Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford Univ. Press, New York, 1971). In field theory this approximation was first used by K. Wilson, Phys. Rev. D 7, 2911 (1973).

[14]Here we disagree with S. Weinberg, Ref. 6, who quotes an error $O(\lambda\beta_c)$. However, we find below, in an approximate summation of graphs, that all $O(\sqrt{\lambda}\beta_c)$ corrections cancel. We have not established if this is generally true.

[15]The approximation we make is equivalent to the Hartee approximation in many-body theory. Recall the exact equation for the propagator:

$$(\Box_x+m_0^2)\langle T\varphi_a(x)\varphi_b(0)\rangle = -i\delta_{ab}\delta^4(x)-\tfrac{1}{6}\lambda\langle T\varphi_a(x)\varphi^2(x)\varphi_b(0)\rangle\,.$$

Replacing the correlation function on the right-hand side by $\langle T\varphi_a(x)\varphi_b(0)\rangle\langle\varphi^2(x)\rangle$ produces the same result as dropping the terms involving T in the Schwinger-Dyson equation of Fig. 7. Moreover, it is known that the Hartee approximation is exact in the many-body version of our large-N limit. Correspondingly, our Eqs. (3.41) and (3.42) have well-known analogs in many-body theory. See S.-k. Ma, Phys. Rev. A 7, 2172 (1973); E. Brêzin and D. J. Wallace, Phys. Rev. B 7, 1967 (1973). We thank Professor P. Martin for explaining this to us. The Hartree approximation has a history of field-theoretic applications, e.g., E. H. Lieb, Proc. R. Soc. A241, 339 (1957); K. Johnson, in Proceedings of Seminar on Unified Theories of Elementary Particles, Rochester, 1963 (unpublished).

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[17]Note that even though the ghosts obey Fermi statistics, their temperature Green's function $S\{\hat{\varphi};k\}$ is that of Bose particles, i.e., the time component of k is $2n\pi/(-i\beta)$. That this is correct follows from the fact that S must cancel against parts of the remaining expression. For further discussion, see Ref. 8.

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Quantum theory of dual relativistic parastring models*

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We paraquantize the classical massless relativistic-string action and find that the resulting theory is Poincaré-invariant in four space-time dimensions if we use para-Bose commutation relations of order 12. More generally, we find that if the dimension D of the space-time and the order q of parabosons are related by the expression $D=2+24/q$, then the quantized theory is Poincaré-invariant. We also construct a fermionic parastring model which is the analog of the Ramond-Neveu-Schwarz model and find that it is invariant in D dimensions if $D=2+8/q$, both the fermions and the bosons being of order q. We show by explicit Klein transformations that these theories are equivalent to "color"-endowed canonically quantized strings with SO$(q-1)$ "color" symmetry. We obtain dual tree amplitudes by suitable choice of vertices. Finally, we consider second-quantized parastring theories and show, by an explicit example, that they can be Poincaré-invariant in four space-time dimensions.

I. INTRODUCTION

The search for the understanding of the fundamental structure that underlies the dual resonance models has been the subject of many interesting investigations in recent years.[1-13] From among various approaches, the one which has reached the status of a bona fide theory is the gauge theory of the relativistic string,[14] which is based on a geometrical description initiated by Nambu.[5] In this case the fundamental structure is a massless relativistic string.

An important feature of the string model is that all of its properties follow from a single action:

$$S=\frac{1}{2\pi\alpha'}\int_0^\pi d^2\eta\sqrt{-g}\ , \tag{1.1}$$

where

$$\begin{aligned} g_{ab} &= \frac{\partial Y^\mu}{\partial \eta^a}\frac{\partial Y_\mu}{\partial \eta^b}\,, \\ Y^\mu(\tau,\theta) &\equiv \text{string variable}\,, \\ g &= \det(g_{ab})\,, \\ \alpha' &= \text{the only dimensional parameter of the theory } (\hbar = c = 1). \end{aligned} \tag{1.2}$$

This action can be arrived at

(a) by a generalization of the action of a free point particle,[5] or alternatively,[14,15]

(b) by requiring that as a function of $Y^\mu(\tau,\theta)$ and its derivatives

(i) it be Poincaré-invariant;

(ii) it be parametrization-invariant;

(iii) Euler-Lagrange equations involve derivatives not higher than the second.

The equations of motion which follow from the action (1.1) are fairly complicated

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial \eta^a}\left(\sqrt{-g}\; g^{ab}\frac{\partial}{\partial \eta^b}\right) Y^\mu(\eta^0,\eta^1) = 0\,. \tag{1.3}$$

It is thus clear that because of the nonlinearities involved a coordinate-invariant quantization of such a theory is out of the question. It is possible, however, to exploit the parametrization invariance of the action (1.1) to cast the theory into a manageable form. The simplest way of seeing this[14] is to note that by a coordinate transformation it is always possible to cast the differential quadratic form of the world sheet

$$ds^2 + g_{00}(d\eta^0)^2 + g_{01}\, d\eta^0\, d\eta^1 + g_{11}(d\eta^1)^2$$

into the form

$$ds^2 = g_{00}(d\tau^2 - d\theta^2)\,, \tag{1.4}$$

which is invariant under conformal transformations. This amounts to imposing the coordinate conditions:

$$\left(\frac{\partial Y^\mu}{\partial u^\pm}\right)^2 = 0\,, \quad u^\pm = \frac{1}{\sqrt{2}}(\tau \pm \theta)\,. \tag{1.5}$$

It is then straightforward to show that the spectrum as well as the ghost-eliminating conditions of the dual resonance model follow from the action (1.1) in the gauge specified by (1.5). One thus arrives at the gauge theory of the relativistic string.[14]

In quantizing a gauge theory, one can either work in a manifestly covariant gauge, in which case one must explicitly prove the absence of ghosts, or one can work in a manifestly ghost-free gauge, in which case one must explicitly prove the Poincaré invariance of the theory. The proof of the absence of ghosts in the manifestly covariant gauge can be carried out as far as showing that the spectrum of states is positive-*semidefinite*. To complete the proof one would have to show, among other things, that the generators of the Poincaré group are well defined in the *positive-definite* sector of the spectrum.

The quantization in one manifestly ghost-free gauge has been studied in detail.[16] It is found that such a quantized theory is Poincaré-invariant not in four but in 26 dimensions. To preserve Poincaré invariance even in a 26-dimensional world, it is further necessary that the ground state be a tachyon.

The free-string theory has also been extended to an interacting theory[17,18] in which the interaction is taken into account by studying the breaking and joining of the strings.[5] This theory is again Poincaré-invariant in 26 space-time dimensions and has a tachyon in the ground state.

Although disappointing, the beauty and the internal consistency of a gauge theory of interacting strings leads one to hope that a modified version of this theory might be free from these difficulties. Attempts at the modification of the theory could be made in various ways. In one of these,[19] the fundamental dynamical variable $Y^\mu(\tau,\theta)$ is replaced by a fermionic dynamical variable. It is found that the resulting theory is tachyon-free, ghost-free, and is Poincaré-invariant in four space-time dimensions. However, since the model departs from strict canonical formalism, the incorporation of interactions is not straightforward. In another recent modification,[20,21] attempts have been made to make the relativistic string massive. Here again strict canonical formalism can be maintained only at 25 space-time dimensions.[20]

The purpose of the present work is to present an alternative quantization of the massless relativistic string which is Poincaré-invariant in four space-time dimensions. This we do by making an essential use of the nonuniqueness of the passage from a classical to a quantum theory. It was shown by Green[22] some time ago that the ordinary canonical quantization is the simplest of a large class of quantization schemes which are now known as paraquantizations. The motivation for the paraquantization of the classical massless relativistic string is manifold. First, as we shall see, it will restore the Poincaré invariance in four space-time dimensions. Second, by leaving the classical theory intact and altering the quantization scheme, one may hope to learn the manner in which the formalism perfected by Mandelstam[18] ought to be altered so as to obtain an interacting theory of parastrings. Moreover, in recent years a number of equivalence theorems have been proved[23,24] connecting parastatistics with internal-symmetry groups. It would then be interesting to see what internal-symmetry groups could emerge from the study of the parastrings. Our approach is to be compared with another recent attempt[25] in which new models are constructed by introducing extra "colored" Fermi and Bose oscillators while leaving the string variable intact. Finally, the results which emerge from a first-quantized parastring theory show how a second-quantized parastring theory could be constructed, which is Poincaré-invariant in four space-time dimensions.

The study of parastatistics in the literature has been to a large extent confined to the second-quantized theories. In applying it to the first quantization of the classical string, we encounter a number of novel features which do not show up in the second-quantized theories. The most important of these is the manner in which the center-of-mass variables of the string are to be handled. On the one hand, the consistent paraquantization of the string variable requires *relative paracommutation* relations between the c.m. variables and the excitation modes; on the other hand, since the c.m. position and momentum operators are conjugate physical operators regardless of the nature of quantization, we must require that they satisfy the usual canonical commutation relations. To make these two requirements compatible, it is necessary to fix a direction in the paraspace of the Green components (see Sec. II) and specify the momenta along this direction. In the language of internal symmetry this amounts to reducing the over-all color symmetry from, say, SO(q) to SO($q-1$) where q is the order of parastatistics. It is interesting to note that Günaydin and Gürsey,[26] in their construction of the representations of the Poincaré group in an octonionic Hilbert space, were also led to specify a direction in that space to ensure that space-time and internal-symmetry transformations commute. Then the internal symmetry is reduced from G_2 to SU(3).

The remaining sections of this work are arranged as follows: In Sec. II we discuss the nonuniqueness of the quantization of a classical theory and review paraquantization from a point of view relevant to the subsequent sections. In Sec. III we paraquantize the classical string and find that the resulting theory is Poincaré-invariant in $D=2+24/q$ space-time dimensions, where q is the order of para-Bose statistics. The intercept remains at $\alpha_0=1$, as in the conventional theory, so that the ground state is still a tachyon. In Sec. IV we demonstrate, via a Klein transformation, the equivalence of the parastring model to a color-endowed canonically quantized string theory and show how dual tree amplitudes can be obtained by a suitable choice of vertex. In Sec. V we discuss second-quantized parastring theories which are Poincaré-invariant in four space-time dimensions. In Sec. VI we similarly construct a fermionic parastring theory which is Poincaré-invariant in $D=2+8/q$ space-time dimensions, where q is the *common order* of the parafermions and parabosons in the theory. Section VII is devoted to concluding remarks. A number of methematical details are relegated to the Appendix.

II. NONUNIQUENESS OF QUANTIZATION AND PARASTATISTICS[22,24,27,28]

To demonstrate the nonuniqueness of the quantization procedure, consider how starting from a classical Lagrangian for a free field one arrives at a quantized theory[29]:

(i) Independent dynamical variables and their canonical momenta are specified.

(ii) Via Noether's theorem conservation laws are derived. In particular, the generators P^μ and $M^{\mu\nu}$ of the Poincaré group are specified.

