how one may substantially alter the infrared behavior of the Green's functions or choosing an inappropriate path by which to approach $E_t \to 0$, $\vec{k}_t \to 0$ or having an unfortunate ratio of renormalized coupling constants. Such trickery did not appear in the ψ^3 theory which possesses a single coupling and provides a richness of solutions not explorable there.

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Feynman rules for gauge theories at finite temperature*

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Feynman's functional formulation of statistical mechanics is used to study general-relativistic quantum field theories at finite temperature. The techniques are then applied to gauge theories. The partition function $\mathrm{Tr}\, e^{-\beta H}$ is discovered to be a gauge-dependent quantity which is meaningless in most gauges. Instead, we define a physically meaningful partition function which is gauge-invariant and only equal to $\mathrm{Tr}\, e^{-\beta H}$ in certain "physical" gauges. Feynman rules for this partition function and for finite-temperature Green's functions are derived for a general gauge.

I. INTRODUCTION

Recently, several authors[1-3] have considered what happens when a system of elementary particles described by a quantum field theory is heated. They have found that symmetries which are spontaneously broken at zero temperature (such as those of the weak interactions) may be restored at sufficiently high temperatures, and have calculated[2,3] the critical temperature at which such a restoration takes place. To do this kind of calculation, one needs to know the Feynman rules for a field theory at finite temperature. For a nongauge theory, these rules can be derived using well-known methods.[4] However, for a gauge theory, a more powerful technique is needed to cope with several new problems that arise. Chief among these is the troublesome fact that the partition function $\mathrm{Tr}\, e^{-\beta H}$ is a gauge-dependent quantity, as we show by an explicit example. The gauge dependence is caused by the appearance, in some gauges, of specious degrees of freedom in H which do not correspond to physical particles. The trace over all states of H is not physically meaningful in these gauges—the specious particles cannot come to equilibrium with a physical heat bath.

It would seem, then, that gauge invariance is completely lost at finite temperature. This is not the case. The functional methods set forth in this paper allow one to calculate the physically meaningful partition function (i.e., $\mathrm{Tr}\, e^{-\beta H}$ in a gauge without specious degrees of freedom—such as the unitarity gauge) using Feynman rules defined in any of the usual gauges (for example, the R_ξ gauges in a spontaneously broken non-Abelian theory). Thus gauge invariance of physical quantities is not lost at finite temperature; we must merely remember that $\mathrm{Tr}\, e^{-\beta H}$ is not in general a quantity with a direct physical interpretation.

The existance of Feynman rules for the true partition function in renormalizable (R_ξ) gauges of a non-Abelian theory is extremely useful in calculation of the critical temperature. It turns out that in these calculations one must keep track of powers of temperature that occur in higher-order perturbation theory—a task which is very difficult in unitarity gauge but fairly simple in a renormalizable gauge.[2]

Section II of this paper presents a somewhat heuristic derivation of the finite-temperature functional formalism for a nongauge field theory, starting from the better-known zero-temperature formalism. This finite-temperature formalism is not new; Feynman first wrote it down for a single nonrelativistic particle in one dimension.[5] Since then, it has been used by many authors for various purposes.[6] Here, we present it in a somewhat different context and with a different motivation: that of finding finite-temperature Feynman rules for a relativistic quantum field theory. The rules for an interacting scalar field theory and the exact partition function for a free scalar theory are worked out as examples.

Section III deals with gauge theories. The lack of gauge invariance of $\mathrm{Tr}\, e^{-\beta H}$ is discussed. A functional integral in a "physical gauge" is set up which can be used to calculate the true partition function; the Faddeev-Popov ansatz[7] is then used to change the functional integral to "nonphysical" gauges. The Feynman rules for general gauges are derived, and the nature of "ghosts" at finite temperature is discussed. Free electrodynamics is used as an example.

Section IV restates some of our conclusions, and an appendix works out a technical detail of the functional formulation.

II. DERIVATION OF THE FUNCTIONAL FORMALISM

Consider a quantum field theory described by a Hamiltonian density $\mathcal{H}(\pi, \varphi)$, where $\varphi(\vec{x}, t)$ is the Heisenberg-picture field operator and $\pi(\vec{x}, t)$ is its conjugate momentum. (The generalization to a theory with many fields will be immediate.) Then $\varphi(\vec{x}, 0)$ is the Schrödinger-picture field operator. Let $|\varphi_0\rangle$ and $|\varphi_1\rangle$ be eigenstates of $\varphi(\vec{x}, 0)$ with eigenvalues $\varphi_0(\vec{x})$ and $\varphi_1(\vec{x})$. Thus

$$\begin{aligned} \varphi(\vec{x}, 0)|\varphi_0\rangle &= \varphi_0(\vec{x})|\varphi_0\rangle, \\ \varphi(\vec{x}, 0)|\varphi_1\rangle &= \varphi_1(\vec{x})|\varphi_1\rangle. \end{aligned} \tag{2.1}$$

