha(N)\ . \quad (27)$$

The moments of $f_{N+1,N}$ read

$$m_\alpha(N+1,N)=g_N^2h(\alpha)+(1-g_N^2)+\cdots, \quad (28)$$

where

$$h(\alpha)=\int\eta^{\alpha-1}h(\eta)d\eta \quad (29)$$

and

$$g_N^2\underset{N\to\infty}{\sim}\bar{g}^2/N, \quad (30)$$

where $\bar{g}$ is a combinatoric factor dependent on the number and types of fields in the theory. The iteration formula Eq. (27) is solved by

$$M_\alpha(N+1)=\prod_{\nu=0}^{N}m_\alpha(\nu+1,\nu)M_\alpha\ . \quad (31)$$

As usual $m_1=1$, so the $\alpha=1$ moment of νW_2 is Q^2 independent. It is convenient to write Eq. (28) in the form

$$m_\alpha=1-c(\alpha)g_N^2, \quad (32)$$

where $c(1)=0$ and $c(\alpha)\to1$ as $\alpha\to\infty$. For large N, Eq. (32) can be approximated by

$$m_\alpha\approx\exp[-c(\alpha)g_N^2] \quad (33)$$

so the iteration procedure Eq. (31) generates

$$\begin{aligned}M_\alpha(N+1)&=[e^{-c(\alpha)\Sigma g_k^2}]M_\alpha\\&=[e^{-c(\alpha)\Sigma^N\bar{g}^2/k}]M_\alpha\\&=[e^{-c(\alpha)\bar{g}^2\ln N}]M_\alpha\\&=(\ln Q^2)^{-\bar{g}^2c(\alpha)}M_\alpha\ . \end{aligned}\quad (34)$$

Therefore,

$$\int_0^1\eta^{\alpha-1}\nu W_2d\eta\sim(\ln Q^2)^{-\bar{g}^2c(\alpha)}M_\alpha\ , \quad (35)$$

where $c(1)=0$. As in the fixed-point theories, the positivity of f ensures that $c(\alpha)$ is a monotonically increasing function. This implies that all the moments for $\alpha>1$ tend to zero as Q^2 grows. The violation of canonical scaling occurs via slowly varying logarithms. At a logarithmic rate νW_2 becomes a δ function at $\eta=0$ with a weight given by its area $\int\nu W_2(\eta Q^2)d\eta$.

2. *Gauge theories*

It has recently been shown that Yang-Mills theories with fermions can also be asymptotically free if the number of fermion types is not too large.[3] As an example suppose that only the fermions carry electric charge as would be the case in a colored-quark model with color being a gauge symmetry.[12] In this case the kernel function is given by the perturbative form of the straggling formula[11]

$$f_{N+1,N}(\eta)=g_N^2(1-\eta)^{-1+g_N^2} \quad (36)$$

for η near unity. Note again that as $g_N^2\sim\bar{g}^2/N\to0$, $f_{N+1,N}(y)$ tends to a δ function at $y=0$. The moments of $f_{N+1,N}$ are

$$m_\alpha(\nu+1,\nu)=g_\nu^2\,\frac{\Gamma(\alpha)\Gamma(g_\nu^2)}{\Gamma(\alpha+g_\nu^2)}\ . \quad (37)$$

Consider the large-ν behavior of $m_\alpha(\nu+1,\nu)$. Then Eq. (37) becomes

$$m_\alpha(\nu+1,\nu)\sim\alpha^{-g_\nu^2}\ . \quad (38)$$

So, in this case Eq. (31) becomes

$$\begin{aligned}M_\alpha(N+1)&\approx\alpha^{-\bar{g}^2\Sigma^N\nu^{-1}}M_\alpha\\&\approx\alpha^{-\bar{g}^2\ln N}M_\alpha\\&\approx(\ln Q^2)^{-\bar{g}^2\ln\alpha}M_\alpha\ .\end{aligned}\quad (39)$$

So, the moments of νW_2 have the form

$$\int\eta^{\alpha-1}\nu W_2(Q^2,\nu)d\eta\sim(\ln Q)^{-d_\alpha}M_\alpha\ , \quad (40)$$

where $d_1=0$ and d_α grows monotonically and approaches the curve $\bar{g}^2\ln\alpha$ asymptotically. Again the violations of canonical scaling are logarithmand, contrary to the nongauge asymptotically free theories, the d_α grow without bound.