(iii) The c-number fields are made into operators satisfying the same field equations.

(iv) The expressions for the generators P^μ, $M^{\mu\nu}$, etc., in terms of the field operators are taken to be the same as the corresponding classical expressions, except for the symmetrization with respect to noncommuting operators.

From these requirements, it then follows that, e.g., for a single free field in momentum space, we have

$$[P^\mu, a_k^\dagger]_- = k^\mu a_k^\dagger\,, \tag{2.1}$$

where

$$P^\mu = \frac{1}{2}\sum_p P^\mu [a_p^\dagger, a_p]_\pm\,, \tag{2.2}$$

with plus (minus) sign corresponding to bosons (fermions). Substituting (2.2) into (2.1) we find

$$[[a_k^\dagger, a_l]_\pm, a_m]_- = -2\delta_{km} a_l\,, \tag{2.3}$$

$$[[a_k, a_l]_\pm, a_m]_- = 0\,. \tag{2.4}$$

Other relations can be obtained from these by Hermitian conjugation and the use of the Jacobi identity.

To recover the ordinary canonical quantization, one must impose two further requirements:

(v) The commutator or the anticommutator of two field operators is a c number.

(vi) All the physical operators such as P^μ must be written in normal-ordered form.

From (v) and (2.4) it follows that

$$[a_k, a_l]_\pm = 0\,.$$

Then (2.3) reduces to

$$[:[a_k^\dagger, a_l]_\pm :, a_m] = -2\delta_{km} a_l$$

or

$$[a_l, a_k^\dagger]_\pm = \delta_{lk}\,.$$

These are just the ordinary canonical commutation rules. It thus follows that the main difference between the ordinary quantization and the more general paraquantization conditions (2.3) and (2.4) is the requirement (v). Since this requirement does not follow from any general physical principles, it need not be imposed. It is therefore clear that there are other inequivalent representations of paracommutation relations (PCR) (2.3) and (2.4) which may be adopted as a basis for quantization, that is, the quantization procedure is not unique.

Of the many inequivalent unitary irreducible representations of the paracommutation relations (2.3) and (2.4), the ones relevant to our work are the Fock-type irreducible representations realized on a Hilbert space with a unique vacuum state annihilated by all a_k:

$$a_k|0\rangle = 0\,. \tag{2.5}$$

These representations can be most conveniently characterized by specifying the order of the parafields through the Green's ansatz

$$a_n = \sum_{\alpha=1}^{q} a_n^\alpha\,, \tag{2.6}$$

where q is the order of the parafield. The Green components a_n^α satisfy the anomalous commutation relations

$$\begin{aligned}
&[a_n^\alpha, a_m^{\alpha\dagger}]_{-\epsilon} = \delta_{nm}\,,\\
&[a_n^\alpha, a_m^\alpha]_{-\epsilon} = 0\,,\\
&[a_n^\alpha, a_m^\beta]_\epsilon = [a_n^\alpha, a_m^{\beta\dagger}]_\epsilon = 0\,, \quad \alpha \neq \beta
\end{aligned} \tag{2.7}$$

where

$$\epsilon = \begin{cases} +1 \text{ for parabosons}\,,\\ -1 \text{ for parafermions}\,.\end{cases}$$

It has been shown by Greenberg and Messiah[27] that all the irreducible Fock representations of Eqs. (2.3)-(2.5) are given by the Green's ansatz (2.6).

Recently, a number of theorems concerning the equivalence of a single parafield of order q with an ordinary field with q additional degrees of freedom have been established.[23,24,28] One version of these theorems given by Ohnuki and Kamefuchi[24] is based on the requirement of strong locality

$$[F(V_I), \psi(X)] = 0\,, \quad x \notin V_I \tag{2.8}$$

where $F(V_I)$ is a Hermitian functional of $\psi(X)$ and $\psi^\dagger(X)$ representing an observable. They are then able to show that $F(V_I)$ must be a functional of the bilinears $[\psi(X), \psi(Y)]$, $[\psi(X), \psi^\dagger(Y)]$, $[\psi^\dagger(X), \psi^\dagger(Y)]$, $X, Y \in V_I$. Then, Klein transformations[30] are utilized to study the nature of the equivalence between a parafield (PF-field) of order q with a single ordinary field with a hidden variable which takes on q values (F-field). For a parafield, one sets

$$\psi^\alpha(X) = K_\alpha \phi^\alpha(X)\,, \tag{2.9}$$

with K_α such that

$$\begin{aligned}
&K_\alpha|0\rangle = |0\rangle\,, \quad K_\alpha^\dagger = K_\alpha^{-1}\\
&[K_\alpha, \phi^\beta(X)] = 0\,, \quad \text{for } \alpha > \beta\\
&\{K_\alpha, \phi^\beta(X)\} = 0\,, \quad \text{for } \alpha \leq \beta\,.
\end{aligned} \tag{2.10}$$

Then if, for example, $\psi^\alpha(x)$ is a parafermion, it can be seen immediately that

$$\begin{aligned}
&\{\phi^\alpha(X), \phi^{\beta\dagger}(Y)\}_{X^0=Y^0} = \delta^{\alpha\beta}\delta^3(X-Y)\,,\\
&\{\phi^\alpha(X), \phi^\beta(Y)\}_{X^0=Y^0} = 0\,.
\end{aligned} \tag{2.11}$$

The bilinear operators mentioned above can be expressed in terms of the Klein-transformed fields $\phi^\alpha(X)$[24]:

$$\begin{aligned}
&[\psi(X), \psi(Y)] = \sum_{\alpha=1}^{q} [\phi^\alpha(X), \phi^\alpha(Y)]\,,\\
&[\psi^\dagger(X), \psi^\dagger(Y)] = \sum_{\alpha=1}^{q} [\phi^\alpha(X), \phi^{\alpha\dagger}(Y)]\,,\\
&[\psi(X), \psi^\dagger(Y)] = \sum_{\alpha=1}^{q} [\phi^{\alpha\dagger}(X), \phi^{\alpha\dagger}(Y)]\,.
\end{aligned} \tag{2.12}$$

The expressions on the right-hand side are invariant under the gauge transformations SO(q) of the form

$$\phi^\alpha(X) = \sum_{\beta=1}^{q} g_{\alpha\beta}\,\phi^\beta(X)\,, \quad g \in \mathrm{SO}(q)\,. \tag{2.13}$$

The relation $[\psi(X), \psi^\dagger(Y)]$ alone is invariant under SU(q) transformations of the form (2.13) with $g \in \mathrm{SU}(q)$. Then two types of equivalence between the PF-fields and the F-fields can be stated as follows[24]:

(a) If the observables are functionals of all the $[\psi, \psi]$, $[\psi^\dagger, \psi^\dagger]$, $[\psi, \psi^\dagger]$, then PF-field $\rightarrow$ F-field for all possible physical consequences, but the converse is not true.

(b) If the observables are restricted to be functionals of $[\psi, \psi^\dagger]$ only, then PF-field $\longleftrightarrow$ F-field for all possible physical consequences.

III. THE RELATIVISTIC PARASTRING MODEL

We now proceed to the paraquantization of the massless classical string. As mentioned in Sec. I, it is always possible[14] to work in a gauge in which the differential quadratic form of the world sheet has the form

$$ds^2 = g_{00}(d\tau^2 - d\theta^2). \tag{3.1}$$

Then in the notation of Ref. 14 the gauge conditions (1.5) can be written as

$$\begin{aligned} &g_{00} + g_{11} = 0, \\ &g_{01} = 0, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} g_{00} &= \left(\frac{\partial Y^\mu}{\partial \tau}\right)^2 \equiv \dot{Y}^2, \\ g_{11} &= \left(\frac{\partial Y^\mu}{\partial \theta}\right) = Y'^2, \\ g_{01} &= \frac{\partial Y^\mu}{\partial \tau}\frac{\partial Y_\mu}{\partial \theta} = \dot{Y}\cdot Y'. \end{aligned} \tag{3.3}$$

In this gauge the equation of motion (1.3) for Y^μ reduces to

$$\left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \theta^2}\right) Y^\mu(\tau,\theta) = 0. \tag{3.4}$$

A. Noncovariant paraquantization

The coordinate constraints (3.2) do not completely specify a gauge. This is because the differential quadratic form (3.1) specifies a *conformally flat* manifold, so that under conformal transformations

$$\tau \to \tau'(\theta,\tau), \quad \theta \to \theta'(\tau,\theta),$$

$$g_{00}(\tau,\theta) \to g'_{00}(\tau',\theta')$$

and the form (3.1) is left invariant. As a result, τ' and θ' must satisfy the Cauchy-Riemann conditions

$$\frac{\partial\theta'}{\partial\theta} = \frac{\partial\tau'}{\partial\tau}, \quad \frac{\partial\theta'}{\partial\tau} = \frac{\partial\tau'}{\partial\theta}.$$

Moreover,

$$\left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \theta^2}\right) \tau'(\theta,\tau) = 0.$$

Since τ' satisfies the same differential equation as Y^μ and can be made to satisfy the same boundary conditions, the suggestion has been made[16] to make one component of Y^μ proportional to τ. Then using Dirac's null-plane dynamics,[31] we set

$$Y^+ = 2\alpha' P^+ \tau, \tag{3.5}$$

where P^+ is the string momentum conjugate to the coordinate X^-.