Then the Feynman functional formula in Hamiltonian form[8] gives the transition amplitude for going from $|\varphi_0\rangle$ at $t=0$ to $|\varphi_1\rangle$ at $t=t_1$:

$$\langle\varphi_1| e^{-iHt_1}|\varphi_0\rangle = N\int [d\pi][d\varphi]\exp\left\{i\int_0^{t_1} dt\int d^3x[\pi\dot{\varphi} - \mathcal{H}(\pi, \varphi)]\right\}, \tag{2.2}$$

where the integral over classical fields, $\int [d\varphi]$, runs over all possible configurations that start at $\varphi_0(\vec{x})$ at $t=0$ and go to $\varphi_1(\vec{x})$ at $t=t_1$. The integral over momenta, $\int [d\pi]$, is unrestricted. $\dot{\varphi}(\vec{x}, t)$ is defined to be $\partial\varphi(\vec{x}, t)/\partial t$. N is a constant normalizing factor. To avoid ambiguity, we require that the momentum integration always be done before the field integration.[9]

Now simply let $it_1=\beta$, where β is the inverse of the temperature. Make the variable change $it=\tau$ in the integrand of the exponent. Note that $\dot{\varphi} = i\partial\varphi/\partial\tau$. We have

$$\langle\varphi_1| e^{-\beta H}|\varphi_0\rangle = N\int [d\pi][d\varphi]\exp\left\{\int_0^{\beta} d\tau\int d^3x[i\pi\dot{\varphi} - \mathcal{H}(\pi, \varphi)]\right\}, \tag{2.3}$$

where $\dot{\varphi}$ is now taken to mean $\partial\varphi/\partial\tau$.

To find the partition function $Z=\mathrm{Tr}\, e^{-\beta H}$, we just allow the $\int [d\varphi]$ to go over all periodic paths, i.e., those that have the same classical field at $\tau=\beta$ as at $\tau=0$. Symbolically

$$\begin{aligned} \mathrm{Tr}\, e^{-\beta H} &= \sum_{\varphi} \langle\varphi| e^{-\beta H}|\varphi\rangle \\ &= N\int [d\pi]\int_{\text{periodic}} [d\varphi]\exp\left\{\int_0^{\beta} d\tau\int d^3x[i\pi\dot{\varphi} - \mathcal{H}(\pi, \varphi)]\right\}. \end{aligned} \tag{2.4}$$

Note that only the field integration, and not the momentum integration, is restricted to periodic orbits.

In the usual cases where $\mathcal{H}$ is no more than a quadratic function of π's, we can do the π integration immediately by completing the square. This merely replaces π in the integrand by its value at the stationary point of the integrand and adds a ghost term if the quadratic term in π is φ-dependent.[9] The stationary point is given by

$$i\dot{\varphi} = \frac{\partial\mathcal{H}(\pi, \varphi)}{\partial\pi}. \tag{2.5}$$

This, together with the addition of a possible ghost term, is just the prescription for going from the Hamiltonian to the usual effective Lagrangian, $\mathcal{L}_{\text{eff}}$, with the additional stipulation that all τ derivatives in $\mathcal{L}_{\text{eff}}$ are multiplied by i. Thus we have

$$\operatorname{Tr} e^{-\beta H} = N'(\beta) \int_{\text{periodic}} [d\varphi] \exp\left[\int_0^\beta d\tau \int d^3x\, \mathcal{L}_{\text{eff}}(\varphi, i\dot{\varphi})\right], \tag{2.6}$$

where $N'(\beta)$ is a new (infinite) normalizing factor which will be determined later. The β dependence of N' comes from a careful evaluation of the $\int [d\pi]$. The same kind of thing happens at zero temperature, but there we are interested in a $t \to \infty$ limit (the S matrix) and so N' is an unimportant infinite constant which is usually ignored.

For a nongauge theory, we can use (2.6) to derive the Feynman rules in the ordinary way. The quadratic part of $\mathcal{L}_{\text{eff}}$ determines the propagators and the nonquadratic part determines the vertices. There are only two basic differences from the zero-temperature rules:

(1) The presence of the $i\dot{\varphi}$ and the absence of an i multiplying $\mathcal{L}_{\text{eff}}$ just accomplish a "Wick rotation" of the rules into Euclidean space. There is no need for an $i\epsilon$ in the propagator denominators.

(2) The requirement of periodicity over the finite range of the τ integration changes all energy integrations into energy sums. For bosons this means replacing the energy k_0 by $2\pi n/\beta$ and summing over integer n instead of integrating over k_0. For fermions the correct prescription is buried in the complications of the integral of anticommuting C numbers. However, since we never have to do anything other than a quadratic fermion integral, we can *define* this integral to give the right rules.[9] These rules are well known[4]; they are the same as the boson rules except for the usual minus sign for loops and the sum over odd, rather than even, multiples of π/β.

As an example of those techniques, we work out the Feynman rules of simple scalar field theory described by

$$\mathcal{L}(\varphi, \dot{\varphi}) = \mathcal{L}_{\text{eff}}(\varphi, \dot{\varphi}) = \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \tfrac{1}{2}m^2\varphi^2 - \lambda\varphi^4. \tag{2.7}$$

There is no additional ghost term in $\mathcal{L}_{\text{eff}}$, since there are no derivative interactions in $\mathcal{L}$, and hence the term in $\mathcal{H}(\pi, \varphi)$ which is quadratic in π has a φ-independent coefficient. We now have from (2.6)

$$\operatorname{Tr} e^{-\beta H} = N'(\beta) \int [d\varphi] \exp\left(\int_0^\beta d\tau \int d^3x \{-\tfrac{1}{2}[(\partial_0\varphi)^2 + (\partial_i\varphi)(\partial_i\varphi) + m^2\varphi^2] - \lambda\varphi^4\}\right). \tag{2.8}$$