The fact that the high moments of νW_2 have the fastest Q^2 dependence suggests investigating the short-distance modifications of form factors in asymptotically free gauge theories. According to arguments given in Ref. 5, the form factor for absorption of a photon of momentum squared Q^2 is given by

$$G_N(Q^2)=G_0(Q^2)g_{1,0}\left(\frac{Q}{K_1}\right)g_{1,2}\left(\frac{Q}{K_2}\right)\cdots \times g_{N,N-1}\left(\frac{Q}{K_N}\right), \tag{41}$$

where $G_0(Q^2)$ is the form factor applicable to the $N=0$ scale (quarks inside hadrons) and K_i is the transverse momentum fluctuation of the wave function of type-i constituents. The $N=\ln Q/\ln\lambda$ factors $g_{i,i-1}$ in Eq. (41) are

$$g_{i+1,i}\left(\frac{Q}{K_{i+1}}\right)=\int_{1-K_{i+1}/Q}^{1} f_{i+1,i}(\eta)d\eta . \tag{42}$$

In Sec. IV we shall argue that $K_i \sim K\lambda^i/i$. For our estimates here the approximation $K_i\sim K\lambda^i$ is sufficient. Roughly speaking, the Q^2 dependence coming from the N factors originates in the fact that more and more structure is uncovered by a virtual photon of greater and greater Q^2. The presence of this additional structure suppresses the elastic process more and more as Q^2 grows. Substituting the straggling formula into Eq. (42) gives

$$g_{i+1,i}\left(\frac{Q}{\lambda^i K}\right)=g_i^2\int_{1-\lambda^i K/Q}^{1}(1-\eta)^{-1+g_i^2}d\eta =\left(\frac{\lambda^i K}{Q}\right)^{g_i^2}. \tag{43}$$

Then the form factor of the hadron becomes

$$\begin{aligned}G(Q^2)&=G_0(Q^2)\prod_{i=0}^{N}\left(\frac{\lambda^i K}{Q}\right)^{g_i^2}\\ &\sim G_0(Q^2)\lambda^{N\bar g^2}\left(\frac{K}{Q}\right)^{\bar g^2\Sigma^N(1/i)}\\ &\sim G_0(Q^2)\lambda^{\bar g^2\ln Q/\ln\lambda}\left(\frac{K}{Q}\right)^{\bar g^2\ln N}\\ &\sim G_0(Q^2)\left(\frac{K}{Q}\right)^{\bar g^2(\ln\ln Q^2-1)}.\end{aligned} \tag{44}$$

Therefore, barring peculiar behavior for $G_0(Q^2)$, the short-distance structural terms in Eq. (41) force the form factor to vanish faster than any power of Q^2 for truly asymptotic values of Q^2. If one identifies $G_0(Q^2)$ with present fits to the nucleon form factor (a dipole, say), then asymptotically free gauge theories predict a measurably faster Q^2 falloff once $\ln Q^2$ becomes a sizable number. Since $\bar g^2$ is a small[6] number for most theories (about $\frac{1}{3}$ for color-quark schemes) this effect may be small in practice. It is surprising, however, that the influence on the form factor is even this significant in an asymptotically free theory. The reason for this behavior can be traced to the absence of a $Z\delta(y)$ term in the kernel function f which which in turn can be understood as the effect of elementary vector fields in the theory.