With the boundary conditions

$$\left.\frac{\partial Y^\mu}{\partial\theta}\right|_{\theta=0,\pi} = 0, \tag{3.6}$$

a general solution of Eq. (3.4) has the form

$$Y^\mu(\tau,\theta) = AX^\mu + BP^\mu\tau + C\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(a_n^\mu e^{-in\tau} + a_n^{\mu\dagger}e^{in\tau})\cos n\theta.$$

In the conventional quantization, the coefficients A, B, and C are fixed by requiring that canonical commutation relations on a_n^μ and $a_n^{\mu\dagger}$ lead to the corresponding relations between $Y^\mu(\tau,\theta)$ and its conjugate variable, and that the independent components of P^μ follow canonically from the Lagrangian. It turns out that in paraquantization the same requirements determine the coefficients A, B, and C uniquely. Anticipating this result, we write

$$Y^\mu(\tau,\theta) = X^\mu + 2\alpha' P^\mu\tau + (2\alpha')^{1/2}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(a_n^\mu e^{-in\tau} + a_n^{\mu\dagger}e^{in\tau})\cos n\theta, \tag{3.7}$$

where, by (3.5)

$$X^+ = 0, \quad a_n^\dagger = 0. \tag{3.8}$$

This means that the momentum P^- conjugate to X^+ as well as the modes a_m^- are not independent dynamical variables. In fact, from the constraint equations (3.2), we find

$$P^- = \frac{1}{2\alpha' P^+}\left\{\frac{1}{4\pi\alpha'}\int_0^\pi d\theta\left[\left(\frac{d\vec{Y}}{d\tau}\right)^2 + \vec{Y}'^2\right]\right\} \tag{3.9}$$

and

$$a_m^- = \frac{i}{(2\alpha')^{1/2}P^+}\frac{T_m}{\sqrt{m}},$$

where

$$T_m = \frac{1}{4\pi\alpha'}\int_0^\pi d\theta\left[\left(\frac{d\vec{Y}}{d\tau}\right)^2 + \vec{Y}'^2\right]\cos m\theta + \frac{i}{2\pi\alpha'}\int_0^\pi d\theta\,\frac{d\vec{Y}}{d\tau}\cdot\vec{Y}'\sin m\theta, \tag{3.10}$$

$$Y^\mu = (Y^-, Y^+, \vec{Y}). \tag{3.11}$$

Before we carry out the paraquantization of the classical string, we state a number of requirements which such a quantization must satisfy: (1) Paraquantization of the string variable must lead to the paraquantization of the Fourier coefficients in its normal-mode expansion, and vice versa. (2) The space-time properties of the string must not be affected by paraquantization. In particular, since momenta are observables, the center-of-mass variables must satisfy the ordinary canonical commutation relations. (3) Since for each value of the index μ, Y^μ is a single dynamical variable, the independent components of X^μ, P^μ, and a_n^μ's must satisfy parastatistics of the same order. Moreover, they must satisfy relative paracommutation relations with each other and for different values of the index μ. This last requirement is crucial to even maintaining three-dimensional rotational invariance in *any* paraquantized theory, not just the string.[27]

With these requirements in mind, we will express our independent dynamical variables in terms of their Green's ansatz:

$$Y^{i}(\tau,\theta)=\sum_{\beta=1}^{q} Y^{i\beta}(\tau,\theta)\,,\quad i=1,2 \tag{3.12}$$

$$X^{-}=\sum_{\beta=1}^{q} X^{-\beta}\,,\quad P^{+}=\sum_{\beta=1}^{q} P^{+\beta}\,, \tag{3.13}$$

where

$$Y^{i\beta}(\tau,\theta)=X^{i\beta}+2\alpha' P^{i\beta}\tau + (2\alpha')^{1/2}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\,(a_n^{i\beta}e^{-in\tau}+a_n^{i\beta\dagger}e^{in\tau})\cos n\theta\,. \tag{3.14}$$

It may appear at first that one can simply apply the paraquantization (2.7) to the independent Green components $X^{i\beta}$, $P^{i\beta}$, $a_n^{i\beta}$, $a_n^{i\beta\dagger}$, $X^{-\beta}$, $P^{+\beta}$. However, it is easy to check that such a procedure will violate one or more of the requirements stated above. In fact, to satisfy these requirements one must specify a direction in the paraspace of the Green's ansatz, say $\beta=1$, and demand that the c.m. coordinates have a component in that direction only. That is,

$$X^{i\beta}=X^{i}\delta_{\beta 1}\,,\quad P^{i\beta}=P^{i}\delta_{\beta 1}\,,\qquad X^{-\beta}=X^{-}\delta_{\beta 1}\,,\quad P^{+\beta}=P^{+}\delta_{\beta 1}\,. \tag{3.15}$$

In the language of internal symmetry this means that from the Klein-transformed analogs of our dynamical variables, one can construct observables which are invariant under those transformations which leave the direction $\beta=1$ invariant. As we mentioned in the Introduction, this way of disentangling the space-time and internal-symmetry tranformations is reminiscent of the work by Günaydin and Gürsey[26] on the octonionic representations of the Poincaré group.

Thus, for the transverse components $Y^{i\beta}(\tau,\theta)$ we have, instead of (3.14)

$$Y^{i\beta}(\tau,\theta)=X^{i}\delta_{\beta 1}+2\alpha' P^{i}\delta_{\beta 1}\tau+(2\alpha')^{1/2}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\,(a_n^{i\beta}e^{-in\tau}+a_n^{i\beta\dagger}e^{in\tau})\cos n\theta\,,\quad i=1,2\,. \tag{3.16}$$

The independent Green components which are to be paraquantized are

$$X^{i},P^{i},X^{-},P^{+},a_n^{i\beta},a_n^{i\beta\dagger}\,,\quad i=1,2\,. \tag{3.17}$$

For these we have

$$\begin{aligned}
&[X^{i},P^{j}]=i\delta^{ij}\,,\quad [X^{-},P^{+}]=-i\,,\\
&[a_n^{i\beta},a_m^{j\beta\dagger}]=\delta^{ij}\delta_{mn}\,,\quad [a_n^{i\beta},a_m^{j\beta}]=0\,,\quad i=1,2\\
&\{a_n^{i\beta},a_m^{j\alpha\dagger}\}=\{a_n^{i\beta},a_m^{j\alpha}\}=0\,,\quad \alpha\neq\beta\\
&[X^{i},a_n^{j1}]=[X^{-},a_n^{j1}]=[P^{i},a_n^{j1}]=[P^{+},a_n^{j1}]=0\,,\\
&\{X^{i},a_n^{j\beta}\}=\{X^{-},a_n^{j1}\}=\{P^{i},a_n^{j\beta}\}=\{P^{+},a_n^{j\beta}\}=0\,,\quad \beta\neq 1\,.
\end{aligned} \tag{3.18}$$

Using these relations we then find

$$[\,Y^{i\beta}(\theta),\Pi^{j\beta}(\theta')]_{\tau=\tau'}=i\,\frac{\delta^{ij}}{\pi}[\,\delta_{\beta 1}-1+\pi\delta(\theta-\theta')]\,, \tag{3.19}$$

where $\Pi^{j\beta}(\theta)$ is the momentum canonically conjugate to $Y^{j\beta}(\theta)$.

Next, we shall express the null-plane Hamiltonian P^{-} and the constraints T_m in terms of the transverse normal modes. In evaluating quadratic expressions such as $(d\vec{Y}/d\tau)^2$ and $\vec{Y}'^2$ we must use symmetrized expressions because the Green component fields satisfy anomalous commutation relations. Thus we write

$$\begin{aligned}
&\left(\frac{d\vec{Y}}{d\tau}\right)^2=\frac{1}{2}\sum_{\alpha,\beta=1}^{q}\{\dot{Y}^{i\alpha},\dot{Y}^{i\beta}\}\,,\\
&\vec{Y}'^2=\frac{1}{2}\sum_{\alpha,\beta=1}^{q}\{Y'^{i\alpha},Y'^{i\beta}\}\,,\\
&\frac{d\vec{Y}}{d\tau}\cdot\vec{Y}'=\frac{1}{2}\sum_{\alpha,\beta=1}^{q}\{\dot{Y}^{i\alpha},Y'^{i\beta}\}\,.
\end{aligned} \tag{3.20}$$

Then using (3.18), it is straightforward to show that

$$T_l=\sum_{\beta=1}^{q} T_l^{\beta}\,, \tag{3.21}$$

where

$$\begin{aligned}
T_l^{\beta}=\alpha'\,\vec{P}^2\delta_{l,0}\delta_{\beta 1}&+\sum_{n=1}^{\infty}[n(n+l)]^{1/2}\,\vec{a}_n^{\,\beta\dagger}\cdot\vec{a}_{n+l}^{\,\beta}\\
&-\frac{1}{2}\sum_{n=1}^{l-1}[n(l-n)]^{1/2}\,\vec{a}_n^{\,\beta}\cdot\vec{a}_{l-n}^{\,\beta}\\
&-i(2\alpha')^{1/2}\sqrt{l}\,\vec{P}\cdot\vec{a}_l^{\,\beta}\delta_{\beta 1}-(\alpha_0/q)\delta_{l,0}\,.
\end{aligned} \tag{3.22}$$

Moreover,

$$T_{-l}^{\beta}\equiv T_l^{\beta\dagger}\,,\quad T_{-l}=T_l^{\dagger}\,. \tag{3.23}$$

The following relations can be easily verified:

$$[T_l^\beta, a_k^{i\alpha}] = -\delta^{\alpha\beta}[k(k+l)]^{1/2} a_{k+l}^{j\alpha},$$

$$[T_l^\beta, a_k^{i\alpha\dagger}] = \delta^{\alpha\beta}[k(k-l)]^{1/2} a_{k-l}^{i\alpha\dagger} - \delta^{\alpha\beta}[k(l-k)]^{1/2} a_{l-k}^{i\alpha} - i(2\alpha')^{1/2}\sqrt{l}\,\delta^{\alpha\beta}\delta_{\beta 1}P^{i\alpha}\delta_{k,l}, \qquad (3.24)$$

$$[T_l^\beta, P^i] = 0, \quad [T_l^\beta, P^+] = [T_l^\beta, X^-] = 0,$$

$$[T_l^\beta, X^i] = -2i\alpha' P^i \delta_{l,0}\delta_{\beta 1} - (2\alpha')^{1/2}\sqrt{l}\, a^{i\beta}\delta_{\beta 1}.$$

From these it follows that

$$[T_l, a_k^j] = -[k(l+k)]^{1/2} a_{k+l}^j,$$

$$[T_l, a_k^{j\dagger}] = [k(k-l)]^{1/2} a_{k-l}^{j\dagger} - [k(l-k)]^{1/2} a_{l-k}^j - i(2\alpha')^{1/2}\sqrt{l}\,P^j\delta_{k,l}, \qquad (3.25)$$

$$[T_l, P^j] = [T_l, P^+] = [T_l, X^-] = 0,$$

$$[T_l, X^j] = -2i\alpha' P^j \delta_{l,0} - (2\alpha')^{1/2}\sqrt{l}\, a_l^{j1},$$

where

$$a_l^{j1} = a_l^{j\beta}|_{\beta=1}. \qquad (3.26)$$

Finally, we have for fixed β

$$[T_n^\beta, T_m^\beta] = (n-m)T_{n+m}^\beta + \tfrac{1}{12}(D-2)\delta_{n+m,0}(n^3-n). \qquad (3.27)$$

Therefore,

$$[T_n, T_m] = (n-m)T_{n+m} + \tfrac{1}{12}q(D-2)(n^3-n)\delta_{n+m,0}, \qquad (3.28)$$

where, explicitly,

$$T_m = \alpha'\vec{\mathrm{P}}^2\delta_{m,0} + \sum_{\beta=1}^{q}\sum_{n=1}^{\infty}[n(n+m)]^{1/2}\,\vec{\mathrm{a}}_n^{\beta\dagger}\cdot\vec{\mathrm{a}}_{n+m}^\beta - \frac{1}{2}\sum_{\beta=1}^{q}\sum_{n=1}^{m-1}[n(m-n)]^{1/2}\,\vec{\mathrm{a}}_n^\beta\cdot\vec{\mathrm{a}}_{m-n}^\beta - i(2\alpha')^{1/2}\sqrt{m}\,\vec{\mathrm{P}}\cdot\vec{\mathrm{a}}_m^1 - \alpha_0\delta_{m,0}. \qquad (3.29)$$