Define the quadratic part of the action by

$$S_0 \equiv -\tfrac{1}{2}\int_0^\beta d\tau \int d^3x[(\partial_0\varphi)(\partial_0\varphi) + (\partial_i\varphi)(\partial_i\varphi) + m^2\varphi^2]. \tag{2.9}$$

Since $\varphi(\vec{x}, \tau)$ is periodic in the interval $0 < \tau < \beta$, we can expand in a Fourier series:

$$\varphi(x, \tau) = (1/\beta)\sum_n \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} e^{i\omega_n\tau}\varphi_n(\vec{k}),$$
$$\varphi_n(\vec{k}) = \int d^3x \int_0^\beta d\tau\, e^{-i\vec{k}\cdot\vec{x}} e^{-i\omega_n\tau}\varphi(x, \tau), \tag{2.10}$$

where $\omega_n \equiv 2\pi n/\beta$. Using the identity

$$\int_0^\beta d\tau\, e^{i(\omega_n - \omega_{n'})\tau} = \beta\delta_{n,n'}, \tag{2.11}$$

with δ the Kronecker delta, we obtain

$$S_0 = -\tfrac{1}{2}(1/\beta)\sum_n \int \frac{d^3k}{(2\pi)^3}(\omega_n^2 + \vec{k}^2 + m^2)\varphi_n(\vec{k})\varphi_{-n}(-\vec{k}). \tag{2.12}$$

If we write $S_0 = -\frac{1}{2}(\varphi, D\varphi)$, where the brackets denote the scalar product on our function space, then

$$D = \omega_n^2 + \vec{k}^2 + m^2, \tag{2.13}$$

in momentum space.

The Feynman propagator, Δ_F, is then just D^{-1}. Thus in momentum space

$$\Delta_F(\omega_n, \vec{k}) = \frac{1}{\omega_n + \vec{k}^2 + m^2}, \tag{2.14a}$$

and in position space

$$\Delta_F(\vec{x} - \vec{x}', \tau - \tau') = (1/\beta)\sum_n \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}') + i\omega_n(\tau - \tau')}}{\omega_n^2 + \vec{k}^2 + m^2}. \tag{2.14b}$$

Just as at zero temperature, we can now expand the exponential of the interaction term, $\lambda\varphi^4$, in a power series to get a diagrammatic expansion for $\operatorname{Tr} e^{-\beta H}$. We use the functional formula

$$\int [d\varphi] e^{-(\varphi, D\varphi)/2}\varphi(\vec{x}_1, \tau_1)\varphi(\vec{x}_2, \tau_2)\varphi(\vec{x}_3, \tau_3)\varphi(\vec{x}_4, \tau_4)\cdots = \text{const} \times (\det D)^{-1/2}(\varphi^{\cdot}(\vec{x}_1, \tau_1)\varphi^{\cdot}(\vec{x}_2, \tau_2)\varphi^{\cdot\cdot}(\vec{x}_3, \tau_3)\varphi^{\cdot\cdot}(\vec{x}_4, \tau_4)) + \text{permutations}, \tag{2.15}$$

where the contraction of two fields is indicated by corresponding sets of superscript dots and is given by the Feynman propagator (2.14b):

$$\varphi^{\cdot}(\vec{x}_1,\tau_1)\varphi^{\cdot}(\vec{x}_2,\tau_2)=\Delta_F(\vec{x}_1-\vec{x}_2,\tau_1-\tau_2)\,. \tag{2.16}$$

We represent the contractions in the usual way by diagrams. Use of (2.11) shows that we have conservation of discrete energy and a factor of β at each vertex, as well as the usual conservation of momentum. The Feynman rules are thus the zero-temperature rules, with the replacements

$$\begin{aligned}&\int\frac{d^4k}{(2\pi)^4}\to\frac{i}{\beta}\sum_n\int\frac{d^3k}{(2\pi)^3},\\&k_0\to i\omega_n,\\&(2\pi)^4\delta^4(k_1+k_2+\cdots)\to\frac{1}{i}(2\pi)^3\beta\delta_{\omega_{n_1}+\omega_{n_2}+\cdots}\\&\qquad\times\delta^3(\vec{k}_1+\vec{k}_2+\cdots)\,,\end{aligned} \tag{2.17}$$

with the factors of i just coming from the rotation to Euclidean space.

If we are calculating $\mathrm{Tr}\,e^{-\beta H}$, we must remember to include $N'(\beta)$ and $(\det D)^{-1/2}$ from (2.8) and (2.15) (see below). However, we can also calculate finite-temperature Green's functions, which are statistical averages of τ-ordered fields:

$$\langle T[\varphi(\vec{x}_1,\tau_1)\varphi(\vec{x}_2,\tau_2)\cdots]\rangle \equiv\frac{\mathrm{Tr}\,e^{-\beta H}T[\varphi(\vec{x}_1,\tau)\varphi(\vec{x}_2,\tau_2)\cdots]}{\mathrm{Tr}\,e^{-\beta H}}, \tag{2.18}$$

where T is the τ-ordering symbol. In this case, $N'(\beta)$, $(\det D)^{-1/2}$, and all parts of diagrams not connected to the external fields are canceled by the denominator of (2.18), just as in the zero-temperature S matrix.