Since the formula for $G(Q^2)$ is somewhat unusual, we shall also obtain it in another more pragmatic fashion. To do this recall the Bloom-Gilman threshold relation[13]

$$\int_{1-m^2/Q^2}^{1}\nu W_2(\eta,Q^2)d\eta\sim|G(Q^2)|^2 \tag{45}$$

which states that the area under the resonance region of the inclusive quantity νW_2 should match the strength of the exclusive (resonant) channels themselves. The integral of νW_2 over the resonance region is isolated by the high moments of νW_2. In detail, it is easy to see that a moment

$$\int_0^1 \eta^{\alpha-1}\nu W_2 d\eta$$

receives all of its contribution from the region between $\eta=1-m^2/Q^2$ and $\eta=1$, if one chooses the moment α to be

$$\alpha\cong\frac{Q^2}{m^2}. \tag{46}$$

However, the high moments of νW_2 are

$$\int_0^1\eta^{\alpha-1}\nu W_2 d\eta\sim(\ln Q^2)^{-\bar g^2\ln\alpha}M_\alpha .$$

Taking $\alpha=Q^2/m^2$ and matching with the Bloom-Gilman relation gives

$$\begin{aligned}G(Q^2)&=G_0(Q^2)(\ln Q^2)^{-(1/2)\bar g^2\ln(Q^2/m^2)}\\ &\sim G_0(Q^2)\left(\frac{1}{Q}\right)^{-\bar g^2\ln\ln Q},\end{aligned} \tag{47}$$

which is the same asymptotic form factor obtained by the more detailed theoretical argument.

Similar arguments may be given for fixed-point theories and asymptotically free nongauge theories. For the fixed-point nongauge theories renormalization effects contribute a power to the asymptotic form factor.[5]

$$G_N(Q^2)=G_0(Q^2)(Q)^{\ln Z/\ln\lambda};$$

in fixed-point gauge theories,

$$G_N(Q^2)=G_0(Q^2)Q^{-B(g*)\ln Q/\ln\lambda},$$

and in asymptotically free nongauge theories,

$$G_N(Q^2)=G_0(Q^2)(\ln Q^2)^{-\bar g^2}.$$

IV. TRANSVERSE-MOMENTUM FLUCTUATIONS OF CONSTITUENTS

In this section we discuss the behavior of the transverse-momentum fluctuations of constituents of type N as N increases. Picture a constituent of type N as a dynamical solution of the Hamiltonian H_{N+1}. This Hamiltonian is a cutoff field-theoretic Hamiltonian characterized by a dimensionless coupling constant g_N. Assuming that N is chosen so large that the mass scales of the theory become unimportant, the Hamiltonian H_N contains no mass terms. If we compute the bound states of scale N and compute the transverse momentum of their constituents the result must be proportional to the cutoff λ^N since no other scale is present. Thus

$$\langle P_T{}^2\rangle_N \approx (\lambda^N)^2 f(g_N{}^2) \ . \tag{48}$$

For theories at a fixed point g_N is independent of N, so $\langle P_T{}^2\rangle_N$ simply grows as λ^N. For asymptotically free theories one can calculate $f(g_N{}^2)$ perturbatively. For example, we can write an N-type constituent of total transverse momentum zero as a superposition of one and two $(N+1)$-type constituents in first-order perturbation theory. The single $(N+1)$-type constituent piece of the wave function does not contribute to transverse-momentum fluctuations. Therefore,

$$f(g_N{}^2) \sim g_N{}^2 \tag{49}$$

for small $g_N{}^2$. It follows that in asymptotically free theories

$$\langle P_T{}^2\rangle \sim \bar{g}^2 \lambda^{2N}/N \ . \tag{50}$$

Two important phenomenological consequences follow from these considerations. In the conventional pointlike-parton model the ratio σ_S/σ_T (the ratio of the longitudinal-photon absorption cross section to the transverse-photon cross section) behaves as[14]

$$\frac{\sigma_S}{\sigma_T} \sim \frac{K^2}{Q^2} \ , \tag{51}$$

where K^2 is the average transverse-momentum fluctuations of the spin-$\frac{1}{2}$ partons. From Eq. (48) the generalization of this result to interacting field theories becomes

$$\frac{\sigma_S}{\sigma_T} \sim \langle P_T{}^2\rangle_N/Q^2 \ , \tag{52}$$

where $\langle P_T{}^2\rangle_N$ is the mean-square transverse-momentum fluctuations of the partons of type N. Since $N \sim \ln Q/\ln\lambda$