The Hamiltonian P^- may now be written as

$$P^- = \frac{1}{2\alpha' P^+}T_0 = \frac{1}{2\alpha' P^+}\left(\alpha'\vec{\mathrm{P}}^2 + \sum_{\beta=1}^{q}\sum_{n=1}^{\infty} n\,\vec{\mathrm{a}}_n^{\beta\dagger}\cdot\vec{\mathrm{a}}_n^\beta - \alpha_0\right). \qquad (3.30)$$

This means that the (mass)^2 operator must have the form

$$M^2 \equiv 2P^+P^- - \vec{\mathrm{P}}^2 = \frac{1}{\alpha'}\sum_{\beta=1}^{q}\sum_{n=1}^{\infty} n\,\vec{\mathrm{a}}_n^{\beta\dagger}\cdot\vec{\mathrm{a}}_n^\beta - \frac{\alpha_0}{\alpha'}. \qquad (3.31)$$

To check the consistency of the quantum parastring model, we must now construct the Lorentz generators and check their commutation relations. The generators which follow from the classical action are

$$M^{\mu\nu} = \tfrac{1}{2}\int_0^\pi d\theta\,(\{Y^\mu, \Pi^\nu\} - \{Y^\nu, \Pi^\mu\}). \qquad (3.32)$$

Then, as was outlined in Sec. II, the quantum-mechanical generators are obtained by replacing Y^μ and Π^μ by the corresponding paraoperators. We then find that the various components of $M^{\mu\nu}$ are (see the Appendix for details)

$$\begin{aligned}
M^{ij} &= X^iP^j - X^jP^i + i\sum_{\beta=1}^{q}\sum_{n=1}^{\infty}(a_n^{i\beta}a_n^{j\beta\dagger} - a_n^{j\beta}a_n^{i\beta\dagger}),\\
M^{i+} &= X^iP^+,\\
M^{+-} &= \tfrac{1}{2}(P^+X^- + X^-P^+),\\
M^{i-} &= \frac{1}{4\alpha' P^+}(X^iT_0 + T_0X^i) - X^-P^i + \frac{1}{(2\alpha')^{1/2}P^+}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(T_n^\dagger a_n^{i1} + a_n^{i1\dagger}T_n).
\end{aligned} \qquad (3.33)$$

It is to be emphasized that these generators are not obtained by simply attaching an extra index to the creation and destruction operators. For $q=1$, these expressions will, of course, reduce to those of the canonical quantization.[16] We have checked various commutators of the above algebra. We find in particular, for $i\neq j$,

$$[M^{i-}, M^{j}] = \frac{1}{\alpha' P^{+2}}\sum_{n=1}^{\infty}(a_n^{i1\dagger}a_n^{j1} - a_n^{j1\dagger}a_n^{i1})\left[\left(\frac{q(D-2)}{24} - 1\right)n^2 - \left(\frac{q(D-2)}{24} - \alpha_0\right)\right]. \qquad (3.34)$$

In order that this commutator vanish we must have

$$\alpha_0 = 1, \qquad D = 2 + \frac{24}{q}. \qquad (3.35)$$

In particular, for $q=12$ the theory is Lorentz-invariant in four space-time dimensions. Generally, the space-time dimensions for which a Lorentz-invariant theory can be constructed are 26, 14, 10, 8, 6, 5, 4, 3, 2. These correspond, respectively, to q values of 1, 2, 3, 4, 6, 8, 12, 24, ∞. We note that for $q=\infty$ the para-Bose statistics become ordinary Fermi statistics.[32]

The above procedure can be repeated for the closed string boundary conditions, with the result that again $D=2+24/q$. We also expect that in a massive parastring model constructed in this fashion, the dimension of space-time is related to the order of paraquantization according to $D=1+24/q$.

B. Covariant paraquantization

The covariant paraquantization can be carried out essentially along the same lines as the ordinary canonical quantization.[14] The main difference is that the quantization must be compatible with the requirements set forth in Sec. III A.

Thus we again write

$$Y^\mu(\tau,\theta)=\sum_{\beta=1}^{q} Y^{\mu\beta}(\tau,\theta)\,, \tag{3.36}$$

where

$$Y^{\mu\beta}(\tau,\theta)=X^\mu\delta_{\beta 1}+2\alpha' P^\mu\delta_{\beta 1}+(2\alpha')^{1/2}\sum_{N=1}^{\infty}\frac{1}{\sqrt{N}}\left(a_N^{\mu\beta}e^{-iN\tau}+a_N^{\mu\beta\dagger}e^{iN\tau}\right)\times\cos N\theta\,, \tag{3.37}$$

with

$$\Pi^\mu(\tau,\theta)=\frac{1}{2\pi\alpha'}\dot{Y}^\mu \tag{3.38}$$

We require equal-time paracommutation relations of the Green components:

$$\begin{aligned}
[Y^{\mu\beta}(\tau,\theta),\Pi^{\nu\beta}(\tau,\theta')] &= -i\frac{g^{\mu\nu}}{\pi}\left[\delta_{\beta 1}-1+\pi\delta(\theta-\theta')\right],\\
[Y^{\mu\beta}(\theta),Y^{\nu\beta}(\theta')] &= [\Pi^{\mu\beta}(\theta),\Pi^{\nu\beta}(\theta')]=0\,,\\
\{Y^{\mu\beta}(\theta),Y^{\nu\alpha}(\theta')\} &= \{Y^{\mu\beta}(\theta),\Pi^{\nu\alpha}(\theta')\}\\
&= \{\Pi^{\mu\beta}(\theta),\Pi^{\nu\alpha}(\theta')\}\\
&= 0,\quad \alpha\neq\beta\,.
\end{aligned} \tag{3.39}$$

The various paracommutators between $X^{\mu\beta}$, $P^{\mu\beta}$, $a_N^{\mu\beta}$, $a_N^{\mu\beta\dagger}$ follow from these in the usual manner. They are similar in structure to those given in (3.18). The constraints (3.2) are now imposed as *weak* operator conditions on states.[14] Defining the Fourier coefficients

$$T_{\pm M}=\frac{1}{4\pi\alpha'}\int_0^\pi d\theta\left[(\dot{Y}^\mu)^2+(Y'^\mu)^2\right]\cos M\theta\pm\frac{i}{2\pi\alpha'}\int_0^\pi d\theta\,\dot{Y}^\mu Y'_\mu\sin M\theta\,, \tag{3.40}$$

we find, with $A^\mu B_\mu=A\cdot B$,

$$\begin{aligned}
T_M=&-\alpha' P^2\delta_{M,0}-\sum_{\beta=1}^{q}\sum_{N=1}^{\infty}[N(N+M)]^{1/2}a_N^{\beta\dagger}\cdot a_{N+M}^{\beta}\\
&+\frac{1}{2}\sum_{\beta=1}^{q}\sum_{N=1}^{M-1}[N(M-N)]^{1/2}a_N^{\beta}\cdot a_{M-N}^{\beta}\\
&+(2\alpha')^{1/2}\sqrt{M}\,P\cdot a_M^1+\alpha_0\delta_{M,0}\,.
\end{aligned} \tag{3.41}$$

The algebra of T_M's has the form

$$[T_N,T_M]=(N-M)T_{M+N}+\tfrac{1}{12}qD(n^3-n)\delta_{n+m,0}\,. \tag{3.42}$$

Thus the weak operator conditions on quantum states are

$$T_M|\psi\rangle=0\,,\quad M\geq 0\,. \tag{3.43}$$

It is easy to check that these conditions are compatible. Because of these constraints, the spectrum of states $|\psi\rangle$ is *positive-semidefinite*.

To show the consistency of the covariant paraquantized theory, it is necessary to show that the matrix elements of all physical operators are well defined in the *positive-definite* sector of the states $|\psi\rangle$. In particular, one must show that the Lorentz generators have support in this sector only. This is at present an open question, just as in the conventional string formalism.

IV. CONNECTION WITH INTERNAL SYMMETRY AND DUAL TREE AMPLITUDES

In this section we will discuss the problem of constructing an interacting theory of parastrings. A complete solution to this problem would be the analog of Mandelstam's solution[18] to the conventional interacting string theory. We shall not undertake that task in this paper. Instead, we will discuss how by suitable choice of vertices the conventional dual tree amplitudes can be obtained from a parastring theory which is Poincaré-invariant in four (as well as a number of other) space-time dimensions. Since the duality properties are more transparent in a canonically quantized version of our theory with internal symmetry than it is in our paraquantized theory, we will first show, by a Klein transformation, that a free parastring model of order q can be recast into a canonically quantized colored string theory with internal SO$(q-1)$ symmetry.

A. Connection with internal symmetry

To show the equivalence of the parastring model with an ordinary colored string with SO$(q-1)$ symmetry, we must implement the Klein transformations (2.9) and (2.10) for our independent dynamical variables. We first write for the normal modes

$$a_m^{j\alpha}=K_\alpha A_m^{j\alpha}\,,\quad j=1,2 \tag{4.1}$$

with K_α such that

$$\begin{aligned}&K_\alpha\,|0\rangle=|0\rangle\,,\quad K_\alpha^\dagger=K_\alpha^{-1}\\&[K_\alpha,A_m^{j\beta}]=0\,,\ \text{for}\ \alpha>\beta\\&\{K_\alpha,A_m^{j\beta}\}=0\,,\ \text{for}\ \alpha\leq\beta\,.\end{aligned} \tag{4.2}$$

Then it is easy to check that

$$[A_m^{i\alpha},A_n^{j\beta\dagger}]=\delta^{ij}\delta_{mn}\delta^{\alpha\beta}\,. \tag{4.3}$$

It now remains to construct the operator K_α explicitly. Consider the operator

$$K_\alpha^0=\exp\left(-i\pi\sum_{n=1}^{\infty}\sum_{\gamma=\alpha}^{q}\vec{A}_n^{\gamma\dagger}\cdot\vec{A}_n^{\gamma}\right), \tag{4.4}$$

which has basically the same form as the Klein operator of Nambu and Han.[28] It clearly satisfies the conditions (4.2).