As an example, we calculate the partition function $Z=\mathrm{Tr}\,e^{-\beta H}$ in the simplest possible case—a free scalar field, with the Lagrangian given by (2.7) with $\lambda=0$. Since the functional integral is now just a Gaussian, we can evaluate it exactly, using the version of (2.15) where there are no fields to contract:

$$\int[d\varphi]e^{-(\varphi,D\varphi)/2}=\mathrm{const}\times(\det D)^{-1/2}\,. \tag{2.19}$$

Thus we have, using (2.6),

$$\begin{aligned}\ln Z=\ln\mathrm{Tr}\,e^{-\beta H}&=-\tfrac{1}{2}\ln[\det D]+\ln N'(\beta)+\mathrm{const}\\&=-\tfrac{1}{2}\mathrm{Tr}\ln D+\ln N'(\beta)+\mathrm{const}\,.\end{aligned}$$

Using (2.13) and ignoring the β-independent constant gives

$$\ln Z=-\tfrac{1}{2}\sum_n\int\frac{d^3k}{(2\pi)^3}\ln(\omega_n^2+\omega_k^2)+\ln N'(\beta)\,, \tag{2.20}$$

where $\omega_k^2\equiv\vec{k}^2+m^2$. We will do the $\sum_n$ in (2.20) first. Write

$$\sum_n\ln(\omega_n^2+\omega_k^2)=\int_{1/\beta^2}^{\omega_k^2}da^2\sum_n\frac{1}{\omega_n^2+a^2}+\sum_n\ln(\omega_n^2+1/\beta^2)\,, \tag{2.21}$$

where the lower limit has been chosen so that it will contribute no β dependence to the final result.

The second term in (2.21) is β-dependent and infinite, but a careful evaluation of the π and φ integrations in (2.4) (see Appendix) shows that its contribution to $\ln Z$ is canceled by $\ln N'(\beta)$ to within a β-independent constant. Using the results of the Appendix, we can write, symbolically

$$\ln N'(\beta)=-(\ln\beta)\times\sum_n\int\frac{d^3k}{(2\pi)^3}, \tag{2.22}$$

which cancels the contribution of the second term in (2.21). It is clear that we get one factor of $N'(\beta)$ for every $\int[d\pi]$ we do in (2.4).

Continuing the evaluation of $\ln Z$, we can do the $\sum_n$ in (2.21) with the standard Regge-type trick of introducing a factor of $\frac{1}{2}\beta\cot(\frac{1}{2}\beta\omega)$ which has poles of residue 1 at $\omega=2\pi n/\beta$, and integrating in the complex ω plane over a contour which includes all the poles. The contour is then continued into the upper and lower half planes to pick up the poles at $\omega=\pm ia$ only. The $\int da^2$ in (2.21) can then be performed. The result is

$$\ln Z=\int\frac{d^3k}{(2\pi)^3}\ln\left(\mathrm{csch}\frac{\beta\omega_k}{2}\right)+\beta\text{-independent constant}\,. \tag{2.23}$$

With a simple rearrangement, (2.23) can be put into the form we would get by evaluating $\ln Z$ by a trace in Fock space:

$$\ln Z=\int\frac{d^3k}{(2\pi)^3}\left[\frac{-\beta\omega_k}{2}-\ln(1-e^{-\beta\omega_k})\right]. \tag{2.24}$$

This is the usual result for an ideal Bose gas, with the zero-point energy of the vacuum included. The presence of the zero-point energy should be no surprise—the functional-integral formalism never does normal ordering for us.

III. GAUGE THEORIES

We can now apply the functional formalism to gauge field theories. Before starting, however, let us show that there is a nontrivial problem involved here. Consider free electrodynamics described by the Lagrangian

$$\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu},\quad\text{with } F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,. \tag{3.1}$$

If we work, for example, in the Coulomb gauge (described by $\vec{\nabla}\cdot\vec{A}=0$) or the axial gauge (described by $A_3=0$), calculate H, and take $\mathrm{Tr}\, e^{-\beta H}$, we come to the usual conclusion that Z describes a massless Bose gas with two degrees of polarization. If, however, we work in the Feynman gauge (described by the Lagrangian $\mathcal{L}=-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu$), calculate H, and take $\mathrm{Tr}\, e^{-\beta H}$, we get obvious nonsense: a Bose gas with three positive- and one negative-metric states. That is,

$$\ln(\mathrm{Tr}\, e^{-\beta H})_{\text{Feynman gauge}} = 3\int \frac{d^3k}{(2\pi)^3}\left[\frac{-\beta\omega_k}{2} - \ln(1-e^{-\beta\omega_k})\right] + \int \frac{d^3k}{(2\pi)^3}\left[\frac{-\beta\omega_k}{2} - \ln(1+e^{-\beta\omega_k})\right], \quad (3.2)$$

with $\omega_k=(\vec{k}^2)^{1/2}$.

The reason for the nonsense of (3.2) is not hard to find. The Hamiltonian in the Coulomb or the axial gauge has just two independent degrees of freedom, whereas in the Feynman gauge there are two extra degrees of freedom: longitudinal and timelike photons. When we take $\mathrm{Tr}\, e^{-\beta H}$ in the Feynman gauge we include these specious states.

The lesson of (3.2) is that $\mathrm{Tr}\, e^{-\beta H}$ is not a physically meaningful quantity in all gauges—in some gauges spurious particles, which could never come to equilibrium with a physical heat bath, are wrongly counted as physical degrees of freedom. Thus the partition function Z must be defined as $\mathrm{Tr}\, e^{-\beta H}$ only in a "physical gauge"—i.e., one with the right number of degrees of freedom. Functional methods may then be used to determine Feynman rules for *this* Z in other gauges, but it is still this physical Z that we calculate.