$$\langle P_T{}^2\rangle_N \sim f(g_N{}^2)\lambda^{2\ln Q/\ln\lambda} \sim f(g_N{}^2)Q^2 \ , \tag{53}$$

and one then expects, roughly,

$$\frac{\sigma_S}{\sigma_T} \sim f(g_N{}^2) \tag{54}$$

if the fundamental charge-carrying fields have spin $\frac{1}{2}$. In the fixed-point theories $f(g_N{}^2) = f(g^*)$ and σ_S/σ_T becomes a constant. If the coupling constant g_N were small, then

$$\sigma_S/\sigma_T \sim g_N{}^2 \ . \tag{55}$$

Therefore, in asymptotically free theories the ratio σ_S/σ_T should approach zero as Q^2 grows at an asymptotic rate,

$$\sigma_S/\sigma_T \sim \bar{g}^2/\ln Q^2 \ . \tag{56}$$

Strictly speaking, this formula applies only to the quantity σ_S/σ_T averaged over η. The logarithmic dependence of σ_S/σ_T should be contrasted with the naive parton model Eq. (51) and the scale-invariant fixed-point results.

The increase of the transverse-momentum fluctuations of constituents of smaller scales should influence the momentum distribution of final-state hadrons. To see this we work in the virtual-photon–struck-parton Breit frame in which the photon has momentum

$$q_\mu = (q_0, q_x, q_y, q_z) = (0, 0, 0, -2\eta P)$$

and the proton momentum is $(P+m^2/P, 0, 0, P)$. The mean-square transverse momentum of the struck parton (of type $N = \ln Q/\ln\lambda$) is $f(g_N{}^2)\lambda^{2N}$. After absorbing the photon momentum the parton recoils with z momentum $=\eta P$ and mean-square transverse momentum $f(g_N{}^2)Q^2$. By transverse-momentum conservation the remnant of the target also has a mean-square transverse momentum which grows as $f(g_N{}^2)Q^2$. Therefore, the total transverse momentum squared of both the target and virtual-photon fragments will grow as

$$f(g^{*2})Q^2 \tag{57}$$

in a fixed-point theory, and as

$$\bar{g}^2Q^2/\ln Q^2 \tag{58}$$

in asymptotically free theories.

If the multiplicity of hadrons is not too great, then the average momentum of *individual* secondary hadrons transverse to the direction of the virtual photon should also grow as a power of Q^2. Only if the multiplicity grows at a rate comparable to the kinematic limit $[\bar{n}(s) = \sqrt{s}\,]$ will the transverse momentum of the struck constituent then not result in a growth of the mean transverse momentum of individual hadrons. Most views of final states of deep-inelastic processes suggest that the multiplicity grows logarithmically[15] with Q^2 or as a small power[16] of Q^2, so the average transverse momentum of individual fragments of the virtual photon should grow roughly as a power of Q^2,

$$\langle P_T^2\rangle \sim (Q^2)^p, \quad (59)$$

where $0<p<1$. This result applies to both scale-invariant and asymptotically free theories.

In the naive parton model one expects the fragments of the virtual photon to have finite transverse momentum. Hence, Eq. (59) represents a potentially strong deviation from naive-parton-model[15] expectations and emphasizes the fact that much of the physics of asymptotically free theories is almost identical to a scale-invariant theory having a small fixed-point coupling constant.

V. CONCLUDING REMARKS

The general way in which canonical scaling breaks down is similar in the various theories discussed here. Consider the area under νW_2 in the interval between η and $\eta+d\eta$. This represents the longitudinal momentum of partons in that element of phase space. Increasing Q^2 resolves each of these partons into smaller structures each carrying less longitudinal fraction than the parent. Therefore, this element of area is shifted towards lower values of η. However, momentum conservation ensures that the total area under νW_2 is preserved. Therefore, the Q^2 dependence of νW_2 should roughly resemble that shown in Fig. 2. It is clear that νW_2 near $\eta=1$ will *fall* as Q^2 grows while near $\eta=0$ it must *rise*. Eventually, at truly asymptotic values of Q^2, νW_2 will become a δ function at $\eta=0$ with a weight given by its area at large but not necessarily infinite Q^2.

