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Ghost neutrinos in general relativity

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An exact solution to the Einstein-Dirac equations is presented for a static, plane-symmetric spacetime generated by neutrinos. We find the neutrino field to be nonzero and correspond to a neutrino current along the symmetry axis of the space. The neutrinos yield a zero energy-momentum tensor and therefore the gravitational field is exactly the same as for the vacuum case. A comparison with other solutions is presented along with a discussion of the possible physical significance of this "ghost neutrino" field.

I. INTRODUCTION

There have been many discussions of neutrinos in general relativity since the classic paper by Brill and Wheeler on the subject.[1] In this paper we present an exact solution to the Einstein-Dirac equations for the case of neutrinos. The field equations we wish to solve are

$$\begin{aligned} G_{jk} &= R_{jk} - \tfrac{1}{2} g_{jk} R \\ &= (8\pi\kappa/c^4) T_{jk} , \end{aligned} \tag{1.1}$$

where T_{jk} is the energy-momentum tensor for the Dirac field Ψ:

$$T_{jk} = -\tfrac{1}{4}\hbar c(\Psi^\dagger \gamma_j \Psi_{;k} - \Psi^\dagger_{;k}\gamma_j \Psi + \Psi^\dagger \gamma_k \Psi_{;j} - \Psi^\dagger_{;j}\gamma_k \Psi) . \tag{1.2}$$

Ψ satisfies the zero-mass Dirac equation

$$\gamma^j \Psi_{;j} = 0 . \tag{1.3}$$

Here the semicolon stands for covariant differentiation. Since for zero-mass Dirac particles the trace of the energy-momentum tensor vanishes, it follows from Eqs. (1.1) that the scalar curvature R also vanishes. Hence, for zero-mass Dirac particles the Einstein equations reduce (just as for the electrovac case) to

$$R_{jk} = (8\pi\kappa/c^4) T_{jk} . \tag{1.4}$$

In this paper we shall study solutions of Eqs. (1.3) and (1.4) in spacetimes of plane symmetry where the metric is defined by[2]

$$ds^2 = e^{2v}(dy^2 + dz^2) + e^{2u}(dx^2 - dt^2) , \tag{1.5}$$

where u and v are functions of (x, t). As is clear from Eq. (1.5), the x axis is the symmetry axis. In this paper we shall restrict our discussion to the static case. Hence, all functions u, v, Ψ, etc. depend on only the one space coordinate x. We shall carry out the calculations in the Cartan orthonormal frame defined by

$$\omega^1 = e^u dx, \quad \omega^2 = e^v dy , \quad \omega^3 = e^v dz , \quad \omega^4 = e^u dt . \tag{1.6}$$

The formulation of the Dirac equation in Cartan frames has been discussed by Brill and Cohen.[3] The essential point of this approach is to identify the Bargmann *vierbein* frame discussed by Brill and Wheeler in Ref. 1 with the Cartan orthonormal frame used to describe the geometry. The tangent vectors ω_i dual to the one forms ω^i in Eqs. (1.6) are given by

$$\begin{aligned} &\omega_1 = e^{-u} e_1, \quad \omega_2 = e^{-v} e_2 , \\ &\omega_3 = e^{-v} e_3 , \quad \omega_4 = e^{-u} e_4 , \end{aligned} \tag{1.7}$$

where

$$e_i = \partial/\partial x^i .$$

In the orthornormal frame the covariant derivatives of the Dirac spinor are given by

$$\Psi_{;j} = \omega_j[\Psi] - \Gamma_j \Psi , \tag{1.8}$$

where Γ_i is the spinor connection defined by

$$\Gamma_j = -\tfrac{1}{4}\Gamma^k_{ij}\gamma_k\gamma^i \tag{1.9}$$

and Γ^k_{ij} are the Ricci rotation coefficients. Since the notation is explained in detail in Refs. 1 and 3 we shall not repeat all of the results presented in these papers. In general our notation follows that of Misner, Thorne, and Wheeler.[4] Our convention on the Dirac γ matrices is the one given by Jauch and Rohrlich,[5] which is also that used in Refs. 1 and 3.