The first step in deriving Feynman rules for a gauge theory is to write down a correct expression for Z in terms of a functional integral in a particular gauge. There are two basic ways of doing this. One way would be to start with (2.4) in a gauge which is physical. For example, consider a pure Yang-Mills theory (fermions or scalars would be trivial to add) described by the Lagrangian

$$\mathcal{L}=-\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}, \quad (3.3)$$

where $F^a_{\mu\nu}=\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc}A^b_\mu A^c_\nu$ and f^{abc} are the structure constants of the group.

Following Coleman,[11] we see that the Hamiltonian for this theory in the axial gauge (defined by $A^a_3=0$) has only two degrees of freedom for each vector-meson field: A^a_1, and A^a_2. Thus, in the axial gauge, (2.4) becomes

$$Z=(\mathrm{Tr}\, e^{-\beta H})_{\text{axial gauge}} = N\int \prod_a [dP^a_1][dP^a_2]\int_{\text{periodic}} [dA^a_1][dA^a_2]\exp\left\{\int_0^\beta d\tau \int d^3x[iP^a_j\dot{A}^a_j - \mathcal{H}(A^a_j, P^a_j)]\right\}, \quad (3.4)$$

where $P^a_{1,2}$ are the conjugate momenta to $A^a_{1,2}$ and $j=1,2$. The integral over P's can be done easily. The result is simply the expression for the Faddeev-Popov ansatz in the axial gauge[11]:

$$Z=[N'(\beta)]^{2n}\int_{\text{periodic}}[dA]\prod_a \delta(A^a_3)\exp\left[\int_0^\beta d\tau\int d^3x\, \mathcal{L}(A, i\dot{A})\right], \quad (3.5)$$

where we have gotten one factor of $N'(\beta)$ for each P integration (a runs from 1 to n), and $[dA]$ means integral over all the vector-meson fields.

There is a second way of proceeding, which I will briefly sketch out. We could start with $\mathrm{Tr}\, e^{-\beta H}$ in a physical gauge, calculate the Feynman rules straightforwardly using the techniques of many-body theory,[4] and then write down a functional integral which gave the same Feynman rules. For example, consider a spontaneously broken gauge theory of the type studied by Weinberg.[12] Weinberg performs a canonical quantization of the theory in unitarity gauge, a physical gauge with no specious degrees of freedom. Thus, we can use his canonically quantized Hamiltonian to derive a set of Feynman rules for Z. These turn out to be just the same as the zero-temperature rules he writes down with the usual substitution of energy sums for integrals. It is then a fairly simple matter to check that his method of "summing the springs" to simplify the Feynman rules goes through exactly the same at finite temperature, unaffected by the switch from integrals to sums. The resultant rules are then just the ones that can be derived from a functional integral with the Faddeev-Popov ansatz in the unitarity gauge, so we can set Z equal to that functional integral.

In any case, using either the first or the second method outlined above, we can arrive at an expression for a gauge theory like (3.5). In general, it will look like[13]

$$Z=[N'(\beta)]^m\int_{\text{periodic}}[dA][d\varphi]\exp\left[\int_0^\beta d\tau\int d^3x\mathcal{L}(A,\varphi,i\dot{A},i\dot{\varphi})\right]\det\left(\frac{\partial F^b}{\partial\omega^c}\right)\prod_b \delta(F^b), \quad (3.6)$$

where m is the total number of physical particles and polarization states in the theory. The integral in (3.6) is over all periodic fields (A stands for the vector-meson fields, φ for any other fields in the theory). The $\prod_b \delta(F^b)$ picks out a surface in the function space of the fields which *corresponds to a physical gauge*. The $\det(\partial F^b/\partial\omega^c)$ is necessary because the equations $F^b(x)=0$ determine the surface only implicitly: $\omega^c(x)$ are the set of functions which parameterize the gauge transformation of $\mathcal{L}$. [The determinant is lacking in (3.5) because, for the axial gauge, it is $\det(\partial/\partial x_3)$, [11] which is a constant independent of fields *and* of β.] If we had used the second method for the unitarity gauge, (3.6) would look slightly differently: Instead of a δ function, there would be an extra term added to $\mathcal{L}$ in the exponent, which is just a slightly different way of choosing a gauge surface.[11]

Now comes the crucial point of the argument. Since the Lagrangian is gauge-invariant, the integral in (3.6) must be independent of how we pick our gauge surface. In other words, we do not have to stick to a physical gauge on the right-hand side of (3.6); we calculate the physical partition function even if the equations $F^b(x)=0$ correspond to a nonphysical gauge. This statement does require one qualification: One surface is as good as another only as long as they both intersect the orbit of any given field under gauge transformations once and only once. This is to ensure that the integral in (3.6) runs equally over all physically distinct fields (i.e., those not connected by a gauge transformation). However, this qualification is not really a problem, since the situation is essentially the same as at zero temperature. There, we know that the specification of the gauge surface by the δ function *and* the boundary condition on the fields (vanishing at spatial and temporal infinity) is enough to define the exact location of a given field on its gauge orbit. At finite temperature, we have merely changed the boundary condition (vanishing at spatial infinity, periodic from $\tau=0$ to $\tau=\beta$).