The details of the breakdown of canonical scaling depend in detail on the nature of the short-distance interactions in the various theories. We summarize here the behaviors for the four cases studied.

1. Nongauge fixed point:

$$\int_0^1 \eta^{\alpha-1}\nu W_2 d\eta \sim (Q^2)^{-d_\alpha} M_\alpha\,,$$

$$d_1=0,\quad d_\alpha \underset{\alpha\to\infty}{\sim} \ln Z/\ln\lambda^2\,.$$

In Ref. 5 it was shown that the constant $\ln Z/\ln\lambda^2$ is twice the anomalous dimension of the charge-carrying field.

2. Gauge fixed point:

$$\int_0^1 \eta^{\alpha-1}\nu W_2(Q^2,\eta)d\eta \sim (Q^2)^{-d_\alpha} M_\alpha\,,$$

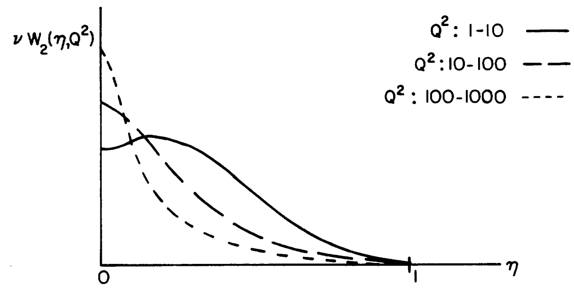

FIG. 2. General behavior of νW_2 as Q^2 increases.

$$d_1=0,\quad d_\alpha \underset{\alpha\to\infty}{\sim} \text{const}\times\ln\alpha\,.$$

3. Asymptotically free nongauge theories:

$$\int_0^1 \eta^{\alpha-1}\nu W_2 d\eta \sim (\ln Q^2)^{-d_\alpha} M_\alpha\,,$$

$$d_1=0,\quad d_\alpha \underset{\alpha\to\infty}{\sim} \text{const.}$$

4. Asymptotically free gauge theories:

$$\int_0^1 \eta^{\alpha-1}\nu W_2 d\eta \sim (\ln Q^2)^{-d_\alpha} M_\alpha\,,$$

$$d_1=0,\quad d_\alpha \underset{\alpha\to\infty}{\sim} \text{const}\times\ln\alpha\,.$$

In particular these scaling laws agree with more formal renormalization group derivations.[6]

In the case of asymptotically free gauge theories we have, roughly,

$$\sigma_S/\sigma_T \sim (\ln Q^2)^{-1}$$

and that the total transverse momentum of the fragments of the virtual photon (or remnants of the target proton) grow as

$$Q^2/\ln Q^2.$$

At finite values of Q^2 where $(\ln Q^2)^{-1}$ cannot be ignored, these last two results are more similar to fixed-point physics than to naive pointlike-parton physics. This is due to the fact that the coupling constant tends to zero very slowly at smaller and smaller distances. Results such as Eq. (56) and Eq. (58) should provide clear distinctions between the conventional parton picture of deep-inelastic scattering and asymptotically free gauge theories.

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Currents and local gauge symmetries*

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The current-algebra properties of gauge field theories are investigated. First, we consider a unified gauge field model of strong, weak, and electromagnetic interactions, which is a natural extension of the σ model combined with the Weinberg-Salam model. The violation of *CP* invariance is forbidden, and isospin is only broken by electromagnetic interactions. The pion is possibly a pseudo-Goldstone boson, which picks up its mass from weak and electromagnetic interactions. In the physical gauge the weak axial-vector currents are not of the canonical form, thus invalidating the current-algebra hypothesis. However, further analysis based on generalized Ward-Takahashi identities shows that the divergence equations are not affected. Furthermore, we discuss in which case the partially conserved axial-vector current approximation can be justified.

I. INTRODUCTION

The current-algebra hypothesis has been one of the most fruitful ideas in the theory of weak and electromagnetic interactions.[1] According to this hypothesis, the weak and electromagnetic currents can be expressed in terms of currents that are directly related to the internal symmetry of the hadronic system. The latter are the so-called canonical, or Noether, currents, which can be gen-