We must now modify K_α^0 so as to have the correct behavior with respect to the c.m. operators $\vec{X}$, $\vec{P}$, X^-, P^+. One way to do this is to express the latter in terms of zero-mode creation and destruction operators:

$$\begin{aligned}&\left.\begin{aligned}X^j&=\tfrac{1}{2}(a_0^j+a_0^{j\dagger})\\P^j&=-\tfrac{1}{2}i\,(a_0^j-a_0^{j\dagger})\end{aligned}\right\}\quad j=1,2\,,\\&X^-=\tfrac{1}{2}(a_0+a_0^\dagger)\,,\\&P^+=+\tfrac{1}{2}i\,(a_0-a_0^\dagger)\,.\end{aligned} \tag{4.5}$$

As in (4.1) we set

$$\begin{aligned}a_0^j&=K_1A_0^j\,,\\a_0&=K_1A_0^j\,.\end{aligned} \tag{4.6}$$

We now construct our Klein operators as follows:

$$\begin{aligned}&K_1=K_1^0\exp\{-\pi[\vec{A}_0^\dagger\cdot\vec{A}_0+A_0^\dagger A_0]\}\,,\\&K_\alpha=K_\alpha^0\,,\quad\alpha=2,\ldots,q\,.\end{aligned} \tag{4.7}$$

Then, in addition to (4.2) we have

$$\begin{aligned}&[K_\alpha,A_0^j]=[K_\alpha,A_0]=0\,,\quad\alpha>1\\&\{K_1,A_0^j\}=\{K_1,A_0\}=0\,.\end{aligned} \tag{4.8}$$

Denoting the Klein-transformed string variable and the c.m. operators by the same symbols $Y^{i\beta}(\tau,\theta)$, X's, and P's, we get

$$\begin{aligned}Y^{i\beta}(\tau,\theta)=X^i\delta_{\beta1}+2\alpha'P^i\delta_{\beta1}&\\+(2\alpha')^{1/2}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}&\,(A_n^{i\beta}e^{-in\tau}+A_n^{i\beta\dagger}e^{in\tau})\\&\times\cos n\theta\,,\qquad i=1,2\end{aligned} \tag{4.9}$$

where now

$$[Y^{i\beta}(\tau,\theta),\Pi^{j\alpha}(\tau,\theta')]=\frac{i}{\pi}\delta^{ij}\delta^{\alpha\beta}[\delta_{\beta1}-1+\pi\delta(\theta-\theta')]\,. \tag{4.10}$$

It is interesting to note that the motivation for the choice of a direction, e.g., $\beta=1$, which appeared quite naturally from the paraquantization point of view, would not have been as clear if we had started from the internal-symmetry point of view. Insisting, nevertheless, that the direction $\beta=1$, remain invariant, we consider color transformations of the type

$$\begin{aligned}Y^{i\alpha}(\tau,\theta)&=\sum_{\beta=2}^{q}g_{\alpha\beta}Y^{i\beta}(\tau,\theta)\,,\\\Pi^{i\alpha}(\tau,\theta)&=\sum_{\beta=2}^{q}g_{\alpha\beta}\Pi^{i\beta}(\tau,\theta)\,.\end{aligned} \tag{4.11}$$

Since both $Y^{i\alpha}$ and $\Pi^{i\alpha}$ are Hermitian we must have

$$g_{\alpha\beta}=g_{\alpha\beta}^\dagger\,. \tag{4.12}$$

Since the functional dependence of the observables on operators which carry the index α, $\alpha=2,\ldots,q$, are of the form

$$(\vec{Y}^\alpha)^2,\ (\vec{\Pi}^\alpha)^2,\ \vec{Y}^\alpha\cdot\vec{\Pi}^\alpha\,,$$

then the color group is SO($q-1$). Thus under the transformations of this gauge group, we can distinguish two types of excitations: the *color-independent excitations*, $\beta=1$, which are invariant under SO($q-1$) transformations and the *color-dependent excitations*, $\beta=2,\ldots,q$, which transform as the fundamental representation of SO($q-1$).

The Klein-transformed operators and their corresponding gauge group for the covariant formalism can be obtained in a similar manner. Consider the Klein operators K_α, with

$$\begin{aligned}K_1&=\exp\left[-i\pi\left(A_0^{\mu\dagger}A_{0\mu}+\sum_{n=1}^{\infty}\sum_{\gamma=\alpha}^{q}A_n^{\mu\gamma\dagger}A_{\mu n}^{\gamma}\right)\right],\\K_\alpha&=\exp\left(-i\pi\sum_{n=1}^{\infty}\sum_{\gamma=\alpha}^{q}A_n^{\mu\gamma\dagger}A_{\mu n}^{\gamma}\right),\quad\alpha>1\,.\end{aligned} \tag{4.13}$$

Then with

$$\begin{aligned}a_m^{\mu\alpha}&=K_1A_m^{\mu\alpha}\,,\\a_0^\mu&=K_1A_0^\mu\,,\end{aligned} \tag{4.14}$$

one finds

$$\begin{aligned}&[K_\alpha,A_m^{\mu\beta}]=0\,,\quad\alpha>\beta\\&\{K_\alpha,A_m^{\mu\beta}\}=0\,,\quad\alpha\leq\beta\\&[K_\alpha,A_0^\mu]=0\,,\quad\alpha>1\\&\{K_1,A_0^\mu\}=0\,.\end{aligned} \tag{4.15}$$

In this form

$$[A_m^{\mu\alpha},A_n^{\nu\beta\dagger}]=-g^{\mu\nu}\delta_{mn}\delta^{\alpha\beta} \tag{4.16}$$

and

$$[Y^{\mu\beta}(\tau,\theta),\Pi^{\nu\alpha}(\tau,\theta')] = \frac{-i}{\pi} g^{\mu\nu}\delta^{\alpha\beta}[\delta_{\beta 1}-1+\pi\delta(\theta-\theta')]\,, \quad (4.17)$$

where

$$Y^{\mu\beta}(\tau,\theta)=X^{\mu}\delta_{\beta 1}+2\alpha' P^{\mu}\delta_{\beta 1}\tau + (2\alpha')^{1/2}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(A_n^{\mu\beta}e^{-in\tau}+A_n^{\mu\beta\dagger}e^{in\tau}) \times \cos n\theta\,. \quad (4.18)$$

All the paraoperators such as T_m's, P^-, M^2, $M^{\mu\nu}$ which we obtained in Sec. III may now be Klein-transformed by replacing the small a's with capital A's. We will leave it to the reader to verify that the algebraic relations among the Klein-transformed operators are the same as those of the corresponding paraoperators.

B. The dual tree amplitudes

The dual tree amplitudes can be written down from the knowledge of the dual vertices $V(k)$ and the propagators D. The propagator D is given by

$$D=(P^2-M^2)^{-1}\,, \quad (4.19)$$

where by (3.31)

$$M^2=\frac{1}{\alpha'}\sum_{n=1}^{\infty}\sum_{\beta=1}^{12} n A_n^{\beta\dagger}\cdot A_n^{\beta}-\frac{\alpha_0}{\alpha'}\,. \quad (4.20)$$

The form of the vertices $V(k)$ depends on the specific properties that the scattering amplitude is expected to have. Here we want to show that the ordinary dual ghost-free amplitudes with $\alpha_0=1$ can be obtained from the interaction of four-dimensional Poincaré-invariant parastrings with external fields. To be explicit, we consider the ground-state vertex $V_0(k)$. Moreover, we require that the dual vertices be *color-invariant*, i.e., invariant under SO(11) transformations. The simplest such possibility is to construct $V_0(k)$ from the color-independent operators. Thus we take

$$V_0(k)=:e^{ik\cdot Y^1(0)}:\,.$$

This amounts to assuming that external momenta excite only the color-independent operators. The dual n-point amplitudes are then given by

$$\langle 0,k_N\,|\,V_0(k_{N-1})D\cdots V_0(k_2)\,|\,0,k_1\rangle\,. \quad (4.21)$$

Although the propagator D involves oscillators with $\beta=1,\ldots,12$, because of the choice of vertices only oscillators with $\beta=1$ contribute to the scattering amplitude. So at the tree level dual amplitudes can be obtained from the parastring model which is relativistic in four space-time dimensions.

The possibility of exciting the color oscillators opens up the way for writing down other vertex operators. They may be useful in particular for the inclusion of the electromagnetic interactions in a manner which avoids Gaussian form factors. In the spirit of our work, the new vertices must still be *color-invariant* functions of the color-dependent oscillators.

V. SECOND-QUANTIZED PARASTRINGS

Recently, attempts have been made[33] to construct second-quantized theories of the string model. It is again found that the free theory, classical or quantum, is Poincaré-invariant only in 26 space-time dimensions. We do not wish to give a detailed discussion of such a theory. We only want to point out how a Poincaré-invariant second-quantized parastring theory can be constructed.

In a second-quantized theory the Poincaré generators are constructed[34] by sandwiching the first-quantized generators between canonical field variables. Therefore, if one merely paraquantizes the field variables and attempts to construct the generators by sandwiching the first-quantized generators of the conventional string model between the parafields, one finds that the dimension problem remains unsolved. This is not surprising, because the lesson to be learned from the preceding sections is that to obtain a relativistic theory in four space-time dimensions, the number of internal coordinates must be increased.

We thus introduce the field functional

$$\Phi[Y^{\mu}]\equiv\Phi[Y^{\mu 1},Y^{\mu 2},\ldots,Y^{\mu q}]\,, \quad (5.1)$$

where

$$Y^{\mu\beta}(\theta)=(Y^{+\beta},Y^{-\beta},\vec{Y}^{\beta})\,.$$

As in the previous sections, we fix a direction in the internal β space, so that only for $\beta=1$ does $Y^{\mu\beta}(\theta)$ have zero-frequency modes $(X^+,X^-,\vec{X})$. Now we identify Y^{+1} with τ and let the Fourier transform with respect to X^- be $\Phi_{P^+}[\vec{Y},\tau]\equiv\Phi_{P^+}(Y)$. As an example of how a relativistic field theory can be constructed, we generalize a recently proposed Lagrangian[33] by writing

$$\mathcal{L}=\int_0^{\infty}dP^+\frac{1}{P^+}\int_0^{P^+}d\theta\left\{i\phi_{P^+}^{\dagger}[Y]\frac{\partial}{\partial X^{\dagger}}\Phi_{P^+}[Y]-\frac{1}{2P^+}\sum_{\beta=1}^{q}\left[\frac{\partial\Phi_{P^+}^{\dagger}}{\partial\vec{Y}^{\beta}(\theta)}\cdot\frac{\partial\Phi_{P^+}^{\dagger}}{\partial\vec{Y}^{\beta}(\theta)}-\Phi_{P^+}^{\dagger}[Y]\,\vec{Y}'^{\beta 2}(\theta)\Phi_{P^+}[Y]\right]\right\}\,. \quad (5.2)$$

Then following the works of Refs. 33 and 34, the Poincaré generators can be obtained by sandwiching *our* *first-quantized generators* given by (3.33) between the canonical field variables as follows:

$$\begin{bmatrix} \vec{\mathrm{P}} \\ P^+ \\ P^- \\ M^{\mu\nu} \end{bmatrix} = \int_0^\infty dk^+ \int \prod_\beta D\vec{\mathrm{Y}}^\beta(\theta)\Phi^\dagger_{k^+}[Y]\frac{1}{k^+}\int_0^{k^+} \begin{bmatrix} -i\dfrac{\partial}{\partial \vec{\mathrm{Y}}^1(\theta)} \\ k^+ \\ \dfrac{1}{2}\displaystyle\sum_{\beta=1}^{q}\left[\frac{\partial}{\partial\vec{\mathrm{Y}}^\beta(\theta)^2}+\vec{\mathrm{Y}}^{\beta\prime}(\theta)\right] \\ Y^\mu\Pi^\nu - Y^\nu\Pi^\mu \end{bmatrix} d\theta\,\Phi_{k^+}[Y], \tag{5.3}$$

where $M^{\mu\nu}$ are the coordinate-space analogs of (3.33). It can be checked directly that the generators satisfy the correct commutation relations in four space-time dimensions if $q=12$ and if the fields are quantized canonically.