Thus, we can change the surface of integration in (3.6) just as at zero temperature, and can write the left-hand side in any of the usual gauges (Feynman gauge, Landau gauge, the R_ξ gauges for a spontaneously broken theory, etc.). The Feynman rules in any of these gauges will be the zero-temperature rules with the simple changes given by (2.17). The important point is that when we use these rules in a given gauge we calculate the true Z, which is not necessarily $\mathrm{Tr}\, e^{-\beta H}$ in that gauge.

An example will show how this works. We return to free electrodynamics described by (3.1). Write down (3.6) with $F(\vec{x},\tau)=\partial_\mu A^\mu - f(\vec{x},\tau)$, $f(\vec{x},\tau)$ an arbitrary function, $0\le\tau\le\beta$:

$$Z=[N'(\beta)]^2\int[dA]\exp\left[\int_0^\beta d\tau\int d^3x(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu})\right]\times\det\left(\frac{\delta(\partial_\mu A^\mu)}{\delta\omega}\right)\delta(\partial^\mu A_\mu - f), \qquad (3.7)$$

where a rotation to Euclidean space of all τ derivatives is understood. Note that there are two factors of $N'(\beta)$ in front, since (3.6) must be derived initially in a physical gauge (such as the axial gauge) where there are two momentum integrations to do [see (3.4)]. Since under gauge transformations $\delta A^\mu = -\partial^\mu\omega$, we have

$$\det\left(\frac{\delta(\partial_\mu A^\mu)}{\delta\omega}\right)=\det(-\Box^2). \qquad (3.8)$$

In addition, since (3.7) is independent of f, multiplying the right-hand side by

$$\exp\left[-\frac{1}{2\alpha}\int_0^\beta d\tau\int d^3x f^2\right]$$

and integrating over $[df]$ merely gives an extra β-independent normalization which can be absorbed into $N'(\beta)$.[14] The integral over $[df]$ is then evaluated trivially with the δ function, and we have

$$Z=[N'(\beta)]^2(\det-\Box^2)\int_{\text{periodic}}[dA]\exp\left\{\int_0^\beta d\tau\int d^3x\,[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-(1/2\alpha)(\partial_\mu A^\mu)^2]\right\}.$$

At zero temperature the $\det(-\Box^2)$ is an unimportant constant and is usually ignored, but here it is β-dependent, so we must keep it. For convenience we work in the Feynman gauge ($\alpha=1$), Z should be (and is) independent of α. With $\alpha=1$, (3.9) becomes

$$Z=[N'(\beta)]^2\det(-\Box^2)\int_{\text{periodic}}[dA]\exp\left[\int_0^\beta d\tau\int d^3x(-\tfrac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu)\right], \qquad (3.10)$$

where a total derivative in the exponent has been dropped using the periodicity of A. It is now trivial to perform the functional integration. We merely have four integrals, one for each ν, each of which is exactly the same as the scalar-field integral done in Sec. II. (The fact that for $\nu=0$ the integrand has the wrong sign is no problem; we merely analytically continue by defining $A_0' = iA_0$. This adds an irrelevant constant to $\ln Z$.) The $\det(-\Box^2)$ can be written

$$\det(-\Box^2) = \exp[\mathrm{Tr}\ln(-\Box^2)]$$

$$= \exp \sum_n \int \frac{d^3k}{(2\pi)^3} \ln(\omega_n{}^2 + \vec{k}^2) , \quad (3.11)$$

where we have used the fact that the determinant is *defined* on the space of periodic functions. Thus, we have from (2.19) and (2.20)

$$\ln Z = 2 \ln N'(\beta) - \sum_n \int \frac{d^3k}{(2\pi)^3} \ln(\omega_n{}^2 + \vec{k}^2), \quad (3.12)$$

where ln det$(-\Box^2)$ has canceled two of the four functional integrals. Using the results of Sec. II, we see that (3.12) can be written

$$\ln Z = 2 \int \frac{d^3k}{(2\pi)^3} \ln\left[-\frac{\beta\omega_k}{2} - \ln(1 - e^{-\beta\omega_k})\right], \quad (3.13)$$

with $\omega_k = [(\vec{k})^2]^{1/2}$. This is the partition function for a zero-mass Bose gas with two polarization states: the correct answer. Comparing (3.13) with (3.2) shows that the functional integral in the Feynman gauge is not equal to $(\mathrm{Tr}\, e^{-\beta H})_{\text{Feynman gauge}}$, which was wrong. Note that the presence of the determinant from the Faddeev-Popov ansatz was crucial; without it, we would have gotten the wrong answer, and not even the same wrong answer as (3.2).

In more complicated cases (such as non-Abelian gauge theories) we would write the determinant as an integral over ghost fields, which are unphysical spin-zero particles which have a fermion-like minus for loops. It is clear from the example, however, that since the ghosts are just a shorthand way of writing a determinant defined in the space of periodic functions, their Feynman rules will involve a sum over even multiples of π/β, as for bosons, and not odd multiples, as for fermions.

Note that the whole Faddeev-Popov procedure also goes through if we calculate finite-temperature Green's functions such as those of (2.18) as long as the operators in the Green's functions are gauge-invariant combinations of fields. In other words, if we define

$$\langle T[B(\vec{x}_1, \tau_1)B(\vec{x}_2, \tau_2) \cdots]\rangle \equiv \left. \frac{\mathrm{Tr}\{e^{-\beta H} T[B(\vec{x}_1, \tau_1)B(\vec{x}_2, \tau_2) \cdots]\}}{\mathrm{Tr}\, e^{-\beta H}} \right|_{\text{physical gauge}}, \quad (3.14)$$

where B's are gauge-invariant operators, then these Green's functions will have the same value when calculated in any gauge by the Feynman rules given here. Of course, even if the B's are not gauge-invariant operators, (3.14) is "gauge-invariant" since it is defined in a particular gauge. But then we will get wrong results if we calculate (3.14) using the Feynman rules of a different gauge.