In this convention the neutrino field Ψ_ν is obtained from the zero-mass Dirac field Ψ by the projections

$$(1 - i\gamma_5)\Psi = 0\,, \tag{1.10}$$

$$\Psi_\nu = \tfrac{1}{2}(1 + i\gamma_5)\Psi\,, \tag{1.11}$$

which yield

$$\Psi_\nu = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ i\Psi_1 \\ i\Psi_2 \end{pmatrix}. \tag{1.12}$$

One could, of course, discuss neutrinos using two-component spinors but this offers very little advantage in the material discussed in this paper.

Besides the energy-momentum tensor, another observable associated with the Dirac field is the conserved current, which is given in our notation by

$$S^j = i\Psi^\dagger\gamma^j\Psi\,. \tag{1.13}$$

Since the x axis is a symmetry axis one would expect only S^1 of the spatial components of S^j to be nonzero.

As the first part of our problem we must solve the Dirac equation for an arbitrary static metric with plane symmetry. This solution is for a test neutrino field in the given geometry. Then we use this solution in the energy-momentum tensor to obtain the energy-momentum tensor generated by neutrinos consistent with the geometry. Finally, we solve the Einstein equations using this energy-momentum tensor to obtain the solution to the coupled Einstein-Dirac equations in which the neutrinos are the source of the gravitational field.

II. SOLUTION OF THE DIRAC EQUATION

The nonzero-spin coefficients Γ_i are given by

$$\Gamma_2 = \tfrac{1}{2}e^{-u}v_{,1}\gamma^1\gamma^2\,, \tag{2.1}$$

$$\Gamma_3 = \tfrac{1}{2}e^{-u}v_{,1}\gamma^1\gamma^3\,, \tag{2.2}$$

$$\Gamma_4 = \tfrac{1}{2}e^{-u}u_{,1}\gamma^4\gamma^1\,, \tag{2.3}$$

where the comma denotes ordinary differentiation. Using the spin coefficients the Dirac equation, Eq. (1.3), can be reduced to the very simple form

$$\Psi_{,1} + (v + \tfrac{1}{2}u)_{,1}\Psi = 0\,, \tag{2.4}$$

with the solution

$$\Psi = e^{-(v+u/2)}\Psi_0\,, \tag{2.5}$$

where Ψ_0 is an arbitrary constant spinor. This gives the solution for a test neutrino field in the given geometry for any $v(x)$, $u(x)$.

III. THE ENERGY-MOMENTUM TENSOR

The solution (2.5) yields the following nonzero components of the energy-momentum tensor

$$T_{24} = -\tfrac{1}{4}\hbar c e^{-u}(v_{,1} - u_{,1})\Psi^\dagger\gamma^1\gamma^2\gamma^4\Psi\,, \tag{3.1}$$

$$T_{34} = -\tfrac{1}{4}\hbar c e^{-u}(v_{,1} - u_{,1})\Psi^\dagger\gamma^1\gamma^3\gamma^4\Psi\,. \tag{3.2}$$

All other components of the energy-momentum tensor vanish identically or via the Dirac equation.

IV. THE EINSTEIN EQUATIONS

For the static metric with plane symmetry the only nonzero components of the Ricci tensor R_{ij} are

$$R_{11} = -e^{-2u}(2v_{,11} + u_{,11} + 2v_{,1}{}^2 - 2v_{,1}u_{,1})\,, \tag{4.1}$$

$$R_{22} = -e^{-2u}(v_{,11} + 2v_{,1}{}^2)\,, \tag{4.2}$$

$$R_{33} = -e^{-2u}(v_{,11} + 2v_{,1}{}^2)\,, \tag{4.3}$$

$$R_{44} = e^{-2u}(u_{,11} + 2v_{,1}u_{,1})\,. \tag{4.4}$$

Therefore only the diagonal components of the Ricci tensor are nonzero. Since the diagonal components of the energy-momentum tensor all vanish, the Einstein equations yield

$$R_{ii} = 0 \quad (\text{no sum on } i)\,. \tag{4.5}$$

Equations (4.5) are exactly the same as the vacuum field equations for this geometry, which were solved by Taub[2] to obtain

$$u = \ln(kx + 1)^{-1/4}\,, \quad v = \ln(kx + 1)^{1/2}\,, \tag{4.6}$$

where k is an arbitrary constant. Also from the Einstein field equations we see that since $R_{24} = R_{34} = 0$, we must have $T_{24} = T_{34} = 0$. Referring to T_{24} and T_{34} in Eqs. (3.1) and (3.2) and using Eqs. (4.6) we find that $(v_{,1} - u_{,1})$ does not vanish. Hence, we are left with the conditions