The significance of the eleven extra internal or relative coordinates in the parastring theory is not transparent. However, as we shall see in the fermionic color string theory, where the color symmetry is SO(3), there are only three extra internal coordinates which can be interpreted as the relative coordinates of the three constituent quarks.

We have singled out the Lagrangian (5.2) purely for illustrative purposes and for comparison with that constructed from a field functional of the conventional string. Clearly, to any field functional of the string, there corresponds a field functional of the color string.

VI. A FERMIONIC PARASTRING MODEL

A. The model

In this section we will present a fermionic parastring model in which the dynamical variables are partly parafermions and partly parabosons. This model bears the same relation to the Ramond-Neveu-Schwarz[35] model as the parastring of Sec. V does to the conventional string model. Because we want to obtain a paraquantized theory, it is convenient to use the Hamiltonian formalism developed by Iwasaki and Kikkawa.[36] Thus, in addition to the string variable $Y^\mu(\tau,\theta)$, we introduce the anticommuting spin variables $S^\mu_a(\tau,\theta)$, $a=1,2$. These dynamical variables satisfy the equations of motion

$$\ddot{Y}-Y''=0, \tag{6.1}$$

$$S-\sigma_3 S'=0,\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{6.2}$$

together with the gauge conditions

$$\dot{Y}\cdot Y'+\tfrac{1}{2}i(S\cdot S'-S'\cdot S+S\cdot\sigma_3\dot{S}-\dot{S}\cdot\sigma_3 S)=0, \tag{6.3}$$

$$\dot{Y}^2+Y'^2+i(S\cdot\dot{S}-\dot{S}\cdot S+S\cdot\sigma_3 S'-S'\cdot\sigma_3 S)=0, \tag{6.4}$$

$$(\dot{Y}+Y')\cdot S_1+S_1\cdot(\dot{Y}+Y')=0, \tag{6.5}$$

$$(\dot{Y}-Y')\cdot S_2+S_2\cdot(\dot{Y}-Y')=0. \tag{6.6}$$

We set, just as in the classical string model,

$$Y^+=2\alpha'P^+\tau, \tag{6.7}$$

$$S^+_a=0,\quad a=1,2. \tag{6.8}$$

The boundary conditions

$$Y'^\mu|_{\theta=0,\pi}=0, \tag{6.9}$$

$$(S_1-S_2)|_{\theta=0}=0,\quad (S_1'-S_2')|_{\theta=\pi}=0 \tag{6.10}$$

lead to the following expansions for Y^μ and S^μ_1:

$$Y^\mu=X^\mu+2\alpha'P^\mu\tau+(2\alpha')^{1/2}\sum_{n=1}^\infty\frac{1}{\sqrt{n}}(a^\mu_n e^{-in\tau}+a^{\mu\dagger}_m e^{in\tau})\cos n\theta, \tag{6.11}$$

$$S^\mu_1=\frac{1}{2}\sum_{k=1/2}^\infty[b^\mu e^{-ik(\tau+\theta)}+b^{\dagger\mu}e^{ik(\tau+\theta)}], \tag{6.12}$$

where, in anticipation of commutation relations, we have fixed the coefficients in the expansions. Observe that the constraint equations (6.3)-(6.6) lead to the relation

$$S^\mu_2(\tau,\theta)=S^\mu_1(\tau,-\theta). \tag{6.13}$$

Taking X^-, P^+, and the transverse components of Y^μ and S^μ as the independent dynamical variables, Eqs. (6.3)-(6.6) may be solved for the dependent variables

$$a^-_n=\frac{i}{(2\alpha')^{1/2}P^+}\frac{T_n}{\sqrt{n}},\quad n=1,2,\ldots \tag{6.14}$$

$$P^-=\frac{1}{2\alpha'P^+}T_0, \tag{6.15}$$

$$b^-_k=\frac{1}{(2\alpha')^{1/2}P^+}G_k,\quad k=\tfrac{1}{2},\tfrac{3}{2},\ldots \tag{6.16}$$

$$a^+_n=b^+_k=X^+=0, \tag{6.17}$$

where

$$T_n=T^{(a)}_n+T^{(b)}_n, \tag{6.18}$$

$$T_n^{(a)}=\frac{1}{4\pi\alpha'}\int_0^\pi d\theta\left\{\left[\left(\frac{d\vec{Y}}{d\tau}\right)^2+\vec{Y}'^2\right]\cos n\theta\right.$$
$$\left.+2i\,\frac{d\vec{Y}}{d\tau}\cdot\vec{Y}'\sin n\theta\right\}, \tag{6.19}$$

$$T_n^{(b)}=\frac{i}{2\pi\alpha'}\int_0^\pi d\theta\left[\left(\vec{S}\cdot\frac{d\vec{S}}{d\tau}-\frac{d\vec{S}}{d\tau}\cdot\vec{S}\right)\cos n\theta\right.$$
$$\left.+i\,(\vec{S}\cdot\vec{S}'-\vec{S}'\cdot\vec{S})\sin n\theta\right], \tag{6.20}$$

and

$$G_k=\frac{1}{2\pi\alpha'}\int_0^\pi d\theta\left[e^{ik\theta}\left(\frac{d\vec{Y}}{d\tau}+\vec{Y}'\right)\cdot\vec{S}_1+(\theta\to-\theta)\right]. \tag{6.21}$$

At this point we impose paraquantization on the independent dynamical variables. It is natural to demand para-Bose commutation relations on the string variables Y^i, and para-Fermi commutation relations on the spin variables S^i. The relative commutation relation between these two sets of variables is determined by the requirement that the invariances of the theory not be disturbed by the paraquantization.[27] In particular, we must have closure of the algebra generated by G_k and T_n. It can be easily seen that the closure depends on the relative commutation relation between the two sets being of para-Bose type, which in turn forces the order of the para-Bose and para-Fermi statistics to be the same.[27] Therefore, in addition to the commutation relations (3.8) on the string variables, for the spin variables S^i we will write

$$S_a^i(\tau,\theta)=\sum_{\alpha=1}^{q}S_a^{i\alpha}(\tau,\theta),$$
$$S_1^{i\alpha}(\tau,\theta)=\frac{1}{2}\sum_{k=1/2}^{\infty}\left(b_k^{i\alpha}e^{-ik(\tau+\theta)}+b_k^{i\alpha\dagger}e^{ik(\tau+\theta)}\right), \tag{6.22}$$
$$S_2^{i\alpha}(\tau,\theta)=S_1^{i\alpha}(\tau_1-\theta),$$

and demand

$$\begin{aligned}
&\{b_k^{i\alpha},b_l^{j\alpha\dagger}\}=\delta^{ij}\delta_{kl}\,,\quad \{b_k^{i\alpha},b_l^{j\alpha}\}=0\,,\\
&[\,b_k^{i\alpha},b_l^{j\beta\dagger}]=[\,b_k^{i\alpha},b_k^{j\beta}]=0\,,\quad \alpha\neq\beta\\
&[\,b_k^{i\alpha},a_n^{j\alpha}]=[\,b_k^{i\alpha},a_n^{j\alpha\dagger}]=0\,,\\
&\{b_k^{i\alpha},a_n^{j\beta}\}=\{b_k^{i\alpha},a_n^{j\beta\dagger}\}=0\,,\\
&[\,b_k^{i1},x^j]=[\,b_k^{i1},P^j]=[\,b_k^{i1},X^-]=[\,b_k^{i1},P^+]=0\,,\\
&\{b_k^{i\alpha},X^j\}=\{b_k^{i\alpha},P^j\}=\{b_k^{i\alpha},X^-\}=\{b_k^{i\alpha},P^+\}=0\,,\quad \alpha\neq 1.
\end{aligned} \tag{6.23}$$

Now the dependent variables a_n^-, b_n^-, and P^-, may be expressed in terms of the independent variables provided we symmetrize the bilinear forms appearing in (6.19) and (6.21), and antisymmetrize the ones in (6.20). The result is

$$T_n=\sum_{\alpha=1}^{q}T_n^\alpha\,,$$

$$T_n^{(a)\alpha}=\sum_{k=1}^{\infty}[\,k(k+n)]^{1/2}\vec{a}_k^{\alpha\dagger}\cdot\vec{a}_{k+n}^{\alpha}$$
$$-\frac{1}{2}\sum_{k=1}^{n-1}[\,k(n-k)]^{1/2}\vec{a}_k^{\alpha}\cdot\vec{a}_{n-k}^{\alpha}-\frac{\alpha_0}{q}\,\delta_{l,0}$$
$$-i\,(2\alpha')^{1/2}\sqrt{n}\;\vec{P}\cdot\vec{a}_n^{1}+\alpha'\,\vec{P}^2\delta_{n,0}\delta^{\alpha 1}, \tag{6.24}$$

$$T_n^{(b)\alpha}=\sum_{k=1/2}^{\infty}(k+\tfrac{1}{2}n)\vec{b}_k^{\alpha\dagger}\cdot\vec{b}_{k+n}^{\alpha}$$
$$+\frac{1}{2}\sum_{k=1/2}^{n-1/2}(\tfrac{1}{2}n-k)\vec{b}_k^{\alpha}\cdot\vec{b}_{n-k}^{\alpha}\,, \tag{6.25}$$

$$G_k=\sum_{\alpha=1}^{q}G_k^\alpha\,,$$

$$G_k^\alpha=i\sum_{n=1}^{\infty}\sqrt{n}\,\vec{a}_n^{\alpha\dagger}\cdot\vec{b}_{k-n}^{\alpha}-i\sum_{l=1/2}^{\infty}(k+l)^{1/2}\vec{b}_l^{\alpha\dagger}\cdot\vec{a}_{k+l}^{\alpha}$$
$$+(2\alpha')^{1/2}\,\vec{P}^1\cdot\vec{b}_k^1\delta^{\alpha 1}. \tag{6.26}$$