IV. CONCLUSIONS

We have seen that $\mathrm{Tr}\, e^{-\beta H}$ is a gauge-dependent quantity which is only physically meaningful in a gauge where there are no unphysical particles. The true partition function Z is then defined to be $\mathrm{Tr}\, e^{-\beta H}$, where H is evaluated in one of these physical gauges. Using functional-integral methods and the Faddeev-Popov ansatz, the Feynman rules for Z can be derived in any convenient gauge—when we use those rules we calculate Z and not, in general, $\mathrm{Tr}\, e^{-\beta H}$ in that gauge. The Feynman rules for a given gauge (for example, any of the R_ξ gauges of a spontaneously broken theory) are just the zero-temperature rules for that gauge, with the substitutions given by (2.17):

$$k_0 \to i\omega_n,$$

$$\int \frac{d^4k}{(2\pi)^4} \to i\frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3},$$

$$(2\pi)^4\delta^4(k_1 + k_2 + \cdots) \to \frac{1}{i}(2\pi)^3\delta^3(\vec{k}_1 + \vec{k}_2 + \cdots) \times \beta\delta_{\omega_{n_1}+\omega_{n_2}+\cdots},$$

where $\omega_n = 2\pi n/\beta$ for bosons and Faddeev–Popov ghosts, and $(2n+1)\pi/\beta$ for fermions.

APPENDIX: DETERMINATION OF THE CONSTANT $N'(\beta)$

A "careful evaluation" of (2.4) involves dividing up the $d\tau$ integration into a Riemann sum of n fields evaluated at τ values separated by ϵ, with $n\epsilon = \beta$, and taking the limit $n \to \infty$, $\epsilon \to 0$. For a scalar theory defined by (2.7) with $\lambda = 0$, we have

$$\mathcal{H}(\pi, \varphi) = \tfrac{1}{2}[\pi^2 + (\vec{\nabla}\varphi)^2 + m^2\varphi^2]. \quad (A1)$$

Thus (2.4) becomes

$$\mathrm{Tr}\, e^{-\beta H} = \lim_{n\to\infty} N \int_{-\infty}^{+\infty} \prod_{i=1}^{n} d\pi_i(\vec{x}) \int_{-\infty\ \text{periodic}}^{+\infty} d\pi_i(\vec{x}) \exp\left(\sum_j \int d^3x \{ i\pi_j(\varphi_j - \varphi_{j-1}) - \tfrac{1}{2}\epsilon[\pi_j{}^2 + (\nabla\varphi_j)^2 + m^2\varphi_j{}^2]\}\right), \quad (A2)$$

where we integrate over n π and φ fields instead of a continuum of them, and where we have replaced $\dot{\varphi}_j$ by $(\varphi_j - \varphi_{j-1})/\epsilon$. The periodicity condition just means identifying $\varphi_0 = \varphi_n$. Each of the $\pi_i(\vec{x})$ integrations is now a trivial Gaussian, and can be done immediately. (We do not have to divide up the space integrations and do them carefully, since the infinite constant we would get would not be β-dependent.) The result is

$$\operatorname{Tr} e^{-\beta H} = \lim_{n\to\infty} \int_{-\infty}^{+\infty} \prod_{i=1}^{n} \frac{d\varphi_i(\vec{x})}{\sqrt{\epsilon}} \exp\left\{-\tfrac{1}{2}\sum_j \int d^3x \left[\frac{(\varphi_j - \varphi_{j-1})^2}{\epsilon} + \epsilon(\vec{\nabla}\varphi_j)^2 + \epsilon m^2 \varphi_j^2\right]\right\}, \tag{A3}$$

where $\varphi_0 = \varphi_n$ and we have ignored all β-independent constants. Already we see the appearance of a β-dependent constant in the product of the $1/\sqrt{\epsilon} = 1/(2\pi\beta/n)^{1/2}$. However, to get $N'(\beta)$ in a more useful form we must proceed in the evaluation of (A3). Upon Fourier transforming the $\vec{x}$ variable to $\vec{k}$, (A3) becomes

$$\operatorname{Tr} e^{-\beta H} = \lim_{n\to\infty} \int_{-\infty}^{+\infty} \prod_{i=1}^{n} \frac{d\varphi_i(\vec{k})}{\sqrt{\epsilon}} \exp\left\{-\tfrac{1}{2}\sum_j \int \frac{d^3k}{(2\pi)^3}\left[\left(\frac{\varphi_j(\vec{k}) - \varphi_{j-1}(\vec{k})}{\epsilon}\right)^2 + \epsilon\omega_k^2[\varphi_j(\vec{k})]^2\right]\right\}, \tag{A4}$$

where $\omega_k^2 = \vec{k}^2 + m^2$. If, for the moment, we keep $\varphi_0 \neq \varphi_n$ and do not perform the $\int d\varphi_n$ integration, (A4) simply becomes the expression for the infinite product (one for each $\vec{k}$) of independent, frequency-ω_k, harmonic-oscillator density matrices, $\rho(\varphi_n(\vec{k}), \varphi_0(\vec{k}))$. The density matrix, ρ, is defined by Feynman, who obtained it by performing functional integration with exactly the normalization of (A4)[15]:

$$\rho(\varphi_n(\vec{k}), \varphi_0(\vec{k})) = \left(\frac{\omega_k}{\sinh\beta\omega_k}\right)^{1/2} \exp\left\{\frac{-\omega_k}{\sinh\beta\omega_k}[(\varphi_n^2 + \varphi_0^2)\cosh(\beta\omega_k) - 2\varphi_n\varphi_0]\right\}, \tag{A5}$$

where we have again ignored all constant factors. Thus (A4) becomes

$$\operatorname{Tr} e^{-\beta H} = \prod_{\vec{k}} \int d\varphi_n(\vec{k}) \left(\frac{\omega_k}{\sinh\beta\omega_k}\right)^{1/2} \exp\left[\frac{-2\omega_k\varphi_n^2}{\sinh\beta\omega_k}(\cosh\beta\omega_k - 1)\right], \tag{A6}$$

where we have now set $\varphi_0 = \varphi_n$. The φ_n integration is trivial, and we now take $\ln \operatorname{Tr} e^{-\beta H}$ for convenience. The result is

$$\ln \operatorname{Tr} e^{-\beta H} = \int \frac{d^3k}{(2\pi)^3} \ln\left(\operatorname{csch}\frac{\beta\omega_k}{2}\right) + \beta\text{-independent constant}. \tag{A7}$$

But this is just the answer obtained in (2.23). Thus we see that the identification of $N'(\beta)$ by (2.22) is the correct one to use in that method of evaluation of the functional integral.

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[13]For an explanation of the notation, see S. Coleman, Ref. 9, section 5.

[14]We have normalized the field variable $[dA]$ so that $\int[dA]\exp[-\frac{1}{2}(A, DA)] = (\beta\text{-independent constant}) \times (\det D)^{1/2}$. With the same normalization for $[df]$, we have

$$\int [df]\exp\left[-\frac{1}{2\alpha}\int_0^\beta d\tau\int d^3x\, f^2(\vec{x},\tau)\right] = \beta\text{-independent constant.}$$

Note also that the δ function is normalized so that $\int[dA]\delta(A) = 1$, so we have

$$\int [df]\exp\left(-\frac{1}{2\alpha}\int_0^\beta d\tau\int d^3x\, f^2\right)\delta(\partial_\mu A^\mu - f) = \exp\left[-\frac{1}{2\alpha}\int_0^\beta d\tau\int d^3x(\partial_\mu A^\mu)^2\right].$$

The presence or absence of β-dependent normalization factors is the trickiest part of the whole business. It is therefore comforting to recall that the normalization factors are irrelevant as long as we calculate Green's functions like (2.18) or (3.12) and stay away from calculating $\mathrm{Tr}\, e^{-\beta H}$ itself.

[15]R. P. Feynman and A. P. Hibbs, Ref. 5, Chap. 10. Equation (A5) comes from Feynman's Eq. (10-44) after the correction of a typographical error.

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Symmetry behavior at finite temperature*

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Spontaneous symmetry breaking at finite temperature is studied. We show that for the class of theories discussed, symmetry is restored above a critical temperature β_c^{-1}. We determine β_c by a functional-diagrammatic evaluation of the effective potential and the effective mass. A formula for β_c is obtained in terms of the renormalized parameters of the theory. By examining a large subset of graphs, we show that the formula is accurate for weak coupling. An approximate gap equation is derived whose solutions describe the theory near the critical point. For gauge theories, special attention is given to ensure gauge invariance of physical quantities. When symmetry is violated dynamically, it is argued that no critical point exists.

I. INTRODUCTION

By drawing an analogy with the Meissner effect, Kirzhnits and Linde[1] have suggested that spontaneous symmetry violation in relativistic field theory will disappear above a critical temperature. They gave qualitative arguments to support this contention in a theory with global symmetry (*not* a gauge theory) and obtained an order-of-magnitude expression for the critical temperature in terms of the parameters of the theory. This problem was next examined by Weinberg, who, in a preliminary investigation,[2] derived a numerical value for the critical temperature in the Kirzhnits-Linde model. He then began to develop a complete analysis of spontaneous symmetry violation and/or persistence at finite temperature, with special emphasis on gauge theories with local symmetries.

It was Weinberg who suggested to us that the diagrammatic-functional methods for evaluating effective potentials in field theory, which had recently been developed,[3-5] might be profitably employed to study temperature effects. We report here the results of our investigation. Weinberg has also presented an analysis of the problem.[6] He uses diagrammatic methods to determine a temperature-dependent mass, as well as operator techniques to compute a temperature-dependent potential. We give a functional-diagrammatic evaluation of these quantities, from which the critical temperature can be deduced. All physical results are in agreement and confirm the qualitative observations of Kirzhnits and Linde.[1]

We examine a field theory at nonzero temperature, or equivalently the ensemble of finite-temperature Green's functions, defined by

$$G_\beta(x_1, \ldots, x_j) = \frac{\mathrm{Tr}\, e^{-\beta H}\, T\varphi(x_1)\cdots\varphi(x_j)}{\mathrm{Tr}\, e^{-\beta H}}. \quad (1.1)$$

Here H is the Hamiltonian governing the dynamics of the field $\varphi(x)$, and β^{-1} is proportional to the temperature. Spontaneous symmetry violation is conveniently studied with the help of the finite-temperature effective action $\Gamma^\beta(\bar{\varphi})$—the generating