$$T_{24} = 0 \rightarrow \Psi^\dagger\gamma^1\gamma^2\gamma^4\Psi = 0\,, \tag{4.7}$$

$$T_{34} = 0 \rightarrow \Psi^\dagger\gamma^1\gamma^3\gamma^4\Psi = 0\,, \tag{4.8}$$

which yield the following two equations:

$$|\Psi_1|^2 - |\Psi_2|^2 + |\Psi_3|^2 - |\Psi_4|^2 = 0 \tag{4.7$'$}$$

and

$$\Psi_1^*\Psi_2 - \Psi_2^*\Psi_1 + \Psi_3^*\Psi_4 - \Psi_4^*\Psi_3 = 0\,, \tag{4.8$'$}$$

which reduce for neutrinos, Eq. (1.12), to the conditions

$$|\Psi_1|^2 - |\Psi_2|^2 = 0\,, \tag{4.7$''$}$$

$$\Psi_1^*\Psi_2 - \Psi_2^*\Psi_1 = 0\,. \tag{4.8$''$}$$

These equations yield the following two solutions:

$$\Psi_\nu = \begin{pmatrix} 1 \\ \pm 1 \\ i \\ \pm i \end{pmatrix} \Psi_1\,, \tag{4.9}$$

where Ψ_1 is a scalar; and since we have previously found the solution for Ψ in Eq. (2.5), we have the neutrino solution

$$\Psi_\nu = e^{-(v+u/2)}\Psi_{\nu 0}\,, \tag{4.10}$$

where

$$\Psi_{\nu 0} = a \begin{pmatrix} 1 \\ \pm 1 \\ i \\ \pm i \end{pmatrix}, \tag{4.11}$$

and a is an arbitrary constant. Using Taub's solutions for u and v from Eqs. (4.6) we find that

$$\Psi_\nu = (kx+1)^{-3/8}\Psi_{\nu 0}\,. \tag{4.12}$$

Calculation of the neutrino current density S^j then yields

$$S^j = 4|a|^2(kx+1)^{-3/4}(\mp\delta_1^j + \delta_4^j)\,. \tag{4.13}$$

We note that this current density corresponds to a flow of neutrinos along the symmetry axis as we had previously predicted.

V. SUMMARY AND CONCLUSIONS

Equations (4.6) and (4.12) represent an exact solution to the Einstein-Dirac equations for a static, plane-symmetric spacetime. The most interesting property of the solution is that the neutrino energy-momentum tensor vanishes, whereas the neutrino field and current density do not vanish. Since the neutrino energy-momentum tensor vanishes the gravitational field is exactly the same as for the vacuum field equations. Hence, there is no way to determine that the neutrinos are present by measurements on the gravitational field. For this reason we refer to these neutrinos as "ghost neutrinos".

Our solution should be compared to the solution of the Einstein-Dirac equations in the Taub universe presented in Ref. 3. There, the Ricci tensor is also diagonal but the only way the energy-momentum tensor can be diagonal is for Ψ itself to vanish. Hence, the neutrino field generates a gravitational field which is not consistent with the geometry of the Taub universe. For our solution we find the neutrino field is consistent with the geometry of the gravitational field but it does not generate it, because it yields a zero energy-momentum tensor.

Our solution is analogous to the constant spinor fields allowed by the zero-mass Dirac equation in flat spacetime. These constant spinor solutions also yield a zero energy-momentum tensor in the flat spacetime. The question of whether "ghost neutrino" fields in general relativity have physical significance is, of course, the most important question to answer. Although these neutrinos yield a zero energy-momentum tensor they do yield a nonzero current density which is in principle observable. Hence, it is conceivable that these neutrinos could be observed, although it is difficult to imagine how they could be observed.

The fact that the Einstein equations allow "ghost neutrinos" opens the possibility that these equations could also allow other "ghost solutions". That is, if we have a solution for a given energy-momentum tensor, we cannot be assured that there are not other fields ("ghost fields") present which give a zero energy-momentum tensor. Of course, if these "ghost fields" are not observable, then they would not be of any importance except as mathematical oddities. We have not found any references in the literature to the possibility of "ghost fields" in general relativity.

Note added in proof. Neutrino solutions yielding a zero energy-momentum tensor are mentioned in C. Collinson and P. Morris, Nuovo Cimento 16B, 273 (1973).

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