It is straightforward to verify that

$$[\,T_n\,,T_m]=(n-m)T_{n+m}+\tfrac{1}{8}q(D-2)\,n(n^2-1)\delta_{n,-m}\,, \tag{6.27}$$

$$[\,T_n\,,G_k]=(\tfrac{1}{2}n-k)G_{n+k}\,, \tag{6.28}$$

$$\{G_k\,,G_l\}=2T_{k+l}+\tfrac{1}{2}q(D-2)\,(k^2-\tfrac{1}{4})\delta_{k,-l}\,, \tag{6.29}$$

where we have defined

$$T_{-n}=T_n^\dagger\,,\quad G_{-k}=G_k^\dagger\,.$$

B. Poincaré invariance

To prove Poincaré invariance of the model we construct generators of the Poincaré group in terms of our dynamical variables. The generators of translations, in the null-plane quantization, are P^i and P^+ and, from (6.15), (6.24), and (6.25),

$$P^-=\frac{1}{2\alpha'P^+}\,T_0$$
$$=\frac{1}{2\alpha'P^+}\left(\alpha'\vec{P}^2+\sum_{\alpha=1}^{q}\sum_{n=1}^{\infty}n\,\vec{a}_n^{\alpha\dagger}\cdot\vec{a}_n^{\alpha}\right.$$
$$\left.+\sum_{\alpha=1}^{q}\sum_{k=1/2}^{\infty}k\,\vec{b}_k^{\alpha\dagger}\cdot\vec{b}_k^{\alpha}-\alpha_0\right). \tag{6.30}$$

This leads to the mass squared operator

$$M^2=2P^+P^- - \vec{P}^2$$
$$=\frac{1}{\alpha'}\left(\sum_{\alpha=1}^{q}\sum_{n=1}^{\infty} n\vec{a}_n^{\alpha\dagger}\cdot\vec{a}_n^{\alpha} + \sum_{\alpha=1}^{q}\sum_{k=1/2}^{\infty} k\vec{b}_k^{\alpha\dagger}\cdot\vec{b}_k^{\alpha}-\alpha_0\right). \tag{6.31}$$

The generators of the homogeneous Lorentz group may be constructed in the usual manner:

$$M^{\mu\nu}=\frac{1}{2}\int_0^{\pi} d\theta\left(\{Y^{\mu},\Pi^{\nu}\}-\{Y^{\nu},\Pi^{\mu}\}-\frac{2i}{\pi\alpha'}[S^{\mu},S^{\nu}]\right). \tag{6.32}$$

In the null frame these generators take the following forms in terms of the independent modes:

$$\begin{aligned} M^{ij}&=X^iP^j-X^jP^i-i\sum_{\alpha=1}^{q}\sum_{n=1}^{\infty}(a_n^{i\alpha\dagger}a_n^{j\alpha}-a_n^{j\alpha\dagger}a_n^{i\alpha})\\ &\quad -i\sum_{\alpha=1}^{q}\sum_{k=1/2}^{\infty}(b_k^{i\alpha\dagger}b_k^{j\alpha}-b_k^{j\alpha\dagger}b_k^{i\alpha}),\\ M^{i\dagger}&=X^iP^+,\\ M^{+-}&=-\tfrac{1}{2}(P^+X^-+X^-P^+), \end{aligned} \tag{6.33}$$

and

$$M^{i-}=\frac{1}{4\alpha'P^+}(X^iT_0+T_0X^i)-X^-P^i+\frac{1}{(2\alpha')^{1/2}P^+}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(T_n^{\dagger}a_n^{i1}+a_n^{i1\dagger}T_n)+\frac{i}{(2\alpha')^{1/2}P^+}\sum_{k=1/2}^{\infty}(G_k^{\dagger}b_k^{i1}-b_k^{i1\dagger}G_k). \tag{6.34}$$

For $q=1$, the expressions for $M^{\mu\nu}$ reduce to the known results.[36,37] It can be seen that our Poincaré group generators close except for the nontrivial commutators

$$\begin{aligned}[M^{i-},M^{j-}]&=\frac{1}{\alpha'(P^+)^2}\sum_{n=1}^{\infty}\left\{n^2\left[\frac{q(D-2)}{16}-\tfrac{1}{2}\right]-\left[\frac{q(D-2)}{16}-\alpha_0\right]\right\}(a_n^{i1\dagger}a_n^{j1}-a_n^{j1\dagger}a_n^{i1})\\ &\quad+\frac{1}{\alpha'(P^+)^2}\sum_{k=1/2}^{\infty}\left\{k^2\left[\frac{q(D-2)}{4}-2\right]-\left[\frac{q(D-2)}{4}-4\alpha_0\right]\right\}(b_k^{i1\dagger}b_k^{j1}-b_k^{j1\dagger}b_k^{i1}). \end{aligned} \tag{6.35}$$

To ensure the vanishing of this commutator for $i\neq j$, we must require

$$\begin{aligned}\alpha_0&=\tfrac{1}{2},\\ D&=2+8/q.\end{aligned} \tag{6.36}$$

For $q=4$, the theory will be Lorentz-invariant in four dimensions. Other possibilities are $D=10$, 6, 3, 2, for which $q=1$, 2, 8, ∞, respectively.

C. Connection with internal symmetry and dual tree amplitudes

In Sec. IV to construct dual tree amplitudes we made a Klein transformation to the colored string picture. In this section we will again construct the Klein operators which transform the fermionic parastring model to a colored fermionic string picture. To be explicit, we do this for the meson sector boundary conditions. We write, in addition to (4.14),

$$b_k^{i\alpha}=K_{\alpha}B_k^{i\alpha}, \quad k=\tfrac{1}{2},\tfrac{3}{2},\ldots, \tag{6.37}$$

where the zero modes are defined by (4.5). Consider the operators

$$K_{\alpha}=K_{\alpha}^{(a)}K_{\alpha}^{(b)}, \tag{6.38}$$

where $K_{\alpha}^{(a)}$ are defined through Eq. (4.7), and

$$K_{\alpha}^{(b)}=\exp\left(-i\pi\sum_{k=1/2}^{\infty}\sum_{\beta=\alpha}^{q}\vec{B}_k^{\beta\dagger}\cdot\vec{B}_k^{\beta}\right). \tag{6.39}$$

Then one can verify that they satisfy the conditions

$$\begin{aligned} &K_{\alpha}^{\dagger}=K_{\alpha}^{-1},\\ &[K_{\alpha},A_n^{i\beta}]=[K_{\alpha},B_l^{i\beta}]=0, \quad \alpha>\beta\\ &\{K_{\alpha},A_n^{i\beta}\}=\{K_{\alpha},B_k^{i\beta}\}=0, \quad \alpha\leq\beta. \end{aligned} \tag{6.40}$$

From these it follows that

$$\begin{aligned} &[A_n^{i\alpha},A_m^{j\beta\dagger}]=\delta^{ij}\delta_{mn}\delta^{\alpha\beta},\\ &\{B_k^{i\alpha},B_l^{j\beta\dagger}\}=\delta^{ij}\delta_{kl}\delta^{\alpha\beta},\\ &[A_n^{i\alpha},B_k^{j\beta\dagger}]=0. \end{aligned} \tag{6.41}$$

Again we can distinguish two types of excitations: the color-independent excitations $\alpha=1$, which are invariant under the gauge group SO(3), and the color-dependent excitations, $\alpha=2,3,4$, which transform according to the fundamental representation of the gauge group SO(3).

Finally, in the spirit of the discussion of Sec. IV, we will write down the tree amplitudes for the meson sector of the colored fermionic string:

$$\langle 0,k_N|V(k_{N-1})D\cdots V(k_2)|0,k_1\rangle. \tag{6.42}$$

In this amplitude,

$$D=(P^2-M^2)^{-1}, \tag{6.43}$$

where by (6.31)

$$M^2=\frac{1}{\alpha'}\Bigg(\sum_{\alpha=1}^{q}\sum_{n=1}^{\infty} nA_n^{\alpha\dagger}\cdot A_n^{\alpha} + \sum_{\alpha=1}^{q}\sum_{k=1/2}^{\infty} kB_k^{\alpha\dagger}\cdot B_k^{\alpha} - \alpha_0\Bigg) \tag{6.44}$$

and

$$V(k)=:k\cdot S_1^1(0)V_0(k):, \tag{6.45}$$

where $V_0(k)$ is given by (4.15). Again, by choosing this vertex, only excitations with $\alpha=1$ contribute to the tree amplitude, even though the underlying theory is Poincaré-invariant in four space-time dimensions.

VII. CONCLUDING REMARKS

The main objective of the present work has been to exploit the nonuniqueness of the quantization of a classical theory and to restore Poincaré invariance to string theories. We have constructed two parastring models, one without and one with fermions. In both cases, we have found that dual tree amplitudes can be obtained from Poincaré-invariant theories in four space-time dimensions. The ground-state particles are still tachyons, however.

We have also shown by Klein transformations that these models are equivalent to color string models. In fact, it is not difficult to see that the color string formalism can be derived from the Lagrangian

$$\mathcal{L}=(-\det g_{ab})^{1/2}, \tag{7.1}$$

where now

$$g_{ab}=\sum_{\beta=1}^{q}\frac{\partial Y_\beta^\mu}{\partial\eta^a}\frac{\partial Y_\mu^\beta}{\partial\eta^b}. \tag{7.2}$$

It is, of course, necessary to specify a direction in the color symmetry space and specify the c.m. coordinates and momenta in that direction, so that they become color singlets under SO($q-1$) transformations. The essential advantage of the paraquantization approach in arriving at these results is that it allows one to make use of the nonuniqueness of the quantization rather than imposing an *ad hoc* color symmetry group. Similarly, one can derive the color fermionic string formalism from the Hamiltonian formalism of Sec. VI by allowing the dynamical variables to carry suitable color indices.

If one describes the degrees of freedom of a composite object in terms of the c.m. and relative coordinates of its constituents, then the properties of our para- (color) string models seem to indicate that there are more than just one relative coordinate involved. For example, in the color fermionic string model, there are four such coordinates. One of these is more intimately connected with the c.m. motion and is distinguished from the other three by being invariant under the transformations of the color symmetry group SO(3). This distinguished relative coordinate also behaves qualitatively different from the other three in that it is excited by external momenta and carries the bulk of the energy and momentum.

In transcribing these ideas to a second-quantized theory, we have shown again that if the field functional depends on more than one relative coordinate, it is indeed possible to construct free field theories which are Poincaré-invariant in four space-time dimensions.

In connection with these results, we wish to bring two important points to the reader's attention. First, by extrapolating from the conventional string model, it has sometimes been thought that for any value of the c-number anomaly in the algebra of the gauge operators, there exists a dimension of space-time for which the theory is Poincaré-invariant. This is not true, in general. For example, in our para- (color) string models, the expression for the c-number anomalies is the same whether or not we break the symmetry in the manner that we have. However, if the symmetry remains intact, the algebra of the Lorentz generators close only for $q=1$ and $D=26$, i.e., no color symmetry at all. Thus to prove the Poincaré invariance of a model, it is not enough to know the expression for the c-number anomalies in the algebra of gauge operators.

Second, as a result of specifying a direction in para- (color) space, the particle states in our dual tree amplitudes are color singlets, i.e., they correspond to particles satisfying not parastatistics but ordinary statistics. It is, of course possible to construct states which are not color singlets, but at least at present they do not seem to be of any physical interest.

There are clearly a number of crucial problems which remain to be investigated. Among them is a systematic study of the interactions of color strings with each other.

Note added in proof. After submitting our manuscript for publication, we learned that parafield excitation has also been utilized by J. F. L. Hopkinson and R. W. Tucker, Phys. Lett. 47B, 519 (1973).

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APPENDIX

Here, we will collect some useful formulas and derive some of the results quoted in Secs. III and VI.

First we derive the expression for M^{i-} in (3.33) and derive (3.34). From (3.23), it is straightforward to find

$$M^{i-}=\tfrac{1}{2}(\{X^i,P^-\}-\{P^i,X^-\})+\frac{i}{2}\sum_{n=1}^{\infty}(\{a_n^i,a_n^{-\dagger}\}-\{a_n^{i\dagger},a_n^-\}). \qquad (A1)$$

Using (3.9), (3.10), and (3.22), Eq. (A1) becomes

$$M^{i-}=\frac{1}{4\alpha' P^+}\{X^i,T_0\}-X^-P^i+\frac{1}{2(2\alpha')^{1/2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left(\left\{a_n^i,\frac{T_n^\dagger}{P^+}\right\}+\left\{a_n^{i\dagger},\frac{T_n}{P^+}\right\}\right).$$

We then observe that P^+ and T_n commute and

$$\left\{a_n^i,\frac{1}{P^+}\right\}=\left\{a_n^{i1},\frac{1}{P^+}\right\}=\frac{2}{P^+}a_n^{i1}.$$

Inserting this into the above expression for M^{i-} and using (3.24) we will recover (3.33).

The commutator (3.34) may be calculated by decomposing $M^{i-}=A^i+B^i$, where

$$A^i=\frac{1}{4\alpha' P^+}\{X^i,T_0\}-X^-P^i,$$
$$B^i=\frac{1}{(2\alpha')^{1/2}P^+}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(T_n^\dagger a_n^{i1}+a_n^{i1\dagger}T_n). \qquad (A2)$$

It is easily checked that

$$[A^i,A^j]=0,\quad i\neq j$$
$$[A^i,B^j]-(i\leftrightarrow j)=\frac{-1}{\alpha'(P^+)^2}\left[\sum_{n=1}^{\infty}(a_n^{i1\dagger}a_n^{j1}-a_n^{j1\dagger}a_n^{i1})\right]T_0-\frac{i}{P^+}(P^iB^j-P^jB^i). \qquad (A3)$$

The calculation of the commutator between two B's is much more tedious owing to appearance of double sums, which cancel to give

$$[B^i,B^j]=\frac{i}{P^+}(P^iB^j-P^jB^i)+\frac{1}{\alpha'(P^+)^2}\sum_{n=1}^{\infty}(a_n^{i1\dagger}a_n^{j1}-a_n^{j1\dagger}a_n^{i1})\left[\left(\frac{q(D-2)}{24}-1\right)n^2+T_0+\alpha_0-\frac{q(D-2)}{24}\right]. \qquad (A4)$$

Utilizing these relations will immediately lead to (3.34).

Next, we derive Eq. (6.34) and (6.35). For this purpose we need the commutation relations of various gauge operators with the independent dynamical variables

$$\begin{aligned}
&[T_n,a_m^{i\alpha}]=-[m(n+m)]^{1/2}a_{n+m}^{i\alpha},\\
&[T_n,a_m^{i\alpha\dagger}]=[m(m-n)]^{1/2}a_{m-n}^{i\alpha\dagger}-[m(n-m)]^{1/2}a_{n-m}^{i\alpha}-i(2\alpha')^{1/2}\sqrt{n}\,P^i\delta_{n,0},\\
&[T_n,b_k^{i\alpha}]=-(k+\tfrac{1}{2}n)b_{k+n}^{i\alpha},\\
&[T_n,b_k^{i\alpha\dagger}]=(k-\tfrac{1}{2}n)b_{k-n}^{i\alpha\dagger}+(k-\tfrac{1}{2}n)b_{n-k}^{i\alpha},\\
&[G_k,a_n^{i\alpha}]=-i\sqrt{n}\,b_{n+k}^{i\alpha},\\
&[G_k,a_n^{i\alpha\dagger}]=-i\sqrt{n}\,b_{n-k}^{i\alpha\dagger}-i\sqrt{n}\,b_{k-n}^{i\alpha},\\
&\{G_k,b_l^{i\alpha}\}=-i(k+l)^{1/2}a_{k+l}^{i\alpha},\\
&\{G_k,b_l^{i\alpha\dagger}\}=i(l-k)^{1/2}a_{l-k}^{i\alpha\dagger}-i(k-l)^{1/2}a_{k-l}^{i\alpha}+(2\alpha')^{1/2}P^i\delta^{\alpha 1}\delta_{kl},\\
&[G_k,X^i]=-i(2\alpha')^{1/2}b_k^{i1},\\
&[G_k,P^i]=[G_k,X^-]=[G_k,P^+]=0.
\end{aligned} \qquad (A5)$$

Now, from (6.32) we can find

$$M^{i-}=M_a^{i-}+M_b^{i-},$$

where M_a^{i-} is given by the same expression as (3.33), and

$$M_b^{i-}=\frac{-i}{2(2\alpha')^{1/2}}\sum_{k=1/2}^{\infty}\left(\left[b_k^i,\frac{G_k^\dagger}{P^+}\right]+\left[b_k^{i\dagger},\frac{G_k}{P^+}\right]\right),$$

where we have used (6.16). We then employ the anticommutator relations between the a's and b's of (A5) and observe that

$$\left\{b_k^i,\frac{1}{P^+}\right\}=\frac{2}{P^+}b_k^{i1}$$

and recover Eq. (6.34). Similarly Eq. (6.35) may be obtained with the calculation of various commutator involved, writing as before

$$M_a^{i-}=A^i+B^i,$$

we find

$$[M_a^{i-}, M_a^{j-}] = \frac{1}{\alpha'(P^+)^2} \sum_{n=1}^{\infty} \left\{ \left[\frac{q(D-2)}{16} - 1 \right] n^2 - \left[\frac{q-(D-2)}{16} \right] \right\} (a_n^{i1\dagger} a_n^{j1} - a_n^{D1\dagger} a_n^{i1}) \tag{A6}$$

$$\begin{aligned}[B^i, M_b^{j-}] - (i \leftrightarrow j) = -\frac{1}{2\alpha'(P^+)^2} \Bigg(\sum_{k=1/2}^{\infty} \sum_{n=1}^{\infty} \{ [b_k^{i1\dagger} b_{n-k}^{j1\dagger} + b_k^{i1\dagger} b_{k-n}^{j1} - (i \leftrightarrow j)] T_n \\ + i\sqrt{n} [a_n^{i1\dagger} b_{n-k}^{j1} + a_n^{i1\dagger} b_{k-n}^{j1\dagger} - a_n^{i1} b_{k+n}^{j1\dagger} - (i \leftrightarrow j)] G_k - \text{H.c.}\} \\ + \sum_{k=1/2}^{\infty} (k^2 - \tfrac{1}{4}) [b_k^{i1\dagger} b_k^{j1} - (i \leftrightarrow j)] - \sum_{n=1}^{\infty} 2n^2 [a_n^{i1\dagger} a_n^{j1} - (i \leftrightarrow j)] \Bigg)\end{aligned} \tag{A7}$$

and

$$\begin{aligned}[M_b^{i-}, M_b^{j-}] &= -\frac{i}{P^+} (P^i M_b^{j-} - P^j M_b^{i-}) \\ &= \frac{1}{2\alpha'(P^+)^2} \Bigg(\sum_{k=1\ 2}^{\infty} \sum_{n=1}^{\infty} \{[b_k^{i1\dagger} b_{n-k}^{j1\dagger} + b_k^{i1\dagger} b_{k-n}^{j1} - (i \leftrightarrow j)] T_n \\ &\quad + i\sqrt{n} [a_n^{i1\dagger} b_{n-k}^{j1} + a_n^{i1\dagger} b_{k-n}^{j1\dagger} + a_n^{i1} b_{k+n}^{j1\dagger} - (i \leftrightarrow j)] G_k - \text{H.c.}\} \\ &\quad + \sum_{k=1/2}^{\infty} [2(T_0 + \alpha_0) - 3k^2 - \tfrac{1}{4} + \tfrac{1}{2} q (D-2)\ (k^2 - \tfrac{1}{4})][b_k^{i1\dagger} b_k^{j1} - (i \leftrightarrow j)] \\ &\quad - \sum_{n=1}^{\infty} n^2 [a_n^{i1\dagger} a_n^{j1} - (i \leftrightarrow j)] \Bigg)\end{aligned} \tag{A8}$$

and

$$[A^i, M_b^{j-}] - (i \leftrightarrow j) = -\frac{1}{\alpha'(P^+)^2} (T_0) \sum_{k=1/2}^{\infty} [b_k^{i1\dagger} b_k^{j1} - (i \leftrightarrow j)] + \frac{i}{P^+} (P^i M_b^{j-} - P^j M_b^{i-}) . \tag{A9}$$

Adding (A6)-(A9) will immediately yield (6.35).

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Gauge and global symmetries at high temperature*

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It is shown how finite-temperature effects in a renormalizable quantum field theory can restore a symmetry which is broken at zero temperature. In general, for both gauge symmetries and ordinary symmetries, such effects occur only through a temperature-dependent change in the effective bare mass of the scalar bosons. The change in the boson bare mass is calculated for general field theories, and the results are used to derive the critical temperatures for a few special cases, including gauge and nongauge theories. In one case, it is found that a symmetry which is unbroken at low temperature can be broken by raising the temperature above a critical value. An appendix presents a general operator formalism for dealing with higher-order effects, and it is observed that the one-loop diagrams of field theory simply represent the contribution of zero-point energies to the free energy density. The cosmological implications of this work are briefly discussed.

I. INTRODUCTION

The idea of broken symmetry was originally brought into elementary-particle physics on the basis of experience with many-body systems.[1] Just as a piece of iron, although described by a rotationally invariant Hamiltonian, may spontaneously develop a magnetic moment pointing in any given direction, so also a quantum field theory may imply physical states and S matrix elements which do not exhibit the symmetries of the Lagrangian.

It is natural then to ask whether the broken symmetries of elementary-particle physics would be restored by heating the system to a sufficiently high temperature, in the same way as the rotational invariance of a ferromagnet is restored by raising its temperature. A recent paper by Kirzhnits and Linde[2] suggests that this is indeed the case. However, although their title refers to a gauge theory, their analysis deals only with ordinary theories with broken global symmetries. Also, they estimate but do not actually calculate the critical temperature at which a broken symmetry is restored.

The purpose of this article is to extend the analysis of Kirzhnits and Linde to gauge theories,[3] and to show how to calculate the critical temperature for general renormalizable field theories, with either gauge or global symmetries. Our results completely confirm the more qualitative conclusions of Kirzhnits and Linde.[2]

The diagrammatic formalism[4] used here is described in Sec. II. Any finite-temperature Green's function is given by a sum of Feynman diagrams, just as in field theory, except that en-